Asymptotic Expansions for Transmission Eigenvalues for Media with Small Inhomogeneities

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Abstract.

We consider the transmission eigenvalue problem for an inhomogeneous medium containing a finite number of diametrically small inhomogeneities of different refractive index. We prove a convergence result for the transmission eigenvalues and eigenvectors corresponding to media with small homogeneities as the diameter of small inhomogeneities goes to zero. In addition we derive rigorously a formula for the perturbations in the real transmission eigenvalues caused by the presence of these small inhomogeneities.

Keywords: Transmission eigenvalues, small inhomogeneities.

1. Introduction

Transmission eigenvalues appear in the study of scattering by inhomogeneous media and are closely related to non-scattering incident waves [9], [5]. Such eigenvalues provide information about material properties of the scattering media [8] and can be determined from scattering data [6]. Hence they can play an important role in a variety of inverse problems in target identification and non-destructive testing [7], [14]. The transmission eigenvalue problem is a non-selfadjoint and nonlinear problem that is not covered by the standard theory of eigenvalue problems for elliptic operators. In the past few years transmission eigenvalues have become an important area of research in inverse scattering theory. Since the first proof of existence of transmission eigenvalues in [8] and [20], the interest in the transmission eigenvalue problem has increased, resulting in a number of important advancements in this area [12], [13]. For an update survey on the topic we refer the reader to [9].

In this paper we consider the transmission eigenvalues corresponding to inhomogeneous media containing a finite number of small volume inhomogeneities with different refractive index. Our goal is to understand how the presence of these small inhomogeneities affect the transmission eigenvalues in the asymptotic regime, i.e. as the volume of the small inhomogeneities goes to zero. These types of questions are addressed in detail for the conductivity problem by many authors (see [2], [3], [10] to name a few). Our analysis essentially follows the approach in [2] and [10]. However, as opposed to the conductivity problem, here we have to deal with a quadratic eigenvalue problem [11] for a fourth order partial differential equation of inhomogeneous biharmonic type [8]. This causes additional technical difficulties. The fundamental result which yields our asymptotic formulas for the transmission eigenvalues is a theorem in [18]. This theorem allows for non-selfadjoint operators and enables us to derive an explicit first order correction expression. However, since the non-selfadjoint compact operator that arises in connection with the transmission eigenvalue problem is a matrix valued operator with complicated terms, a major part of our analysis deals with deriving asymptotic formulae for each of the operators involved. One of the key results of the paper is the proof that the operators whose eigenvalues coincide with transmission eigenvalues for the media with small inhomogeneities converge in the norm to the those from the corresponding media without small inhomogeneities. Furthermore, we obtain an asymptotic formula for the perturbation of real transmission eigenvalues with an explicit correction term involving the location, refractive index and the size of the small inhomogeneities; and the transmission eigenfunctions for the unperturbed media. Interestingly this expression involves a scalar term which acts in the place of what was a polarization tensor in the case of small volume conductivity inhomogeneities. Our asymptotic formula could potentially be used to obtain information about the location, refractive index or the size of the small inhomogeneities from the measured real transmission eigenvalues for the perturbed medium and the computed transmission eigenvalues for the unperturbed media.

2. Formulation of the Problem

Let $D \subset \mathbb{R}^d$, $d \geq 2$ be a bounded connected region with smooth boundary ∂D and let ν denote the unit normal vector oriented outward to D. We consider a real valued function $n(x)$ defined in D, such that $n \in C^2(D)$ and $n_0(x) \geq n_0 > 0$. Furthermore we assume that inside D there are m small subregions $\epsilon B_i \subset D$, $i = 1 \dots m$ where each $B_i \subset \mathbb{R}^d$ is a bounded connected reference domain which is a smooth deformation of a ball centered at $z_i \in D$, and $\epsilon > 0$ is sufficiently small so that these domains are well separated from each other and the boundary. We denote $W_\epsilon := \bigcup_{i=1}^m \epsilon B_i$. In each small subregion we consider a real valued function n_i, $i = 1 \dots m$ where again $n_i \in C^2(\epsilon B_i)$ is such that $n_i(x) \geq n_i > 0$ and $n_i(x) \neq n_0(x)$ for $x \in \epsilon B_i$. Let us denote by

$$n_\epsilon(x) := \begin{cases} n_0(x) & x \in D \setminus W_\epsilon \\ n_i(x) & x \in \epsilon B_i,\ i = 1 \dots m \end{cases} \tag{1}$$

We now consider the scattering of time harmonic (acoustic in $\mathbb{R}^3$ or electromagnetic in $\mathbb{R}^2$) waves by the inhomogeneity D with refractive index n_ϵ embedded in a homogeneous background normalized to 1. After factoring out time dependent factor $e^{-i\omega t}$ and denoting by k the corresponding wave number, we obtain that the scattered field u^s

due to the incident field u^i satisfies

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in } \mathbb{R}^d \setminus \overline{D} \tag{2}$$

$$\Delta u + k^2 n_\epsilon(x) u = 0 \qquad \text{in } D \tag{3}$$

$$u^s + u^i = u \qquad \text{on } \partial D \tag{4}$$

$$\frac{\partial (u^s + u^i)}{\partial \nu} = \frac{\partial u}{\partial \nu} \qquad \text{on } \partial D \tag{5}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0. \tag{6}$$

The transmission eigenvalues associated with the inhomogeneity D, n_ϵ are the values of k for which the homogeneous interior transmission problem

$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{7}$$

$$\Delta w + k^2 n_\epsilon(x) w = 0 \qquad \text{in } D \tag{8}$$

$$w = v \qquad \text{on } \partial D \tag{9}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D \tag{10}$$

has nontrivial solution $v \in L^2(D)$, $w \in L^2(D)$ such that $w - v \in H^2(D)$. It is well-known that an infinite set of *real* transmission eigenvalues exist provided that either $n_\epsilon(x) - 1 \geq \alpha > 0$ or $0 < \beta \leq 1 - n_\epsilon(x) < 1$ [8], [9]. Since the transmission eigenvalue problem is non-selfadjoint, complex eigenvalues may exists, but their existence up to date has been proven only for the spherically stratified media [16]. To fix our ideas, in this paper we consider in detail only the case when $n_\epsilon(x) - 1 \geq \alpha > 0$. A similar analysis holds true for the case when $0 < \beta \leq 1 - n_\epsilon(x) < 1$. The main goal of the paper is to understand the asymptotic behavior of real transmission eigenvalues with respect to the small parameter $\epsilon > 0$. To this end we write the transmission eigenvalue problem in the operator form. Under the assumption on n_ϵ, it is now possible to write (7)-(10) as an equivalent eigenvalue problem for $u = w - v \in H_0^2(D)$ as solution of the fourth order homogeneous equation [8]

$$\left(\Delta + k^2 n_\epsilon\right) \frac{1}{n_\epsilon - 1} \left(\Delta + k^2\right) u = 0 \tag{11}$$

which in variational form, after integration by parts, is formulated as finding a nonzero function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n_\epsilon - 1} (\Delta u + k^2 u)(\Delta \overline{v} + k^2 n_\epsilon \overline{v})\, dx = 0 \qquad \text{for all } v \in H_0^2(D) \tag{12}$$

where

$$H_0^2(D) := \left\{ u \in H^2(D) \quad \text{such that } u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \right\}.$$

The functions v and w are related to u through

$$v = -\frac{1}{k^2(n_\epsilon - 1)}(\Delta u + k^2 u) \qquad \text{and} \qquad w = -\frac{1}{k^2(n_\epsilon - 1)}(\Delta u + k^2 n_\epsilon u).$$

To understand the structure of the interior transmission eigenvalue problem we first observe that, setting $k^2 := \tau$, (12) can be written as

$$\mathbb{A}_\epsilon u + \tau \mathbb{B}_\epsilon u + \tau^2 \mathbb{C}_\epsilon u = 0, \tag{13}$$

where $\mathbb{A}_\epsilon : H_0^2(D) \to H_0^2(D)$ is the bounded, self-adjoint and positive definite operator defined by means of the Riesz representation theorem

$$(\mathbb{A}_\epsilon u, v)_{H^2(D)} = \int_D \frac{1}{n_\epsilon - 1} \Delta u \, \Delta \overline{v} \, \mathrm{d}x \qquad \text{for all } u, v \in H_0^2(D), \tag{14}$$

(note that the $H^2(D)$ norm of a function in $H_0^2(D)$ is equivalent to the $L^2(D)$ norm of its Laplacian), $\mathbb{B}_\epsilon : H_0^2(D) \to H_0^2(D)$ is the bounded compact self-adjoint operator defined by means of the Riesz representation theorem

$$\begin{aligned} (\mathbb{B}_\epsilon u, v)_{H^2(D)} &= \int_D \frac{1}{n_\epsilon - 1} \left(\Delta u \, \overline{v} + n_\epsilon u \, \Delta \overline{v} \right) dx \\ &= \int_D \frac{1}{n_\epsilon - 1} \left(\Delta u \, \overline{v} + u \ \Delta \overline{v} \right) dx + \int_D u \Delta \overline{v} \, dx \\ &= -\int_D \left[\nabla u \cdot \nabla \left(\frac{1}{n_\epsilon - 1} \overline{v} \right) + \nabla \left(\frac{1}{n_\epsilon - 1} u \right) \cdot \nabla \overline{v} \right] dx + \int_D \nabla u \cdot \nabla \overline{v} \, dx \end{aligned} \tag{15}$$

for $u, v \in H_0^2(D)$, and $\mathbb{C}_\epsilon : H_0^2(D) \to H_0^2(D)$ is the bounded compact non-negative self-adjoint operator defined by means of the Riesz representation theorem

$$(\mathbb{C}_\epsilon u, v)_{H^2(D)} = \int_D \frac{n}{n-1} u \, \overline{v} \, dx \qquad \text{for all } u, v \in H_0^2(D). \tag{16}$$

Compactness of $\mathbb{B}_\epsilon$ and $\mathbb{C}_\epsilon$ is a consequence of the compact embedding of $H_0^2(D)$ and $H_0^1(D)$ into $L^2(D)$. Setting $U := \left(u, \tau \, \mathbb{C}_\epsilon^{1/2} u \right)$, the interior transmission eigenvalue problem becomes the eigenvalue problem

$$(\mathbf{K}_\epsilon - \lambda \mathbf{I}) \, U = 0, \qquad U \in H_0^2(D) \times H_0^2(D), \quad \lambda := \frac{1}{\tau}$$

for the compact non-selfadjoint operator $\mathbf{K}_\epsilon : H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$ given by

$$\mathbf{K}_\epsilon := \begin{pmatrix} -\mathbb{A}_\epsilon^{-1} \mathbb{B}_\epsilon & -\mathbb{A}_\epsilon^{-1} \mathbb{C}_\epsilon^{1/2} \\ \mathbb{C}_\epsilon^{1/2} & 0 \end{pmatrix}. \tag{17}$$

(Note that if a generic operator T is a bounded, positive, compact and self-adjoint on a Hilbert space U, the operator $T^{1/2}$ is defined by $T^{1/2} = \int_0^\infty \lambda^{1/2} dE_\lambda$ where dE_λ is the spectral measure associated with T. It is easy to show that $T^{1/2}$ is also compact and self-adjoint). We remark that in our analysis we also consider the medium D without the inhomogeneities, i.e. with refractive index $n_0(x)$. The corresponding operators will be denoted by $\mathbb{A}_0$, $\mathbb{B}_0$, $\mathbb{C}_0$ and $\mathbf{K}_0$.

3. Convergence of the spectrum

The goal of this section is to show that the spectrum of the operator $\mathbf{K}_\epsilon$ (transmission eigenvalues and eigenvectors corresponding to the medium with small inhomogeneities) converges to the spectrum of the $\mathbf{K}_0$ (the transmission eigenvalues and eigenvectors corresponding to the reference medium without inhomogeneities) as ϵ goes to zero. To this end, we need to study the convergence properties of each of the operators $\mathbb{A}_\epsilon^{-1}$, $\mathbb{B}_\epsilon$, $\mathbb{A}_\epsilon^{-1}\mathbb{C}_\epsilon^{1/2}$ that appear in the definition of $\mathbf{K}_\epsilon$. In the following we work in the Hilbert space $H_0^2(D)$, equipped with the inner product

$$(u,v)_{H_0^2} = \int_D \Delta u \Delta \overline{v} dx.$$

We start by first studying the operators

$$\mathbb{B}_0, \mathbb{B}_\epsilon : H_0^2(D) \to H_0^2(D)$$

which we recall are given by

$$(\mathbb{B}_0 u, \phi)_{H_0^2} = \int_D \left(\frac{1}{n_0 - 1} \Delta u \cdot \overline{\phi} + \frac{n_0}{n_0 - 1} u \Delta \overline{\phi} \right) dx \tag{18}$$

and

$$(\mathbb{B}_\epsilon u, \phi)_{H_0^2} = \int_D \left(\frac{1}{n_\epsilon - 1} \Delta u \cdot \overline{\phi} + \frac{n_\epsilon}{n_\epsilon - 1} u \Delta \overline{\phi} \right) dx \tag{19}$$

for all $\phi \in H_0^2(D)$. Note that $\mathbb{B}_0$ and $\mathbb{B}_\epsilon$ are compact, self-adjoint operators. In what follows we may leave off the subscript H_0^2 in the inner product.

Lemma 3.1 *Let $\mathbb{B}_0$ and $\mathbb{B}_\epsilon$ be defined by (18) and (19) respectively. Then $\mathbb{B}_\epsilon \to \mathbb{B}_0$ in the operator norm, and for $d = 2, 3$ we have that*

$$\|\mathbb{B}_\epsilon - \mathbb{B}_0\| \leq C \epsilon^{d/2}$$

for some C independent of ϵ. Furthermore, if $u, \phi \in H_0^2(D) \bigcap C^2(D)$ then

$$\begin{aligned}
&((\mathbb{B}_\epsilon - \mathbb{B}_0)u, \phi) \\
&\quad = \sum_{j=1}^N \epsilon^d |B_j| \left[\left(\frac{1}{n_j(z_j) - 1} - \frac{1}{n_0(z_j) - 1} \right) \Delta u(z_j) \overline{\phi(z_j)} \right. \\
&\qquad \left. + \left(\frac{n_j(z_j)}{n_j(z_j) - 1} - \frac{n_0(z_j)}{n_0(z_j) - 1} \right) u(z_j) \Delta \overline{\phi(z_j)} \right] + o(\epsilon^d)
\end{aligned}$$

Proof. By combining terms we see that

$$((\mathbb{B}_\epsilon - \mathbb{B}_0)u, \phi) = \int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1} \right) \Delta u \, \overline{\phi} \, dx + \int_D \left(\frac{n_\epsilon}{n_\epsilon - 1} - \frac{n_0}{n_0 - 1} \right) u \, \Delta \overline{\phi} \, dx. \tag{20}$$

For the norm estimate we observe that

$$\begin{aligned}
((\mathbb{B}_\epsilon - \mathbb{B}_0)u, \phi) &\leq \|\Delta u\|_{L^2(D)} \left\| \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1} \right) \overline{\phi} \right\|_{L^2(D)} \\
&\quad + \left\| \left(\frac{n_\epsilon}{n_\epsilon - 1} - \frac{n_0}{n_0 - 1} \right) u \right\|_{L^2(D)} \|\Delta \overline{\phi}\|_{L^2(D)} \\
&= \|\Delta u\|_{L^2(D)} \left\| \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1} \right) \overline{\phi} \right\|_{L^2(W_\epsilon)} \\
&\quad + \left\| \left(\frac{n_\epsilon}{n_\epsilon - 1} - \frac{n_0}{n_0 - 1} \right) u \right\|_{L^2(W_\epsilon)} \|\Delta \overline{\phi}\|_{L^2(D)}
\end{aligned} \tag{21}$$

since the support of the difference of coefficients is only in W_ϵ. Now we use the Sobolev embedding theorem; for $d = 2, 3$,

$$\begin{aligned}
\left\| \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1} \right) \overline{\phi} \right\|_{L^2(W_\epsilon)} &\leq \max \left| \frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1} \right| \|\phi\|_{C^0(D)} \sqrt{|W_\epsilon|} \\
&\leq C \max \left| \frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1} \right| \|\phi\|_{H_0^2(D)} \epsilon^{d/2}.
\end{aligned}$$

Likewise

$$\| \left(\frac{n_\epsilon}{n_\epsilon - 1} - \frac{n_0}{n_0 - 1} \right) u\|_{L^2(W_\epsilon)} \leq C \max \left| \frac{n_\epsilon}{n_\epsilon - 1} - \frac{n_0}{n_0 - 1} \right| \|u\|_{H_0^2(D)} \epsilon^{d/2}$$

for some C independent of ϵ, u, or ϕ. Inserting these bounds into (21) we have

$$\begin{aligned}
&((\mathbb{B}_\epsilon - \mathbb{B}_0)u, \phi) \\
&\qquad \leq C \max \left| \frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1} \right| \|u\|_{H_0^2(D)} \|\phi\|_{H_0^2(D)} \epsilon^{d/2} \\
&\qquad\quad + C \max \left| \frac{n_\epsilon}{n_\epsilon - 1} - \frac{n_0}{n_0 - 1} \right| \|u\|_{H_0^2(D)} \|\phi\|_{H_0^2(D)} \epsilon^{d/2}
\end{aligned} \tag{22}$$

which yields the norm convergence result. (For larger dimension d, we have Sobolev embedding of $H_0^2(D)$ functions into L^p, which will also give strong norm convergence but with a lower power in ϵ). To show the asymptotic formula, we first assume for simplicity that we have one inhomogeneity centered at the origin, that is

$$W_\epsilon = \epsilon B$$

where B is a smooth region containing the origin. This gives that

$$((\mathbb{B}_\epsilon - \mathbb{B}_0)u, \phi) = \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1} \right) \Delta u \, \overline{\phi} \, dx + \int_{\epsilon B} \left(\frac{n_1}{n_1 - 1} - \frac{n_0}{n_0 - 1} \right) u \, \Delta \overline{\phi} \, dx.$$

We make the change of variables

$$y = \frac{x}{\epsilon}$$

to get

$$((\mathbb{B}_\epsilon - \mathbb{B}_0)u, \phi) = \epsilon^d \int_B \left(\frac{1}{n_1(\epsilon y) - 1} - \frac{1}{n_0(\epsilon y) - 1} \right) \Delta_x u(\epsilon y) \overline{\phi}(\epsilon y) dy$$
$$+ \epsilon^d \int_B \left(\frac{n_1(\epsilon y)}{n_1(\epsilon y) - 1} - \frac{n_0(\epsilon y)}{n_0(\epsilon y) - 1} \right) u(\epsilon y) \Delta \overline{\phi}(\epsilon y) dy.$$

So, clearly, if the integrand is continuous we have

$$((\mathbb{B}_\epsilon - \mathbb{B}_0)u, \phi) = \epsilon^d |B| \left(\frac{1}{n_1(0) - 1} - \frac{1}{n_0(0) - 1} \right) \Delta u(0) \overline{\phi}(0)$$
$$+ \epsilon^d |B| \left(\frac{n_1(0)}{n_1(0) - 1} - \frac{n_0(0)}{n_0(0) - 1} \right) u(0) \Delta \overline{\phi}(0) + o(\epsilon^d)$$

By using translation and summing a finite number of such inhomogeneities, the asymptotic result follows. □

Similarly, we look at

$$\mathbb{C}_0, \mathbb{C}_\epsilon : H^2_0(D) \to H^2_0(D)$$

which we recall are given by

$$(\mathbb{C}_0 u, \phi)_{H^2_0} = \int_D \frac{n_0}{n_0 - 1} u \, \overline{\phi} \, dx \tag{23}$$

and

$$(\mathbb{C}_\epsilon u, \phi)_{H^2_0} = \int_D \frac{n_\epsilon}{n_\epsilon - 1} u \, \overline{\phi} \, dx \tag{24}$$

for all $\phi \in H^2_0(D)$. Note that we also have that $\mathbb{C}_0$ and $\mathbb{C}_\epsilon$ are compact, self-adjoint operators.

Lemma 3.2 *Let $\mathbb{C}_0$ and $\mathbb{C}_\epsilon$ be defined by (23) and (24) respectively. Then $\mathbb{C}_\epsilon \to \mathbb{C}_0$ in the operator norm, and for $d = 2, 3$ we have that*

$$\|\mathbb{C}_\epsilon - \mathbb{C}_0\| \leq C\epsilon^d$$

for some C independent of ϵ. Furthermore, if $u, \phi \in H^2_0(D) \bigcap C^2(D)$ then

$$((\mathbb{C}_\epsilon - \mathbb{C}_0)u, \phi) = \sum_{j=1}^{N} \epsilon^d |B_j| \left(\frac{n_j(z_j)}{n_j(z_j) - 1} - \frac{n_0(z_j)}{n_0(z_j) - 1} \right) u(z_j) \overline{\phi(z_j)} + o(\epsilon^d)$$

Proof. We combine terms to obtain

$$((\mathbb{C}_\epsilon - \mathbb{C}_0)u, \phi) = \int_D \left(\frac{n_\epsilon}{n_\epsilon - 1} - \frac{n_0}{n_0 - 1} \right) u \, \overline{\phi} \, dx \tag{25}$$
$$\leq \| \frac{n_\epsilon}{n_\epsilon - 1} - \frac{n_0}{n_0 - 1} \|_{L^1(D)} \|u\|_{C^0(D)} \|\overline{\phi}\|_{C^0(D)}$$
$$\leq C\epsilon^d \|u\|_{H^2_0(D)} \|\overline{\phi}\|_{H^2_0(D)} \tag{26}$$

in dimensions $d = 2,3$ by Sobolev embedding and the small volume nature of the inhomogeneities. For higher dimensions, we use the embedding into L^p and the power of ϵ will be less. This shows the norm convergence result.

For the asymptotical formula, we proceed with a rescaling of (25) as in the previous proof to obtain, for one inhomogeneity centered at the origin,

$$((\mathbb{C}_\epsilon - \mathbb{C}_0)u, \phi) = \epsilon^d |B| \left(\frac{n_1(0)}{n_1(0)-1} - \frac{n_0(0)}{n_0(0)-1} \right) u(0)\overline{\phi}(0) + o(\epsilon^d)$$

from which we obtain the result by translation and summing over a finite number of inhomogeneities. □

Now we consider the operators $\mathbb{A}_0, \mathbb{A}_\epsilon : H_0^2(D) \to H_0^2(D)$, also defined via the Riesz Representation Theorem by

$$(\mathbb{A}_0 u, \phi)_{H_0^2} = \int_D \frac{1}{n_0 - 1} \Delta u \cdot \Delta\overline{\phi}\, dx \tag{27}$$

and

$$(\mathbb{A}_\epsilon u, \phi)_{H_0^2} = \int_D \frac{1}{n_\epsilon - 1} \Delta u \cdot \Delta\overline{\phi}\, dx \tag{28}$$

for all $\phi \in H_0^2(D)$. Note that $\mathbb{A}_0$ and $\mathbb{A}_\epsilon$ are invertible, bounded linear operators and are indeed not compact. We can show that $\mathbb{A}_\epsilon^{-1}$ converges point-wise in the operator sense, but not in norm, to A_0^{-1}. However, when preceded by compact operators as in our case, the convergence is in norm.

Lemma 3.3 *Let $\mathbb{A}_0$ and $\mathbb{A}_\epsilon$ be defined by (27) and (28) respectively, and let $\mathbb{B}_0, \mathbb{B}_\epsilon, \mathbb{C}_0, \mathbb{C}_\epsilon$ be defined by (18), (19), (23) and (24) respectively. Then both*

$$\mathbb{A}_\epsilon^{-1}\mathbb{B}_\epsilon \to \mathbb{A}_0^{-1}\mathbb{B}_0$$

and

$$\mathbb{A}_\epsilon^{-1}\mathbb{C}_\epsilon^{1/2} \to \mathbb{A}_0^{-1}\mathbb{C}_0^{1/2}$$

in the operator norm.

Proof. We first show a point-wise operator convergence result for $\mathbb{A}_\epsilon^{-1}$. Assume that f is smooth, and define

$$w_\epsilon = \mathbb{A}_\epsilon^{-1} f \qquad \text{and} \qquad w_0 = \mathbb{A}_0^{-1} f. \tag{29}$$

Consider

$$\begin{aligned}\int_D \frac{1}{n_\epsilon - 1}\Delta(w_\epsilon - w_0)\Delta\overline{v}\, dx &= \int_D \frac{1}{n_\epsilon - 1}\Delta w_\epsilon \Delta\overline{v}\, dx - \int_D \frac{1}{n_0 - 1}\Delta w_0 \Delta\overline{v}\, dx \\ &\quad + \int_D \left(\frac{1}{n_0 - 1} - \frac{1}{n_\epsilon - 1} \right) \Delta w_0 \Delta\overline{v}\, dx \\ &= \int_D \left(\frac{1}{n_0 - 1} - \frac{1}{n_\epsilon - 1} \right) \Delta w_0 \Delta\overline{v}\, dx, \end{aligned} \tag{30}$$

where the last equality follows from the definitions of the operators and (29). Hence we have that for any $v \in H_0^2(D)$

$$\int_D \frac{1}{n_\epsilon - 1}\Delta(w_\epsilon - w_0)\Delta\overline{v}\,dx \leq \left\| \left(\frac{1}{n_0-1} - \frac{1}{n_\epsilon - 1}\right)\Delta w_0 \right\|_{L^2(W_\epsilon)} \|v\|_{H_0^2(D)}. \tag{31}$$

So, by plugging in $v = (w_\epsilon - w_0)$ we can obtain, for some C independent of ϵ

$$\begin{aligned}\|w_\epsilon - w_0\|^2_{H_0^2(D)} &\leq C\int_D \frac{1}{n_\epsilon - 1}\Delta(w_\epsilon - w_0)\Delta\overline{(w_\epsilon - w_0)}dx \\ &\leq C\left\| \left(\frac{1}{n_0-1} - \frac{1}{n_\epsilon - 1}\right)\Delta w_0 \right\|_{L^2(W_\epsilon)} \|w_\epsilon - w_0\|_{H_0^2(D)} \end{aligned}\tag{32}$$

which of course implies

$$\|w_\epsilon - w_0\|_{H_0^2(D)} \leq C\left\| \left(\frac{1}{n_0-1} - \frac{1}{n_\epsilon - 1}\right)\Delta w_0 \right\|_{L^2(W_\epsilon)}, \tag{33}$$

or

$$\|\mathbb{A}_\epsilon^{-1}f - \mathbb{A}_0^{-1}f\|_{H_0^2(D)} \leq \left\| \left(\frac{1}{n_0-1} - \frac{1}{n_\epsilon - 1}\right)\Delta \mathbb{A}_0^{-1}f \right\|_{L^2(W_\epsilon)}. \tag{34}$$

Note that the right hand side goes to zero for any fixed $f \in H_0^2(D)$, but this does not imply operator norm convergence. Next consider, for $g \in H_0^2(D)$,

$$\begin{aligned}\|\mathbb{A}_\epsilon^{-1}\mathbb{B}_\epsilon g - \mathbb{A}_0^{-1}\mathbb{B}_0 g\|_{H_0^2(D)} &\leq \|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})(\mathbb{B}_0 g)\|_{H_0^2(D)} + \|\mathbb{A}_\epsilon^{-1}(\mathbb{B}_\epsilon - \mathbb{B}_0)g\|_{H_0^2(D)} \\ &\leq C\left\| \left(\frac{1}{n_0-1} - \frac{1}{n_\epsilon - 1}\right)\Delta \mathbb{A}_0^{-1}\mathbb{B}_0 g \right\|_{L^2(W_\epsilon)} + \|\mathbb{A}_\epsilon^{-1}\|\|\mathbb{B}_\epsilon - \mathbb{B}_0\|\|g\|_{H_0^2(D)} \\ &\leq \tilde{C}\|\Delta\mathbb{A}_0^{-1}\mathbb{B}_0 g\|_{L^2(W_\epsilon)} + \|\mathbb{A}_\epsilon^{-1}\|\|\mathbb{B}_\epsilon - \mathbb{B}_0\|\|g\|_{H_0^2(D)} \end{aligned}\tag{35}$$

where we have used (34) and the $\tilde{C}$ now incorporates the maximum contrast. Now, note that $\mathbb{B}_0$ is smoothing by two orders order. Indeed, (18) is the weak form of

$$\Delta\Delta\mathbb{B}_0 u = \Delta\left(\frac{n_0}{n_0-1}u\right) + \frac{1}{n_0-1}\Delta u, \qquad u \in H_0^2(D).$$

Since $n_0 \in C^2(D)$, classic regularity results [17], [24], imply that $\mathbb{B}_0 u \in H^4(D) \cap H_0^2(D)$. Thus, we have that

$$\|\mathbb{A}_0^{-1}\mathbb{B}_0 g\|_{H^3(D)} \leq C\|g\|_{H^2(D)},$$

(note that by the same argument as above we can see that $\mathbb{A}_0$ is a zero order operator) and therefore

$$\|\Delta\mathbb{A}_0^{-1}\mathbb{B}_0 g\|_{H^1(D)} \leq C\|g\|_{H^2(D)}.$$

By Sobolev embedding, this implies that

$$\|\Delta\mathbb{A}_0^{-1}\mathbb{B}_0 g\|_{L^{\hat{p}}(D)} \leq C\|g\|_{H^2(D)}$$

for some $\hat{p} > 2$. So, if we let $p = \hat{p}/2 > 1$, and q be its Hölder dual,

$$\begin{aligned}\int_{W_\epsilon}(\Delta\mathbb{A}_0^{-1}\mathbb{B}_0 g)^2 dx &\leq \|(\Delta\mathbb{A}_0^{-1}\mathbb{B}_0 g)^2\|_{L^p(D)}\|\chi_{W_\epsilon}\|_{L^q(W_\epsilon)} \qquad (36)\\ &= \|\Delta\mathbb{A}_0^{-1}\mathbb{B}_0 g\|^2_{L^{\hat{p}}(D)}|W_\epsilon|^{1/q} \qquad (37)\end{aligned}$$

where χ_{W_ϵ} is the characteristic function of the support of the inhomogeneities. Note that since $p > 1$, $\frac{1}{q} > 0$ so that the bound has a positive power of ϵ. Hence

$$\|\Delta \mathbb{A}_0^{-1}\mathbb{B}_0 g\|_{L^2(W_\epsilon)} \leq \|\Delta \mathbb{A}_0^{-1}\mathbb{B}_0 g\|_{L^{\hat{p}}(D)} |W_\epsilon|^{1/2q} \tag{38}$$

$$\leq C\|g\|_{H^2(D)} |W_\epsilon|^{1/2q}. \tag{39}$$

This, combined with (35) and Lemma 3.1 yield the first result. The second result follows from exactly the same proof where we replace $\mathbb{B}_0, \mathbb{B}_\epsilon$ with $\mathbb{C}_0^{1/2}, \mathbb{C}_\epsilon^{1/2}$. Note that we use that Lemma 3.2 implies convergence of the square roots [21], and $\mathbb{C}_0^{1/2}$ is also smoothing of two orders since from $\mathbb{C}_0$ is smoothing to four orders. The latter follows from the fact that (23) is the weak form of

$$\Delta\Delta \mathbb{C}_0 u = \frac{n_0}{n_0 - 1} u \qquad u \in H_0^2(D)$$

which again from classic regularity results implies that $\mathbb{C}_0 u \in H^6(D) \cap H_0^2(D)$ for smooth enough boundary (at least in $H^4(D) \cap H_0^2(D)$ for C^1-boundary which suffices for our proof). □

Again, from Lemma 3.2, we also have norm convergence of $\mathbb{C}_\epsilon^{1/2}$ (see for example [21]). Combining all of the above Lemmas, we have proven the following theorem.

Theorem 3.1 *Let $\mathbf{K}_\epsilon$ and $\mathbf{K}_0$ be defined by (17) for $\epsilon > 0$ and $\epsilon = 0$, respectively. Then $\mathbf{K}_\epsilon$ converges to $\mathbf{K}_0$ in the operator norm for any dimension d, and hence their eigenvalues and eigenvectors also converge.*

4. Asymptotic formulae for operators

Having proven the convergence of the spectrum, we now want to obtain asymptotic formulae for the real transmission eigenvalues. To this end, we need to obtain explicit formula for the first term in the asymptotic expansion for the operator $\mathbf{K}_\epsilon$ and consequently for each of the operators involved in the definition of $\mathbf{K}_\epsilon$. Lemma 3.1 and Lemma 3.2 provide the asymptotic formulae for the operators $\mathbb{B}_\epsilon$ and $\mathbb{C}_\epsilon$ respectively. Next we need to obtain such formula for the operator $\mathbb{A}_\epsilon^{-1}$. Since the structure of the operator $\mathbb{A}_\epsilon^{-1}$ is more complicated, for sake of simplicity of presentation we now assume that the refractive index in the hosting media (i.e. refractive index n_0) as well as in each of the small inhomogeneities are constants. However we remark that it is possible to generalize the same type of analysis to the case of non-constant inhomogeneities and hosting media (see [10] for the case of the conductivity problem).

To this end, we start by assuming that there is only one inhomogeneity of the form ϵB with constant refractive index n_1, centered at the origin inside the homogeneous media D with constant refractive index n_0. Let us make the scaling

$$y = x/\epsilon,$$

and denote by

$$\tilde{D} = \frac{1}{\epsilon} D, \quad \tilde{n}(y) = n_\epsilon(x).$$

We define the auxiliary function v_B of y to satisfy the variational equation

$$\int_{\mathbb{R}^d} \frac{1}{\tilde{n}-1} \Delta_y v_B(y) \Delta_y \overline{\phi}(y) dy = \int_B \Delta_y \overline{\phi} dy \tag{40}$$

for any ϕ in some space that includes appropriate decaying conditions at infinity and contains zero extensions of $H_0^2(\tilde{D})$ functions, where $\tilde{n}$ is defined to be equal to n_0 outside $\tilde{D}$. Following [4] we define the Sobolev space

$$W_0^2(\mathbb{R}^d) = \left\{ \begin{array}{r} u \in \mathcal{D}'(\mathbb{R}^d) : \, 0 \le |m| \le k, \, \rho^{|m|-2}(\ln \omega)^{-1} D^{|m|} u \in L^2(\mathbb{R}^d) \\ k+1 \le |m| \le 2, \, \rho^{|m|-2} D^{|m|} u \in L^2(\mathbb{R}^d). \end{array} \right\}$$

where $\rho := (1+|x|^2)^{1/2}$, $\omega := (2+|x|^2)$ and $k = 1$ if $d = 2$, $k = 0$ if $d = 4$ and $k = -1$ if $d \neq 2, 4$ equipped with the usual H^2-norm with the indicated weights. Hence we look for $v_B \in W_0^2(\mathbb{R}^d)$ satisfying (40) for all $\phi \in W_0^2(\mathbb{R}^d)$. As it is shown in [4], for $d = 2, 3, 4$, the behavior at infinity of functions in $W_0^2(\mathbb{R}^d)$ includes constants and first order polynomials which satisfy the homogeneous version of (40). Since the compactly supported distribution defined by the right-hand side of (40) which is in $W_0^{-2}(\mathbb{R}^d)$, is orthogonal to first order polynomials, we have that there exists a unique solution v_B of (40) in the quotient space $W_0^2(\mathbb{R}^d) \backslash \mathcal{P}_{(2-d/2)}$ where $\mathcal{P}_{(2-d/2)}$ is the space of polynomials of order $(2-d/2)$. With the notations $q = \left[\frac{\partial}{\partial \nu}(\Delta v_B)\right]$ and $p = [\Delta v_B]$, where for a generic function f, $[f]$ denotes the jump $\frac{1}{n_1-1} f^+ - \frac{1}{n_0-1} f^-$ across ∂B, the exclusion of constants and the first order polynomial in the case of $d = 2$ relies on the facts that [19]

$$\int_{\partial B} q \, ds = 0, \qquad \int_{\partial B} (qx + p\nu_1) \, ds = 0, \qquad \int_{\partial B} (qy + p\nu_2) \, ds = 0,$$

whereas the exclusion of constants for the case of $d = 3, 4$ relies on the fact that

$$\int_{\partial B} q \, ds = 0.$$

These conditions guarantee the existence of a solution to (40) in $W_0^2(\mathbb{R}^d)$ for every $d \ge 2$ that satisfies the following behavior at infinity

$$v_B(y) = o(|y|^{2-d/2}), \qquad \nabla \cdot v_B(y) = o(|y|^{1-d/2}), \qquad D^2 v_B(y) = o(|y|^{-d/2}). \tag{41}$$

Having defined the solution to (40), we introduce the correction as follows

$$w_\epsilon \approx w_0 + \epsilon^2 \left(\frac{1}{n_0-1} - \frac{1}{n_1-1} \right) \Delta w_0(0) v_B(x/\epsilon)$$

where w_ϵ and w_0 are defined by (29). The goal is to be able to discard the term

$$\int_{\epsilon B} \Delta(w_\epsilon - (w_0 + \epsilon^2 \left(\frac{1}{n_0-1} - \frac{1}{n_1-1} \right) \Delta w_0(0) v_B(x/\epsilon))) \Delta \overline{\phi} \, dx.$$

To this end, let us first define $\tilde{v}_B(y) \in H_0^2(\tilde{D})$ to satisfy the variational equality

$$\int_{\tilde{D}} \frac{1}{\tilde{n}-1} \Delta_y \tilde{v}_B(y) \Delta_y \overline{\phi}(y) dy = \int_B \Delta_y \overline{\phi} dy \tag{42}$$

for any $\phi \in H^2_0(\tilde{D})$. Clearly $\tilde{v}_B$ exists and is unique for any fixed ϵ. We then have that

$$\int_D \frac{1}{n_\epsilon - 1}\Delta(w_\epsilon - (w_0 + \epsilon^2 \left(\frac{1}{n_0 - 1} - \frac{1}{n_1 - 1}\right) \Delta w_0(0)\tilde{v}_B(x/\epsilon)))\Delta\overline{\phi}\, dx$$
$$= \int_D \left(\frac{1}{n_0 - 1} - \frac{1}{n_\epsilon - 1}\right) (\Delta w_0 - \Delta w_0(0))\, \Delta\overline{\phi}\, dx \quad (43)$$

for any $\phi \in H^2_0(D)$. Now we have the following Lemma.

Lemma 4.1 *Let $\tilde{z}_\epsilon$ be defined by*

$$\tilde{z}_\epsilon = w_\epsilon - \left(w_0 + \epsilon^2 \left(\frac{1}{n_0 - 1} - \frac{1}{n_1 - 1}\right) \Delta w_0(0)\tilde{v}_B(x/\epsilon)\right).$$

Then

$$\|\Delta \tilde{z}_\epsilon\|_{L^2(D)} \leq C\epsilon^{\frac{d}{2}+1}\|\nabla\Delta w_0\|_{L^\infty(D)}$$

for some C independent of ϵ.

Proof. From the variational definitions (30) and (42) we have directly that

$$\int_D \frac{1}{n_\epsilon - 1}\Delta(w_\epsilon - \left(w_0 + \epsilon^2 \left(\frac{1}{n_0 - 1} - \frac{1}{n_1 - 1}\right) \Delta w_0(0)\tilde{v}_B(x/\epsilon)\right))\Delta\overline{\phi}\, dx$$
$$= \int_D \left(\frac{1}{n_0 - 1} - \frac{1}{n_\epsilon - 1}\right) (\Delta w_0 - \Delta w_0(0))\, \Delta\overline{\phi}\, dx$$
$$\leq C\epsilon\|\nabla\Delta w_0\|_\infty \int_{\epsilon B} |\Delta\phi|\, dx$$
$$\leq C\epsilon\|\nabla\Delta w_0\|_\infty \|\Delta\phi\|_{L^2(D)}\|\chi_{\epsilon B}\|_{L^2(D)}$$
$$\leq C\epsilon^{\frac{d}{2}+1}\|\nabla\Delta w_0\|_\infty\|\Delta\phi\|_{L^2(D)} \quad (44)$$

for any $\phi \in H^2_0(D)$. Plugging in

$$\phi = w_\epsilon - \left(w_0 + \epsilon^2 \left(\frac{1}{n_0 - 1} - \frac{1}{n_1 - 1}\right) \Delta w_0(0)\tilde{v}_B(x/\epsilon)\right)$$

this yields the result. □

We will need the following lemma.

Lemma 4.2 *Let $v_B(y)$ be the solution to (40) and $\tilde{v}_B(y)$ be the solution to (42). Then we have that*

$$\|\Delta_y(\tilde{v}_B - v_B)\|_{L^2(D)} \leq o(\epsilon^{\frac{d}{2}}),$$

that is,

$$\|\Delta_y(\tilde{v}_B - v_B)\|_{L^2(\tilde{D})} \leq o(1)$$

for some C independent of ϵ . (Note that the first integral is dx and the second is dy.)

Proof. Let $w_B(x) = \epsilon^2(\tilde{v}_B(x/\epsilon) - v_B(x/\epsilon))$ in D. The variational equalities (40) and (42) imply that $\tilde{v}_B(y)$ and $v_B(y)$ are weak solutions to the same fourth order differential equation in $\tilde{D}$ with the same transmission condition across ∂B. So, scaling to the fixed domain we that

$$\Delta_x \frac{1}{n_\epsilon - 1} \Delta_x w_B = 0 \text{ in } \ D.$$

The estimates (41) for the decay of v_B together with the zero boundary condition for $\tilde{v}_B$ yield that on ∂D, we have

$$w_B = o(\epsilon^{d/2}), \quad \frac{\partial w_B}{\partial \nu_x} = o(\epsilon^{d/2}).$$

Here, we have used interior elliptic regularity for the solution of biharmonic equation v_B in a neighborhood of ∂D [24] (note that the coefficient in the equation for v_B is constant near the boundary and we have assumed smooth enough boundary). Hence a priori estimates for w_B imply

$$\|\Delta_x w_B\|_{L^2(D)} \leq C \left(\|w_B(x/\epsilon)\|_{H^{3/2}(\partial D)} + \left\| \frac{\partial}{\partial \nu_x} w_B(x/\epsilon) \right\|_{H^{1/2}(\partial D)} \right)$$

with some $C > 0$ independent of ϵ. Since we also have

$$\epsilon^2 D_x^2 v_B = o(\epsilon^{d/2}),$$

interpolation yields

$$\|\Delta_x w_B\|_{L^2(D)} = o(\epsilon^{d/2}).$$

This of course implies that

$$\|\Delta_y(\tilde{v}_B - v_B)\|_{L^2(D)} = o(\epsilon^{d/2})$$

which proves the lemma. □

The above estimates now allow us to prove the following estimate on the correction term.

Lemma 4.3 *Let z_ϵ be defined by*

$$z_\epsilon = w_\epsilon - \left(w_0 + \epsilon^2 \left(\frac{1}{n_0 - 1} - \frac{1}{n_1 - 1} \right) \Delta w_0(0) v_B(x/\epsilon) \right).$$

Then, if $\phi \in C^2(D) \bigcap H_0^2(D)$,

$$\left| \int_{\epsilon B} \Delta z_\epsilon \Delta \overline{\phi} dx \right| \leq o(\epsilon^d) \|\nabla \Delta w_0\|_{L^\infty(D)} \|\Delta \phi\|_{L^\infty(D)}$$

for some C independent of ϵ.

Proof. Note that

$$z_\epsilon - \tilde{z}_\epsilon = \epsilon^2 \left(\frac{1}{n_0 - 1} - \frac{1}{n_1 - 1} \right) \Delta w_0(0) \left(v_B - \tilde{v}_B \right).$$

By using Cauchy-Schwartz and the previous lemmas

$$\begin{aligned} \left| \int_{\epsilon B} \Delta z_\epsilon \Delta \overline{\phi} dx \right| &\le \|\Delta z_\epsilon\|_{L^2(D)} \|\Delta \phi\|_{L^2(\epsilon B)} \\ &\le \left(\|\Delta (z_\epsilon - \tilde{z}_\epsilon)\|_{L^2(D)} + \|\Delta \tilde{z}_\epsilon\|_{L^2(D)} \right) \|\Delta \phi\|_{L^2(\epsilon B)} \\ &\le o(\epsilon^{\frac{d}{2}}) \|\nabla \Delta w_0\|_{L^\infty(D)} \|\Delta \phi\|_{L^2(\epsilon B)}. \end{aligned}$$

Finally, using the small volume of ϵB and the smoothness of ϕ we have

$$\left| \int_{\epsilon B} \Delta z_\epsilon \Delta \overline{\phi} dx \right| \le o(\epsilon^{\frac{d}{2}}) \|\nabla \Delta w_0\|_{L^\infty(D)} \epsilon^{\frac{d}{2}} \|\Delta \phi\|_{L^\infty(D)}$$

which proves the result. □

Now we have all the ingredients to prove the asymptotic formula for the operator $\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1}$.

Lemma 4.4 *Let $\mathbb{A}_0$ and $\mathbb{A}_\epsilon$ be defined by (27) and (28) respectively, and assume that $n_0, n_i, i = 1 \dots m$ are all constant. For a given $u \in H_0^2(D)$, let*

$$w_\epsilon = \mathbb{A}_\epsilon^{-1} u \qquad \text{and} \qquad w_0 = \mathbb{A}_0^{-1} u.$$

Then, if $w_0 \in C^3(D)$, for any $\phi \in H_0^2(D) \bigcap C^2(D)$,

$$\begin{aligned} (w_\epsilon - w_0, \phi) &= \sum_{j=1}^{N} \epsilon^d |B_j| \left(1 - \frac{n_0 - 1}{n_j - 1} \right) \Delta w_0(z_j) \Delta \overline{\phi}(z_j) \\ &+ \sum_{j=1}^{N} \epsilon^d \left(1 - \frac{n_0 - 1}{n_j - 1} \right)^2 \frac{1}{n_0 - 1} \Delta w_0(z_j) \Delta \overline{\phi}(z_j) \int_{\partial B_j} \frac{\partial v_{B_j}}{\partial \nu_y} d\sigma_y + o(\epsilon^d). \end{aligned} \tag{45}$$

Proof. Note that by the definition (29) of the operators we have

$$\begin{aligned} \int_D \frac{1}{n_0 - 1} \Delta (w_\epsilon - w_0) \Delta \overline{\phi} \, dx &= \int_D \frac{1}{n_\epsilon - 1} \Delta w_\epsilon \Delta \overline{\phi} \, dx - \int_D \frac{1}{n_0 - 1} \Delta w_0 \Delta \overline{\phi} \, dx \\ + \int_D \left(\frac{1}{n_0 - 1} - \frac{1}{n_\epsilon - 1} \right) \Delta w_\epsilon \Delta \overline{\phi} \, dx &= \int_D \left(\frac{1}{n_0 - 1} - \frac{1}{n_\epsilon - 1} \right) \Delta w_\epsilon \Delta \overline{\phi} \, dx. \end{aligned} \tag{46}$$

Let us again assume we have one inhomogeneity of the form ϵB, centered at the origin, so that

$$\begin{aligned} \int_D \Delta (w_\epsilon - w_0) \Delta \overline{\phi} \, dx &= (n_0 - 1) \int_D \left(\frac{1}{n_0 - 1} - \frac{1}{n_\epsilon - 1} \right) \Delta w_\epsilon \Delta \overline{\phi} \, dx \\ &= \left(1 - \frac{n_0 - 1}{n_1 - 1} \right) \int_{\epsilon B} \Delta w_\epsilon \Delta \overline{\phi} \, dx \end{aligned} \tag{47}$$

Now we apply Lemma 4.3 to obtain

$$\begin{aligned}
&\int_D \Delta(w_\epsilon - w_0)\Delta\overline{\phi}\,dx \\
&= \left(1 - \frac{n_0-1}{n_1-1}\right)\int_{\epsilon B}\Delta\left(w_0 + \epsilon^2\left(\frac{1}{n_0-1} - \frac{1}{n_1-1}\right)\Delta w_0(0)v_B(x/\epsilon)\right)\Delta\overline{\phi}\,dx \\
&\qquad\qquad + \left(1 - \frac{n_0-1}{n_1-1}\right)\int_{\epsilon B}\Delta z_\epsilon \Delta\overline{\phi}\,dx \\
&= \left(1 - \frac{n_0-1}{n_1-1}\right)\int_{\epsilon B}\Delta\left(w_0 + \epsilon^2\left(\frac{1}{n_0-1} - \frac{1}{n_1-1}\right)\Delta w_0(0)v_B(x/\epsilon)\right)\Delta\overline{\phi}\,dx + o(\epsilon^d) \\
&= \epsilon^d\left(1 - \frac{n_0-1}{n_1-1}\right)|B|\Delta w_0(0)\Delta\overline{\phi}(0) \\
&\qquad\qquad + \epsilon^d\left(1 - \frac{n_0-1}{n_1-1}\right)^2\frac{1}{n_0-1}\Delta w_0(0)\int_B \Delta_y v_B(y)\Delta\overline{\phi}(\epsilon y)\,dy + o(\epsilon^d) \\
&= \epsilon^d\left(1 - \frac{n_0-1}{n_1-1}\right)|B|\Delta w_0(0)\Delta\overline{\phi}(0) \\
&\qquad\qquad + \epsilon^d\left(1 - \frac{n_0-1}{n_1-1}\right)^2\frac{1}{n_0-1}\Delta w_0(0)\Delta\overline{\phi}(0)\int_B \Delta_y v_B(y)\,dy + o(\epsilon^d).
\end{aligned}$$

Simple integration by parts yields

$$\int_B \Delta_y v_B(y)\,dy = \int_{\partial B}\frac{\partial v_B}{\partial \nu_y}d\sigma_y,$$

from which the result follows. □

Remark 4.1 *The asymptotic formula can also be written as*

$$(w_\epsilon - w_0, \phi) = \sum_{j=1}^{N}\epsilon^d\left(1 - \frac{n_0-1}{n_j-1}\right)m_j\Delta w_0(z_j)\Delta\overline{\phi}(z_j) + o(\epsilon^d)$$

where

$$m_j = |B_j| + \left(\frac{1}{n_0-1} - \frac{1}{n_j-1}\right)\int_{\partial B_j}\frac{\partial v_{B_j}}{\partial \nu_y}d\sigma_y$$

acts in place of what was a polarization tensor in the case of small volume conductivity inhomogeneities. Note that here we no longer have a tensor, but we have a scalar m_j with a form similar to the polarization tensor in [2].

5. Eigenvalue expansion

In this section we provide an explicit formula for the asymptotic expansion of real eigenvalues in dimensions $d = 2, 3$. Similar formulae can be obtained for complex eigenvalues as well as all higher dimensions. However for practical purposes since only real eigenvalues can be measured from scattering data [6], we limit ourselves to the case of real eigenvalues for a simpler exposition. Also, the same type of analysis can carried over to higher dimensions $d > 3$, but the approach (in particular for the estimates of

Lemma 5.1) in this case must be carefully tuned since one needs to use L^p-Sobolev embeddings results. Before we proceed with the formula for the eigenvalue expansion, we will need some estimates we prove in the next Lemma. Note that if $u \in H_0^2(D)$ is the eigenfunction corresponding to eigenvalue k and n_0 constant, we have that it satisfies

$$\Delta\Delta u + k^2(n_0+1)\Delta u + k^4 n_0 u = 0.$$

Standard elliptic regularity [1] results imply that $u \in H^4(D) \cap H_0^2(D)$.

Lemma 5.1 *For n_0 and n_1 constants and $d = 2, 3$ we have that*

$$\|(\mathbb{B}_\epsilon - \mathbb{B}_0)u\|_{H_0^2(D)} \leq C\epsilon^d, \tag{48}$$

$$\|(\mathbb{C}_\epsilon^{1/2} - \mathbb{C}_0^{1/2})u\|_{H_0^2(D)} \leq C\epsilon^d, \tag{49}$$

$$\|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{B}_0 u\|, \|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{C}_0 u\| \leq C\epsilon^{d/2}, \tag{50}$$

and

$$\|\mathbb{B}_0(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})u\|_{H_0^2(D)}, \|\mathbb{C}_0^{1/2}(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})u\|_{H_0^2(D)} \leq C\epsilon^{d/2+\alpha}, \tag{51}$$

for some $\alpha > 0$ where u is any function in $H^4(D) \cap H_0^2(D)$.

Proof. For n constant $\mathbb{B}_0$ takes the form

$$(\mathbb{B}_0 u, \phi) = \int_D \frac{n_0+1}{n_0-1}\Delta u \overline{\phi}\, dx.$$

Hence for any $\phi \in H_0^2(D)$ we have

$$((\mathbb{B}_\epsilon - \mathbb{B}_0)u, \phi)_{H^2} = \int_D \left(\frac{n_\epsilon+1}{n_\epsilon-1} - \frac{n_0+1}{n_0-1}\right)\Delta u\overline{\phi}\, dx = \left(\frac{n_1+1}{n_1-1} - \frac{n_0+1}{n_0-1}\right)\int_{\epsilon B}\Delta u\overline{\phi}\, dx.$$

Sobolev embedding theorems imply that both Δu and ϕ are Hölder continuous and thus we obtain

$$|((\mathbb{B}_\epsilon - \mathbb{B}_0)u, \phi)_{H^2}| \leq \left|\frac{n_1+1}{n_1-1} - \frac{n_0+1}{n_0-1}\right| \max_D |\Delta u| \max_D |\phi|\epsilon^d. \tag{52}$$

This proves (48).

To prove the second identity, we recall that in Lemma 3.2 we have shown that $\mathbb{C}_\epsilon$ converges to $\mathbb{C}_0$ as $O(\epsilon^d)$. Using the square root lemma in [21] (Theorem VI .9) and the fact that $\mathbb{C}_\epsilon^n$ converges to $\mathbb{C}_0^n$ at the same order $O(\epsilon^d)$ (this can be shown based on $\mathbb{C}_\epsilon^n - \mathbb{C}_0^n = \mathbb{C}_\epsilon^{n-1}(\mathbb{C}_\epsilon - \mathbb{C}_0) + (\mathbb{C}_\epsilon^{n-1} - \mathbb{C}_0^{n-1})\mathbb{C}_0$ and induction), we can conclude that $\mathbb{C}_\epsilon^{1/2}$ converges to $\mathbb{C}_0^{1/2}$ at the same order $O(\epsilon^d)$. Hence (49) holds.

Now to show (50), we go back to formula (34) with $f = \mathbb{B}_0 u$ or $f = \mathbb{C}_0 u$. In either case the regularity of f implies that the norm error is bounded by that of the characteristic function of W_ϵ, that is, $\epsilon^{d/2}$. It therefore remains only to show (51). Let us apply a slight modification of the proof of Lemma 4.1 to $w_\epsilon = \mathbb{A}_\epsilon^{-1}u$ and $w_0 = \mathbb{A}_0^{-1}u$ which gives that

$$\|\tilde{z}_\epsilon\|_{H_0^2(D)} \leq C\epsilon^{d/2} \sup_{x\in\epsilon B} |\Delta w_0(x) - \Delta w_0(0)|.$$

Since u is in $H^4(D)$, Δw_0 is in $H^2(D)$, which in dimensions $d = 2, 3$ embeds into $C^\alpha(D)$ for some $\alpha > 0$. This yields

$$\|\tilde{z}_\epsilon\|_{H^2_0(D)} \leq C\epsilon^{d/2+\alpha},$$

and the same estimate holds for the L^2 norm of $\tilde{z}_\epsilon$. In the $L^2(D)$ norm, the correction term is order ϵ^2, and hence

$$\|w_\epsilon - w_0\|_{L^2(D)} \leq C\epsilon^{d/2+\alpha}.$$

Since $\mathbb{B}_0$ is two orders smoothing, we have that

$$\|\mathbb{B}_0(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})u\|_{H^2_0(D)} \leq C\|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})u\|_{L^2(D)},$$

and the result follows. The same proof works for $\mathbb{C}_0^{1/2}$. $\Box$

Finally we are in a position to derive the eigenvalue expansion using the theorem of Osborn [18], which we state here for convenience to the reader. The theorem is valid for non-selfadjoint operators and also yields a correction term. The actual result is more general, but we state it here for the case of norm convergence on a Hilbert space.

Suppose X is a Hilbert space and $K_n : X \to X$ is a sequence of compact linear operators such that $K_n \to K$ in norm. It then follows that the adjoint operators also converge in norm. Let μ be a nonzero eigenvalue of K of algebraic multiplicity m. It is well known that for n large enough, there exist m eigenvalues of K_n, $\mu_1^n, \dots \mu_m^n$ (counted according to algebraic multiplicity) such that $\mu_j^n \to \mu$ as $n \to \infty$, for each $1 \leq j \leq m$. Let E be the spectral projection onto the generalized eigenspace of K corresponding to eigenvalue μ. The space X can be decomposed in terms of the range and null space of E as $X = R(E) \oplus N(E)$.

Theorem 5.1 (Osborn) *Let $\phi_1, \phi_2, \dots \phi_m$ be a normalized basis for $R(E)$. Then there exists a constant C such that*

$$\left|\mu - \frac{1}{m}\sum_{j=1}^m \mu_j^n - \frac{1}{m}\sum_{j=1}^m \langle (K - K_n)\phi_j, \phi_j\rangle\right| \leq C\|(K-K_n)|_{R(E)}\|\cdot\|(K^*-K_n^*)|_{R(E^*)}\|. \quad (53)$$

Our operators $\mathbf{K}_\epsilon, \mathbf{K}_0$ are compact on $H^2_0(D) \times H^2_0(D)$ and we have seen that $\mathbf{K}_\epsilon \to \mathbf{K}_0$ in norm. We know that $\mathbb{A}_0$ is positive. Let us therefore equip $H^2_0(D) \times H^2_0(D)$ with the inner product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle := (\mathbb{A}_0 u, w)_{H^2_0(D)} + (v, z)_{H^2_0(D)}$$

instead of the usual one. Since $\mathbb{A}_0$ is positive, invertible, bounded above and below, and self adjoint, this yields an equivalent inner product. Note that if $\mathbb{A}_0$ is negative instead (which would correspond to $n_0 < 1$), we could do the same with $-\mathbb{A}_0$, but we will not consider that here.

For the sake of simplicity, let us first assume that μ is a simple eigenvalue of $\mathbf{K}_0$. Then we have for each ϵ small enough, some eigenvalue μ_ϵ of $\mathbf{K}_\epsilon$ is such that $\mu_\epsilon \to \mu$. Let

U be a normalized eigenvector of $\mathbf{K}_0$ corresponding to μ. The fact that U is normalized with this new inner product means that

$$\langle U, U\rangle = \left\langle \begin{pmatrix} u \\ \tau\mathbb{C}_0^{1/2}u \end{pmatrix}, \begin{pmatrix} u \\ \tau\mathbb{C}_0^{1/2}u \end{pmatrix} \right\rangle = 1$$

which can be restated as

$$(\mathbb{A}_0 u, u)_{H_0^2(D)} + \tau^2(\mathbb{C}_0 u, u)_{H_0^2(D)} = 1$$

or

$$\int_D \left(\frac{1}{n_0 - 1}|\Delta u|^2 + \tau^2 \frac{n_0}{n_0 - 1}|u|^2 \right) dx = 1 \tag{54}$$

using the definitions of the operators. First we show the remainder term in Osborn's theorem is small. Let's compute $\mathbf{K}_\epsilon U$:

$$\mathbf{K}_\epsilon U = \begin{pmatrix} -\mathbb{A}_\epsilon^{-1}\mathbb{B}_\epsilon u - \tau\mathbb{A}_\epsilon^{-1}\mathbb{C}_\epsilon^{1/2}C_0^{1/2}u \\ \mathbb{C}_\epsilon^{1/2}u \end{pmatrix} \tag{55}$$

$$= \begin{pmatrix} -\mathbb{A}_0^{-1}\mathbb{B}_\epsilon u - \tau\mathbb{A}_0^{-1}\mathbb{C}_\epsilon^{1/2}\mathbb{C}_0^{1/2}u \\ \mathbb{C}_\epsilon^{1/2}u \end{pmatrix} + \begin{pmatrix} -(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})(\mathbb{B}_0 u + \tau\mathbb{C}_0 u) \\ 0 \end{pmatrix} + o(\epsilon^d)$$

where we have used (48),(49) and the estimate (34) to discard the remainder. Thus we have

$$(\mathbf{K}_\epsilon - \mathbf{K}_0)U \tag{56}$$

$$= \begin{pmatrix} -\mathbb{A}_0^{-1}(\mathbb{B}_\epsilon - \mathbb{B}_0)u - \tau\mathbb{A}_0^{-1}(\mathbb{C}_\epsilon^{1/2} - \mathbb{C}_0^{1/2})C_0^{1/2}u - (\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})(\mathbb{B}_0 u + \tau\mathbb{C}_0 u) \\ (\mathbb{C}_\epsilon^{1/2} - \mathbb{C}_0^{1/2})u \end{pmatrix} + o(\epsilon^d).$$

Clearly from this expression and Lemma 5.1, we have that

$$\|(\mathbf{K}_\epsilon - \mathbf{K}_0)U\| = O(\epsilon^{d/2}).$$

This is not small enough to discard the remainder in Osborn's theorem, so noting that all of the individual operators are self adjoint, we compute

$$(\mathbf{K}_\epsilon^* - \mathbf{K}_0^*)U^*$$

$$= \begin{pmatrix} -(\mathbb{B}_\epsilon - \mathbb{B}_0)\mathbb{A}_0^{-1}u + \mathbb{B}_0(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})u - \tau(\mathbb{C}_\epsilon^{1/2} - \mathbb{C}_0^{1/2})\mathbb{C}_0^{1/2}u \\ -(\mathbb{C}_\epsilon^{1/2} - \mathbb{C}_0^{1/2})\mathbb{A}_0^{-1}u - \mathbb{C}_0(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})u \end{pmatrix} + o(\epsilon^d). \tag{57}$$

Using Lemma 5.1, noting in particular (51), we now have that

$$\|(\mathbf{K}_\epsilon^* - \mathbf{K}_0^*)U^*\| = O(\epsilon^{d/2+\alpha}).$$

Hence Osborn's theorem gives us (for a simple eigenvalue)

$$\mu - \mu_\epsilon = \langle(\mathbf{K}_0 - \mathbf{K}_\epsilon)U, U\rangle + o(\epsilon^d).$$

To obtain the correction term we compute

$$\begin{aligned}\langle \mathbf{K}_\epsilon U, U\rangle &= -(\mathbb{B}_\epsilon u, u) - \tau(\mathbb{C}_\epsilon^{1/2}\mathbb{C}_0^{1/2}u, u) - \tau(\mathbb{A}_0(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})(\mathbb{B}_0 u + \tau\mathbb{C}_0 u), u)\\ &\quad + (\mathbb{C}_\epsilon^{1/2}u, \tau\mathbb{C}_0^{1/2}u) + o(\epsilon^d)\\ &= -(\mathbb{B}_\epsilon u, u) - (\mathbb{A}_0(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})(\mathbb{B}_0 u + \tau\mathbb{C}_0 u), u) + o(\epsilon^d)\end{aligned}$$

by the fact that $\mathbb{C}_\epsilon^{1/2}$ is self-adjoint. Recall

$$\mathbb{A}_0 u + \tau\mathbb{B}_0 u + \tau^2\mathbb{C}_0 u = 0$$

so that

$$\langle \mathbf{K}_\epsilon U, U\rangle = -(\mathbb{B}_\epsilon u, u) + \frac{1}{\tau}((\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{A}_0 u, \mathbb{A}_0 u) + o(\epsilon^d)$$

using the fact that $\mathbb{A}_0$ is self-adjoint. (Note that this simplification may not hold for generalized eigenvectors, but similar expressions can be obtained.) Similarly, we calculate that we have very simply

$$\langle \mathbf{K}_0 U, U\rangle = -(\mathbb{B}_0 u, u),$$

which yields

$$\langle (\mathbf{K}_0 - \mathbf{K}_\epsilon)U, U\rangle = ((\mathbb{B}_\epsilon - \mathbb{B}_0)u, u) - \frac{1}{\tau}((\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{A}_0 u, \mathbb{A}_0 u) + o(\epsilon^d)$$

and hence

$$\mu - \mu_\epsilon = ((\mathbb{B}_\epsilon - \mathbb{B}_0)u, u) - \frac{1}{\tau}((\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{A}_0 u, \mathbb{A}_0 u) + o(\epsilon^d).$$

Now, let us assume that n_0 is constant and plug in our asymptotic expansions from the previous lemmas. We have that

$$\begin{aligned}&((\mathbb{B}_\epsilon - \mathbb{B}_0)u, u)\\ &= \sum_{j=1}^N \epsilon^d |B_j| \left[\left(\frac{1}{n_j - 1} - \frac{1}{n_0 - 1}\right)\Delta u(z_j)\overline{u(z_j)} + \left(\frac{n_j}{n_j - 1} - \frac{n_0}{n_0 - 1}\right)u(z_j)\Delta\overline{u(z_j)}\right]\\ &\qquad + o(\epsilon^d)\end{aligned}$$

Recall that for simplicity we assume u is real, so that this is

$$((\mathbb{B}_\epsilon - \mathbb{B}_0)u, u) = \sum_{j=1}^N \epsilon^d |B_j| \left(\frac{1+n_j}{n_j - 1} - \frac{1+n_0}{n_0 - 1}\right)\Delta u(z_j) u(z_j) + o(\epsilon^d)$$

We also have that

$$((\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{A}_0 u, \mathbb{A}_0 u) = \frac{1}{n_0 - 1}\sum_{j=1}^N \epsilon^d \left(1 - \frac{n_0 - 1}{n_j - 1}\right) m_j |\Delta u(z_j)|^2 + o(\epsilon^d)$$

where

$$m_j = |B_j| + \left(\frac{1}{n_0 - 1} - \frac{1}{n_j - 1}\right)\int_{\partial B_j} \frac{\partial v_{B_j}}{\partial \nu_y} d\sigma_y.$$

This finally yields the formula (when n_0 and each n_j is constant and everything is real)

$$\mu - \mu_\epsilon = \sum_{j=1}^{N} \epsilon^d |B_j| \left(\frac{1+n_j}{n_j - 1} - \frac{1+n_0}{n_0-1} \right) \Delta u(z_j) u(z_j) + \frac{1}{\tau} \sum_{j=1}^{N} \epsilon^d \left(\frac{1}{n_0-1} - \frac{1}{n_j-1} \right) m_j |\Delta u(z_j)|^2 + o(\epsilon^d). \quad (58)$$

The case of multiple eigenvalues yields a similar expression for the correction term where one now takes the averaged sum of the correction term for each eigenfunction in the generalized eigenspace.

We have proven the following theorem:

Theorem 5.2 *Assume that $d = 2, 3$, the background n_0 and all n_j are constant, and τ is a real transmission eigenvalue corresponding to the background n_0 (i.e. in the absence of small inhomogeneities). Let $\{u_p\}$ be a basis for the corresponding generalized eigenspace, normalized according to (54). Then*

(i) If τ is simple, for each $\epsilon > 0$ small enough, there exists a transmission eigenvalue τ_ϵ corresponding to medium (1) (in the presence of small inhomogeneities) such that

$$\frac{1}{\tau} - \frac{1}{\tau_\epsilon} = \sum_{j=1}^{N} \epsilon^d |B_j| \left(\frac{1+n_j}{n_j - 1} - \frac{1+n_0}{n_0-1} \right) \Delta u(z_j) u(z_j) + \frac{1}{\tau} \sum_{j=1}^{N} \epsilon^d \left(\frac{1}{n_0-1} - \frac{1}{n_j-1} \right) m_j |\Delta u(z_j)|^2 + o(\epsilon^d). \quad (59)$$

(ii) If τ has algebraic multiplicity m, there exists m transmission eigenvalues $\tau_\epsilon^1, \dots, \tau_\epsilon^m$ of (1) counted according to multiplicity, with

$$\frac{1}{\tau} - \frac{1}{m} \sum_{p=1}^{m} \frac{1}{\tau_\epsilon^p} = \frac{1}{m} \sum_{p=1}^{m} \left(\sum_{j=1}^{N} \epsilon^d |B_j| \left(\frac{1+n_j}{n_j - 1} - \frac{1+n_0}{n_0-1} \right) \Delta u_p(z_j) u_p(z_j) + \frac{1}{\tau} \sum_{j=1}^{N} \epsilon^d \left(\frac{1}{n_0-1} - \frac{1}{n_j-1} \right) m_j |\Delta u_p(z_j)|^2 \right) + o(\epsilon^d). \quad (60)$$

The first term in the asymptotic expansion of Theorem 5.2 contains information about the size, location and refractive indices of unknown inhomogeneities in terms of the known refractive index and the computable eigenvalue and corresponding eigenfunction of the unperturbed medium [23]. Since the real (or near the real axis) transmission eigenvalues corresponding to the perturbed media can be measured from the scattering data [6], the equation (59) or (60) can potentially be used to obtain information about small inhomogeneities (see for example [3]).

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