The Inverse Scattering Problem for a Penetrable Cavity with Internal Measurements

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Abstract. We consider the inverse scattering problem for a cavity that is bounded by a penetrable inhomogeneous medium of compact support and seek to determine the shape of the cavity from internal measurements on a curve or surface inside the cavity. We prove uniqueness and establish a linear sampling method for determining the shape of the cavity. A central role in our analysis is played by an unusual non-selfadjoint eigenvalue problem which we call the exterior transmission eigenvalue problem.

Dedicated to Gunther Uhlmann on the occasion of his 60th birthday

1. Introduction

The use of sampling methods and transmission eigenvalues has played an important role in inverse scattering theory for the past fifteen years and for a survey of recent results in this area we refer the reader to [**3**] and [**6**]. These methods are concerned with the inverse scattering problem for an inhomogeneous medium and seek to determine the support and bounds on the constitutive parameters of the scattering object by solving a linear integral equation of the first kind called the far field equation. A central role in this approach is an investigation of a class of non-selfadjoint eigenvalue problems called interior transmission eigenvalue problems. On the other hand, in the case of scattering by an impenetrable obstacle with Dirichlet, Neumann or impedance boundary conditions, there has been a recent interest in the inverse scattering problem with measured data inside a cavity [**12**], [**13**], [**21**]-[**23**]. In this class of problems the object is to determine the shape of the cavity from the use of sources and measurements along a curve or surface inside the cavity. A possible motivation for studying such a problem is to determine the shape of an underground reservoir by lowering receivers and transmitters into the reservoir through a bore hole drilled from the surface of the earth. In this paper we will combine the above two directions of research and consider the inverse scattering problem for a cavity that is bounded by a penetrable inhomogeneous medium

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of compact support and seek to determine the shape of the cavity from internal measurements. Of particular interest in this investigation is the central role played by an unusual non-selfajoint eigenvalue problem called the *exterior transmission eigenvalue problem*.

The plan of our paper is as follows. In the next section we will formulate both the direct and inverse scattering problems for a cavity bounded by a penetrable inhomogeneous medium of compact support. In Section 3 we will formulate the exterior transmission eigenvalue problem and establish the Fredholm property of the associated exterior transmission problem. These results will prove to be central to our analysis of both the uniqueness of the solution to our inverse scattering problem (discussed in Section 4) as well as our use of the linear sampling method in Section 5 to recover the shape of the cavity.

2. The direct and inverse scattering problems

Let $D \subset \mathbb{R}^d$, $d = 2, 3$, be a simply connected bounded region of $\mathbb{R}^d$ with Lipshitz boundary ∂D and denote by ν the outward unit normal to ∂D. We assume the medium inside D is homogeneous with refractive index scaled to one and denote by k the corresponding wave number. The medium outside D is assumed to be inhomogeneous and possibly anisotropic such that outside a large ball B_R it is homogeneous with the same wave number as the medium in D. More specifically, the physical properties of the medium in $\mathbb{R}^d \setminus \overline{D}$ are described by the $d \times d$ symmetric matrix valued function with $L^\infty(\mathbb{R}^d \setminus \overline{D})$ entries and the bounded function $n \in L^\infty(\mathbb{R}^d \setminus \overline{D})$ such that $\overline{\xi} \cdot \Re(A)\xi \geq \alpha\|\xi\|^2$, $\overline{\xi} \cdot \Im(A)\xi \leq 0$, for all $\xi \in \mathbb{C}$ and $n \geq n_0 > 0$ in $\mathbb{R}^d \setminus \overline{D}$. Furthermore we assume that $A \equiv I$ and $n \equiv 1$ in $\mathbb{R}^d \setminus B_R$ where B_R is a large ball containing D.

In acoustic scattering ($d = 3$) or electromagnetic scattering ($d = 2$, for an H-polarized infinite cylinder) D represents the support of a cavity filled e.g. with air which is assumed to be the reference media with wave number k. Let $\Phi(x, y)$ be a point source located at a point $y \in D$ inside the cavity given by

$$\Phi(x,y) = \begin{cases} H_0^{(1)}(k|x-y|) & \text{in } \mathbb{R}^2 \\ \dfrac{e^{ik|x-y|}}{|x-y|} & \text{in } \mathbb{R}^3. \end{cases} \tag{2.1}$$

and consider the scattering of this point source by the inhomogeneous media. The total filed $u = u^s + \Phi(\cdot, y)$ inside the cavity satisfies

$$\Delta_x u + k^2 u = \delta(x - y) \qquad x \in D$$

whereas the total field w outside the cavity satisfies

$$\nabla \cdot A(x)\nabla w + k^2 n(x) w = 0 \qquad x \in \mathbb{R}^d \setminus \overline{D} \tag{2.2}$$

and across the interface ∂D both the total field and its normal derivative are continuos, i.e.

$$u = w \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu_A} \qquad \text{on } \partial D \tag{2.3}$$

together with the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial w}{\partial r} - ikw\right) = 0$$

uniformly with respect to $\hat{x} = x/r$, $r = |x|$, where $\frac{\partial w}{\partial \nu_A} := A\nabla w \cdot \nu$. We recall that $\text{supp}(A-I) \subset B_R \setminus D$ and $\text{supp}(n-1) \subset B_R \setminus D$. Written in terms of the scattered field u^s in D and the total field w in $\mathbb{R}^d \setminus \overline{D}$ the above scattering problem is a particular case of the following boundary value problem in $\mathbb{R}^d$: Find $u^s \in H^1(D)$ and $u \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ such that

$$\nabla \cdot A\nabla w + k^2 n w = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{2.4}$$

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad D \tag{2.5}$$

$$w - u^s = f \quad \text{on} \quad \partial D \tag{2.6}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = h \quad \text{on} \quad \partial D \tag{2.7}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial w}{\partial r} - ikw\right) = 0 \tag{2.8}$$

where $f := \Phi(\cdot, y)|_{\partial D}$ and $h := \frac{\partial \Phi(\cdot,y)}{\partial \nu}|_{\partial D}$. In general $f \in H^{\frac{1}{2}}(\partial D)$ and $h \in H^{-\frac{1}{2}}(\partial D)$. Using a variational approach it is shown in [**3**] that the forward scattering problem (2.4)-(2.8) has a unique solution which depends continuously on the data f, h.

Now assume that C is a smooth $(d-1)$-manifold entirely included in D which is referred to as the measurement manifold. We place the point source at every $y \in C$ and measure the corresponding scattered field $u^s(x) := u^s(x, y)$ for $x \in C$. The *inverse problem* we consider in this paper is for fixed (but not necessarily known) A and n satisfying the above assumptions, determine the boundary of the cavity ∂D from a knowledge of $u^s(x, y)$ for all $x, y \in C$. (Note that if C is chosen to be an analytic manifold by the analyticity of the solution u^s to the Helmholtz equation in D it suffices to know $u^s(x, y)$ for x, y on a open arc $C_0 \subset C$.) Throughout this paper we make the following assumption:

Assumption 2.1. The measurement manifold C is such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in the bounded region circumscribed by C.

Note that since the interrogating wave number k is known, it is easy to choose C to satisfy Assumption 2.1. The main goal of this paper is to prove a uniqueness result and develop a solution method for solving the above inverse problem.

3. The exterior transmission eigenvalue problem

In this section we will formulate and study the so-called exterior transmission problem which will play a fundamental role in our uniqueness proof and the justification of the linear sampling method. To make an analogy with the exterior scattering problem, the exterior transmission problem here plays the same role as the interior transmission problem does for the exterior scattering problem for an inhomogeneous media [**4**], [**6**], [**11**]. As a physical motivation of the exterior transmission problem we ask the question if it is possible to send an outgoing incident field u^i from inside the cavity D that does not produce any scattered field in D and all the energy is transmitted to the exterior of D. Since this outgoing incident field satisfies the Helmholtz equation outside D, the scattering problem (2.4)-(2.8) implies that $v := u^i|_{\mathbb{R}^d\setminus\overline{D}}$ and the total field w satisfies the homogenous problem

$$\nabla\cdot A\nabla w + k^2 nw = 0 \qquad \text{in} \quad \mathbb{R}^d\setminus\overline{D} \tag{3.1}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \quad \mathbb{R}^d\setminus\overline{D} \tag{3.2}$$

$$w = v \qquad \text{on} \quad \partial D \tag{3.3}$$

$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \qquad \text{on} \quad \partial D \tag{3.4}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial w}{\partial r} - ikw) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial v}{\partial r} - ikv) = 0 \tag{3.5}$$

Definition 3.1. *Values of $k \in \mathcal{C}$ with $\Re(k) > 0$ for which the homogeneous problem (3.1)-(3.5) has a nontrivial solution are called exterior transmission eigenvalues.*

In next section we will see that exterior transmission eigenvalues are related to the injectivity of the near field (data) operator. For later use we need the non-homogeneous version of (3.1)-(3.5) which we formulate in the following.

The *exterior transmission problem* is given $f \in H^{\frac{1}{2}}(\partial D)$, $h \in H^{\frac{1}{2}}(\partial D)$, $\ell_1 \in L^2(B_R\setminus\overline{D})$ and $\ell_2 \in L^2(B_R\setminus\overline{D})$, find $w \in H^1_{loc}(\mathbb{R}^d\setminus\overline{D})$, $v \in H^1_{loc}(\mathbb{R}^d\setminus\overline{D})$ such that

$$\nabla\cdot A\nabla w + k^2 nw = \ell_1 \qquad \text{in} \quad \mathbb{R}^d\setminus\overline{D} \tag{3.6}$$

$$\Delta v + k^2 v = \ell_2 \qquad \text{in} \quad \mathbb{R}^d\setminus\overline{D} \tag{3.7}$$

$$w - v = f \qquad \text{on} \quad \partial D \tag{3.8}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{3.9}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial w}{\partial r} - ikw) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial v}{\partial r} - ikv) = 0, \tag{3.10}$$

where ℓ_1 and ℓ_2 vanish in $\mathbb{R}^d\setminus\overline{B_R}$ and R is the radius of the ball B_R outside of which $A = I$ and $n = 1$. We use a variational approach to study this problem. To this end we take $v_l \in H^1_{loc}(\mathbb{R}^d\setminus\overline{D})$ to be the unique solution of the exterior Dirichlet problem

$$\Delta v_l + k^2 v_l = 0 \qquad \textit{in} \quad \mathbb{R}^d\setminus\overline{D}$$

$$v_l = f \qquad \textit{on} \quad \partial D$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial v_l}{\partial r} - ikv_l) = 0$$

and set $v_0 = v + v_l$. Then (w, v_0) satisfies (3.6)-(3.10) with $(f,h) = (0, \tilde{h} := h - \frac{\partial v_l}{\partial \nu})$. Therefore it suffices to study (3.6)-(3.10) with $f = 0$. We can now rewrite (3.6)-(3.10) as an equivalent problem in the bounded domain $B_R\setminus\overline{D}$, namely find $w \in$

$H^1(B_R\backslash\overline{D})$, $v \in H^1(B_R\backslash\overline{D})$ such that

$$\nabla\cdot A\nabla w + k^2 nw = \ell_1 \quad \text{in} \quad B_R\backslash\overline{D} \tag{3.11}$$

$$\Delta v + k^2 v = \ell_2 \quad \text{in} \quad B_R\backslash\overline{D} \tag{3.12}$$

$$w - v = 0 \quad \text{on} \quad \partial D \tag{3.13}$$

$$\frac{\partial w}{\partial\nu_A} - \frac{\partial v}{\partial\nu} = h \quad \text{on} \quad \partial D \tag{3.14}$$

$$\frac{\partial w}{\partial\nu} = T_k w \quad \text{on} \quad \partial B_R \tag{3.15}$$

$$\frac{\partial v}{\partial\nu} = T_k v \quad \text{on} \quad \partial B_R \tag{3.16}$$

where $T_k : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ is the exterior Dirichlet to Neuman map defined by

$$T_k : g \to \frac{\partial v}{\partial\nu}|_{\partial B_R}, \qquad g \in H^{\frac{1}{2}}(\partial B_R)$$

where u is the radiating solution to the Helmholtz equation $\Delta u + k^2 u = 0$ outside B_R with boundary data $u = g$ on ∂B_R, and ν is the outward unit normal to ∂B_R [**3**]. Next we define

$$H = \{(w, v) \in H^1(B_R\backslash\overline{D}) \times H^1(B_R\backslash\overline{D}), w - v = 0 \quad on \quad \partial D\}.$$

Taking a test function $(w', v') \in H$, multiplying both sides of (3.11) by w' and (3.12) by v', and integrating by parts we obtain

$$\int\limits_{\partial B_R} T_k w\overline{w'}ds - \int\limits_{\partial D} \frac{\partial w}{\partial\nu_A}\overline{w'}ds - \int\limits_{B_R\backslash\overline{D}} A\nabla w\cdot\nabla\overline{w'}dx + \int\limits_{B_R\backslash\overline{D}} nk^2 w\overline{w'}dx = \int\limits_{B_R\backslash\overline{D}} \ell_1\overline{w'}dx,$$

and

$$\int\limits_{\partial B_R} T_k v\overline{v'}ds - \int\limits_{\partial D} \frac{\partial v}{\partial\nu}\overline{v'}ds - \int\limits_{B_R\backslash\overline{D}} \nabla v\cdot\nabla\overline{v'}dx + \int\limits_{B_R\backslash\overline{D}} k^2 v\overline{v'}dx = \int\limits_{B_R\backslash\overline{D}} \ell_2\overline{v'}dx,$$

respectively. Now taking the difference and and using the fact that $w' = v'$ on ∂D together with (3.14) we have that

$$\begin{aligned}\int\limits_{B_R\backslash\overline{D}} A\nabla w\cdot\nabla\overline{w'}dx - \int\limits_{B_R\backslash\overline{D}} \nabla v\cdot\nabla\overline{v'}dx + \int\limits_{B_R\backslash\overline{D}} (-nk^2 w\overline{w'} + k^2 v\overline{v'})dx \\ - \int\limits_{\partial B_R} T_k w\overline{w'}ds + \int\limits_{\partial B_R} T_k v\overline{v'}ds = -\int_{\partial D} h\overline{w'} - \int\limits_{B_R\backslash\overline{D}} \ell_1\overline{w'}dx + \int\limits_{B_R\backslash\overline{D}} \ell_1\overline{v'}dx.\end{aligned} \tag{3.17}$$

We define the sesquilinear form $a_k(\cdot,\cdot) : H \times H \to \mathbb{C}$ by

$$\begin{aligned}a_k((w, v), (w', v')) = \int\limits_{B_R\backslash\overline{D}} A\nabla w\cdot\nabla\overline{w'}dx - \int\limits_{B_R\backslash\overline{D}} \nabla v\cdot\nabla\overline{v'}dx \\ + \int\limits_{B_R\backslash\overline{D}} (-nk^2 w\overline{w'} + k^2 v\overline{v'})dx - \int\limits_{\partial B_R} T_k w\overline{w'} + \int_{\partial B_R} T_k v\overline{v'}dx\end{aligned}$$

and the conjugate linear functional $F(\cdot): H \to \mathbb{C}$ by

$$F(w',v') := -\int_{\partial D} h\overline{w'} - \int_{B_R\setminus\overline{D}} \ell_1 \overline{w'} dx + \int_{B_R\setminus\overline{D}} \ell_1 \overline{v'} dx.$$

Conversely, assume that $(w,v) \in H$ satisfies $a_k((w,v),(w',v')) = F(w',v')$ for all $(w',v') \in H$. Taking $v' = 0$, $w' \in C_0^\infty(B_R\setminus\overline{D})$, we have (3.11) and in a similar way we have (3.12). Taking $(w',v') \in H$ such that $w' = v' = 0$ on ∂B_R, we recover (3.12). Finally, a choice of $(w',0) \in H$ implies (3.15) and in a similar way we obtain (3.16). Hence we have proven the following theorem.

Theorem 3.2. *The exterior transmission problem (3.1)-(3.5) is equivalent to the following problem: Find $(w,v) \in H$ such that for all $(w',v') \in H$*

$$a_k((w,v),(w',v')) = F(w',v'). \tag{3.18}$$

Note that by means of the Riesz representation theorem we can define the operator $\mathcal{A}_k : H \to H$ by

$$(\mathcal{A}_k(w,v),(w',v'))_H = a_k((w,v),(w',v')) \qquad \text{for all } ((w,v),(w',v')) \in H \times H.$$

We would like to show that $\mathcal{A}_{i\kappa} : H \to H$ for $\kappa > 0$ is invertible. To prove this we use the T-coercivity approach introduced in [**2**] and [**7**], following the ideas in [**1**]. The idea behind the T-coercivity method is to consider an equivalent formulation of (3.18) where a_k is replaced by $a_k^{\mathcal{T}}$ defined by

$$a_k^{\mathcal{T}}((w,v),(w',v')) := a_k((w,v),\mathcal{T}(w',v')), \quad \forall ((w,v),(w',v')) \in H \times H, \tag{3.19}$$

with $\mathcal{T}$ being an *ad hoc* isomorphism of H. Indeed, $(w,v) \in H$ satisfies

$$a_k((w,v),(w',v')) = 0 \qquad \text{for all} \qquad (w',v') \in H$$

if, and only if, it satisfies $a_k^{\mathcal{T}}((w,v),(w',v')) = 0$ for all $(w',v') \in H$. Assume that $\mathcal{T}$ and k are chosen so that $a_k^{\mathcal{T}}$ is coercive. Then using the Lax-Milgram theorem and the fact that $\mathcal{T}$ is an isomorphism of H, one deduces that $\mathcal{A}_k$ is an isomorphism on H.

In the following, in addition to the assumptions on A and n stated at the beginning of Section 2, we assume that there exists a neighborhood Ω of ∂D where both $\Im(A) = 0$ and $\Im(n) = 0$ in $B_R\setminus\overline{D}\cap\Omega$. Setting $\mathcal{N} := B_R\setminus\overline{D}\cap\Omega$, we denote by

$$\begin{aligned} A_* &:= \inf_{x\in\mathcal{N}} \inf_{|\xi|=1} \overline{\xi}\cdot A(x)\xi > 0, \quad & A^* &:= \sup_{x\in\mathcal{N}} \sup_{|\xi|=1} \overline{\xi}\cdot A(x)\xi < \infty, \\ n_* &:= \inf_{x\in\mathcal{N}} n(x) > 0, & n^* &:= \sup_{x\in\mathcal{N}} n(x) < \infty. \end{aligned} \tag{3.20}$$

for $\xi \in \mathbb{C}^d$. Then we can prove the following result.

Lemma 3.1. *Assume that either $A^* < 1$ and $n^* < 1$ or $A_* > 1$ and $n_* > 1$. Then there exists $\kappa > 0$ such that $\mathcal{A}_{i\kappa}$ is invertible.*

Proof. We first consider the case when $A^* < 1$ and $n^* < 1$. Take $\chi \in C^\infty(\overline{B_R\setminus\overline{D}})$ to be a cut off function equal to 1 in a neighbourhood of ∂D with support in $\mathcal{N} := (B_R\setminus\overline{D})\cap\Omega$ and let $\mathcal{T}(w,v) = (w - 2\chi v, -v)$. We then have that

$$\begin{aligned} a_{i\kappa}^{\mathcal{T}}((w,v),(w,v)) \quad = \quad & (A\nabla w,\nabla w)_{B_R\setminus\overline{D}} + (\nabla v,\nabla v)_{B_R\setminus\overline{D}} - 2(A\nabla w,\nabla(\chi v))_{B_R\setminus\overline{D}} \\ & + \kappa^2((nw,w)_{B_R\setminus\overline{D}} + (v,v)_{B_R\setminus\overline{D}} - 2(nw,\chi v)_{B_R\setminus\overline{D}}) \\ & - (T_{i\kappa}w,w)_{\partial B_R} - (T_{i\kappa}v,v)_{\partial B_R} + 2(T_{i\kappa}w,\chi v)_{\partial B_R} \end{aligned}$$

where $(\cdot,\cdot)_X$ denotes the L^2-inner product in the generic space X. By Young's inequality we have

$$\begin{aligned}2|(A\nabla w,\nabla\chi v)_{B_R\backslash\overline{D}}| &\leq 2|(\chi A\nabla w,\nabla v)_{\mathcal{N}}|+2|(A\nabla w,\nabla(\chi)v)_{\mathcal{N}}|\\ &\leq \alpha(A\nabla w,\nabla w)_{\mathcal{N}}+\alpha^{-1}(A\nabla v,\nabla v)_{\mathcal{N}}\\ &+ \beta(A\nabla w,\nabla w)_{\mathcal{N}}+\beta^{-1}(A\nabla(\chi)v,\nabla(\chi)v)_{\mathcal{N}},\end{aligned}$$

and

$$2|(nw,\chi v)_{B_R\backslash\overline{D}}|\leq 2|(nw,v)_{\mathcal{N}}|\leq\eta(nw,w)_{\mathcal{N}}+\eta^{-1}(nv,v)_{\mathcal{N}}$$

for some $\alpha>0,\beta>0,\eta>0$. Recall that A and n are real in $\mathcal{N}$. Furthermore, due to the exponential decay of w and v at ∞ we have that

$$-(T_{i\kappa}w,w)_{\partial B_R}=\int\limits_{\mathbb{R}^d\backslash\overline{B_R}}\left(|\nabla w|^2+\kappa^2|w|^2\right)\,dx$$

with a similar expression for $-(T_{i\kappa}v,v)_{\partial B_R}$. Note also that

$$(T_{i\kappa}w,\chi v)_{\partial B_R}=0.$$

Using all the above estimates we finally obtain that

$$\begin{aligned}|a^{\mathcal{T}}_{i\kappa}((w,v),(w,v))| &\geq \Re\left(a^{\mathcal{T}}_{i\kappa}((w,v),(w,v))\right)\\ &\geq \Re(A\nabla w,\nabla w)_{\{B_R\backslash\overline{D}\}\backslash\overline{\Omega}}+(\nabla v,\nabla v)_{\{B_R\backslash\overline{D}\}\backslash\overline{\Omega}}\\ &+ \kappa^2\left(\Re(nw,w)_{\{B_R\backslash\overline{D}\}\backslash\overline{\Omega}}+(v,v)_{\{B_R\backslash\overline{D}\}\backslash\overline{\Omega}}\right)\\ &+ (1-\alpha-\beta)(A\nabla w,\nabla w)_{\mathcal{N}}+((I-\alpha^{-1}A)\nabla v,\nabla v)_{\mathcal{N}}\\ &+ \kappa^2(1-\eta)(nw,w)_{\mathcal{N}}+(\kappa^2(1-\eta^{-1}n)-sup|\nabla\chi|^2A_+)v,v)_{\mathcal{N}}.\end{aligned}$$

Taking α,β,η,κ such that $A^*<\alpha$, $n^*<\eta$, $\beta<1-\alpha$, and κ large enough yields that $a^S_{i\kappa}$ is coercive.

The case when $A^*>1$ and $n^*>1$ can be proven the same way using $\mathcal{T}(w,v)=(w,-v+2\chi w)$. □

REMARK 3.3. *In Lemma 3.1 the assumption that A and n are real in a neighborhood $\mathcal{N}$ of ∂D can be relaxed. In particular, the proof of Lemma 3.1 goes through if we only assume that $-\Im(A)<\Re(A)$ snd $\Im(n)<\Re(n)$ in $\mathcal{N}$.*

THEOREM 3.4. *Assume that A and n satisfies the assumptions of Lemma 3.1. Then if k is not an exterior transmission eigenvalue the exterior transmission problem (3.6)-(3.10) has a unique solution which depends continuously on the data f, h, ℓ_1 and ℓ_2.*

PROOF. From Lemma 3.1, we can choose κ such that $\mathcal{A}_{i\kappa}$ is invertible. Since the embedding from H to $L^2(B_R\backslash\overline{D})\times L^2(B_R\backslash\overline{D})$ is compact and $T_k-T_{i\kappa}$ is a compact operator from $H^{\frac{1}{2}}(\partial B_R)$ to $H^{-\frac{1}{2}}(\partial B_R)$ [**3**], we can conclude that $\mathcal{A}_k-\mathcal{A}_{i\kappa}$ is a compact, and hence the result follows from the Fredholm alternative. □

We can now prove the following discreteness result for exterior transmission eigenvalues.

THEOREM 3.5. *Assume that A and n satisfies the assumptions of Lemma 3.1. Then the set of exterior transmission eigenvalues is discrete.*

PROOF. Since T_k depends analytically on $k \in \mathbb{C}$, $\Re(k) > 0$, we have the mapping $\mathcal{A}_k - \mathcal{A}_{i\kappa} : k \to \mathcal{L}(H)$ is analytic. We can choose κ such that $\mathcal{A}_{i\kappa}$ is invertible. The theorem follows from the analytic Fredholm theory [**8**]. □

4. Uniqueness of the inverse problem

In this section we prove that the boundary of the cavity is uniquely determined from a knowledge of the scattered field $u^s(x, y)$ for all $x, y \in C$ where C is the measurement manifold introduced in Section 2. It is not necessary to know the physical properties of the inhomogeneous exterior medium as long as they satisfy appropriate a priori assumptions. The proof of uniqueness for the inverse penetrable cavity is more complicated than for the case of scattering by an impenetrable cavity considered in [**22**]. The idea of the uniqueness proof for the inverse medium scattering problem originates from [**14**], [**15**]. Here we make use of the exterior transmission problem inspired by the idea in [**11**]. Since we are using some regularity results, in this section we assume more regularity of the boundary ∂D and material properties A and n than in previous sections.

Let C be the smooth closed $d-1$ manifold of measurement satisfying Assumption 2.1 and let us define the admissible set of cavities

$$\mathbb{S} := \{D \subset \mathbb{R}^d : \partial D \text{ is of class } C^1,\ D \text{ contains } C \text{ in its interior.}\}$$

Furthermore, we assume that the media outside the cavity has the material properties (A, n) which belong to

$$\mathcal{A} := \left\{ \begin{array}{l} A, n \in C^1(\Omega_{\partial D} \setminus \overline{D}) \cap L^\infty(\mathbb{R}^d \setminus \overline{D}),\ \Omega_{\partial D} \text{ is a neighborhood of } \partial D \\ \text{and } A, n \text{ satisfy the assumptions in Section 2 and in Theorem 3.4.} \end{array} \right\}$$

We begin with a simple lemma.

LEMMA 4.1. *Assume that $A, n \in \mathcal{A}$. Let $\{v_n, w_n\} \in H^1(\mathbb{R}^d \setminus \overline{D}) \times H^1(\mathbb{R}^d \setminus \overline{D})$, $n \in \mathbb{N}$, be a sequence of solutions to the exterior transmission problem (3.6)-(3.10) with boundary data $f_n \in H^{\frac{1}{2}}(\partial D)$, $h_n \in H^{-\frac{1}{2}}(\partial D)$. If the sequences $\{f_n\}$ and $\{h_n\}$ converge in $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$ respectively, and if the sequences $\{v_n\}$ and $\{w_n\}$ are bounded in $H^1(B_R \setminus \overline{D})$, then there exists a subsequence $\{v_{n_k}\}$ which converges in $H^1(B_R \setminus \overline{D})$.*

PROOF. Let $\{v_n, w_n\}$ be as in the statement of the lemma. Due to the compact imbedding of $H^1(B_R \setminus \overline{D})$ into $L^2(B_R \setminus \overline{D})$ we can select L^2-convergent subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$. Hence, $\{v_{n_k}\}$ and $\{w_{n_k}\}$ satisfy

$$\nabla \cdot A\nabla w_{n_k} - \kappa^2 n w_{n_k} = -(\kappa^2 + k^2) w_{n_k} \quad \text{in} \quad B_R \setminus \overline{D} \tag{4.1}$$

$$\Delta v_{n_k} - \kappa^2 v_{n_k} = -(\kappa^2 + k^2) v_{n_k} \quad \text{in} \quad B_R \setminus \overline{D} \tag{4.2}$$

$$w_{n_k} - v_{n_k} = f_{n_k} \quad \text{on} \quad \partial D \tag{4.3}$$

$$\frac{\partial w_{n_k}}{\partial \nu_A} - \frac{\partial v_{n_k}}{\partial \nu} = h_{n_k} \quad \text{on} \quad \partial D \tag{4.4}$$

$$\frac{\partial w_{n_k}}{\partial \nu} - T_{i\kappa} w_{n_k} = (T_k - T_{i\kappa}) w_{n_k} \quad \text{on} \quad \partial B_R \tag{4.5}$$

$$\frac{\partial v_{n_k}}{\partial \nu} - T_{i\kappa} v_{n_k} = (T_k - T_{i\kappa}) v_{n_k} \quad \text{on} \quad \partial B_R \tag{4.6}$$

for $\kappa > 0$ chosen as in Lemma 3.1. Note that the left hand side of (4.1)-(4.6) in the variational setting is equivalent to the bounded invertible map $\mathcal{A}_{i\kappa}$. Thus

v_{n_k} and w_{n_k} are bounded by the right hand side with respect to the appropriate norm. Now, due to compactly embedding of H^1 into L^2, there is a subsequence of the right hand sides of (4.1) and (4.2) that converge in L^2. Since $T_k - T_{i\kappa}$ is a compact operator there is a subsequence of the right hand side of (4.5) and (4.6) that converge in $H^{-\frac{1}{2}}(\partial B_R)$. Hence the result follows from the boundeness of $\mathcal{A}_{ik}$. □

Note that Lemma 4.1 allows us to prove the uniqueness result without assuming that k is not an exterior transmission eigenvalue.

THEOREM 4.1. *Assume that $D_1, D_2 \in \mathbb{S}$ are two penetrable cavities having material properties $A_1, n_1 \in \mathcal{A}$ and $A_2, n_2 \in \mathcal{A}$ in the exterior of D_1 and D_2, respectively, such that the corresponding scattered fields coincide on C for all point sources located in C and any fixed wave number k. Then $D_1 = D_2$.*

PROOF. We denote by G the connected component of $D_1 \cap D_2$ which contains the region bounded by C. Let $u_j^s(\cdot, z)$ be the solution of (2.4)-(2.8) corresponding to D_j, A_j, n_j, $j = 1, 2$. We have that $u_1^s(x, z) = u_2^s(x, z)$ for $x, z \in C$. Following the argument in [**21**], the latter implies that $u_1^s(x, z) = u_2^s(x, z)$ for $x, z \in \overline{G}$. Next, assume that $\bar{D}_1$ is not included in $\bar{D}_2$. We can find a point $z \in \partial D_1$ and $\epsilon > 0$ with the following properties, where $\Omega_\delta(z)$ denotes the ball of radius δ centered at z:

(1) $\Omega_{8\epsilon}(z) \cap \bar{D}_2 = \emptyset$.
(2) The intersection $\bar{D}_1 \cap \Omega_{8\epsilon}(z)$ is contained in the connected component of $\bar{D}_1$ to which z belongs.
(3) There are points from this connected component of $\bar{D}_1$ to which z belongs which are not contained in $\bar{D}_1 \cap \bar{\Omega}_{8\epsilon}(z)$.
(4) The points $z_n := z + \dfrac{\epsilon}{n}\nu(z)$ lie in G for all $n \in \mathbb{N}$, where $\nu(z)$ is the innerward unit normal to ∂D_1 at z.

Due to the singular behavior of $\Phi(\cdot,\, z_n)$ at the point z_n, it is easy to show that

$$\|\Phi(\cdot,\, z_n)\|_{H^1(B_R\setminus\overline{D_1})} \to \infty \quad \text{as} \quad n \to \infty$$

where B_R is a large ball of radius R containing D_1 and D_2. We now define

$$v_n(x) := \frac{1}{\|\Phi(\cdot,\, z_n)\|_{H^1(B_R\setminus\overline{D_1})}}\Phi(x,\, z_n), \qquad x \in \mathbb{R}^d \setminus \overline{G}$$

and let $w_{1,n}, u_{1,n}^s$ and $w_{2,n}, u_{2,n}^s$ be the solutions of the scattering problem (2.4)-(2.8) with boundary data $f := v_n$ and $h := \partial v_n/\partial\nu$ corresponding to D_1 and D_2, respectively. Note that for each n, v_n is a radiating solution of the Helmholtz equation outside D_1 and D_2. Our aim is to prove that if $\bar{D}_1 \not\subset \bar{D}_2$ then the equality $u_1(\cdot, z) = u_2(\cdot, z)$ for $z \in G$ allows the selection of a subsequence $\{v_{n_k}\}$ from $\{v_n\}$ that converges to zero with respect to $H^1(B_R \setminus \overline{D_1})$. This certainly contradicts the definition of $\{v^n\}$ as a sequence of functions with $H^1(B_R \setminus \overline{D_1})$-norm equal to one. Note that as mentioned above we have $u_{1,n}^s = u_{2,n}^s$ in G.

We begin by noting that, since the functions $\Phi(\cdot,\, z_n)$ together with their derivatives are uniformly bounded in every compact subset of $\mathbb{R}^2 \setminus \Omega_{2\epsilon}(z)$ and $\|\Phi(\cdot,\, z_n)\|_{H^1(B_R\setminus\overline{D_1})} \to \infty$ as $n \to \infty$, then $\|v_n\|_{H^1(B_R\setminus\overline{D_2})} \to 0$ as $n \to \infty$. Hence, $\|u_{2,n}^s\|_{H^1(D_2)} \to 0$ as $n \to \infty$ from the well-posedness of the forward scattering problem. Since $u_{1,n}^s = u_{2,n}^s$ in G then $\|u_{1,n}\|_{H^1(G)} \to 0$ as $n \to \infty$ as well. Now, with the help of a cutoff function $\chi \in C_0^\infty(\Omega_{8\epsilon}(z))$ satisfying $\chi(x) = 1$ in $\Omega_{7\epsilon}(z)$, we see that $\|u_{1,n}\|_{H^1(G)} \to 0$ implies that

$$(\chi u_{1,n}) \to 0, \quad \frac{\partial(\chi u_{1,n})}{\partial \nu} \to 0, \qquad \text{as } n \to \infty \tag{4.7}$$

with respect to the $H^{\frac{1}{2}}(\partial D_1)$-norm and $H^{-\frac{1}{2}}(\partial D_1)$-norm, respectively. Indeed, for the first convergence we simply apply the trace theorem while for the convergence of $\partial(\chi u_{1,n})/\partial\nu$, we first deduce the convergence of $\Delta(\chi u_{1,n})$ in $L^2(D_1)$, which follows from $\Delta(\chi u_{1,n}) = \chi\Delta u_{1,n} + 2\nabla\chi\cdot\nabla u_{1,n} + u_{1,n}\Delta\chi$, and then apply Theorem 5.5 in [**3**]. Note here that we need conditions 2 and 4 on z to ensure $\Omega_{8\epsilon}(z) \cap D_1^e = \Omega_{8\epsilon}(z) \cap G$.

We next note that in the exterior of $\Omega_{2\epsilon}(z)$ the $H^2(\Omega_R \setminus \Omega_{2\epsilon}(z))$-norms of v_n remain uniformly bounded. Then using the interior elliptic regularity and localization techniques as in Theorem 8.8 in [**10**] we can conclude that $u^s_{1,n}$ is uniformly bounded with respect to the $H^2((\Omega_{\partial D} \cap D_1) \setminus \Omega_{4\epsilon}(z))$-norm, where $\Omega_{\partial D}$ is an open neighborhood of ∂D. Therefore, using the compact imbedding of H^2 into H^1, we can select a $H^1(\Omega_{\partial D} \cap D_1)$ convergent subsequence $\{(1-\chi)u^s_{1,n_k}\}$ from $\{(1-\chi)u^s_{1,n}\}$. Hence, $\{(1-\chi)u^s_{1,n_k}\}$ is a convergent sequence in $H^{\frac{1}{2}}(\partial D_1)$ and similarly to the above reasoning we also have that $\{\partial((1-\chi)u^s_{1,n_k})/\partial\nu\}$ converges in $H^{-\frac{1}{2}}(\partial D_1)$. This, together with (4.7), implies that the sequences

$$\{u^s_{1,n_k}\} \qquad \text{and} \qquad \left\{\frac{\partial u^s_{1,n_k}}{\partial \nu}\right\}$$

converge in $H^{\frac{1}{2}}(\partial D_1)$ and $H^{-\frac{1}{2}}(\partial D_1)$, respectively.

Finally, the functions v_{n_k} and w_{1,n_k} are solutions to the exterior transmission problem (3.6)-(3.10) for the domain $\mathbb{R}^d \setminus \overline{D}_1$ with boundary data $f = u^s_{1,n_k}$ and $h = \partial u^s_{1,n_k}/\partial\nu$. Since, the $H^1(B_r \setminus \overline{D_1})$-norms of v_{n_k} and w_{1,n_k} remain uniformly bounded, from Lemma 4.1 we can select a subsequence of $\{v_{n_k}\}$, denoted again by $\{v_{n_k}\}$, which converges in $H^1(B_r \setminus \overline{D_1})$ to some v. As H^1-limit of weak solutions to the Helmholtz equation, v is a distributional solution to the Helmholtz equation. We also have that $v|_{B_R\setminus(D_1\cup\Omega_{2\epsilon}(z))} = 0$ because the functions v_{n_k} converge uniformly to zero in the exterior of $\Omega_{2\epsilon}(z)$. Hence, v must be zero in all of $B_R \setminus \overline{D_1}$ (here we make use of condition 3). This contradicts the fact that $\|v_{n_k}\|_{H^1(B_R\setminus\overline{D_1})} = 1$. Hence the assumption $\bar{D}_1 \not\subset \bar{D}_2$ is false.
Since we can derive the analogous contradiction for the assumption $\bar{D}_2 \not\subset \bar{D}_1$, we have proved that $D_1 = D_2$. □

REMARK 4.2. *The assumptions of Theorem 3.4 required for A and n can be replaced by any other assumptions that guaranty the well-posedness of the exterior transmission problem. Also the assumption that ∂D is smooth can be relaxed as long as it guaranties $H^{1+\epsilon}$-regularity near the boundary of the solution of the corresponding transmission problem (e.g. piecewise smooth* [**9**]*).*

5. The solution of inverse problem

Now we turn our attention to reconstructing the boundary of the cavity D from a knowledge of the scattered field $u^s(x, y)$ for $x \in C$ corresponding to all point sources for $y \in C$. We will develop the linear sampling method which allows us to reconstruct D without any a priori knowledge about the physical properties of the media outside D, i.e. of A and n. The basic assumptions are Assumption

2.1, the assumptions of Theorem 3.4 and that k is not an exterior transmission eigenvalue.

Our data set defines the data operator N: $L^2(C) \to L^2(C)$ by

$$(Ng)(x) = \int_C u^s(x,y)g(y)ds(y) \quad g \in L^2(C),\, x \in C \tag{5.1}$$

which is obviously compact since it is an integral operator with analytic kernel. If we define the single layer potential v_g by

$$v_g(x) := \int_C \Phi(x,y)g(y)ds(y), \quad x \in \mathbb{R}^2 \setminus C, \tag{5.2}$$

then by linearity Ng is the scattered field evaluated on C due to v_g as incident field.

Theorem 5.1. *$N : L^2(C) \to L^2(C)$ is injective with dense range if and only if there does not exist a non-zero $g \in L^2(C)$ such that $w \in H^1_{loc}(\mathbb{R}^d \backslash \overline{D})$ and $v := v_g$ solve the homogeneous exterior transmission problem, i.e. (3.6)-(3.10) with $f = 0$ and $h = 0$.*

Proof. In a similar way as in Theorem 2.1 in [**21**], we can prove that the scattered field u^s satisfies the reciprocity condition $u^s(x,y) = u^s(y,x)$ for $x, y \in C$. Indeed

$$u^s(x,y) = \int\limits_{\partial D} \left\{ \frac{\partial u^s(\cdot,y)}{\partial \nu}\Phi(x,\cdot) - u^s(\cdot,y)\frac{\partial \Phi(x,\cdot)}{\partial \nu} \right\} ds, \qquad x \in C \tag{5.3}$$

and

$$u^s(y,x) = \int\limits_{\partial D} \left\{ \frac{\partial u^s(\cdot,x)}{\partial \nu}\Phi(y,\cdot) - u^s(\cdot,x)\frac{\partial \Phi(y,\cdot)}{\partial \nu} \right\} ds, \qquad y \in C. \tag{5.4}$$

Applying Green's second identity we have that

$$0 = \int\limits_{\partial D} \left\{ \frac{\partial u^s(\cdot,y)}{\partial \nu}u^s(\cdot,x) - u^s(\cdot,y)\frac{\partial u^s(\cdot,x)}{\partial \nu} \right\} ds \tag{5.5}$$

and since $\Phi(\cdot,\cdot)$ satisfies the radiation condition

$$0 = \int\limits_{\partial D} \left\{ \frac{\partial \Phi(x,\cdot)}{\partial \nu}\Phi(y,\cdot) - \Phi(x,\cdot)\frac{\partial \Phi(y,\cdot)}{\partial \nu} \right\} ds \tag{5.6}$$

Since $\Phi(\cdot,\cdot)$ is symmetric, subtracting (5.4) from (5.3) and adding to the result the sum of (5.5) and (5.6) we obtain

$$u^s(y,x) - u^s(x,y) = \int\limits_{\partial D} \left\{ \frac{\partial u(\cdot,y)}{\partial \nu}u(\cdot,x) - u(\cdot,y)\frac{\partial u(\cdot,x)}{\partial \nu} \right\} ds$$

where u is the total field. Now using the transmission conditions (2.3), the fact that A is symmetric, the assumptions that $A - I$ and $n - 1$ are zero in $\mathbb{R}^d \setminus B_R$ and the equation (2.2) we have that

$$\begin{aligned} u^s(y,x) - u^s(x,y) &= \int_{\partial D} \left\{ \frac{\partial w(\cdot,y)}{\partial \nu_A} w(\cdot,x) - w(\cdot,y) \frac{\partial w(\cdot,x)}{\partial \nu_A} \right\} ds \\ &= - \int_{B_R \setminus \overline{D}} \{ A\nabla w(\cdot,y) \cdot \nabla w(\cdot,x) - A\nabla w(\cdot,x) \cdot \nabla w(\cdot,y) \} dv \\ &\quad - \int_{B_R \setminus \overline{D}} \{ \nabla \cdot A\nabla w(\cdot,y) w(\cdot,x) - \nabla A \nabla w(\cdot,x) w(\cdot,y) \} dv \\ &\quad + \int_{\partial B_R} \left\{ \frac{\partial w(\cdot,y)}{\partial \nu} w(\cdot,x) - w(\cdot,y) \frac{\partial w(\cdot,x)}{\partial \nu} \right\} ds = 0, \end{aligned} \tag{5.7}$$

since the first volume integral is zero due to the symmetry of A, the second volume integral is zero due the fact that $w(\cdot,x)$ and $w(\cdot,y)$ satisfy the same equation and the last integral is zero due to the fact that $w(\cdot,x)$ and $w(\cdot,y)$ are radiating solutions to the Helmholtz equation outside B_R.

The symmetry property of u^s implies that $N^*h = \overline{N\overline{h}}$, where N^* is the L^2-adjoint of N. Hence N is injective if and only if N^* is injective, Since $\text{Ker}(N^*)^\perp = \overline{\text{Range}(N)}$ to prove the theorem we must only prove that N is injective. To this end, let a non-zero $g \in L^2(C)$ be such that $(Ng)(x) = 0, x \in C$. Let $v_g(x) = \int_C \phi(x,z) g(z) ds(z)$, and consider $(\tilde{w}, \tilde{v})$ the unique solution of (3.6)-(3.10) with $f := v_g$ and $g := \frac{\partial v_g}{\partial \nu}$. By superposition $\tilde{v}(x) = (Ng)(x)$, which means that $(Ng)(x) = 0, x \in C$, is equivalent to the fact that $\tilde{v}(x) = 0, x \in C$. Furthermore we have $\Delta \tilde{v} + k^2 \tilde{v} = 0$ in the domain bounded by C and since k satisfies Assumption 2.1 we have $\tilde{v} = 0$ inside C. But $\Delta \tilde{v} + k^2 \tilde{v} = 0$ in D and hence by analyticity $\tilde{v} = 0$ in D. The latter implies that $\tilde{w}$ and v_g satisfy the homogeneous exterior transmission problem. This proves the theorem. □

The above theorem implies:

Corollary 5.1. *If k is not an exterior transmission eigenvalue then the operator $N : L^2(C) \to L^2(C)$ is injective with dense range*

For the rest of the paper we need to assume that k is not an exterior transmission eigenvalue in addition to Assumption 2.1.

We now introduce the data equation

$$(Ng)(x) = \Phi(x,z) \quad \forall \quad x \in C \quad \textit{and} \quad x \neq z \tag{5.8}$$

where z is a sampling point in $\mathbb{R}^d$. This is an ill-posed linear equation whose regularized solution will be the indicator function of the cavity. To this end we investigate the solvability of (5.8).

We first define $\overline{\mathcal{U}}$ to be the closure of the set

$$\mathcal{U} := \left\{ \int_C \phi(\cdot,z) g(y) ds(y),\ g \in L^2(C) \right\} \text{ with respect to } H^1_{loc}(\mathbb{R}^d \setminus \overline{D}).$$

Lemma 5.1. *Let*

$$\mathcal{U}_0 = \left\{ u \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D}) :\ \Delta u + k^2 u = 0 \ in\ \mathbb{R}^d \setminus \overline{D},\ \lim_{r\to\infty} r^{\frac{d-1}{2}} (\frac{\partial v}{\partial r} - ikv) = 0 \right\}.$$

Then $\overline{\mathcal{U}} = \mathcal{U}_0$

Proof. By the well-posedness of the problem

$$\Delta u + k^2 u = 0 \qquad in \quad \mathbb{R}^d\backslash\overline{D}$$
$$u = g \qquad on \quad \partial D$$
$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial v}{\partial r} - ikv) = 0$$

we have $||u||_{H^1_{loc}(\mathbb{R}^d\backslash\overline{D})} \leq c||g||_{H^{\frac{1}{2}}(\partial D)}$, where c is some constant. Then $\mathcal{U}$ is dense in $\mathcal{U}_0$ if we can show that $\{\int_C \phi(\cdot,z)g(z)ds(z)|_{\partial D}, g \in L^2(C)\}$ is dense in $H^{\frac{1}{2}}(\partial D)$. In fact, let $f \in H^{-\frac{1}{2}}(\partial D)$ be such that for any $g \in L^2(C)$

$$\int_{\partial D}\int_C \phi(x,y)g(y)ds(y)f(x)ds(x) = 0,$$

that is

$$\int_C\int_{\partial D} \phi(x,y)f(x)ds(x)g(y)ds(y) = 0.$$

Then

$$\int_{\partial D} \phi(x,y)f(x)ds(x) = 0, \qquad \forall y \in C. \tag{5.9}$$

Then the single layer potential $v_f(x) = \int_{\partial D}\phi(x,y)f(y)ds(y)$ satisfies $v_f|_C = 0$ and the Helmholtz equation in the bounded domain circumscribed by C. Since k satisfies Assumption 2.1, we have that $v_f = 0$ inside C and by analyticity we have that $v_f = 0$ in D. From the jump conditions across ∂D

$$v_f^- = v_f^+ \qquad on \quad \partial D$$
$$f = \frac{\partial v_f^-}{\partial\nu} - \frac{\partial v_f^+}{\partial\nu} \qquad on \quad \partial D$$

where $+$ and $-$ denote approaching the boundary from outside and inside ∂D, respectively, we now have that $v_f^+ = 0$. Since v_f is a radiating solution to the Helmholtz equation, from uniqueness of the exterior Dirichlet problem we have $v_f = 0$ in $\mathbb{R}^d\backslash\overline{D}$ and hence we have $f = \frac{\partial v_f^-}{\partial\nu} - \frac{\partial v_f^+}{\partial\nu} = 0$. Thus the set $\{v_g,\, g \in L^2(C)\}$, v_g defined by (5.2), is dense in $H^{\frac{1}{2}}(\partial D)$. Finally, note that since $\mathcal{U}_0$ is closed in $H^1_{loc}(\mathbb{R}^d\backslash\overline{D})$ and $\{v_g, g \in L^2(C)\}$ is dense in $\mathcal{U}_0$ we have that $\overline{\mathcal{U}} = \mathcal{U}_0$. □

Now we define $\mathcal{U}(\partial D) := \{(u|_{\partial D}, \frac{\partial u}{\partial\nu}|_{\partial D}), u \in \overline{\mathcal{U}}\}$.

Lemma 5.2. *$\mathcal{U}(\partial D)$ is closed in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ and hence is a Hilbert space.*

Proof. Let $(f,h) \in H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$. If $(f,h) \in \overline{\mathcal{U}}(\partial D)$ then there exists a sequence $\{u_n\}_{n=1}^\infty$ in $\mathcal{U}$ such that

$$\left(u_n|_{\partial D}, \frac{\partial u_n}{\partial\nu}|_{\partial D}\right) \to (f,h) \quad as \quad n \to \infty.$$

Clearly, $(u_n|_{\partial D}, \frac{\partial u_n}{\partial\nu}|_{\partial D})$ is bounded in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ and u_n satisfies

$$\Delta u_n + k^2 u_n = 0 \quad in \quad \mathbb{R}^d\backslash\overline{D}$$
$$u_n = u_n|_{\partial D} \quad on \quad \partial D$$
$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial u_n}{\partial r} - iku_n) = 0.$$

Thus, from the well-posedness of the exterior Dirichlet problem $||u_n||_{H^1_{loc}(\mathbb{R}^d\backslash\overline{D})}$ is bounded by $||u_n||_{H^{\frac{1}{2}}(\partial D)}$ and therefore $\{u_n\}$ is bounded in $H^1_{loc}(\mathbb{R}^d\backslash\overline{D})$. Hence there exists $u \in \overline{\mathcal{U}}$ such that u_n converges to u weakly. Since the trace operator $H^1_{loc}(\mathbb{R}^d\backslash\overline{D}) \to H^{\frac{1}{2}}(\partial D)$ and $H^1_{loc,\Delta}(\mathbb{R}^d\backslash\overline{D}) \to H^{-\frac{1}{2}}(\partial D)$ are bounded [**3**], we obtain

$$(u_n|_{\partial D}, \frac{\partial u_n}{\partial \nu}|_{\partial D}) \text{ converges to } (u|_{\partial D}, \frac{\partial u}{\partial \nu}|_{\partial D}) \text{ weakly.}$$

Hence $f = u|_{\partial D}, h = \frac{\partial u}{\partial \nu}|_{\partial D}$, which implies that $\mathcal{U}(\partial D)$ is closed in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$. □

Definition 5.2. *The operator $B : \mathcal{U}(\partial D) \to L^2(C)$ maps $\big(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}\big)$, $v \in \overline{\mathcal{U}}$, to $u^s_v|_C$ where (u^s_v, w_v) is the unique solution of (2.4)-(2.8) with $f := v|_{\partial D}$ and $h := \frac{\partial v}{\partial \nu}|_{\partial D}$.*

Theorem 5.3. *Assume k is not an exterior transmission eigenvalue. Then $B : \mathcal{U}(\partial D) \to L^2(C)$ is compact, injective and has dense range in $L^2(C)$.*

Proof. The solution $u^s_v \in H^1(D)$ depends continuously on $\big(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}\big)$. Since $u^s_v|_C \in H^{\frac{1}{2}}(C)$ and the imbedding $H^{\frac{1}{2}}(C) \to L^2(C)$ is compact, we have B is compact.

Next, if $B(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}) = 0$, we have $u^s_v|_C = 0$. But in addition we have $\Delta u^s_v + k^2 u^s_v = 0$ inside C and since k satisfies Assumption 2.1 we have $u^s_v = 0$ inside C, and by the unique continuation principle $u^s_v = 0$ in D. Then w_v and v satisfy

$$\nabla \cdot A \nabla w_v + k^2 n w_v = 0 \qquad in \quad \mathbb{R}^d\backslash\overline{D}$$
$$\Delta v + k^2 v = 0 \qquad in \quad \mathbb{R}^d\backslash\overline{D}$$
$$w_v = v \qquad on \quad \partial D$$
$$\frac{\partial w_v}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \qquad on \quad \partial D$$
$$\lim_{r\to+\infty} r^{\frac{d-1}{2}}(\frac{\partial w_v}{\partial r} - ikw_v) = \lim_{r\to+\infty} r^{\frac{d-1}{2}}(\frac{\partial v}{\partial r} - ikv) = 0.$$

Since k is not an exterior transmission eigenvalue, we have $v = 0$ in $\mathbb{R}^d\backslash\overline{D}$ and thus $(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}) = 0$. Hence B is injective.

Finally, since Range$(N) \subset$ Range(B), from Corollary 5.1 we can conclude that the range of B is also dense in $L^2(C)$. □

Theorem 5.4. *Assume that k is not an exterior transmission eigenvalue. Then $\Phi(\cdot, z)$ is in the range of B if and only if $z \in \mathbb{R}^d\backslash\overline{D}$.*

Proof. If $z \in \mathbb{R}^d\backslash\overline{D}$ and k is not an exterior transmission eigenvalue then from Theorem 3.4, we have that the exterior transmission problem

$$\nabla \cdot A \nabla w_z + k^2 n w_z = 0 \qquad in \quad \mathbb{R}^d\backslash\overline{D} \tag{5.10}$$

$$\Delta v_z + k^2 v_z = 0 \qquad in \quad \mathbb{R}^d\backslash\overline{D} \tag{5.11}$$

$$w_z - v_z = \Phi(\cdot, z) \qquad on \quad \partial D \tag{5.12}$$

$$\frac{\partial w_z}{\partial \nu} - \frac{\partial v_z}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \qquad on \quad \partial D \tag{5.13}$$

$$\lim_{r\to+\infty} r^{\frac{d-1}{2}}(\frac{\partial v_z}{\partial r} - ikv_z) = \lim_{r\to+\infty} r^{\frac{d-1}{2}}(\frac{\partial w_z}{\partial r} - ikw_z) = 0 \tag{5.14}$$

has a unique solution $(w_z, v_z) \in H^1_{loc}(\mathbb{R}^d\backslash\overline{D}) \times H^1_{loc}(\mathbb{R}^d\backslash\overline{D})$. Then $(w_z, \Phi(\cdot, z))$ satisfies (2.4)-(2.8) with $(f, h) = (v_z, \frac{\partial v_z}{\partial \nu})|_{\partial D}$. Since $v_z \in \overline{\mathcal{U}}$, we have $B(v_z, \frac{\partial v_z}{\partial \nu}) = \Phi(\cdot, z)|_C$, which means that $\Phi(x, z)$ for $x \in C$ is in the range of B.

Now assume that, for $z \in D$, $\Phi(\cdot, z)$ is in the range of B. Then there exists $v \in \overline{\mathcal{U}}$ such that

$$B(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}) = \Phi(x, z), \quad x \in C.$$

Let w_v, u^s_v be the solution to (2.4)-(2.8) with $(f, h) = (v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D})$. By definition of B, $u^s_v = \Phi(\cdot, z)$ in D but this is not possible since $\Phi(\cdot, z) \notin H^1(D_0)$. □

As the last ingredient to the main theorem of this section we define the bounded linear operator $S : L^2(C) \to \mathcal{U}(\partial D)$ by

$$(Sg)(x) = \left(v_g|_{\partial D}, \frac{\partial v_g}{\partial \nu}|_{\partial D}\right), \qquad \text{where } v_g \text{ is defined by (5.2).}$$

Obviously we have that the data operator N can be factorized as

$$Ng = BSg.$$

We can prove the following denseness result for the operator S.

THEOREM 5.5. *The bounded linear operator $S : L^2(C) \to \mathcal{U}(\partial D)$ is injective with dense range.*

PROOF. If g is such that $Sg = 0$ then $v_g(x) = \int_C \phi(x, y)g(y)ds(y)$ satisfies

$$\Delta v_g + k^2 v_g = 0 \qquad in \quad \mathbb{R}^d\backslash\overline{D}$$

$$v_g = 0 \quad on \quad \partial D$$

$$\lim_{r\to+\infty} r^{\frac{d-1}{2}}(\frac{\partial v_g}{\partial r} - ikv_g) = 0.$$

Then $v_g = 0$ in $\mathbb{R}^d\backslash\overline{D}$, and since $\Delta v_g + k^2 v_g = 0$ in $\mathbb{R}^d\backslash C$, by the unique continuation principle $v_g = 0$ outside C. In particular the single layer boundary integral operator

$$\int_C \phi(x, y)g(y)ds(y), \qquad g \in L^2(C), \qquad x \in C$$

is invertible as long as k^2 is not Dirichlet eigenvalue for $-\Delta$ inside C [**19**]. Hence $g = 0$.

Next, since $\{v_g, \quad g \in L^2(C)\}$ is dense in $\overline{\mathcal{U}}$ by definition, we have that S has dense range in $\mathcal{U}(\partial D)$. □

Now we are ready to prove the main theorem of this section which provides the basis for the linear sampling method.

THEOREM 5.6. *Assume that k is not an exterior transmission eigenvalue eigenvalue and satisfies Assumption 2.1. Let u^s be the scattered field corresponding to the scattering problem (2.4)-(2.8) and N is the associated data operator. Then the following hold:*

(1) *For* $z \in \mathbb{R}^d \setminus \bar{D}$ *and a given* $\epsilon > 0$ *there exists a function* $g_z^\epsilon \in L^2(C)$ *such that*
$$\|Ng_z^\epsilon - \Phi(\cdot, z)\|_{L^2(C)} < \epsilon,$$
and as $\epsilon \to 0$, *the potential* $v_{g_z^\epsilon}$ *given by (5.2) with kernel* g_z^ϵ *converges to the solution* v_z *in the* $H^1(B_R \setminus \overline{D})$*-norm where* (w_z, v_z) *is the solution of (5.10)-(5.14).*

(2) *For* $z \in D \setminus C$ *and a given* $\epsilon > 0$, *every* $g_z^\epsilon \in L^2(C)$ *that satisfies*
$$\|Ng_z^\epsilon - \Phi(\cdot, z)\|_{L^2(C)} < \epsilon,$$
is such that
$$\lim_{\epsilon \to 0} \|v_{g_z^\epsilon}\|_{H^1(B_R \setminus \overline{D})} = \infty.$$

Proof. (1) Let $z \in \mathbb{R}^d \setminus \overline{D}$. Then from Theorem 5.4, $\Phi(\cdot, z)$ is in the range of B and
$$B(v_z|_{\partial D_0}, \frac{\partial v_z}{\partial \nu}|_{\partial D}) = \Phi(\cdot, z),$$
where (w_z, v_z) is the solution of (5.10)-(5.14). Now, for $\epsilon > 0$, since S has dense range in $\mathcal{U}(\partial D)$ by Theorem 5.5, there exists $g_z^\epsilon \in L^2(C)$ satisfying
$$\left\| Sg_z^\epsilon - \left(v_z|_{\partial D}, \frac{\partial v_z}{\partial \nu}|_{\partial D} \right) \right\|_{H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)} < \frac{\epsilon}{||B||} \tag{5.15}$$
which yields
$$\left\| BSg_z^\epsilon - B(v_z|_{\partial D}, \frac{\partial v_z}{\partial \nu}|_{\partial D}) \right\|_{L^2(C)} < \epsilon.$$
The latter can be re-written as
$$\|Ng_z^\epsilon - \Phi(\cdot, z)\|_{L^2(C)} < \epsilon.$$
Furthermore,
$$\lim_{\epsilon \to 0} \left\| Sg_z^\epsilon - \left(v_z|_{\partial D}, \frac{\partial v_z}{\partial \nu}|_{\partial D} \right) \right\|_{H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)} = 0$$
and hence
$$\lim_{\epsilon \to 0} ||v_{g_z^\epsilon} - v_z||_{H^1_{loc}(B_R \setminus \overline{D})} = 0.$$

Furthermore for a fixed $\epsilon > 0$, we observe that $u^s := \Phi(\cdot, z)$ and $w := w_z$ satisfy the scattering problem (2.4)-(2.8) with data $f := v_z|_{\partial D}$ and $h := \frac{\partial v_z}{\partial \nu}|_{\partial D}$. From the well-posedness of (2.4)-(2.8) and the fact that $\|\Phi(\cdot, z)\|_{H^1(D)}$ goes to infinity as $z \to \partial D$, we obtain that
$$\lim_{z \to \partial D} \left\| \left(v_z|_{\partial D}, \frac{\partial v_z}{\partial \nu}|_{\partial D} \right) \right\|_{H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)} = \infty$$
and hence
$$\lim_{z \to \partial D} ||Sg_z^\epsilon||_{H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)} = \infty.$$
Since $||Sg_z^\epsilon||_{H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)}$ is bounded bellow by $||v_{g_z^\epsilon}||_{H^1(B_R \setminus \overline{D})}$, we can conclude that
$$\lim_{z \to \partial D} ||v_{g_z^\epsilon}||_{H^1(B_R \setminus \overline{D})} = \infty \quad \text{and} \quad \lim_{z \to \partial D} ||g_z^\epsilon||_{L^2(C)} = \infty.$$

(2). In order to prove the second statement, for $z \in D \setminus C$ assume to the contrary that there exists a sequence $\{\epsilon_n\} \to 0$ and corresponding functions v_{g_n} with kernels $g_n := g_z^{\epsilon_n}$ satisfying $\|Ng_n - \Phi(\cdot, z)\|_{L^2(C)} < \epsilon_n$ (i.e. $Ng_n \to \Phi(\cdot, z)$

in $L^2(C)$ as $n \to \infty$) such that $\|v_n\|_{H^1(B_R\setminus\overline{D})}$ remains bounded. Then without loss of generality we may assume weak convergence v_n to some $v \in H^1(B_R\setminus\overline{D})$. Let us define $\tau : v \to \big(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}\big)$ which is obviously a bounded operator from $H^1(B_R\setminus\overline{D})$ to $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$. Since $B\tau$ is also bounded, we can conclude the weak convergence $(B\tau)v_{g_n} \rightharpoonup (B\tau)v$ in $L^2(C)$ as $n \to \infty$. But $(B\tau)v_{g_n} = Ng_n$ converges strongly to $\Phi(\cdot, z)|_C$ as $n \to \infty$, which means $\Phi(\cdot, z) = B(\tau v)$. This contradicts Theorem 5.4. □

This theorem can be used to reconstruct the boundary ∂D, since roughly it says that if g_z^ϵ is the approximate solution of $Ng_z^\epsilon = \Phi(\cdot, z)$ provided by Theorem 5.6 then $\|v_{g_z^\epsilon}\|_{H^1(B_R\setminus\overline{D})}$ is large z in D and small for z outside D, for fixed ϵ. Unfortunately, $\|v_{g_z^\epsilon}\|_{H^1(B_R\setminus\overline{D})}$ can not be used as indicator function for D since it depends on D. Instead in practice we use the indicator function $I(z) := \|g_z^\epsilon\|_{L^2(C)}$. Since the data equation (5.8) is ill-posed, it is necessary to use regularization techniques, e.g. Tikhonov regularization. The question if the Tikhonov regularized solution of (5.8) captures the approximate solution g_z^ϵ provided by Theorem 5.6 remains open.

The linear sampling method for the reconstruction of ∂D can now be described as follows.

- Choose a set of sampling points in a region covering the expected obstacle.
- For each sampling point z, solve the regularized version of the data equation,
$$\alpha g + N^*Ng = N^*\Phi(\cdot, z)$$
with a regularization parameter $\alpha > 0$.
- Calculate the indicator function $I(z)$.
- Plot $I(z)$. Then the cavity D is the region containing points z for which $I(z) > C$ for a cut-off value C chosen by ad-hoc procedure (some procedures for choosing C are available in the literature (see e.g. [**5**] and the references therein).

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