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The inverse electromagnetic scattering problem for a mixed boundary value problem for screens

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Dedicated to Professor David Colton on the occasion of his 60th birthday!

Abstract

We consider the inverse scattering problem of determining the shape of mixed perfectly conducting-impedance screens from a knowledge of the incident time harmonic electromagnetic plane wave and the electric far field pattern of the scattered wave. We adapt the linear sampling method invented by Colton and Kirsch (Inverse Problems 12 (1996) 383) for the case of scattering by obstacles with nonempty interior. Numerical examples are given for mixed screens in $\mathbb{R}^3$.

Keywords: Electromagnetic inverse scattering; Screens; Mixed boundary value problem

1. Introduction

The inverse scattering problem we consider in this paper is to determine the shape of a scattering object from a knowledge of the incident time harmonic electromagnetic plane wave and the electric far field pattern of the scattered wave. In some applications the scatterer is very thin, for which the thickness is small compared to the wavelength and other characteristic lengths. Furthermore the scatterer may be a perfect conductor on one side and coated by a dielectric on the other side. It is then convenient to model it by an open surface in $\mathbb{R}^3$, called a *screen*, that satisfies a perfectly conducting boundary condition on one side and an impedance boundary condition on the other side. The difficulty in solving the inverse

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scattering problem for mixed screens by using iterative methods such as the Newton method [18] etc., lies in the amount of a priori information on the geometrical and physical properties of the scatterer needed in order to implement the inversion scheme. The *linear sampling method*, introduced by Colton and Kirch in 1996 [12] for the Helmholtz equation with Dirichlet boundary conditions and further developed for more complicated boundary conditions and Maxwell equations (see e.g. [8,5,11]), is very suitable to arrive at the solution of the inverse problem for screens with mixed type boundary conditions. The linear sampling method was first adapted to obstacles with empty interior by Colton and one of the present authors in [6] for the case of mixed cracks in $\mathbb{R}^2$ and then a modified version was used in [7] to reconstruct perfectly conducting screens in $\mathbb{R}^3$.

The goal of this paper is to establish the validity of the linear sampling method for the solution of the three dimensional electromagnetic inverse scattering problem for screens with mixed perfectly conducting-impedance boundary condition. As in [6] in order to establish this goal it is first necessary to establish the well posedness of the corresponding direct problem. To our knowledge in the case of Maxwell's equation the existing literature covers only the direct scattering problem for perfectly conducting screens [1,2]. In particular, for screens with mixed boundary conditions the integral equation approach used in [1,2,6] becomes rather complicated. In Section 2 we use a variational method in suitable Hilbert spaces to solve the direct problem. In this section, using the ideas of [8], we also establish an approximation property of the traces of electromagnetic Herglotz pairs which is necessary for the inversion scheme given in Section 3. To this end we investigate the trace on both the sides of the screen of functions in the solution space of the forward scattering problem. The analysis of the linear sampling method is based on a proper factorization of the far field operator, an a priori estimate for the solution of the forward problem in terms of the boundary data and on the regularization theory of ill-posed equations. Finally, in Section 4, we present some numerical examples that establish the viability of our approach.

2. The direct scattering problem

We consider the scattering of a time-harmonic electromagnetic plane wave by a very thin perfectly conducting obstacle in $\mathbb{R}^3$ that is coated on one side with a dielectric material. The positive valued function λ describes the material properties of this coating. In particular let Γ be a bounded, simply connected, orientated, piecewise smooth open surface in $\mathbb{R}^3$ bounded by a piecewise smooth boundary curve l. We consider Γ as part of a piecewise smooth boundary ∂D of some bounded domain $D \subset \mathbb{R}^3$. Let ν denote the normal vector to Γ that coincides with the outward normal vector defined almost everywhere on ∂D. Furthermore, for a vector field u, we denote by $\nu\times u^+|_\Gamma$, $\gamma_T^+ u|_\Gamma$ and $\nu\cdot u^+|_\Gamma$, ($\nu\times u^-|_\Gamma$, $\gamma_T^- u|_\Gamma$ and $\nu\cdot u^-|_\Gamma$) the restriction to Γ of the traces $\nu\times u|_{\partial D}$, $\gamma_T u|_{\partial D}$ and $\nu\cdot u|_{\partial D}$ respectively, from the outside (from the inside) of ∂D where $\gamma_T u := \nu\times(u\times\nu)$ is the tangential component of u.

The scattering of electromagnetic waves by a partially coated open surface Γ (the screen) leads to the following boundary value problem for the total electric field E:

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{\Gamma}, \tag{1a}$$

$$\gamma_T^- E = 0 \quad \text{on } \Gamma, \tag{1b}$$

$$\nu\times\operatorname{curl} E^+ - i\lambda\gamma_T^+ E = 0 \quad \text{on } \Gamma, \tag{1c}$$

$$E = E^{\mathrm{s}} + E^{\mathrm{i}}, \tag{1d}$$

where E^{i} is the given incident electric field and E^{s} is the scattered electric wave. The scattered field E^{s}, satisfies the Silver–Müller radiation condition

$$\lim_{r\to\infty}(\operatorname{curl} E^{\mathrm{s}} \times x - \mathrm{i}kr E^{\mathrm{s}}) = 0 \tag{2}$$

uniformly in $\hat{x} = x/|x|$, where $r = |x|$ and we consider incident plane electromagnetic wave given by

$$E^{\mathrm{i}}(x; d, p) := \frac{\mathrm{i}}{k} \operatorname{curl}\operatorname{curl} p\, \mathrm{e}^{\mathrm{i}kd\cdot x} = \mathrm{i}k(d \times p) \times d\, \mathrm{e}^{\mathrm{i}kd\cdot x},$$

$$H^{i}(x; d, p) := \operatorname{curl} p\, \mathrm{e}^{\mathrm{i}kd\cdot x} = \mathrm{i}kd \times p\, \mathrm{e}^{\mathrm{i}kd\cdot x},$$

where $k > 0$ is the wave number, $d \in \Omega := \{x \in \mathbb{R}^3 : \ |x| = 1\}$ is a unit vector giving the direction of the incident plane wave and $p \in \mathbb{R}^3$ is the polarization. We assume that $\lambda \in L_\infty(\Gamma)$ and $\lambda(x) \geqslant \lambda_0 > 0$.

2.1. Solution of the forward problem

Let us first recall the definition of the following Sobolev spaces

$$H(\operatorname{curl}, B_R\backslash\overline{\Gamma}) := \left\{u \in (L^2(B_R\backslash\overline{\Gamma}))^3 : \ \operatorname{curl} u \in (L^2(B_R\backslash\overline{\Gamma}))^3\right\},$$

$$L^2_t(\Gamma) := \left\{u \in (L^2(\Gamma))^3 : \ v \cdot u = 0 \text{ on } \Gamma\right\},$$

where B_R is a ball that contains Γ and denote by $H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3\backslash\overline{\Gamma})$ the space of $u \in H(\operatorname{curl}, B_R\backslash\overline{\Gamma})$ for all B_R with radius R large enough. Then we define the Sobolev space

$$X_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3\backslash\overline{\Gamma}) := \{u \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3\backslash\overline{\Gamma}) \ : \ v \times u^+|_\Gamma \in L^2_t(\Gamma)\}, \tag{3}$$

equipped with the natural norm

$$\| u\|^2_{X(\operatorname{curl}, B_R\backslash\overline{\Gamma})} := \| u\|^2_{H(\operatorname{curl}, B_R\backslash\overline{\Gamma})} + \| v \times u^+\|^2_{L^2(\Gamma)}. \tag{4}$$

Now we can precisely formulate the forward scattering problem: Given $E^{\mathrm{i}} \in X_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3\backslash\overline{\Gamma})$ find $E \in X_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3\backslash\overline{\Gamma})$ satisfying (1a)–(1d) and (2). We will refer to this problem as (MSP).

Theorem 2.1. (MSP) *has at most one solution*.

Proof. Let ∂D be a closed surface containing Γ and enclosing the bounded domain D. We first apply the vector Green's formula for the solution $E \in X_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3\backslash\overline{\Gamma})$ and $H = \frac{1}{\mathrm{i}k} \operatorname{curl} E$ of (MSP) corresponding to $E^{\mathrm{i}} = 0$, in D and in $D_e \cap B_R$ where D_e is exterior domain $D_e := \mathbb{R}^3\backslash\overline{D}$ and B_R is a ball of radius $R > 0$ containing D. Since $E \in H(\operatorname{curl}, B_R\backslash\overline{\Gamma})$ and from the equation $\operatorname{curl} E \in H(\operatorname{curl}, B_R\backslash\overline{\Gamma})$ we have that $v \times E$ and $v \times H$ are continuous across $\partial D\backslash\overline{\Gamma}$. Hence using the impedance condition on Γ^+ we obtain

$$\int_D \left(|\operatorname{curl} E|^2 - k^2|E|^2\right) \mathrm{d}v - \int_{D_e\cap B_R} \left(|\operatorname{curl} E|^2 - k^2|E|^2\right) \mathrm{d}v$$
$$+ \mathrm{i}k \int_{S_R} (v \times \overline{E}) \cdot H \,\mathrm{d}s + \mathrm{i}\int_\Gamma \lambda\, |\gamma_T^+ E|^2 \,\mathrm{d}s = 0. \tag{5}$$

Taking the imaginary part of (5) we now obtain

$$\mathrm{Re}\int_{S_R}(\nu\times\overline{E})\cdot H\,\mathrm{d}s=-\frac{1}{k}\int_{\Gamma}\lambda\,|\gamma_T^+E|^2\,\mathrm{d}s\leqslant 0.$$

Hence the uniqueness follows from [13], Theorem 6.10, and the unique continuation principle.

We now prove the existence of a solution to (MSP) by using a variational method. To this end let ∂D be a closed surface containing Γ and enclosing the bounded domain D, $D_e:=\mathbb{R}^3\backslash\overline{D}$ and define

$$X^0(\mathrm{curl},B_R\backslash\overline{\Gamma}):=\{u\in X(\mathrm{curl},B_R\backslash\overline{\Gamma}):\ \nu\times u^-|_\Gamma=0\}.$$

Integrating by parts in D and D_e, and using the continuity of $\nu\times E$ and $\nu\times\mathrm{curl}\,E$ across $\partial D\backslash\overline{\Gamma}$ (see the proof of Theorem 2.1), we obtain an equivalent variation formulation for (MSP) as follows: Find $E\in X^0(\mathrm{curl},B_R\backslash\overline{\Gamma})$ such that

$$\begin{aligned}&\int_D\left(\mathrm{curl}\,E\cdot\mathrm{curl}\,\overline{\phi}-k^2E\cdot\overline{\phi}\right)\mathrm{d}v+\int_{D_e\cap B_R}\left(\mathrm{curl}\,E\cdot\mathrm{curl}\,\overline{\phi}-k^2E\cdot\overline{\phi}\right)\mathrm{d}v\\&\quad-\mathrm{i}\int_\Gamma\lambda\gamma_T^+E\cdot\gamma_T^+\overline{\phi}\,\mathrm{d}s+\mathrm{i}k\int_{S_R}G_e(\hat{x}\times E)\cdot\left[\hat{x}\times(\overline{\phi}\times\hat{x})\right]\mathrm{d}\hat{x}\\&\quad=-\mathrm{i}\int_\Gamma\lambda\gamma_T^+E^{\mathrm{i}}\cdot\gamma_T^+\overline{\phi}\,\mathrm{d}s+\mathrm{i}k\int_{S_R}\left[G_e(\hat{x}\times E^{\mathrm{i}})-\hat{x}\times H^i\right]\cdot\left[\hat{x}\times(\overline{\phi}\times\hat{x})\,\mathrm{d}\hat{x}\right]\mathrm{d}\hat{x}\end{aligned}\tag{6}$$

for every test function $\phi\in X^0(\mathrm{curl},B_R\backslash\overline{\Gamma})$. Here G_e is the exterior Calderon operator (c.f. [17,22]) which maps a tangential vector field λ on S_R to $\hat{x}\times H$ where (E,H) satisfies

$$\begin{aligned}\nabla\times E-ikH=0&\quad\text{in }\mathbb{R}^3\backslash\overline{B}_R\\\nabla\times H+ikE=0&\quad\text{in }\mathbb{R}^3\backslash\overline{B}_R\\\hat{x}\times E=\lambda&\quad\text{on }S_R\\\lim_{r\to\infty}(H\times x-rE)=0.&\quad\square\end{aligned}$$

In the following we denote by $(\cdot,\cdot)$ the L^2 inner product and by $\langle\cdot,\cdot\rangle$ the duality pairing between a space and its dual. To establish the existence of a solution of the above variational problem we need the following technical lemmas.

Lemma 2.2. *Let us define*

$$S:=\{H^1(B_R\backslash\overline{\Gamma}):p^-|_\Gamma=0\ \textit{and}\ p^+|_\Gamma=c\}$$

and

$$\tilde{X}^0:=\left\{u\in X^0(\mathrm{curl},B_R\backslash\overline{\Gamma}):-k^2(u,\nabla q)_{B_R}+ik\langle G_e(\hat{x}\times u),\nabla_{S_R}q\rangle=0\ \forall q\in S\right\}.$$

Then ∇S *is a closed subset of* $X^0(\mathrm{curl},B_R\backslash\overline{\Gamma})$ *and*

$$X^0(\mathrm{curl},B_R\backslash\overline{\Gamma})=\tilde{X}^0\oplus\nabla S.$$

This lemma is known as the Helmholtz decomposition and the proof is entirely classical (see e.g. [4,9,17,22]).

Lemma 2.3. *The space $\tilde{X}^0$ is compactly imbedded in $L^2(B_R)$.*

Proof. Consider a bounded sequence $\{u_j\}_{j=1}^{\infty}$ in $\tilde{X}^0$. Each function in u_j can be extended to all $\mathbb{R}^3$ by solving the exterior Maxwell problem

$$\nabla \times (\nabla \times v_j) - k^2 v_j = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{B}_R,$$

$$\hat{x} \times v_j = \hat{x} \times v_j \quad \text{on } \partial B_R,$$

together with the Silver–Müller radiation condition at infinity. The extended function u_j^e defined by

$$u_j^e = \begin{cases} u_j & \text{on } B_R, \\ v_j & \text{on } \mathbb{R}^3 \backslash \overline{B}_R \end{cases}$$

is in $H_{\mathrm{loc}}(\mathrm{curl}, B_R \backslash \overline{\Gamma})$ since the tangential components are continuous across S_R. Noting that the condition in $\tilde{X}_0$ is a weak form of

$$\begin{cases} \nabla \cdot u = 0 & \text{in } B_R \backslash \overline{\Gamma} \\ k^2 \hat{x} \cdot u = ik \nabla_{S_R} \cdot G_e(\hat{x} \times u) & \text{on } S_R, \end{cases} \tag{7}$$

we have that the extended function has a well-defined divergence and

$$\nabla \cdot (u_j^e) = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{\Gamma}.$$

Now we choose a cutoff function $\chi \in C_0^{\infty}(\mathbb{R}^3)$ such that $\chi = 1$ in $\overline{B}_R$ and χ is supported in $\mathcal{O} \supset \overline{B}_R$. From a result of Costabel [15] we have that a function $u \in L^2(\mathcal{O})$ such that $\mathrm{curl}\, u \in L^2(\mathcal{O} \backslash \overline{\Gamma})$, $\mathrm{div}\, u \in L^2(\mathcal{O} \backslash \overline{\Gamma})$, and $v \times u^{\pm}|_{\Gamma} \in L_t^2(\Gamma)$ is continuously imbedded in $H^{1/2-\varepsilon}(\mathcal{O} \backslash \overline{\Gamma})$ for every $\varepsilon > 0$. This proves the lemma. □

Now we can look for a solution of (6) in the form $E = W + \nabla p$ with $W \in \tilde{X}^0$ and $p \in S$. Hence by using a standard argument (see e.g. [8,17,22]), Lemmas 2.2 and 2.3 together with the Lax–Milgram lemma imply that the Fredholm alternative can be applied to (6). Hence the uniqueness theorem 2.1 implies the existence result. We summarize the above analysis in the following theorem.

Theorem 2.4. *For any incident field $E^{\mathrm{i}} \in X_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \backslash \overline{\Gamma})$ there exists a unique solution $E \in X_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \backslash \overline{\Gamma})$ of* (MSP) *which depends continuously on E^{i}.*

We remark that the above variation approach shows that the mixed screen problem for $\lambda = 0$ is still well posed. In particular if $\lambda = 0$, the mixed screen problem has a unique solution $E \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \backslash \overline{\Gamma})$.

For the analysis of the inverse problem we need the trace spaces of $v \times E^{\pm}|_{\Gamma}$ and $\gamma_T^{\pm} E$ for a $E \in H(\mathrm{curl}, B_R \backslash \overline{\Gamma})$. Letting $H^s(\partial D)$, $s \in \mathbb{R}$, denote the standard trace spaces [21] on a closed surface

∂D we define the following trace spaces on a portion Γ of ∂D

$$H^{1/2}(\Gamma) := \left\{u|_\Gamma : u \in H^{1/2}(\partial D)\right\}$$

$$\tilde{H}^{1/2}(\Gamma) := \{u \in H^{1/2}(\Gamma) : \operatorname{supp} u \subseteq \overline{\Gamma}\}.$$

We denote by $H^{-1/2}(\Gamma)$ the dual space of $\tilde{H}^{1/2}(\Gamma)$ and $\tilde{H}^{-1/2}(\Gamma)$ the dual space of $H^{1/2}(\Gamma)$ with $L^2(\Gamma)$ as the pivot space. Note that $\tilde{H}^{-1/2}(\Gamma)$ can also be identified as the space of distributions

$$\tilde{H}^{-1/2}(\Gamma) := \{u \in H^{-1/2}(\Gamma) : \operatorname{supp} u \subseteq \overline{\Gamma}\}.$$

Now we are in the position to define [9]

$$H^{-1/2}_{\mathrm{div}}(\Gamma) := \{u \in (H^{-1/2}(\Gamma))^3,\ \ v \cdot u = 0, \operatorname{div}_{\partial D} u \in H^{-1/2}(\Gamma)\}$$

$$H^{-1/2}_{\mathrm{curl}}(\Gamma) := \{u \in (H^{-1/2}(\Gamma))^3,\ \ v \cdot u = 0,\ \ \operatorname{curl}_{\partial D} u \in H^{-1/2}(\Gamma)\}.$$

Let us denote by $\tilde{H}^{-1/2}_{\mathrm{div}}(\Gamma)$ the dual space of $H^{-1/2}_{\mathrm{curl}}(\Gamma)$ in the duality pairing $\langle v \times u, \gamma_T v\rangle$ for $u \in H^{-1/2}_{\mathrm{curl}}(\Gamma)$ and $v \in \tilde{H}^{-1/2}_{\mathrm{div}}(\Gamma)$. This space contains tangential fields u such that $u \in (\tilde{H}^{-1/2}(\Gamma))^3$, $\operatorname{div}_{\partial D} u|_\Gamma \in \tilde{H}^{-1/2}(\Gamma)$ and

$$\int_\Gamma u \cdot \operatorname{grad}_{\partial D} v \,\mathrm{d}s + \int_\Gamma \operatorname{div}_{\partial D} u\, v \,\mathrm{d}s = 0$$

for every $v \in H^{3/2}(\Gamma)$. The latter means that the normal trace of u at the edge l of Γ is well defined and is zero, that is $v_l \cdot u|_l = 0$ where v_l is the exterior normal vector at the boundary l of Γ (for smooth screens see [1,9, p. 47]). Note also that a function $u \in \tilde{H}^{-1/2}_{\mathrm{div}}(\Gamma)$ can be extended by zero to a function in $H^{-1/2}_{\mathrm{div}}(\partial D)$. It is known that the trace operators $v \times u^{\pm}|_\Gamma$ and $\gamma_T^{\pm}|_\Gamma$ map $H(\mathrm{curl}, B_R\backslash\overline{\Gamma})$ into $H^{-1/2}_{\mathrm{div}}(\Gamma)$ and $H^{-1/2}_{\mathrm{curl}}(\Gamma)$, respectively. We remark that for piecewise smooth open surfaces the definition of the above trace spaces needs a more careful investigation. These spaces are fully characterized (note different notations $H^{-1/2}_{\|\mathrm{div}}(\Gamma)$ and $H^{-1/2}_{\perp\mathrm{curl}}\Gamma)$, etc. are used!), the continuity and surjectivity of the trace operators is proved and the duality pairing is interpreted in [3,4] and [2]. However, for simplicity of our presentation, in general we will keep the same notations for the trace spaces as for smooth open surfaces.

Since the solution space for (MSP) is $X_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3\backslash\overline{\Gamma})$ we need to specify the space of $\gamma_T^- E$ for $E \in X_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3\backslash\overline{\Gamma})$ which is obviously a closed subspace of $H^{-1/2}_{\mathrm{curl}}(\Gamma)$ since $X(\mathrm{curl}, B_R\backslash\overline{\Gamma})$ is a closed subspace of $H(\mathrm{curl}, B_R\backslash\overline{\Gamma})$. (Note that $\gamma_T^+ E \in H^{-1/2}_{\mathrm{curl}}(\Gamma) \cap L^2_t(\Gamma)$.) To this end we introduce

$$Y(\Gamma) := \left\{ f \in (H^{-1/2}(\Gamma))^3 \ : \ \exists\, u \in H(\mathrm{curl}, B_R\backslash\overline{\Gamma}), \begin{array}{l} \gamma_T^+ u|_\Gamma \in L^2_t(\Gamma) \\ \text{and } f = \gamma_T^- u|_\Gamma \end{array} \right\}.$$

It is easy to show that $Y(\Gamma)$ is a Banach space with respect to the norm

$$\| f\|^2_{Y(\Gamma)} := \inf\{\| u\|^2_{H(\mathrm{curl}, B_R\backslash\overline{\Gamma})} + \| v \times u\|^2_{L^2(\Gamma_I)}\}, \tag{8}$$

where the infimum is taken over all functions $u \in H(\mathrm{curl}, B_R\backslash\overline{\Gamma})$ such that $\gamma_T^+ u|_\Gamma \in L^2_t(\Gamma)$ and $f = \gamma_T^- u|_\Gamma$. Let again ∂D be a closed surface containing Γ, B_R a large ball containing D and let $u \in H(\mathrm{curl}, B_R\backslash\overline{\Gamma})$ be such that $\nu \times u|_{\partial B_R} = 0$, $\gamma_T^+ u|_\Gamma \in L_t^2(\Gamma)$ and $f = \gamma_T^- u|_\Gamma$. Applying integration by parts in D and $B_R \backslash \overline{D}$ and using the fact that the tangential component of functions in $H(\mathrm{curl}, B_R \backslash \overline{\Gamma})$ is continuous across $\partial D \backslash \overline{\Gamma}$ we obtain

$$\begin{aligned}\langle f, \phi \rangle &:= \int_\Gamma (\nu \times u^-) \cdot (\gamma_T^- \phi)\, \mathrm{d}s \\ &= -\int_{B_R} (\mathrm{curl}\, u \cdot \phi - u \cdot \mathrm{curl}\, \phi)\, \mathrm{d}v + \int_\Gamma (\nu \times u^+) \cdot (\gamma_T^+ \phi)\, \mathrm{d}s. \qquad (9)\end{aligned}$$

Here $\phi \in X(\mathrm{curl}, B_R \backslash \overline{\Gamma})$ such that $\nu \times \phi|_{\partial B_R} = 0$ and the surface integral on the right-hand side is understood in the L^2 sense. In particular (9) defines a duality relation and characterizes the dual space $Y'(\Gamma)$ of $Y(\Gamma)$. Hence $\| \cdot \|_{Y(\Gamma)}$ is equivalent to the norm

$$|||f||| := \sup_\phi \frac{|\langle f, \phi \rangle|}{\| \phi \|_{X(\mathrm{curl}, B_R \backslash \overline{\Gamma})}}$$

for $\phi \in X(\mathrm{curl}, B_R \backslash \overline{\Gamma})$ such that $\nu \times \phi|_{\partial B_R} = 0$.

Using the surjectivity and the continuity of the trace operator we can reformulate Theorem 2.4. In particular, for any $f \in Y(\Gamma)$ and $h \in L_t^2(\Gamma)$ there exists a unique solution $E^{\mathrm{s}} \in X_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \backslash \overline{\Gamma})$ of the problem

$$\begin{aligned} \mathrm{curl}\,\mathrm{curl}\, E^{\mathrm{s}} - k^2 E^{\mathrm{s}} = 0 \quad & \text{in } \mathbb{R}^3 \backslash \overline{\Gamma} \\ \gamma_T^- E^{\mathrm{s}} = f \quad & \text{on } \Gamma \\ \nu \times \mathrm{curl}\, E^{\mathrm{s}+} - \mathrm{i}\lambda \gamma_T^+ E^{\mathrm{s}} = h \quad & \text{on } \Gamma \\ \lim_{r \to \infty} (\mathrm{curl}\, E^{\mathrm{s}} \times x - \mathrm{i}kr E^{\mathrm{s}}) = 0 \quad & \qquad (10) \end{aligned}$$

which satisfies

$$\| E^{\mathrm{s}} \|_{X(\mathrm{curl}, B_R \backslash \overline{\Gamma})} \leqslant C \left(\| f \|_{Y(\Gamma)} + \| h \|_{L_t^2(\Gamma)} \right) \qquad (11)$$

with a constant $C > 0$ independent of f and h. The new formulation (10) and the a priori estimate (11) will play an important role in the analysis of the linear sampling method in Section 3.1. In particular it will allow us to introduce a bounded solution operator that takes the boundary data to the solution of the corresponding forward problem.

2.2. An approximation property

An *electromagnetic Herglotz pair* is defined to be a pair of vector fields of the form

$$E_g(x) = \int_\Omega \mathrm{e}^{\mathrm{i}kx \cdot d} g(d)\, \mathrm{d}s(d), \qquad H_g(x) = \frac{1}{\mathrm{i}k} \mathrm{curl}\, E_g(x), \qquad (12)$$

where the kernel g is a tangential vector field in $L_t^2(\Omega)$. It is easily seen that E_g, H_g is a solution of the Maxwell equations in $\mathbb{R}^3$.

We now define an operator $\mathscr{H} : L^2_t(\Omega) \to Y(\Gamma) \times L^2_t(\Gamma)$ by

$$\mathscr{H} g := \begin{cases} \gamma_T^- E_g \\ v \times \operatorname{curl} E_g^+ - \mathrm{i}\lambda\gamma_T^+ E_g \end{cases} \quad \text{on } \Gamma, \tag{13}$$

where E_g is the electric field of an electromagnetic Herglotz pair with kernel $g \in L^2_t(\Omega)$ defined by (12).

Theorem 2.5. *The range of $\mathscr{H}$ is dense in $Y(\Gamma) \times L^2_t(\Gamma)$.*

Proof. By the change of variables $d \to -d$ it suffices to consider the operator $\mathscr{H}$ with E_g written as

$$E_g(x) = \int_\Omega \mathrm{e}^{-\mathrm{i}kx\cdot d} g(d)\,\mathrm{d}s(d).$$

Let $H := Y(\Gamma) \times L^2_t(\Gamma)$ with dual $H^* := Y'(\Gamma) \times L^2_t(\Gamma)$ in the component-wise duality pairing. Note that $L^2_t(\Gamma)$ is considered as the dual space of itself with respect to the L^2 scalar product. The dual operator $\mathscr{H}^\top : H^* \to L^2_t(\Omega)$ of the operator $\mathscr{H}$ is such that for every $(\alpha, \beta) \in H^*$ and $g \in L^2_t(\Omega)$ we have

$$\langle \mathscr{H} g, (\alpha, \beta)\rangle_{H,H^*} = \langle g, \mathscr{H}^\top[\alpha, \beta]\rangle_{L^2_t(\Omega), L^2_t(\Omega)}.$$

It is enough to show that the dual operator $\mathscr{H}^\top$ is injective. Then the result follows from the fact that the range of $\mathscr{H}$ can be characterized as (see [21, p. 23])

$$\overline{(\text{Range } \mathscr{H})} = {}^a\text{Kern } \mathscr{H}^\top,$$

where

$${}^a\text{Kern } \mathscr{H}^\top := \left\{(p_1, p_2) \in H : \langle (p_1, p_2), (q_1, q_2)\rangle_{H,H^*} = 0 \ \forall\, (q_1, q_2) \in \text{Kern } \mathscr{H}^\top \right\}.$$

In particular, the injectivity of $\mathscr{H}^\top$ implies that $\overline{(\text{Range } \mathscr{H})} = H$. Simple computations shows that the dual operator $\mathscr{H}^\top$ is defined by

$$\mathscr{H}^\top[\alpha, \beta] = d \times \left\{ \int_\Gamma \mathrm{e}^{-\mathrm{i}kx\cdot d} \alpha\,\mathrm{d}s \right. \\ \left. - \mathrm{i}k\, d \times \int_\Gamma \mathrm{e}^{-\mathrm{i}kx\cdot d}(v \times \beta)\,\mathrm{d}s - \mathrm{i}\lambda \int_\Gamma \mathrm{e}^{-\mathrm{i}kx\cdot d}\beta\,\mathrm{d}s \right\} \times d.$$

Note that α and β are tangential fields defined on γ. One sees that $\mathscr{H}^\top[\alpha, \beta]$ coincides with the far field pattern of the combined electric and magnetic dipole distributions

$$P(z) = \frac{1}{k^2} \operatorname{curl}\operatorname{curl} \int_\Gamma \Phi(x, z)\alpha(x)\,\mathrm{d}s_x + \operatorname{curl} \int_\Gamma \Phi(x, z)(v \times \beta(x))\,\mathrm{d}s_x \\ - \mathrm{i}\lambda \frac{1}{k^2} \operatorname{curl}\operatorname{curl} \int_\Gamma \Phi(x, z)\beta(x)\,\mathrm{d}s_x, \quad z \notin \overline{\Gamma}$$

where

$$\Phi(x, z) := \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k|x-z|}}{|x-z|}, \quad x \neq z \quad \text{and} \quad x, z \in \mathbb{R}^3. \tag{14}$$

The potential $P(z)$ is well defined for $z \in \mathbb{R}^3 \backslash \overline{\Gamma}$ and satisfies curl curl $P - k^2 P = 0$. Now, let us assume that $\mathscr{H}^\top[a_1, a_2] = 0$. This means that the far field pattern of P is zero and from Rellich's lemma $P = 0$ in $\mathbb{R}^3 \backslash \overline{\Gamma}$. If $z \to \Gamma$ the following jump relations hold

$$\nu \times P^+ - \nu \times P^-|_\Gamma = \nu \times \beta, \tag{15}$$

$$\nu \times \text{curl } P^+ - \nu \times \text{curl } P^-|_\Gamma = \alpha - i\lambda\beta. \tag{16}$$

The above jump relations are well defined in the sense of L^2 limit (see [13, p. 172]) due to the relation (9) and the fact that β is a square integrable tangential field. Hence from (15) and (16) we conclude that $\alpha = \beta = 0$. Thus $\mathscr{H}^\top$ is injective which proves the theorem.

We remark that Theorem 2.5 claims that any pair $(f, g) \in Y(\Gamma) \times L^2_t(\Gamma)$ can be approximated arbitrarily closely by the mixed trace of the same electric Herglotz function E_g.

3. Inverse scattering problem

It is known [13] that the scattered electric field E^{s} has the asymptotic behavior

$$E^{\mathrm{s}}(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|} \left\{ E_\infty(\hat{x}) + \mathrm{O}\left(\frac{1}{|x|}\right) \right\}$$

as $|x| \to \infty$, where the tangential field E_∞ is defined on the unit sphere Ω and is known as the *electric far field pattern*.

We now consider the scattering of an electromagnetic plane wave by a very thin obstacle represented by Γ which on one side behaves like a perfect conductor and on the other side like a dielectric material with surface impedance λ. We indicate the dependence of the electric far field on the incident direction d and polarization p by writing $E_\infty(\hat{x}, d, p)$. The *inverse scattering problem* we will consider in this paper is to determine Γ from the knowledge of the electric far field $E_\infty(\hat{x}; d, p)$ for $\hat{x}, d \in \Omega$ and three linearly independent polarizations. (Note that we do not assume a priori knowledge of λ. In particular the screen can be a perfect conductor on both sides). By using the ideas of [5] we could easily consider the limited aperture case where $\hat{x}, -d \in \Omega_0 \subset \Omega$.

Theorem 3.1. *Let B denote either a perfectly conducting boundary condition or a mixed boundary condition. Assume that Γ_1 andΓ_2 are two open surfaces with boundary conditions B_1 and B_2 such that the far field patterns coincide for all incident directions D, all observation directions $\hat{x}$, and three linearly independent polarization p. Then $\Gamma_1 = \Gamma_2$.*

Proof. The proof of this theorem is based on the idea of Kirsh and Kress [16]. A general and simplified framework for the uniqueness proof for inverse electromagnetic obstacle scattering is given in [20, Theorem 1]. For the sake of completeness we present here a sketch of the proof following exactly the lines of Theorem 1 in [20].

First by Rellich's lemma from the coincidence of the far field patterns it follows that the corresponding scattered waves coincides in $G = \mathbb{R}^3 \backslash (\overline{\Gamma}_1 \cup \overline{\Gamma}_2)$. Next we consider the electric field of the electric dipole

$E_e^{\mathrm{i}}(x, z, p)$ given by

$$E_e^{\mathrm{i}}(x, z, p) = \frac{\mathrm{i}}{k}\mathrm{curl}_x\, \mathrm{curl}_x\, p\, \Phi(x, z)$$

with $\Phi(x, z)$ given by (14). Let $E_{e,1}^{\mathrm{s}}(\cdot, z, p)$ and $E_{e,2}^{\mathrm{s}}(\cdot, z, p)$ be the scattered electric fields corresponding to the scattering of $E_e^i(x, z, p)$ by Γ_1 and Γ_2, respectively. We can conclude (see [13,19,20]) that

$$E_{e,1}^{\mathrm{s}}(x, z, p) = E_{e,2}^{\mathrm{s}}(x, z, p)$$

for all $x, z \in G$ and all polarizations p.

Now assume that $\Gamma_1 \neq \Gamma_2$. Then we can find a $x^* \in \Gamma_1$ and $x^* \notin \Gamma_2$ such that $\nu(x^*)$ is defined, and consider $z_n = x^* + \frac{1}{n}\nu(x^*) \in G$. Then in view of well-posedness of the direct scattering problem for Γ_2 with boundary condition B_2, on one hand we obtain that

$$\lim_{n\to\infty} \| B_1(E_{e,2}^{\mathrm{s}}(x, z_n, p))\|_{X_1} = \lim_{n\to\infty} \| B_1(E_{e,2}^{\mathrm{s}}(x, x*, p))\|_{X_1},$$

where X_1 is the boundary data space corresponding to Γ_1 with boundary condition B_1. On the other hand we find that

$$\lim_{n\to\infty} \| B_1(E_{e,2}^{\mathrm{s}}(x, z_n, p))\|_{X_1} = \lim_{n\to\infty} \| B_1(E_{e,1}^{\mathrm{s}}(x, z_n, p))\|_{X_1} = \infty$$

because the boundary condition of $E_{e,1}^{\mathrm{s}}(x, z_n, p)$ are given in terms of the electric dipole and the traces of $E_e^i(x, x^*, p)$ on Γ_1 do not belong to the boundary data space due to the singularity at $z = x^*$. We have arrived at a contradiction and hence $\Gamma_1 = \Gamma_2$. $\square$

Our main concern in this paper is with presenting an algorithm to reconstruct the screen from the above (measured) data. To this end we will use the *linear sampling method* as was done in [6] for the scalar case.

3.1. The linear sampling method

The electric far field pattern defines the *electric far field operator* $F : L_t^2(\Omega) \to L_t^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, \mathrm{d}s(d), \quad \hat{x} \in \Omega, \tag{17}$$

for $g \in L_t^2(\Omega)$. Note that by superposition Fg is the electric far field pattern of (MSP) corresponding to the electromagnetic Herglotz pair with kernel ikg as incident field. We consider the *far field equation*

$$(Fg)(\hat{x}) = E_\infty^e(\hat{x}) \quad \hat{x} \in \Omega, \tag{18}$$

where E_∞^e is the far field pattern of a suitable (to be defined later) radiating solution to Maxwell's equations. The main idea is to characterize the screen Γ by the behavior of an approximate solution g of the far field equation (18). To understand the far field operator better we consider the operator $\mathscr{S}$ which maps the data $(f,h)\in Y(\Gamma)\times L^2_t(\Gamma)$ to the far field pattern of the radiating solution to the corresponding scattering problem (10). Hence F and $\mathscr{S}$ are related through the following relation:

$$(Fg)=-\mathrm{i}k\mathscr{S}(\mathscr{H}g), \tag{19}$$

where $\mathscr{H}$ is given by (13).

Lemma 3.2. *The linear operator* $\mathscr{S}: Y(\Gamma)\times L^2_t(\Gamma)\to L^2_t(\Omega)$ *is injective, bounded, compact and has dense range.*

Proof. The injectivity follows from the uniqueness of the scattering problem and Rellich's lemma whereas (11) and the fact that the far field pattern depends continuously on the scattered field imply that $\mathscr{S}$ is bounded.

Furthermore $\mathscr{S}$ can be seen as the composition of the bounded operator that takes the boundary data to the scattered solution on a large sphere ∂B_R of radius R and the compact operator (see [13, Theorem 6.8]) that maps data on ∂B_R to the corresponding far field. Hence $\mathscr{S}$ is compact.

Next we prove that the range of $\mathscr{S}$ is dense. To this end we consider the dual operator $\mathscr{S}^\top: L^2_t(\Omega)\to Y'(\Gamma)\times L^2_t(\Gamma)$

$$\langle\mathscr{S}(f,h),g\rangle_{L^2_t(\Omega),L^2_t(\Omega)}=\langle(f,h),\mathscr{S}^\top g\rangle.$$

From [13, Theorem 6.8], we obtain that

$$S(f,h):=E_\infty=\frac{\mathrm{i}k}{4\pi}\hat{x}\times\int_\Gamma\left\{[v\times E^{\mathrm{s}}]+\frac{1}{\mathrm{i}k}[v\times\operatorname{curl}E^{\mathrm{s}}]\times\hat{x}\right\}\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}ds,$$

where $E^{\mathrm{s}}\in X_{\mathrm{loc}}(D_e,\Gamma)$ is the electric scattered field corresponding to the boundary data (f,h) and $[u]$ denotes the jump u^+-u^- of u across Γ. Hence by changing the order of integration we can write

$$\begin{aligned}\langle\mathscr{S}(f,h),g\rangle=&\frac{\mathrm{i}k}{4\pi}\int_\Gamma\int_\Omega\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\left\{[\hat{x}\times(v\times E^{\mathrm{s}})]\cdot g(\hat{x})\right.\\&\left.+\frac{1}{\mathrm{i}k}\hat{x}\times[v\times\operatorname{curl}E^{\mathrm{s}}]\times\hat{x}\cdot g(\hat{x})\right\}\mathrm{d}s(\hat{x})\,\mathrm{d}s.\end{aligned}\tag{20}$$

Let

$$E_g(y):=\int_\Omega g(\hat{x})\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\,\mathrm{d}s(\hat{x})$$

denote the electric Herglotz wave function with kernel $g\in L^2_t(\Omega)$. Simple calculations show that

$$\operatorname{curl}_y E_g(y)=\mathrm{i}k\int_\Omega(g(\hat{x})\times\hat{x})\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\,\mathrm{d}s(\hat{x})$$

$$\operatorname{curl}_y\operatorname{curl}_y E_g(y)=k^2\int_\Omega(\hat{x}\times(g(\hat{x})\times\hat{x}))\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\,\mathrm{d}s(\hat{x}).$$

By using the fact that $\operatorname{curl}\operatorname{curl}E_g=k^2E_g$ we can rewrite (20) as

$$\langle\mathscr{S}(f,h),g\rangle=\frac{1}{4\pi}\int_\Gamma[v\times E^{\mathrm{s}}]\cdot\operatorname{curl}E_g+[v\times\operatorname{curl}E^{\mathrm{s}}]\cdot E_g\,\mathrm{d}s. \tag{21}$$

Note that the jump of $[\nu \times E]$ and $[\nu \times \operatorname{curl} E^{\mathrm{s}}]$ are supported in Γ and can be extended by zero to functions in $H^{1/2}_{\mathrm{div}}(\partial D)$ and $H^{1/2}_{\mathrm{curl}}(\partial D)$, respectively, where again ∂D is a closed surface containing Γ and enclosing the domain D. Now let $\tilde{E} \in X_{\mathrm{loc}}(D_e, \Gamma)$ be the solution of (10) with boundary data

$$\gamma_T^- \tilde{E} = \gamma_T E_g \quad \text{on } \Gamma$$
$$\nu \times \operatorname{curl} \tilde{E}^+ - \mathrm{i}\lambda\gamma_T^+ E = \nu \times \operatorname{curl} E_g - \mathrm{i}\lambda\gamma_T E_g \quad \text{on } \Gamma. \tag{22}$$

Using the boundary relations (22), we obtain

$$\begin{aligned}\langle \mathscr{S}(f,h), g\rangle = &-\frac{1}{4\pi}\int_{\Gamma^-} (\nu \times E^{\mathrm{s}}) \cdot \operatorname{curl} E_g + (\nu \times \operatorname{curl} E^{\mathrm{s}}) \cdot \tilde{E}\, \mathrm{d}s\\ &+ \frac{1}{4\pi}\int_{\Gamma^+} (\nu \times E^{\mathrm{s}}) \cdot [\operatorname{curl} \tilde{E} + \mathrm{i}\lambda(\nu \times \tilde{E})] - \mathrm{i}\lambda(\nu \times E^{\mathrm{s}}) \cdot (\nu \times E_g)\, \mathrm{d}s\\ &+ \frac{1}{4\pi}\int_{\Gamma^+} (\nu \times \operatorname{curl} E^{\mathrm{s}}) \cdot E_g\, \mathrm{d}s,\end{aligned} \tag{23}$$

where for simplicity Γ^- and Γ^+ indicates the negative and positive boundary traces, respectively. Using the relation

$$\begin{aligned}&\int_{\Gamma^+} (\nu \times \operatorname{curl} E^{\mathrm{s}}) \cdot \tilde{E}\, \mathrm{d}s + \int_{\Gamma-} (\nu \times \operatorname{curl} \tilde{E}) \cdot E^{\mathrm{s}}\, \mathrm{d}s\\ &\quad = \int_{\Gamma^-} (\nu \times \operatorname{curl} \tilde{E}) \cdot E^{\mathrm{s}}\, \mathrm{d}s + \int_{\Gamma^+} (\nu \times \operatorname{curl} E^{\mathrm{s}}) \cdot \tilde{E}\, \mathrm{d}s,\end{aligned}$$

(which is obtained by applying Green's formula in D and $\mathbb{R}^3\backslash\overline{D}$ using the continuity of the tangential components of E^{s}, $\tilde{E}$ across $\partial D\backslash\overline{\Gamma}$), and rearranging the terms we have

$$\begin{aligned}\langle \mathscr{S}(f,h), g\rangle = &\frac{1}{4\pi}\int_{\Gamma^-} (\nu \times E^{\mathrm{s}}) \cdot (\operatorname{curl} E_g - \operatorname{curl} \tilde{E})\, \mathrm{d}s\\ &+ \frac{1}{4\pi}\int_{\Gamma^+} [\nu \times \operatorname{curl} E^{\mathrm{s}} - \mathrm{i}\lambda(\nu \times E^{\mathrm{s}}) \times \nu] \cdot (E_g - \tilde{E})\, \mathrm{d}s.\end{aligned}$$

Finally the boundary condition for E^{s} implies

$$\begin{aligned}\langle \mathscr{S}(f,h), g\rangle = &\frac{1}{4\pi}\int_{\Gamma} f \cdot (\nu \times \operatorname{curl} \tilde{E}^- - \nu \times \operatorname{curl} E_g^-)\, \mathrm{d}s\\ &+ \frac{1}{4\pi}\int_{\Gamma} h \cdot (\gamma_T^+ E_g - \tilde{\gamma}_T^+ E)\, \mathrm{d}s.\end{aligned}$$

Hence

$$4\pi\mathscr{S}^\top g = \begin{cases} (\nu \times \operatorname{curl} \tilde{E}^- - \nu \times \operatorname{curl} E_g^-) \in Y(\Gamma)' \\ (\gamma_T^+ E_g - \gamma_T^+ \tilde{E}) \in L_t^2(\Gamma). \end{cases} \tag{24}$$

Let now $\mathscr{S}^\top g \equiv 0$. Then (24) and (22) imply that $\nu \times (\tilde{E} - E_g)^\pm = 0$ and $\nu \times (\operatorname{curl} \tilde{E} - \operatorname{curl} E_g)^\pm = 0$. But since $\tilde{E}$ is a radiating solution while E_g is an entire solution, we now see that E_g must be identically zero which can happen only if the kernel $g \equiv 0$. Hence, $\mathscr{S}^\top$ is injective which implies that $\mathscr{S}$ has dense range, which ends the proof of the lemma.

Note that the case of perfectly conducting screens the above proof works, if and only if, there does not exist an electromagnetic Herglotz pair such that the tangential component of the electric field vanishes on Γ. This condition is observed by Kress in [18] in the scalar case and in [7] in the vector case.

The following lemma will help us to choose the right-hand side of the far field equation (18) appropriately. We denote by $C^\infty_{\mathrm{comp}}(L)$ the space of C^∞ functions with support compact in L.

Lemma 3.3. *For any open surface L and two tangential fields* $\alpha_L, \beta_L \in \left(C^\infty_{\mathrm{comp}}(L)\right)^3$ *we define* $E^L_\infty \in L^2_t(\Omega)$ *by*

$$E^L_\infty := \hat{x} \times \left(\int_L \alpha_L(y)\, \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\, \mathrm{d}s_y + \hat{x} \times \int_L \beta_L(y) \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\, \mathrm{d}s_y \right) \times \hat{x}. \tag{25}$$

Then, $E^L_\infty \in \mathrm{Range}(\mathscr{S})$ *if and only if* $L \subset \Gamma$.

Proof. First assume that $L \subset \Gamma$ and let α_L, β_L be two C^∞ tangential fields, i.e. $\nu \cdot \alpha_L = \nu \cdot \beta_L = 0$, with support contained in L. Again we consider a closed boundary ∂D that contains Γ. We notice that (25) is the far field pattern of the potential V defined by

$$V(x) := \frac{1}{k^2} \operatorname{curl}\operatorname{curl} \int_L \alpha_L(y) \Phi(x, y)\, \mathrm{d}s_y + \frac{\mathrm{i}}{k} \operatorname{curl} \int_L \beta_L(y) \Phi(x, y)\, \mathrm{d}s_y.$$

Since the extensions $\tilde{\alpha}_L$ and $\tilde{\beta}_L$ of α_L and β_L, respectively, by zero to the whole boundary ∂D are C^∞ functions we have that $V \in X_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \backslash \overline{\Gamma})$ and satisfies $\operatorname{curl}\operatorname{curl} V - k^2 V = 0$. Moreover, using the jump relations of the vector potentials across ∂D, we have that V satisfies the following boundary conditions on Γ

$$f := \gamma_T^- V = -\frac{\mathrm{i}}{2k} \tilde{\beta}_L \times \nu + \frac{1}{k^2}(A^- \tilde{\alpha}_L) + \frac{\mathrm{i}}{k}(B^- \tilde{\beta}_L) \in Y(\Gamma)$$

$$\begin{aligned} h := \left(\nu \times \operatorname{curl} V^+ - \mathrm{i}\lambda \gamma_T^+ V\right) &= \frac{1}{2}\tilde{\alpha}_L + \frac{\lambda}{2k}\tilde{\beta}_L \times \nu + \nu \times (B^+ \tilde{\alpha}_L) \\ &\quad + \frac{\mathrm{i}}{k} \nu \times (A^+ \tilde{\beta}_L) - \frac{\mathrm{i}\lambda}{k^2}(A^+ \tilde{\alpha}_L) + \frac{\lambda}{k}(B^- \tilde{\beta}_L) \in L^2_t(\Gamma), \end{aligned}$$

where the boundary operator A and B are given by

$$(A^\pm \phi)(x) = \gamma_T^\pm \operatorname{curl}\operatorname{curl} \int_{\partial D} \phi(y) \Phi(x, y) \mathrm{d}s_y$$

$$(B^\pm \phi)(x) = \gamma_T^\pm \operatorname{curl} \int_{\partial D} \phi(y) \Phi(x, y)\, \mathrm{d}s_y, \quad x \in \partial D.$$

For the mapping properties of A and B see e.g. [21]. Hence E^L_∞ is in the range of $\mathscr{S}$.

Now let $S \not\subset \Gamma$ and assume, on the contrary, that $E^{\mathrm{s}}_\infty \in \mathrm{Range}(\mathscr{S})$, i.e. there exists $f \in Y(\Gamma)$ and $h \in L^2_t(\Gamma)$ such that $E^L_\infty = E^{\mathrm{s}}_\infty$ where E^{s}_∞ is the far field pattern of the radiating solution E^{s} to (10) corresponding to the boundary data f, h. Hence by Rellich's lemma and the unique continuation principle we have that $E^s(x)$ and

$$V(x) := \frac{1}{k^2} \operatorname{curl} \operatorname{curl} \int_L \alpha_L(y)\Phi(x, y)\, \mathrm{d}s_y + \frac{\mathrm{i}}{k} \operatorname{curl} \int_L \beta_S(y)\Phi(x, y)\, \mathrm{d}s_y$$

coincide for $x \in \mathbb{R}^3 \backslash (\overline{\Gamma} \cup \overline{L})$. Now let $x_0 \in L$, $x_0 \notin \Gamma$, and let $B_\varepsilon(x_0)$ be a small ball with center at x_0 such that $B_\varepsilon(x_0) \cap \Gamma = \emptyset$. Hence E^s is analytic in $B_\varepsilon(x_0)$ while V has a singularity at x_0 which is a contradiction. This proves the lemma. $\square$

Since $E^L_\infty \notin \mathrm{Range}\, \mathscr{S}$ in the case when $L \not\subset \Gamma$, by applying the regularization techniques [13] to the compact and injective operator $\mathscr{S}$ with dense range, we have the following result:

Lemma 3.4. *Consider the equation*

$$\mathscr{S}(f, h) = E^L_\infty, \quad (f, h) \in Y(\Gamma) \times L^2_t(\Gamma)$$

and let $L \not\subset \Gamma$. Then for every $\delta > 0$ there exists (f_α, h_α) depending on the regularization parameter $\alpha > 0$ such that

$$\| \mathscr{S}(f_\alpha, h_\alpha) - E^L_\infty \|_{L^2_t(\Omega)} < \delta$$

and

$$\lim_{\alpha \to 0} \| (f_\alpha, h_\alpha) \|_{Y(\Gamma) \times L^2_t(\Gamma)} = \infty.$$

Note that in the above lemma $\alpha \to 0$ as $\delta \to 0$.

We have now all the ingredients to prove the main theoretical result of this paper. Let us denote by $\mathscr{W}$ the set of piecewise smooth open surfaces L and consider the far field equation

$$(Fg)(\hat{x}) = E^L_\infty(\hat{x}), \quad L \in \mathscr{W}. \tag{26}$$

We remark that there are other possible choices for the function on the right-hand side of (26). The criteria for choosing is to characterize the screen Γ from whether or not the right-hand side of (26) is in the range of $\mathscr{S}$. Combining Lemmas 3.3 and 3.4, using the factorization (19) of the far field operator F and the fact that any pair $(f, g) \in Y(\Gamma) \times L^2_t(\Gamma)$ can be approximated arbitrarily closely by $\mathscr{H}g$ with $g \in L^2_t(\Omega)$ (Theorem 2.5), and finally the continuity of the operator $\mathscr{S}$ we can prove the following main theorem.

Theorem 3.5. *Assume that Γ is a bounded, oriented, piecewise smooth open surface. Then if F is the far field operator corresponding to* (*MSP*) *we have that*

(1) *if $L \subset \Gamma$ then for every $\varepsilon > 0$ there exists a solution $g^L_\varepsilon \in L^2_t(\Omega)$ of the inequality*

$$\| Fg^L_\varepsilon - E^L_\infty \|_{L^2_t(\Omega)} < \varepsilon.$$

(2) *if $L \not\subset \Gamma$ then for every $\varepsilon > 0$ and $\delta > 0$ there exists a solution $g^L_{\varepsilon,\delta} \in L^2_t(\Omega)$ of the inequality*

$$\| Fg^L_{\varepsilon,\delta} - E^L_\infty \|_{L^2_t(\Omega)} \leqslant \varepsilon + \delta$$

such that

$$\lim_{\delta\to 0} \| g^L_{\varepsilon,\delta}\|_{L^2_t(\Omega)} = \infty \quad \textit{and} \quad \lim_{\delta\to 0} \| E_{g^L_{\varepsilon,\delta}}\|_{H(\mathrm{curl},B_R)} = \infty,$$

where $E_{g^L_{\varepsilon,\delta}}$ is the electric part of the electromagnetic Herglotz pair with kernel $g^L_{\varepsilon,\delta}$.

In particular, if $L \subset \Gamma$ we can find a bounded solution to the far field equation (26) with discrepancy ε whereas if $L \not\subset \Gamma$ then there exists solutions of the far field equation with discrepancy $\varepsilon + \delta$ with arbitrary large norm in the limit as $\delta \to 0$. For numerical purposes we need to replace E^L_∞ in the far field equation (26) by an expression independent of L. To this end, we note that as L degenerates to a point z with α_L, and β_L an appropriate delta sequence we have that the integral in (25) approaches

$$\frac{\mathrm{i}k}{4\pi}\left[(\hat{x} \times q) \times \hat{x}\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z} + (\hat{x} \times q)\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}\right],$$

where q is a constant vector. Note that the first term is the electric far field of an electric dipole and the second term is the magnetic far field of an electric dipole. Roughly speaking the screen Γ will now be characterized as the set of points where the L^2_t norm of an approximate (regularized) solution of the far field equation

$$(Fg)(\hat{x}) = \frac{\mathrm{i}k}{4\pi}\left[(\hat{x} \times q) \times \hat{x}\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z} + (\hat{x} \times q)\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}\right] \tag{27}$$

becomes very large.

Following the remark below (26) we notice that one can replace either β_L or α_L by zero. More generally in principle one can also consider two independent delta sequences for β_L and α_L. In this case E^L_∞ is replaced by

$$\frac{\mathrm{i}k}{4\pi}\left[(\hat{x} \times q_1) \times \hat{x}\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z} + (\hat{x} \times q_2)\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}\right]$$

where q_1, q_2 are two constant vectors.

We end this section by remarking that, for the sake of presentation, we have considered only the case when one side of the screen is a perfect conductor and the other side is coated. Exactly the same analysis holds true if the material properties change on the same side as well. Note also that the same far field equation is solved to reconstruct perfectly conducting screens [7] or coated (possibly partially!) obstacles with nonempty interior [8]. This enhances the strength of the linear sampling method for solving the inverse obstacle scattering problems, i.e. it does not rely on any a priori knowledge of the geometry or physical properties of the scatterer.

4. Numerical examples

The numerical examples in this section are computed in the same way as discussed in [10,11]. To unify our approach for mixed screens with the approach used in [7] for perfectly conducting screens, we take $\beta_L = 0$ and solve the following far field equation

$$(Fg)(\hat{x}) = \frac{\mathrm{i}k}{4\pi}(\hat{x} \times q) \times \hat{x}\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z} \tag{28}$$

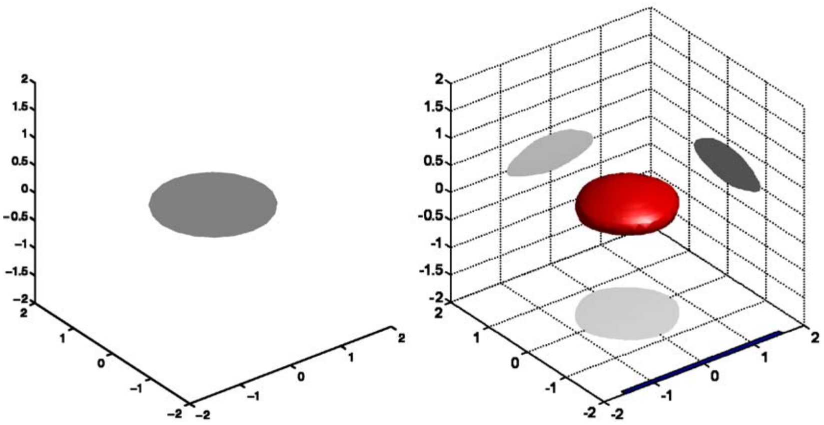

Fig. 1. Exact and reconstructed disk. Both sides satisfy the impedance boundary condition with $\lambda = 2$.

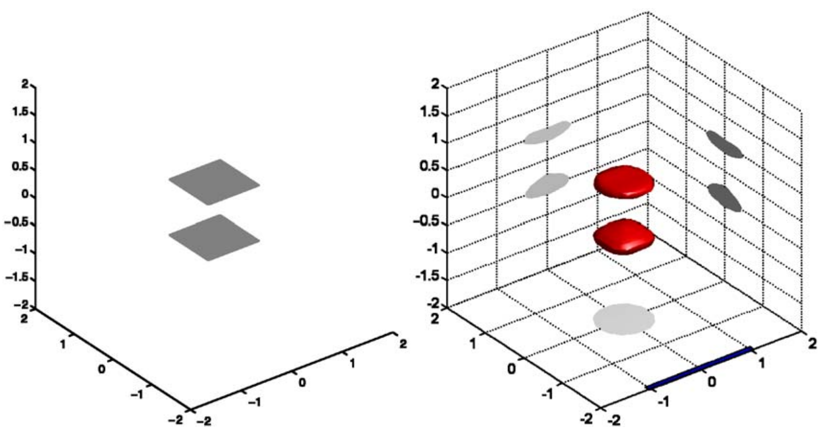

Fig. 2. Exact and reconstructed squares. The upper square is a perfect conductor on both sides. The upper side of the lower square satisfies the impedance boundary condition with $\lambda = 2$ while the other side is a perfect conductor.

for three linearly independent vectors q. The far field data is computed by solving the forward problem using a finite element code with mesh refinement near the edges of the screen. The far field data is then perturbed by random noise and is used in the discrete version of the far field equation (27). We use a uniform triangular meshing of the unit sphere Ω containing $N = 42$ vertices that corresponds to the directions of the incoming waves and the measurement points. All presented examples correspond to full aperture data. We use Tikhonov regularization and the Morozov discrepancy principle to compute the regularization parameter as introduced in [14]. We choose z on a uniform grid in the region we are sampling for a scatterer. In all of our examples a $51 \times 51 \times 51$ uniform grid is used. The noise level added to the synthetic data is 1%. For details and other numerical considerations the reader is referred to [10,11].

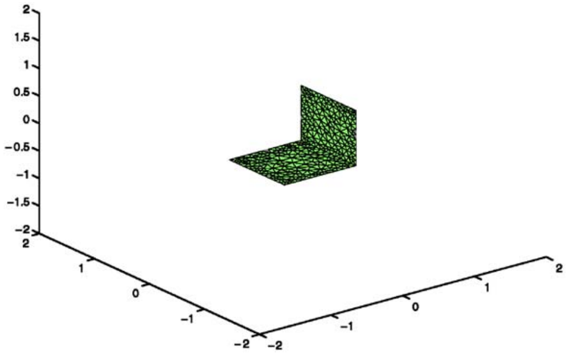

Fig. 3. Exact perpendicular squares.

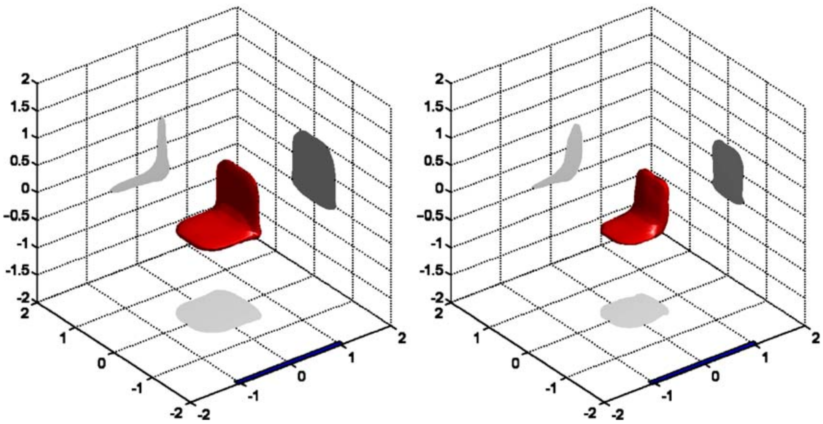

Fig. 4. Two examples of reconstructed perpendicular squares. The screen in the left figure is a perfect conductor. The screen in the right figure satisfies a perfectly conducting boundary condition on all sides except for the inner side of the vertical square which satisfies the impedance boundary condition with $\lambda = 2$.

An important parameter is the contour level at which we draw the iso-surface of the reconstruction. We define

$$G(z) = \frac{1}{3}\left(\frac{1}{\| g(\cdot, z, q_1)\|_{L^2_t(\Omega)}} + \frac{1}{\| g(\cdot, z, q_2)\|_{L^2_t(\Omega)}} + \frac{1}{\| g(\cdot, z, q_3)\|_{L^2_t(\Omega)}}\right),$$

where $g(\cdot, z, q_i)$, $i = 1, 2, 3$ is an approximate solution to the far field equation (28) corresponding to the source point location z and polarization q_i, $i = 1, 2, 3$ of the electric and magnetic dipole source. The iso-surface is then the set of points z such that

$$\mathscr{G}(z) = 0.5 \max_z G(z),$$

where the factor 0.5 is chosen to give the best results for a disk and then is kept fixed for all our other numerical examples. For interesting numerical tests regarding this issue in the case of obstacles with nonempty interior we refer the reader to [10].

We consider three scatterers: a disc, two parallel squares (as an example of disconnected objects) and two perpendicular squares (as an example of piecewise smooth surfaces). Numerical examples for perfectly conducting screens can be found in [7].

4.1. Reconstruction of a disk

The exact geometry is presented in the left graph of Fig. 1. On both sides of the disk we assume impedance boundary condition with $\lambda = 2$. In this reconstruction $k = 2$ (the wavelength is denoted by the bold line). As expected, the reconstruction in this case is worse than the reconstruction of the same disk with perfectly conducting boundary conditions on both sides (see Fig. 1 in [7]).

4.2. Reconstruction of two parallel squares

This example shown in Fig. 2 demonstrates that the linear sampling method can easily reconstruct disconnected objects without knowing a priori how many components there are or the boundary conditions on each component. In particular, we allow impedance boundary condition only on the upper side of the lower square. One can clearly see the effect of the coating in the reconstruction. The reconstructed perfectly conducting square is much thinner compared to the mixed square. In this example $k = 3$.

4.3. Reconstruction of two perpendicular squares

In Fig. 4 are presented two examples of reconstructions of a piecewise smooth screen (the exact geometry is given in Fig. 3) with different boundary conditions (as explained in the text for Fig. 4). The edge is sharply captured in the case of a perfectly conducting boundary condition while it is rounded in the presence of a coating. Here again $k = 3$.

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