Inverse Problems

PAPER

The factorization method for a cavity in an inhomogeneous medium

To cite this article: Shixu Meng *et al* 2014 *Inverse Problems* **30** 045008

IOP Publishing Inverse Problems

Inverse Problems **30** (2014) 045008 (20pp) doi:10.1088/0266-5611/30/4/045008

The factorization method for a cavity in an inhomogeneous medium

Shixu Meng[1], **Houssem Haddar**[2] **and Fioralba Cakoni**[1]

[1] Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716, USA
[2] CMAP, Ecole Polytechnique, Route de Saclay, F-91128 Palaiseau Cedex, France

E-mail: sxmeng@math.udel.edu, Houssem.Haddar@inria.fr and cakoni@math.udel.edu

Received 29 November 2013, revised 6 February 2014
Accepted for publication 7 February 2014
Published 5 March 2014

Abstract

We consider the inverse scattering problem for a cavity that is bounded by a penetrable anisotropic inhomogeneous medium of compact support and seek to determine the shape of the cavity from internal measurements on a curve or surface inside the cavity. We derive a factorization method which provides a rigorous characterization of the support of the cavity in terms of the range of an operator which is computable from the measured data. The support of the cavity is determined without *a priori* knowledge of the constitutive parameters of the surrounding anisotropic medium provided they satisfy appropriate physical as well as mathematical assumptions imposed by our analysis. Numerical examples are given showing the viability of our method.

Keywords: inverse scattering, factorization method, anisotropic medium, interior scattering problem, exterior transmission eigenvalues

(Some figures may appear in colour only in the online journal)

1. Introduction

The use of the so-called qualitative methods has played an important role in inverse scattering theory for the past 15 years and for a survey of these methods we refer the reader to [2] and [10]. The most rigorous approach in this class of methods is the factorization method which rigorously characterize the support of the scatterer in terms of the range of an operator computable from the measured data. Consequently this method provides a simply computable indicator function for the support and in addition provides a uniqueness result for the scatter's support in terms of the given data. The factorization method has been extensively studied for a variety of scattering problems for obstacle as well as inhomogeneous media. On the other hand, particularly in the case of scattering by an impenetrable obstacle with Dirichlet,

0266-5611/14/045008+20$33.00

Neumann or impedance boundary conditions, there has been a recent interest in the inverse scattering problem with measured data inside a cavity [4, 6–8, 15, 17–19]. In this class of problems the object is to determine the shape of the cavity from the use of sources and measurements along a curve or surface inside the cavity. A possible motivation for studying such a problem is to determine the shape of an underground reservoir by lowering receivers and transmitters into the reservoir through a bore hole drilled from the surface of the earth. The first paper to consider the scattering problem for a cavity surrounded by penetrable inhomogeneous media of compact support is [1]. There the authors proved a uniqueness result for the support of the cavity (without *a priori* knowledge of the constitutive parameters of the surrounding anisotropic medium provided they satisfy appropriate physical as well as mathematical assumptions) and established a linear sampling method for determining the shape of the cavity. A central role in their analysis is played by an unusual non-self-adjoint eigenvalue problem so-called the exterior transmission eigenvalue problem. In this paper we consider again the same problem as in [1] and derive a factorization method which provides a rigorous characterization of the support of the cavity in terms of the range of an operator which is computable from the measured data inside. The support of the cavity is determined without any *a priori* knowledge of the properties of the surrounding anisotropic medium. The factorization method for a non-penetrable cavity with Dirichlet, or Neumann, or impedance boundary conditions is considered in [13].

The plan of our paper is as follows. In the next section we will formulate both the direct and inverse scattering problems for a cavity bounded by a penetrable (possibly) inhomogeneous medium of compact support. In section 3 we derive an appropriate factorization of the data operator which is used in section 4 to establish the factorization method following the approach in [12]. The exterior transmission eigenvalue problem and an eigenvalue problem defined in the anisotropic layer appear in our analysis and the results are true provided that the wave number is not an eigenvalue of these eigenvalue problems. However we show that for practical purposes it is possible to avoid the exterior transmissions eigenvalue problem. In our analysis we use the non-physical incident waves which is the conjugate of point sources. However, we explain that in practice it is possible to consider the scattering problem in a bounded region where in the artificial boundary large enough to contain the anisotropic layer we can put zero Neumann boundary condition. In this case the factorization method can be derived exactly in the same way for physical point sources and incident waves. We conclude the paper by presenting some examples of reconstructions to show the viability of the method.

2. Direct and inverse problems

Let $D \subset \mathbb{R}^d$, $d = 2, 3$, be a simply connected bounded region of $\mathbb{R}^d$ with Lipshitz boundary ∂D and denote by ν the outward unit normal to ∂D. We assume the medium inside D is homogeneous with refractive index scaled to one and denote by k the corresponding wave number. The medium outside D is assumed to be inhomogeneous and anisotropic such that outside $D_1 \subset \mathbb{R}^d$ it is homogeneous with the same wave number as the medium in D, where $D \subset D_1$. More specifically, the physical properties of the medium in $D_1 \backslash \overline{D}$ are described by the $d \times d$ symmetric matrix valued function A with $L^\infty(D_1 \setminus \overline{D})$ entries and the bounded function $n \in L^\infty(D_1 \setminus \overline{D})$ such that $\overline{\xi} \cdot \mathrm{Re}(A)\xi \geqslant \alpha \|\xi\|^2$, $\overline{\xi} \cdot \mathrm{Im}(A)\xi \leqslant 0$, for all $\xi \in \mathbb{C}$ and some $\alpha > 0$, and $\mathrm{Re}(n) \geqslant n_0 > 0$, $\mathrm{Im}(n) \geqslant 0$ in $D_1 \setminus \overline{D}$. Furthermore we assume that $A \equiv I$ and $n \equiv 1$ in $\mathbb{R}^d \setminus \overline{D_1}$ and D (see figure 1).

In acoustic scattering ($d = 3$) or electromagnetic scattering ($d = 2$, for an H-polarized infinite cylinder) D represents the support of a cavity filled e.g. with air which is assumed to

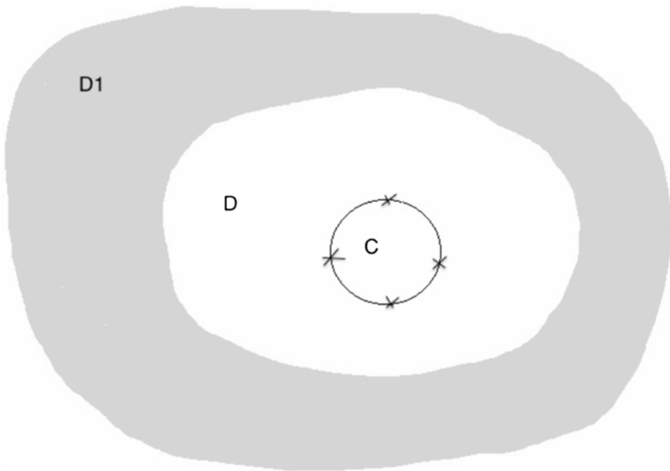


Figure 1. Example of the geometry of the problem.

be the reference media with wave number k. Let $\Phi(x, y)$ be a point source located at a point $y \in D$ inside the cavity given by

$$\Phi(x, y) = \begin{cases} \dfrac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) & \text{in } \mathbb{R}^2 \\ \dfrac{1}{4\pi} \dfrac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} & \text{in } \mathbb{R}^3 \end{cases} \tag{1}$$

and consider the scattering of $\overline{\Phi(\cdot, y)}$ by the inhomogeneous media. Note that this is non-physical incident wave and we briefly discuss this issue in section 6. The scattered field $u^s(\cdot, y)$ inside the cavity D and outside D_1 satisfies

$$\Delta_x u^s(x, y) + k^2 u^s(x, y) = 0 \qquad x \in D \quad \text{and} \quad \mathbb{R}^d \setminus \overline{D_1}$$

and the total field $u(\cdot, y) := u^s(\cdot, y) + \overline{\Phi(\cdot, y)}$ in the inhomogeneous media $D_1 \backslash \overline{D}$ satisfies

$$\nabla \cdot A(x) \nabla u(x, y) + k^2 n(x) u(x, y) = 0 \qquad x \in D_1 \backslash \overline{D}$$

and across the interface ∂D and ∂D_1 both the total field and its normal derivative are continuous, i.e.

$$u^s(\cdot, y) + \overline{\Phi(\cdot, y)} = u \qquad \text{and} \qquad \frac{\partial u^s(\cdot, y)}{\partial \nu} + \frac{\partial \overline{\Phi(\cdot, y)}}{\partial \nu} = \frac{\partial u}{\partial \nu_A}$$

on ∂D and ∂D_1, where $\frac{\partial w}{\partial \nu_A} := A \nabla w \cdot \nu$, and u^s satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - \mathrm{i} k u^s \right) = 0$$

uniformly with respect to $\hat{x} = x/r$, $r = |x|$. We recall that $\mathrm{supp}(A - I) \subset D_1 \setminus \overline{D}$ and $\mathrm{supp}(n - 1) \subset D_1 \setminus \overline{D}$. Written in terms of the radiating scattered field $u^s(\cdot, y)$ the scattering problem becomes

$$\nabla \cdot A \nabla u^s(\cdot, y) + k^2 n u^s(\cdot, y) = \nabla (I - A) \nabla \overline{\Phi(\cdot, y)} + k^2 (1 - n) \overline{\Phi(\cdot, y)} \quad \text{in} \quad \mathbb{R}^d. \tag{2}$$

Using a variational approach (see e.g. [2]), it is well known that the forward scattering problem (2) has a unique solution in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ which depends continuously on $\overline{\Phi(\cdot, y)}$.

Now assume that C is an open region in D such that $\overline{C} \subset D$. We place the artificial point source $\overline{\Phi(\cdot, y)}$ at every $y \in \partial C$ and measure the corresponding scattered field $u^s(x, y)$ for $x \in \partial C$ (see figure 1). The *inverse problem* we consider in this paper is for fixed (but not necessarily known) A and n satisfying certain assumptions, determine the boundary of the cavity ∂D from a knowledge of $u^s(x, y)$ for all $x, y \in \partial C$. Throughout this paper we make the following assumption.

Assumption 2.1. *The open region C is such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in C.*

Note that since the wave number k is known, it is easy to choose C to satisfy assumption 2.1. In this paper we develop the factorization method for solving the inverse problem following [10] and [12].

3. Factorization of the data operator

Our data set defines the data operator N: $L^2(\partial C) \to L^2(\partial C)$ by

$$(Ng)(x) = \int_{\partial C} u^s(x, y) g(y)\, \mathrm{d}s(y) \quad g \in L^2(\partial C), \qquad x \in \partial C \tag{3}$$

where $u^s(\cdot, y)$ is the radiating solution to (2). In order to factorize the data operator, we need to define various operators. To this end, let us define the bounded linear operator H: $L^2(\partial C) \to H^1(D_1\backslash\overline{D})$ by

$$(Hg)(x) := \int_{\partial C} \overline{\Phi(x, y)} g(y)\, \mathrm{d}s(y), \quad x \in D_1\backslash\overline{D}, \tag{4}$$

and the bounded linear operator $G : H^1(D_1\backslash\overline{D}) \to L^2(\partial C)$ which map w_0 to the trace of radiating solution w^* on ∂C, where $w^* \in H^1_{\mathrm{loc}}(R^d)$ is the radiating solution

$$\nabla \cdot A\nabla w^* + k^2 n w^* = \nabla(I - A)\nabla w_0 + k^2(1 - n)w_0 \quad \text{in} \quad \mathbb{R}^d. \tag{5}$$

By definition we obviously have $N = GH$.

Next let B_R be a sufficiently large ball that contains D_1, and let $T_k : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ be the exterior Dirichlet to Neumann map defined by

$$T_k : g \to \frac{\partial u}{\partial \nu}|_{\partial B_R}, \qquad g \in H^{\frac{1}{2}}(\partial B_R) \tag{6}$$

where u is the radiating solution to the Helmholtz equation $\Delta u + k^2 u = 0$ outside B_R with boundary data $u = g$ on ∂B_R, and ν is the outward unit normal to ∂B_R (see e.g. [2]).

Lemma 3.1. *The adjoint operator $H^* : H^1(D_1\backslash\overline{D}) \to L^2(\partial C)$ is given by*

$$(H^* v_0)(x) = \int_{\partial C} \frac{\partial \Phi(x, y)}{\partial \nu_y} v(y)\, \mathrm{d}s(y) - \frac{1}{2} v(x) \quad \textit{for} \quad x \in \partial C \tag{7}$$

where $v \in H^1(B_R\backslash\overline{C})$ is uniquely determined by the variational formula

$$-\int_{B_R\backslash\overline{C}} \nabla v \cdot \nabla\overline{\psi}\, \mathrm{d}x + k^2 \int_{B_R\backslash\overline{C}} v\, \overline{\psi}\, \mathrm{d}x + \int_{\partial B_R} T_k v\, \overline{\psi}\, \mathrm{d}x = (v_0, \psi|_{D_1\backslash\overline{D}})_{H^1(D_1\backslash\overline{D})} \tag{8}$$

$\forall \psi \in H^1(B_R\backslash\overline{C})$.

Proof. First we remark that based on Lax–Milgram lemma and the properties of the Dirichlet to Neumann operator T_k (see e.g. [2]), it is easy to see that (8) has a unique solution $v \in H^1(B_R\backslash\overline{C})$.

Now, let $u = \int_{\partial C} \Phi(x, y)\overline{g(y)}\, \mathrm{d}s(y)$ in $B_R\backslash\overline{C}$. Then $u \in H^1(B_R\backslash\overline{C})$ satisfies

$$
\begin{aligned}
&\Delta u + k^2 u = 0 && \text{in} \quad B_R\backslash\overline{C}\\
&\frac{\partial u^+}{\partial \nu} = \frac{\partial(\overline{Hg})^+}{\partial \nu} && \text{on} \quad \partial C\\
&\frac{\partial u}{\partial \nu} = T_k u && \text{on} \quad \partial B_R
\end{aligned}
$$

and $\overline{u} = Hg$ in $D_1\backslash\overline{D}$. From (8) and the above equation for u, we obtain that

$$
\begin{aligned}
(H^* v_0, g)_{L^2(\partial C)} &= (v_0, Hg)_{H^1(D_1\backslash\overline{D})} = (v_0, \overline{u})_{H^1(D_1\backslash\overline{D})}\\
&= -\int_{B_R\backslash\overline{C}} \nabla v\cdot\nabla u\, \mathrm{d}x + k^2\int_{B_R\backslash\overline{C}} vu\, \mathrm{d}x + \int_{\partial B_R} T_k v u\, \mathrm{d}s\\
&= -\int_{B_R\backslash\overline{C}} \nabla v\cdot\nabla u\, \mathrm{d}x + k^2\int_{B_R\backslash\overline{C}} vu\, \mathrm{d}x + \int_{\partial B_R} T_k u v\, \mathrm{d}s\\
&= \int_{\partial C} \frac{\partial u^+}{\partial \nu} v\, \mathrm{d}s = \int_{\partial C}\left[\int_{\partial C} \frac{\partial \Phi(x, y)}{\partial \nu_x}\overline{g(y)}\, \mathrm{d}s(y) - \frac{1}{2}\overline{g(x)}\right] v(x)\, \mathrm{d}s(x)\\
&= \left(\int_{\partial C} \frac{\partial \Phi(x, \cdot)}{\partial \nu_x} v(x)\, \mathrm{d}s(x) - \frac{1}{2}v, g\right)_{L^2(\partial C)}.
\end{aligned}
$$

Therefore, we have that

$$
(H^* v_0)(x) = \int_{\partial C} \frac{\partial \Phi(x, y)}{\partial \nu_y} v(y)\, \mathrm{d}s(y) - \frac{1}{2}v(x) \quad \text{for} \quad x \in \partial C
$$

which ends the proof. □

Before we factorize the data operator we need to define one more operator in $H^1(D_1 \setminus \overline{D})$. To this end, for a given $w_0 \in H^1(D_1\backslash\overline{D})$, let us consider the second kind integral equation

$$
\int_{\partial C} \frac{\partial \Phi(x, y)}{\partial \nu_y}\varphi(y)\, \mathrm{d}s(y) - \frac{1}{2}\varphi(x) = w^*(x) \quad \text{for} \quad x \in \partial C \tag{9}
$$

where w^* is the radiating solution to (5) with this w_0. Since k^2 is not Dirichlet eigenvalue for $-\Delta$ in C, and C is smooth, the above second kind integral equation has a unique solution $\varphi \in H^{\frac{1}{2}}(\partial C)$ (see e.g. [11, 14]). Then we define $v \in H^1(B_R\backslash\overline{C})$ by the double layer potential

$$
v(x) = \int_{\partial C} \frac{\partial \Phi(x, y)}{\partial \nu_y}\varphi(y)\, \mathrm{d}s(y) - w^*(x) \quad \text{for} \quad x \in B_R\backslash\overline{C}. \tag{10}
$$

(Note that the jump relation of the double layer potential implies that $\varphi := v|_{\partial C}$.) Having defined $v \in H^1(B_R\backslash\overline{C})$ we can now define the unique $v_0 \in H^1(D_1\backslash\overline{D})$ by means of Riesz representation theorem as

$$
(v_0, \psi)_{H^1(D_1\backslash\overline{D})} = -\int_{D_1\backslash\overline{D}} \nabla v\cdot\nabla\overline{\psi}\, \mathrm{d}x + k^2\int_{D_1\backslash\overline{D}} v\overline{\psi}\, \mathrm{d}x + \int_{\partial D_1} \frac{\partial v^+}{\partial \nu}\overline{\psi}\, \mathrm{d}s - \int_{\partial D} \frac{\partial v^-}{\partial \nu}\overline{\psi}\, \mathrm{d}s. \tag{11}
$$

Hereon the subscripts '+' and '−' indicate that we approach the boundary from outside and inside the enclosed region, respectively. Also hereon the integrals over $d - 1$ dimensional manifolds are defined in the sense of duality between $H^{1/2}$ and $H^{-1/2}$.

Definition 3.1. *The bounded linear operator $S : H^1(D_1\backslash\overline{D}) \to H^1(D_1\backslash\overline{D})$ is defined by*

$$
S : w_0 \mapsto v_0
$$

where v_0 is given by (11) corresponding to v defined by (10) with w^* satisfying (5) for the given w_0.

For later use let us derive an explicit formula for $(Sw_0, u_0)_{H^1(D_1\backslash\overline{D})}$. To this end, we recall the double layer potential

$$\mathcal{D}(\varphi)(\cdot) = \int_{\partial C} \frac{\partial\Phi(\cdot, y)}{\partial\nu_y}\varphi(y)\,\mathrm{d}s(y) \qquad \text{in} \quad B_R\backslash\partial C$$

and for a given w_0 let w^*, v and v_0 in definition 3.1. Then

$$\begin{aligned}
(v_0, u_0)_{H^1(D_1\backslash\overline{D})} &= -\int_{D_1\backslash\overline{D}} \nabla v\cdot\nabla\overline{u_0}\,\mathrm{d}x + k^2\int_{D_1\backslash\overline{D}} v\overline{u_0}\,\mathrm{d}x + \int_{\partial D_1}\frac{\partial v^+}{\partial\nu}\overline{u_0}\,\mathrm{d}s - \int_{\partial D}\frac{\partial v^-}{\partial\nu}\overline{u_0}\,\mathrm{d}s\\
&= \int_{D_1\backslash\overline{D}} \nabla w^*\cdot\nabla\overline{u_0}\,\mathrm{d}x - k^2\int_{D_1\backslash\overline{D}} w^*\overline{u_0}\,\mathrm{d}x - \int_{\partial D_1}\frac{\partial (w^*)^+}{\partial\nu}\overline{u_0}\,\mathrm{d}s\\
&\quad + \int_{\partial D}\frac{\partial (w^*)^-}{\partial\nu}\overline{u_0}\,\mathrm{d}x - \int_{D_1\backslash\overline{D}} \nabla\mathcal{D}(v)\cdot\nabla\overline{u_0}\,\mathrm{d}x + k^2\int_{D_1\backslash\overline{D}}\mathcal{D}(v)\overline{u_0}\,\mathrm{d}x\\
&\quad + \int_{\partial D_1}\frac{\partial\mathcal{D}(v)^+}{\partial\nu}\overline{u_0}\,\mathrm{d}s - \int_{\partial D}\frac{\partial\mathcal{D}(v)^-}{\partial\nu}\overline{u_0}\,\mathrm{d}s\\
&= -\int_{\partial D_1}\frac{\partial (w^*)^+}{\partial\nu}\overline{u_0}\,\mathrm{d}s + \int_{D_1\backslash\overline{D}}\nabla w^*\nabla\overline{u_0}\,\mathrm{d}x\\
&\quad - k^2\int_{D_1\backslash\overline{D}} w^*\overline{u_0}\,\mathrm{d}x + \int_{\partial D}\frac{\partial (w^*)^-}{\partial\nu}\overline{u_0}\,\mathrm{d}s
\end{aligned}$$

which gives that

$$\begin{aligned}
(Sw_0, u_0)_{H^1(D_1\backslash\overline{D})} &= -\int_{\partial D_1}\frac{\partial (w^*)^+}{\partial\nu}\overline{u_0}\,\mathrm{d}x + \int_{\partial D}\frac{\partial (w^*)^-}{\partial\nu}\overline{u_0}\,\mathrm{d}x\\
&\quad + \int_{D_1\backslash\overline{D}}\nabla w^*\cdot\nabla\overline{u_0}\,\mathrm{d}x - k^2\int_{D_1\backslash\overline{D}} w^*\overline{u_0}\,\mathrm{d}x.
\end{aligned} \tag{12}$$

Now we are ready to construct the main factorization of our data operator.

Theorem 3.1. *The data operator* N: $L^2(\partial C) \to L^2(\partial C)$ *can be factorized as* $N = H^*SH$ *where* $H : L^2(\partial C) \to H^1(D_1\backslash\overline{D})$ *is defined by* (4), $S : H^1(D_1\backslash\overline{D}) \to H^1(D_1\backslash\overline{D})$ *is defined by definition* 3.1*, and* $H^* : H^1(D_1\backslash\overline{D}) \to L^2(\partial C)$ *is given by lemma* 3.1*.*

Proof. Given $g \in L^2(\partial C)$ and let $w_0 = Hg$ we have that $Ng = w^*|_{\partial C}$. From (10), we have that v satisfies Helmholtz equation in $B_R\backslash\overline{D_1}$ and $D\backslash\overline{C}$ and satisfies radiation condition, whence from (11) for any $\psi \in H^1(B_R\backslash\overline{C})$,

$$\begin{aligned}
(v_0, \psi|_{D_1\backslash\overline{D}})_{H^1(D_1\backslash\overline{D})} &= -\int_{D_1\backslash\overline{D}}\nabla v\cdot\nabla\overline{\psi}\mathrm{d}x + k^2\int_{D_1\backslash\overline{D}} v\overline{\psi}\,\mathrm{d}x\\
&\quad + \int_{\partial D_1}\frac{\partial v^+}{\partial\nu}\overline{\psi}\,\mathrm{d}s - \int_{\partial D}\frac{\partial v^-}{\partial\nu}\overline{\psi}\,\mathrm{d}s\\
&= -\int_{B_R\backslash\overline{C}}\nabla v\cdot\nabla\overline{\psi}\,\mathrm{d}x + k^2\int_{B_R\backslash\overline{C}} v\overline{\psi}\,\mathrm{d}x\\
&\quad + \int_{\partial B_R} T_k v\overline{\psi}\,\mathrm{d}s - \int_{\partial C}\frac{\partial v^+}{\partial\nu}\overline{\psi}\,\mathrm{d}s.
\end{aligned}$$

Next we show $\frac{\partial v^+}{\partial\nu} = 0$ on ∂C. From (9) and jump properties of double layer potential, we have that

$$\left[\int_{\partial C}\frac{\partial\Phi(x, y)}{\partial\nu_y}v(y)\,\mathrm{d}s(y)\right]^- = w^*(x) \quad \text{for} \quad x \in \partial C.$$

Next, since both w^* and the double layer potential satisfy Helmholtz equation in C, the fact that they have the same Dirichlet boundary data on ∂C implies

$$\int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y)\,\mathrm{d}s(y) = w^*(x) \quad \text{for} \quad x \in C,$$

by making use that k^2 is not Dirichlet eigenvalue for $-\Delta$ in C. (Note that w^* is an H^1-solution of the Helmholtz equation in D, and therefore its normal derivative is continuous across ∂C.) Therefore

$$\frac{\partial}{\partial \nu_x}\left[\int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y)\,\mathrm{d}s(y)\right]^- = \frac{\partial w^*(x)}{\partial \nu_x} \quad \text{for} \quad x \in \partial C.$$

From the expression

$$v(x) = \int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y)\,\mathrm{d}s(y) - w^*(x) \quad \text{for} \quad x \in B_R \backslash \overline{C}$$

and the fact that the normal derivative of the double layer potential is continuous, we obtain that

$$\frac{\partial v^+}{\partial \nu} = 0 \quad \text{on} \quad \partial C$$

which now implies that

$$(v_0, \psi|_{D_1\backslash\overline{D}})_{H^1(D_1\backslash\overline{D})} = -\int_{B_R\backslash\overline{C}} \nabla v \cdot \nabla\overline{\psi}\,\mathrm{d}x + k^2 \int_{B_R\backslash\overline{C}} v\overline{\psi}\,\mathrm{d}x + \int_{\partial B_R} T_k v \overline{\psi}\,\mathrm{d}s. \tag{13}$$

Therefore from the definition of H^*, we have that

$$H^* v_0(\cdot) = \int_{\partial C} \frac{\partial \Phi(\cdot,y)}{\partial \nu_y} v(y)\,\mathrm{d}s(y) - \frac{1}{2} v(\cdot) \quad \text{on} \quad \partial C.$$

Finally (10) and the jump properties of double layer potential yield

$$\int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y)\mathrm{d}s(y) - \frac{1}{2} v(x) = w^*(x) \quad \text{for} \quad x \in \partial C$$

which means that $H^* v_0 = w^*|_C$. Thus $H^*SHg = H^*v_0 = w^*|_C = Ng$ and this holds for any $g \in L^2(\partial C)$, therefore we can conclude that $N = H^*SH$. □

4. The factorization method

The above factorization of the data operator will enable us to characterize the cavity D in terms of the range of an operator know from the (measured) data operator. To do so we recall theorem 2.1 in [12] which provides the theoretical basis of the factorization method that we use for our problem. For sake of reader's convenience we state this theorem below and for the proof we refer the reader to [12]. For a generic bounded linear operator K between two Banach spaces, we define its real and imaginary parts by $\mathrm{Re}(K) = \frac{K+K^*}{2}$ and $\mathrm{Im}(A) = \frac{K-K^*}{2\mathrm{i}}$ where K^* is the adjoint of K.

Theorem 4.1. *Let $X \subset U \subset X^*$ be a Gelfand triple with Hilbert space U and reflexive Banach space X such that the embeddings are dense. Furthermore, let V be a second Hilbert space and $F : V \to V$, $H : V \to X$ and $T : X \to X^*$ be linear bounded operators with $F = H^*TH$. Assume*

(a) H is compact and injective,
(b) $Re(T) = T_0 + T_1$ with some positive definite self-adjoint operator T_0 and some compact operator $T_1 : X \to X^*$,

(c) $(Im(T\phi), \phi) \geqslant 0$ *for all* $\phi \in X$.

Furthermore, assume that one of the following two conditions is satisfied.

(d) *T is injective.*
(e) *$Im(T)$ is positive on the (finite-dimensional) null space of $Re(T)$, i.e $(Im(T\phi), \phi) > 0$ for all $\phi \neq 0$ such that $Re(T\phi) = 0$.*

Then the operator $F_\# := |Re(F)| + Im(F)$ is positive definite and the range of $H^* : X^* \to V$ and the range of $F_\#^{1/2} : V \to V$ coincide.

We will apply this theorem to our near field operator $N = H^*SH$ and the rest of the paper is to make sure that the operator H, S and H^* fulfill the assumptions of the above theorem. To this end we make the following assumption on the wave number k.

Assumption 4.1. *The wave number $k > 0$ is such that there does not exist a nonzero $w_0 \in H^1(D_1\backslash\overline{D})$ satisfying*

$$\int_{D_1\backslash\overline{D}} (I - A)\nabla w_0 \cdot \nabla\overline{\psi}\, dx - k^2 \int_{D_1\backslash\overline{D}} (1 - n)w_0\overline{\psi}\, dx = 0, \qquad \forall\, \psi \in H^1(D_1\backslash\overline{D}).$$

Theorem 4.2. *The operators H, S, H^* have the following properties.*

(i) *H is compact and injective.*
(ii) *The imaginary part $Im(S)$ of S is non-negative.*
(iii) *S is injective on $H^1(D_1\backslash\overline{D})$ provided that $k > 0$ satisfies assumption 4.1.*

Proof. (i) Since the embedding of $H^2(D_1\backslash\overline{D})$ to $H^1(D_1\backslash\overline{D})$ is compact, and from the regularity of single layer potential aways from ∂C we obviously have that H is compact. Furthermore if $Hg = 0$ then since Hg solves the Helmholtz equation up to ∂C we have that $Hg|_{\partial C} = 0$. Now since k^2 is not a Dirichlet eigenvalue for $-\Delta$ and by the continuity of single layer potential we have that $Hg = 0$ in C. Now the jump relation gives that $g = 0$ and hence H is injective.

(ii) From (12) we have that,

$$(Sw_0, u_0)_{H^1(D_1\backslash\overline{D})} = -\int_{\partial D_1} \frac{\partial (w^*)^+}{\partial \nu}\overline{u_0}\, ds + \int_{\partial D} \frac{\partial (w^*)^-}{\partial \nu}\overline{u_0}\, dx + \int_{D_1\backslash\overline{D}} \nabla w^* \cdot \nabla\overline{u_0}\, dx - k^2 \int_{D_1\backslash\overline{D}} w^*\overline{u_0}\, dx. \tag{14}$$

Multiplying both sides (5) by $\overline{u_0}$ and integrating by parts yield

$$\begin{aligned}\int_{\partial D_1} \frac{\partial (w^*)^-}{\partial \nu_A}\overline{u_0}\, ds - \int_{\partial D} \frac{\partial (w^*)^+}{\partial \nu_A}\overline{u_0}\, ds - \int_{D_1\backslash\overline{D}} A\nabla w^* \cdot \nabla\overline{u_0}\, dx + k^2 \int_{D_1\backslash\overline{D}} nw^*\overline{u_0}\, dx \\ = \int_{\partial D_1} \frac{\partial w_0^-}{\partial \nu_{(I-A)}}\overline{u_0}\, ds - \int_{\partial D} \frac{\partial w_0^+}{\partial \nu_{(I-A)}}\overline{u_0}\, ds - \int_{D_1\backslash\overline{D}} (I - A)\nabla w_0 \cdot \nabla\overline{u_0}\, dx \\ + k^2 \int_{D_1\backslash\overline{D}} (1 - n)w_0\overline{u_0}\, dx.\end{aligned}$$

Now using the boundary conditions in the above

$$\frac{\partial (w_0)^+}{\partial \nu_{A-I}} = \frac{\partial (w^*)^-}{\partial \nu} - \frac{\partial (w^*)^+}{\partial \nu_A} \qquad \text{on} \qquad \partial D \tag{15}$$

$$\frac{\partial (w_0)^-}{\partial \nu_{I-A}} = -\frac{\partial (w^*)^+}{\partial \nu} + \frac{\partial (w^*)^-}{\partial \nu_A} \qquad \text{on} \qquad \partial D_1, \tag{16}$$

we have that

$$-\int_{\partial D_1}\frac{\partial (w^*)^+}{\partial \nu}\overline{u_0}\,\mathrm{d}s+\int_{\partial D}\frac{\partial (w^*)^-}{\partial \nu}\overline{u_0}\,\mathrm{d}s=-\int_{D_1\backslash\overline{D}}A\nabla w^*\cdot\nabla\overline{u_0}\,\mathrm{d}x+k^2\int_{D_1\backslash\overline{D}}nw^*\overline{u_0}\,\mathrm{d}x$$
$$+\int_{D_1\backslash\overline{D}}(I-A)\nabla w_0\cdot\nabla\overline{u_0}\,\mathrm{d}x-k^2\int_{D_1\backslash\overline{D}}(1-n)w_0\overline{u_0}\,\mathrm{d}x.$$

Let u^* be the solution of (5) corresponding to u_0 in the right-hand side. Plugging the above expression in (14) we can now get

$$\begin{aligned}(Sw_0,u_0)_{H^1(D_1\backslash\overline{D})}=&-\int_{D_1\backslash\overline{D}}A\nabla w^*\cdot\nabla\overline{u_0}\,\mathrm{d}x+k^2\int_{D_1\backslash\overline{D}}nw^*\overline{u_0}\,\mathrm{d}x\\&+\int_{D_1\backslash\overline{D}}(I-A)\nabla w_0\cdot\nabla\overline{u_0}\,\mathrm{d}x-k^2\int_{D_1\backslash\overline{D}}(1-n)w_0\overline{u_0}\,\mathrm{d}x\\&+\int_{D_1\backslash\overline{D}}\nabla w^*\cdot\nabla\overline{u_0}\,\mathrm{d}x-k^2\int_{D_1\backslash\overline{D}}w^*\overline{u_0}\,\mathrm{d}x.\end{aligned}\tag{17}$$

The latter can be re-written as

$$\begin{aligned}(Sw_0,u_0)_{H^1(D_1\backslash\overline{D})}=&-\int_{D_1\backslash\overline{D}}(A-I)\nabla(w^*+w_0)\cdot\nabla\overline{(u^*+u_0)}\,\mathrm{d}x\\&+k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{(u^*+u_0)}\,\mathrm{d}x\\&+\int_{D_1\backslash\overline{D}}(A-I)\nabla(w^*+w_0)\cdot\nabla\overline{u}^*\,\mathrm{d}x\\&-k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{u}^*\,\mathrm{d}x.\end{aligned}\tag{18}$$

Next, noting that

$$-\nabla\cdot(A-I)\nabla(w^*+w_0)-k^2(n-1)(w^*+w_0)=\Delta w^*+k^2w^*\qquad\text{in}\quad D_1\backslash\overline{D},$$

multiplying both sides by $\overline{u^*}$ and integrating by parts we obtain

$$\begin{aligned}-\int_{D_1\backslash\overline{D}}\nabla w^*\cdot\nabla\overline{u^*}\,\mathrm{d}x+k^2\int_{D_1\backslash\overline{D}}w^*\overline{u^*}\,\mathrm{d}x+\int_{\partial D_1}\frac{\partial (w^*)^-}{\partial\nu}\overline{u^*}\,\mathrm{d}x-\int_{\partial D}\frac{\partial (w^*)^+}{\partial\nu}\overline{u^*}\,\mathrm{d}x\\=-\int_{\partial D_1}\frac{\partial (w^*+w_0)^-}{\partial\nu_{(A-I)}}\overline{u^*}\mathrm{d}s+\int_{\partial D}\frac{\partial (w^*+w_0)^+}{\partial\nu_{(A-I)}}\overline{u^*}\,\mathrm{d}s\\+\int_{D_1\backslash\overline{D}}(A-I)\nabla(w^*+w_0)\cdot\nabla\overline{u}^*\,\mathrm{d}x-k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{u}^*\,\mathrm{d}x.\end{aligned}\tag{19}$$

Next using the boundary conditions (15) and (16) along with (19) in (18) yield

$$\begin{aligned}(Sw_0,u_0)_{H^1(D_1\backslash\overline{D})}=&-\int_{D_1\backslash\overline{D}}(A-I)\nabla(w^*+w_0)\cdot\nabla\overline{(u^*+u_0)}\,\mathrm{d}x\\&+k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{(u^*+u_0)}\mathrm{d}x-\int_{D_1\backslash\overline{D}}\nabla w^*\cdot\nabla\overline{u^*}\,\mathrm{d}x\\&+k^2\int_{D_1\backslash\overline{D}}w^*\overline{u^*}\,\mathrm{d}x+\int_{\partial D_1}\frac{\partial (w^*)^+}{\partial\nu}\overline{u^*}\,\mathrm{d}s-\int_{\partial D}\frac{\partial (w^*)^-}{\partial\nu}\overline{u^*}\,\mathrm{d}s.\end{aligned}$$

Since w^* satisfies Helmholtz equation in D and outside D_1, we can rewrite the above expression as

$$\begin{aligned}(Sw_0, u_0)_{H^1(D_1\backslash\overline{D})} = &-\int_{D_1\backslash\overline{D}} (A-I)\nabla(w^*+w_0)\cdot\nabla\overline{(u^*+u_0)}\,\mathrm{d}x \\ &+k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{(u^*+u_0)}\,\mathrm{d}x - \int_{B_R\backslash\overline{D_1}}\nabla w^*\cdot\nabla\overline{u^*}\,\mathrm{d}x \\ &-\int_{D_1\backslash\overline{D}}\nabla w^*\cdot\nabla\overline{u^*}\,\mathrm{d}x + k^2\int_{D_1\backslash\overline{D}} w^*\overline{u^*}\,\mathrm{d}x + \int_{\partial B_R} T_k w^*\overline{u^*}\,\mathrm{d}x \\ &+k^2\int_{B_R\backslash\overline{D_1}} w^*\overline{u^*}\,\mathrm{d}x - \int_D \nabla w^*\cdot\nabla\overline{u^*}\,\mathrm{d}x + k^2\int_D w^*\overline{u^*}\,\mathrm{d}x\end{aligned}$$

which can be finally transformed to

$$\begin{aligned}(Sw_0, u_0)_{H^1(D_1\backslash\overline{D})} = &-\int_{D_1\backslash\overline{D}} (A-I)\nabla(w^*+w_0)\cdot\nabla\overline{(u^*+u_0)}\,\mathrm{d}x \\ &+k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{(u^*+u_0)}\,\mathrm{d}x \\ &-\int_{B_R}\nabla w^*\cdot\nabla\overline{u^*}\,\mathrm{d}x + k^2\int_{B_R} w^*\overline{u^*}\,\mathrm{d}x + \int_{\partial B_R} T_k w^*\overline{u^*}\,\mathrm{d}s. \end{aligned}\tag{20}$$

Now taking the imaginary part of S, we can see that

$$\begin{aligned}(\mathrm{Im}(S)w_0, w_0)_{H^1(D_1\backslash\overline{D})} = &\,\mathrm{Im}\bigg(\int_{\partial B_R} T_k w^*\overline{w^*}\,\mathrm{d}s - \int_{D_1\backslash\overline{D}}(A-I)\nabla(w^*+w_0)\cdot\nabla\overline{(w^*+w_0)}\,\mathrm{d}x \\ &+k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{(w^*+w_0)}\,\mathrm{d}x\bigg) \\ \geqslant &\, k\int_{S^2}|w^*_\infty|^2\,\mathrm{d}s - \int_{D_1\backslash\overline{D}}\mathrm{Im}(A)\nabla(w^*+w_0)\cdot\nabla\overline{(w^*+w_0)}\,\mathrm{d}x \\ &+k^2\int_{D_1\backslash\overline{D}}\mathrm{Im}(n)|w^*+w_0|^2\,\mathrm{d}x \geqslant 0\end{aligned}$$

since $\mathrm{Im}(A)\leqslant 0$ and $\mathrm{Im}(n)\geqslant 0$, where the far field pattern w^*_∞ of the radiating solution w^* is defined from the asymptotic expansion

$$w^*(x) = \frac{\mathrm{e}^{\mathrm{i}kr}}{r^{\frac{d-1}{2}}}w^*_\infty(\hat{x}) + O\big(r^{-\frac{d+1}{2}}\big), \qquad r=|x|,\ \hat{x}=x/|x|.$$

(iii) To prove the third part we assume that $Sw_0=0$. Then for any $\psi\in H^1(D_1\backslash\overline{D})$ from (12) we have that

$$\int_{\partial D}\frac{\partial(w^*)^-}{\partial\nu}\overline{\psi}\,\mathrm{d}s - \int_{\partial D_1}\frac{\partial(w^*)^+}{\partial\nu}\overline{\psi}\,\mathrm{d}s + \int_{D_1\backslash\overline{D}}\nabla w^*\cdot\nabla\overline{\psi}\mathrm{d}x - k^2\int_{D_1\backslash\overline{D}} w^*\overline{\psi}\,\mathrm{d}x = 0$$

which means that w^* satisfies

$$\Delta w^* + k^2 w^* = 0 \quad \text{in} \quad D_1\backslash\overline{D}$$

and the transmission conditions

$$\frac{\partial(w^*)^+}{\partial\nu} = \frac{\partial(w^*)^-}{\partial\nu} \quad \text{on} \quad \partial D_1 \qquad \text{and} \qquad \frac{\partial(w^*)^+}{\partial\nu} = \frac{\partial(w^*)^-}{\partial\nu} \quad \text{on} \quad \partial D.$$

Therefore from (5), we can conclude that $w^*\in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a radiating solution to the Helmholtz equation in $\mathbb{R}^d$, and hence $w^*=0$. Now multiplying both sides of (5) by $\overline{\psi}$ and integrating by parts, we obtain that w_0 satisfies

$$\int_{D_1\backslash\overline{D}}(I-A)\nabla w_0\cdot\nabla\overline{\psi} - k^2\int_{D_1\backslash\overline{D}}(1-n)w_0\overline{\psi} = 0, \quad \forall \quad \psi\in H^1(D_1\backslash\overline{D})$$

whence $w_0 = 0$ providing that $k > 0$ satisfies assumption 4.1, which implies that under this assumption S is injective. □

Theorem 4.3. *The operator S satisfies in addition the following property.*

(i) If $Re(A) > I$ then $-Re(S)$ is the sum of a compact operator and a self-adjoint positive definite operator.

(ii) If $I - Re(A) - \alpha|Im(A)| > 0$ and $Re(A) - \frac{1}{\alpha}|Im(A)| \geqslant 0$ for some $\alpha > 0$, then $Re(S)$ is the sum of a compact operator and a self-adjoint positive definite operator.

Proof. (i) From (20) the real part of the operator S is given by

$$\begin{aligned}(\mathrm{Re}(S)w_0, u_0)_{H^1(D_1\backslash\overline{D})} = &-\int_{D_1\backslash\overline{D}} (\mathrm{Re}(A) - I)\nabla(w^* + w_0)\cdot\nabla\overline{(u^* + u_0)} \\ &+ k^2\int_{D_1\backslash\overline{D}} (\mathrm{Re}(n) - 1)(w^* + w_0)\overline{(u^* + u_0)}\,\mathrm{d}x \\ &- \int_{B_R} \nabla w^*\cdot\nabla\overline{u^*}\,\mathrm{d}x + k^2\int_{B_R} w^*\overline{u^*}\,\mathrm{d}x + \int_{\partial B_R} \mathrm{Re}(T_k)w^*\overline{u^*}\,\mathrm{d}x.\end{aligned}$$

In the case when $\mathrm{Re}(A) > I$ we define the operator $K : H^1(D_1\backslash\overline{D}) \to H^1(D_1\backslash\overline{D})$ by

$$\begin{aligned}(Kw_0, u_0)_{H^1(D_1\backslash\overline{D})} = &-\int_{D_1\backslash\overline{D}} (\mathrm{Re}(A) - I)\nabla(w^* + w_0)\cdot\nabla\overline{(u^* + u_0)}\,\mathrm{d}x \\ &- \int_{D_1\backslash\overline{D}} w_0\overline{u_0}\,\mathrm{d}x - \int_{B_R} \nabla w^*\cdot\nabla\overline{u^*}\,\mathrm{d}x + \int_{\partial B_R} \mathrm{Re}(T_k)w^*\overline{u^*}\,\mathrm{d}x \end{aligned} \tag{21}$$

which is obviously self-adjoint. Using the known fact that the real part of the Dirichlet to Neumann operator $\mathrm{Re}(T_k)$ is non-positive (see e.g. [16] in $\mathbb{R}^3$) and applying Young's inequality yield

$$\begin{aligned}(-Kw_0, w_0)_{H^1(D_1\backslash\overline{D})} &\geqslant (1-\alpha)\left((\mathrm{Re}(A) - I)\nabla w_0, \nabla w_0\right)_{L^2(D_1\backslash\overline{D})} + (w_0, w_0)_{L^2(D_1\backslash\overline{D})} \\ &\quad + \left(1 - \frac{1}{\alpha}\right)\left((\mathrm{Re}(A) - I)\nabla w^*, \nabla w^*\right)_{L^2(D_1\backslash\overline{D})} + (\nabla w^*, \nabla w^*)_{L^2(B_R)} \\ &\geqslant c||w_0||^2_{H^1(D_1\backslash\overline{D})}\end{aligned}$$

where $0 < \alpha < 1$ is such that $(1 - \frac{1}{\alpha})\sup_{D_1\backslash\overline{D}}(\mathrm{Re}(A) - I) + 1 > 0$, and c is some positive constant depending on A. Now, the fact that $\mathrm{Re}(S) - K$ is compact thanks to the compactly imbedding of $H^1(D_1\backslash\overline{D})$ into $L^2(D_1\backslash\overline{D})$, proves the first claim.

(ii) Next, we consider the case when $\mathrm{Re}(A) < I$. To prove the claim we need to derive a new expression for $(Sw_0, u_0)_{H^1(D_1\backslash\overline{D})}$. To this end from the expression (17) we have

$$\begin{aligned}(Sw_0, u_0)_{H^1(D_1\backslash\overline{D})} = &-\int_{D_1\backslash\overline{D}} A\nabla w^*\cdot\nabla\overline{u_0}\,\mathrm{d}x + k^2\int_{D_1\backslash\overline{D}} nw^*\overline{u_0}\,\mathrm{d}x \\ &+ \int_{D_1\backslash\overline{D}} (I - A)\nabla w_0\cdot\nabla\overline{u_0}\,\mathrm{d}x - k^2\int_{D_1\backslash\overline{D}} (1-n)w_0\overline{u_0}\,\mathrm{d}x + \int_{D_1\backslash\overline{D}} \nabla w^*\cdot\nabla\overline{u_0} \\ &- k^2\int_{D_1\backslash\overline{D}} w^*\overline{u_0}\,\mathrm{d}x = \int_{D_1\backslash\overline{D}} (I - A)\nabla w^*\cdot\nabla\overline{u_0}\,\mathrm{d}x - k^2\int_{D_1\backslash\overline{D}} (1-n)w^*\overline{u_0}\,\mathrm{d}x \\ &+ \int_{D_1\backslash\overline{D}} (I - A)\nabla w_0\cdot\nabla\overline{u_0}\,\mathrm{d}x - k^2\int_{D_1\backslash\overline{D}} (1-n)w_0\overline{u_0}\,\mathrm{d}x.\end{aligned} \tag{22}$$

For given $u_0 \in H^1(D_1\backslash\overline{D})$ let u^* be the radiating solution of (5). Multiplying both sides of

$$\nabla\cdot A\nabla u^* + k^2nu^* = \nabla(I - A)\cdot\nabla u_0 + k^2(1-n)u_0 \quad \text{in} \quad \mathbb{R}^d$$

by $\overline{w^*}$ and integrating by parts, we obtain

$$\int_{\partial D_1} \frac{\partial (u^*)^-}{\partial \nu_A}\overline{w^*}\,\mathrm{d}x - \int_{\partial D} \frac{\partial (u^*)^+}{\partial \nu_A}\overline{w^*}\,\mathrm{d}x - \int_{D_1\setminus\overline{D}} A\nabla u^* \cdot \nabla\overline{w^*}\,\mathrm{d}x + k^2\int_{D_1\setminus\overline{D}} nu^*\overline{w^*}\,\mathrm{d}x$$
$$= \int_{\partial D_1} \frac{\partial (u_0)^-}{\partial \nu_{I-A}}\overline{w^*}\mathrm{d}x - \int_{\partial D} \frac{\partial (u_0)^+}{\partial \nu_{I-A}}\overline{w^*}\,\mathrm{d}x$$
$$- \int_{D_1\setminus\overline{D}} (I-A)\nabla u_0 \cdot \nabla\overline{w^*}\,\mathrm{d}x + k^2\int_{D_1\setminus\overline{D}} (1-n)u_0\overline{w^*}\,\mathrm{d}x.$$

Therefore, from the transmission conditions (15) and (16) for u^* and u_0 the above expression can be re-written as

$$\int_{D_1\setminus\overline{D}} (I-A)\nabla u_0 \cdot \nabla\overline{w^*}\,\mathrm{d}x - k^2\int_{D_1\setminus\overline{D}} (1-n)u_0\overline{w^*}\,\mathrm{d}x$$
$$= -\int_{\partial D_1} \frac{\partial (u^*)^+}{\partial \nu}\overline{w^*}\,\mathrm{d}s + \int_{\partial D} \frac{\partial (u^*)^-}{\partial \nu}\overline{w^*}\,\mathrm{d}s + \int_{D_1\setminus\overline{D}} A\nabla u^* \cdot \nabla\overline{w^*}\,\mathrm{d}x - k^2\int_{D_1\setminus\overline{D}} nu^*\overline{w^*}\,\mathrm{d}x$$
$$= -\int_{\partial B_R} T_k u^*\overline{w^*}\,\mathrm{d}s + \int_{B_R} \nabla u^* \cdot \nabla\overline{w^*}\,\mathrm{d}s - k^2\int_{B_R} u^*\overline{w^*}\,\mathrm{d}x + \int_{D_1\setminus\overline{D}} (A-I)\nabla u^* \cdot \nabla\overline{w^*}\,\mathrm{d}x$$
$$-k^2\int_{D_1\setminus\overline{D}} (n-1)u^*\overline{w^*}\,\mathrm{d}x$$

where $T_k : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is the exterior Dirichlet to Neumann operator defined by (6). Conjugating the above expression we obtain

$$\int_{D_1\setminus\overline{D}} (I-\overline{A})\nabla w^* \cdot \nabla\overline{u_0}\,\mathrm{d}x - k^2\int_{D_1\setminus\overline{D}} (1-\overline{n})w^*\overline{u_0}\,\mathrm{d}x$$
$$= -\int_{\partial B_R} \overline{T_k u^*}w^*\,\mathrm{d}s + \int_{B_R} \nabla w^* \cdot \nabla\overline{u^*}\,\mathrm{d}x - k^2\int_{B_R} w^*\overline{u^*}\,\mathrm{d}x$$
$$+ \int_{D_1\setminus\overline{D}} (\overline{A}-I)\nabla w^* \cdot \nabla\overline{u^*}\,\mathrm{d}x - k^2\int_{D_1\setminus\overline{D}} (\overline{n}-1)w^*\overline{u^*}\,\mathrm{d}x \tag{23}$$

and substituting (23) in (22) yields

$$(Sw_0, u_0)_{H^1(D_1\setminus\overline{D})} = \int_{D_1\setminus\overline{D}} (I-A)\nabla w_0 \cdot \nabla\overline{u_0}\,\mathrm{d}x - k^2\int_{D_1\setminus\overline{D}} (1-n)w_0\overline{u_0}\,\mathrm{d}x$$
$$- \int_{\partial B_R} \overline{T_k u^*}w^*\,\mathrm{d}s + \int_{B_R} \nabla w^* \cdot \nabla\overline{u^*}\mathrm{d}x - k^2\int_{B_R} w^*\overline{u^*}\,\mathrm{d}x$$
$$+ \int_{D_1\setminus\overline{D}} (\overline{A}-I)\nabla w^* \cdot \nabla\overline{u^*}\,\mathrm{d}x - k^2\int_{D_1\setminus\overline{D}} (\overline{n}-1)w^*\overline{u^*}\,\mathrm{d}x$$
$$+ \int_{D_1\setminus\overline{D}} (\overline{A}-A)\nabla w^* \cdot \nabla\overline{u_0}\,\mathrm{d}x - k^2\int_{D_1\setminus\overline{D}} (\overline{n}-n)w^*\overline{u_0}\,\mathrm{d}x.$$

Hence, taking the real part of S, i.e. computing $(S+S^*)/2$

$$(\mathrm{Re}(S)w_0, u_0)_{H^1(D_1\setminus\overline{D})} = \int_{D_1\setminus\overline{D}} (I-\mathrm{Re}(A))\nabla w_0 \cdot \nabla\overline{u_0}\,\mathrm{d}x - k^2\int_{D_1\setminus\overline{D}} (1-\mathrm{Re}(n))w_0\overline{u_0}\,\mathrm{d}x$$
$$+ \mathrm{i}\int_{D_1\setminus\overline{D}} (-\mathrm{Im}(A)\nabla w^* \cdot \nabla\overline{u_0} + \mathrm{Im}(A)\nabla\overline{u^*} \cdot \nabla w_0)\,\mathrm{d}x$$
$$- \mathrm{i}k^2\int_{D_1\setminus\overline{D}} (-\mathrm{Im}(n)w^*\overline{u_0} + \mathrm{Im}(n)\overline{u^*}w_0)\,\mathrm{d}x$$
$$+ \int_{B_R} \mathrm{Re}(A)\nabla w^* \cdot \nabla\overline{u^*}\,\mathrm{d}x - k^2\int_{B_R} \mathrm{Re}(n)w^*\overline{u^*}\,\mathrm{d}x$$
$$- \int_{\partial B_R} \overline{\mathrm{Re}(T_k)u^*}w^*\,\mathrm{d}s.$$

Now let us define K by

$$(Kw_0, u_0)_{H^1(D_1\setminus\overline{D})} = \int_{D_1\setminus\overline{D}} (I - \mathrm{Re}(A))\nabla w_0 \cdot \nabla\overline{u_0}\,\mathrm{d}x$$
$$+\int_{D_1\setminus\overline{D}} w_0\overline{u_0}\,\mathrm{d}x + \int_{B_R} \mathrm{Re}(A)\nabla w^* \cdot \nabla\overline{u^*}\,\mathrm{d}x$$
$$+\mathrm{i}\int_{D_1\setminus\overline{D}} \left(-\mathrm{Im}(A)\nabla w^* \cdot \nabla\overline{u_0} + \mathrm{Im}(A)\nabla\overline{u^*} \cdot \nabla w_0\right)\mathrm{d}x - \int_{\partial B_R} \overline{\mathrm{Re}(T_k)u^*}w^*\,\mathrm{d}s$$

which obviously is a self-adjoint. Again, using that the real part of the Dirichlet to Neumann operator $\mathrm{Re}(T_k)$ is non-positive and applying Young's inequality yield

$$(Kw_0, w_0)_{H^1(D_1\setminus\overline{D})} \geqslant ((I - \mathrm{Re}(A) - \alpha|\mathrm{Im}(A)|)\nabla w_0, \nabla w_0)_{L^2(D_1\setminus\overline{D})}$$
$$+\left((\mathrm{Re}(A) - \frac{1}{\alpha}|\mathrm{Im}(A)|)\nabla w^*, \nabla w^*\right)_{L^2(D_1\setminus\overline{D})}$$
$$+(w_0, w_0)_{L^2(D_1\setminus\overline{D})} \geqslant c||w_0||^2_{H^1(D_1\setminus\overline{D})}$$

where α is such that $I - \mathrm{Re}(A) - \alpha|\mathrm{Im}(A)| > 0$, $\mathrm{Re}(A) - \frac{1}{\alpha}|\mathrm{Im}(A)| \geqslant 0$, and c is some constant depending on A, n only.

Finally the difference $\mathrm{Re}(S) - K$ is compact due to the compactly imbedding of $H^1(D_1\setminus\overline{D})$ into $L^2(D_1\setminus\overline{D})$. □

Remark 4.1. Injectivity of the operator $S : H^1(D_1\setminus\overline{D}) \to H^1(D_1\setminus\overline{D})$ holds true if assumption 4.1 is satisfied. Based on the analytic Fredholm theory it is easy to show that such $k > 0$ form at most a discrete set with $+\infty$ as the only possible accumulation point. It is easy to see that if $\mathrm{Im}(A) \leqslant 0$ and $\mathrm{Im}(n) > 0$ in $D_1 \setminus \overline{D}$, or $\mathrm{Im}(A) < 0$ and $n-1$ does not change sign in $D_1 \setminus \overline{D}$ (more generally it suffices that $\int_{D_1\setminus\overline{D}}(n-1)\,\mathrm{d}x \neq 0$), then assumption 4.1 holds for all real $k > 0$. In addition, the latter is also the case when A and n are real valued and the contrasts $(A - I)$ and $n - 1$ have the opposite signs.

Using the factorization in theorem 3.1 along with theorems 4.2 and 4.3, and applying theorem 4.1 to the data operator N we can conclude the following range characterization result.

Corollary 4.1. *Under the assumptions of theorems 4.2 and 4.3, the range of the operator* $N_\#^{1/2} : L^2(\partial C) \to L^2(\partial C)$ *and the range of the operator* $H^* : H^1(D_1 \setminus \overline{D}) \to L^2(\partial C)$ *coincide, where* $N_\# := |Re(N)| + Im(N)$.

The last step of our approach is to characterize the range of H^* in term of the support of the cavity D. At this point we introduce the so-called exterior transmission eigenvalue problem which in the current settings is a slight modification of the problem considered in [1] due to the fact that the incident field is the complex conjugate of the point source. This problem reads as: find $w \in H^1_{\mathrm{loc}}(\mathbb{R}^d\setminus\overline{D})$, $v \in H^1_{\mathrm{loc}}(\mathbb{R}^d\setminus\overline{D})$ such that

$$\nabla \cdot A\nabla w + k^2 nw = 0 \quad \text{in} \quad \mathbb{R}^d\setminus\overline{D} \tag{24}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad \mathbb{R}^d\setminus\overline{D} \tag{25}$$

$$w - v = f \quad \text{on} \quad \partial D \tag{26}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \quad \text{on} \quad \partial D \tag{27}$$

$$\lim_{r\to+\infty} r^{\frac{d-1}{2}}\left(\frac{\partial(w-v)}{\partial r}-\mathrm{i}k(w-v)\right)=0 \tag{28}$$

$$\lim_{r\to+\infty} r^{\frac{d-1}{2}}\left(\frac{\partial v}{\partial r}+\mathrm{i}kv\right)=0 \tag{29}$$

for $f\in H^{1/2}(\partial D)$ and $h\in H^{-1/2}(\partial D)$. Values of $k>0$ for which the homogeneous exterior transmission problem (i.e (24)–(29) with $f=0$ and $h=0$) has non-trivial solution are called *exterior transmission eigenvalues*. Using the same technique as in [1], it can be proven that the problem (24)–(29) satisfies the Fredholm alternative and the exterior transmission eigenvalues form at most a discrete set with $+\infty$ as the only possible accumulation point. Hence one can prove that provided that $k>0$ is not an exterior transmission eigenvalue the problem (24)–(29) has a unique solution $w\in H^1_{\mathrm{loc}}(\mathbb{R}^d\backslash\overline{D})$, $v\in H^1_{\mathrm{loc}}(\mathbb{R}^d\backslash\overline{D})$ that depends continuously on f and h.

Assumption 4.2. *The wave number $k>0$ is not an exterior transmission eigenvalue corresponding to (24)–(29).*

We can now prove the following theorem that relates the range of H^* with the support of the cavity D.

Theorem 4.4. *Suppose that assumption 4.2 holds. Then for $z\in\mathbb{R}^d\backslash\overline{C}$ we have that $\Phi(\cdot,z)$ is in the range of H^* if and only if $z\in\mathbb{R}^d\backslash\overline{D}$.*

Proof. Let $z\in\mathbb{R}^d\backslash\overline{D}$ and since k is not an exterior transmission eigenvalue we can construct the unique solution $w_z\in H^1_{\mathrm{loc}}(\mathbb{R}^d\backslash\overline{D})$, $v_z\in H^1_{\mathrm{loc}}(\mathbb{R}^d\backslash\overline{D})$ of (24)–(29) with $f:=\Phi(\cdot,z)$ and $h:=\frac{\partial\Phi(\cdot,z)}{\partial\nu}$. Setting $u_z=w_z-v_z$, we have that from (28) u_z is an outgoing radiating solution of

$$\nabla\cdot A\nabla u_z+k^2nu_z=\nabla(I-A)\cdot\nabla v_z+k^2(1-n)v_z \qquad \text{in}\quad \mathbb{R}^d\backslash\overline{D}$$

satisfying $u_z:=\Phi(\cdot,z)$ and $\frac{\partial u_z}{\partial\nu}=\frac{\partial\Phi(\cdot,z)}{\partial\nu}$ on ∂D from (26) and (27). Define $u:=u_z$ in $\mathbb{R}^d\backslash\overline{D}$ and $u:=\Phi(\cdot,z)$ in D. The continuity of the Cauchy data guarantees that $u\in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and in addition u is an outgoing radiating solution of

$$\nabla\cdot A\nabla u+k^2nu=\nabla\cdot(I-A)\nabla v_z+k^2(1-n)v_z \qquad \text{in}\quad \mathbb{R}^d$$

which from the definition of operator $G:H^1(D_1\backslash\overline{D})\to L^2(\partial C)$ means that $\Phi(\cdot,z)|_{\partial C}=Gv_z$. Note that $v_z\in H^1(D_1\backslash\overline{D})$ satisfies the Helmholtz equation and the incoming radiation condition and therefore it is in the closure of the range of H. Finally since $G=H^*S$, we now have that $\Phi(\cdot,z)$ is in the range of H^*.

Next assume that $z\in\overline{D}\backslash\overline{C}$ and to the contrary that $\Phi(\cdot,z)|_{\partial C}$ is in the range of H^*. Let $v_0\in H^1(D_1\backslash\overline{D})$ be such that $H^*v_0=\Phi(\cdot,z)$. Then there is $v\in H^1(\mathbb{R}^d\backslash\overline{C})$ uniquely determined by (8) such that

$$(H^*v_0)(x)=\int_{\partial C}\frac{\partial\Phi(\cdot,y)}{\partial\nu_y}v(y)\,\mathrm{d}s(y)-\frac{1}{2}v(x) \quad \text{for}\quad x\in\partial C.$$

From the jump property of the double layer potential we have that

$$\left[\int_{\partial C}\frac{\partial\Phi(\cdot,y)}{\partial\nu_y}v(y)\,\mathrm{d}s(y)\right]^-=\Phi(\cdot,z) \quad \text{on}\quad \partial C$$

approaching ∂C from inside. From (8), we can also see that v satisfies the Helmholtz equation in $D\backslash\overline{C}$ and $\frac{\partial v^+}{\partial \nu}|_{\partial C} = 0$ where $+$ indicates that ∂C is approached from outside C. Now define

$$w(\cdot) = \begin{cases} \displaystyle\int_{\partial C} \frac{\partial \Phi(\cdot, y)}{\partial \nu_y} v(y)\, \mathrm{d}s(y) & \text{in} \quad C \\ \displaystyle\int_{\partial C} \frac{\partial \Phi(\cdot, y)}{\partial \nu_y} v(y)\, \mathrm{d}s(y) - v(\cdot) & \text{in} \quad D\backslash\overline{C} \end{cases}$$

then $w \in H^1(D)$, satisfies Helmholtz equation in D, and $w^- = \Phi(\cdot, z)$ on ∂C. Hence the assumption (2.1) guarantees that $w = \Phi(\cdot, z)$ in C since both satisfy the same Dirichlet boundary value problem for the Helmholtz equation in C. Now, if $z \in D$, by analytic continuation we have that $w = \Phi(\cdot, z)$ in $D\backslash z$, but since $\Phi(x, z)$ has singularity at $x = z$ whereas w is analytic, we arrive at a contradiction. Furthermore, if $z \in \partial D$, then equality of w and $\Phi(\cdot, z)$ up to the boundary ∂D requires that $\Phi(\cdot, z) \in H^{\frac{1}{2}}(\partial D)$, in the sense of the trace, which is not true, whence we again arrive at a contradiction. Therefore we can conclude that for $z \in \overline{D}\backslash\overline{C}$, $\Phi(\cdot, z)$ is not in the range of H^*. □

Theorem 4.4 can be modified to remove assumption 4.2.

Theorem 4.5. *For $z \in D_1\backslash\overline{C}$ we have that $\Phi(\cdot, z)$ is in the range of H^* if and only if $z \in D_1\backslash\overline{D}$.*

Proof. We only need to prove the statement for $z \in D_1\backslash\overline{D}$ since the complimentary case holds under no restriction on the wave and is proven in the second part of theorem 4.4. To this end, for $z \in D_1\backslash\overline{D}$, we need to show that there exists $v_0 \in H^1(D_1\backslash\overline{D})$ such that $H^*v_0 = \Phi(\cdot, z)$. Fix $\epsilon > 0$ small enough and consider $w^* := \Phi(\cdot, z)\chi_\epsilon$, where χ_ϵ is a cut-off function such that $\chi_\epsilon = 0$ in $B(z, \epsilon)$ and $\chi_\epsilon = 1$ outside $B(z, 2\epsilon)$ where $B(z, \epsilon)$ is a ball centered at z with radius ϵ, and $B(z, 2\epsilon) \subset D_1\backslash\overline{D}$. Obviously, $w^* \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$. Let now $v \in H^1(B_R\backslash\overline{C})$ be defined by (10) and $v_0 \in H^1(D_1\backslash\overline{D})$ be defined by (11). We need to show that v_0, v satisfy (8). Indeed, by constructions, w^* satisfies Helmholtz equation in $D\backslash\overline{C}$ and $\mathbb{R}^d\backslash\overline{D_1}$ and so does v. Therefore

$$\int_{\partial D_1} \frac{\partial v^+}{\partial \nu}\overline{\psi}\, \mathrm{d}s = \int_{\partial B_R} T_k v\overline{\psi}\, \mathrm{d}s - \int_{B_R\backslash\overline{D_1}} \nabla v \cdot \nabla\overline{\psi} \mathrm{d}x + k^2 \int_{B_R\backslash\overline{D_1}} v\overline{\psi}\, \mathrm{d}x$$

and

$$\int_{\partial D} \frac{\partial v^-}{\partial \nu}\overline{\psi}\, \mathrm{d}s = \int_{\partial C} \frac{\partial v^+}{\partial \nu}\overline{\psi}\, \mathrm{d}s + \int_{D\backslash\overline{C}} \nabla v \cdot \nabla\overline{\psi}\, \mathrm{d}x - k^2 \int_{D\backslash\overline{C}} v\overline{\psi}\, \mathrm{d}x.$$

Plugging both the above equations in (11), we have that for any $\psi \in H^1(B_R\backslash\overline{C})$

$$(v_0, \psi|_{D_1\backslash\overline{D}})_{H^1(D_1\backslash\overline{D})} = -\int_{B_R\backslash\overline{C}} \nabla v \cdot \nabla\overline{\psi}\, \mathrm{d}x + k^2 \int_{B_R\backslash\overline{C}} v\overline{\psi} \mathrm{d}x + \int_{\partial B_R} T_k v\overline{\psi}\, \mathrm{d}x - \int_{\partial C} \frac{\partial v^+}{\partial \nu}\overline{\psi}\, \mathrm{d}s.$$

From the definition of v and using jump properties of double layer potential we have that

$$\left[\int_{\partial C} \frac{\partial \Phi(x, y)}{\partial \nu_y} v(y)\, \mathrm{d}s(y)\right]^- = w^*(x) \quad \text{for} \quad x \in \partial C$$

where '$-$' indicates approaching ∂C from inside C. Then

$$\frac{\partial}{\partial \nu_x}\left[\int_{\partial C} \frac{\partial \Phi(\cdot, y)}{\partial \nu_y} v(y)\, \mathrm{d}s(y)\right]^- = \frac{\partial w^*}{\partial \nu} \quad \text{on} \quad \partial C$$

and another application of the jump properties of double layer potential implies

$$\frac{\partial}{\partial \nu_x}\left[\int_{\partial C} \frac{\partial \Phi(\cdot, y)}{\partial \nu_y} v(y)\, \mathrm{d}s(y)\right]^+ = \frac{\partial w^*}{\partial \nu} \quad \text{on} \quad \partial C$$

whence by construction of v we have that $\frac{\partial v^+}{\partial \nu} = 0$ on ∂C, where '+' indicates approaching ∂C from outside C. Therefore (8) holds for v and v_0, hence by definition of H^* (7) holds true. From the construction of v and jump properties of the double layer potential we have that

$$\int_{\partial C} \frac{\partial \Phi(\cdot, y)}{\partial \nu_y} v(y)\, \mathrm{d}s(y) - \frac{1}{2} v(x) = w^*(x) \quad \text{for} \quad x \in \partial C$$

and therefore $H^* v_0 = w^*$. Now since $w^* = \Phi(\cdot, z)$ in D we finally obtain $H^* v_0 = \Phi(\cdot, z)$ on ∂C. □

Now we are ready to state the main theorem of the paper. Let us recall the compact data operator $N : L^2(\partial C) \to L^2(\partial C)$ given by (3) and define $\mathrm{Re}(N) = \frac{N+N^*}{2}$, $\mathrm{Im}(N) = \frac{N-N^*}{2\mathrm{i}}$ and $N_\# := |\mathrm{Re}(N)| + \mathrm{Im}(N)$ which is also compact. In addition $N_\#$ is also self-adjoint. We denote by $(\phi_j, \lambda_j)_{j\in\mathbb{N}}$ an orthonormal eigen-system for $N_\#$. Then we have the following result.

Theorem 4.6. *Suppose that all assumptions 2.1, 4.1 and 4.2 are valid for the wave number $k > 0$, and either $Re(A) > I$, or $I - Re(A) - \alpha|Im(A)| > 0$ and $Re(A) - \frac{1}{\alpha}|Im(A)| \geqslant 0$ for some $\alpha > 0$. Then for $z \in \mathbb{R}^d \backslash \overline{C}$*

$$z \in \mathbb{R}^d \backslash \overline{D} \quad \textit{if and only if} \quad \sum_j \frac{|\left(\Phi_z, \phi_j\right)|^2}{\lambda_j} < \infty$$

where $\Phi_z := \Phi(\cdot, z)|_{\partial C}$, with $\Phi(\cdot, z)$ being the fundamental solution of the Helmholtz equation given by (1).

Proof. The result follows from corollary 4.1 and theorem 4.4 along with an application of the Picard's theorem [2] and [3]. □

Using now theorem 4.5 instead of theorem 4.4 we can drop assumption 4.2. Note it is more difficult to handle the existence of exterior transmission eigenvalues than checking whether the wave number $k > 0$ satisfies assumption 4.1.

Theorem 4.7. *Suppose that both assumptions 2.1 and 4.1 are valid for the wave number $k > 0$, and either $Re(A) > I$, or $I - Re(A) - \alpha|Im(A)| > 0$ and $Re(A) - \frac{1}{\alpha}|Im(A)| \geqslant 0$ for some $\alpha > 0$. Then for $z \in D_1 \backslash \overline{C}$*

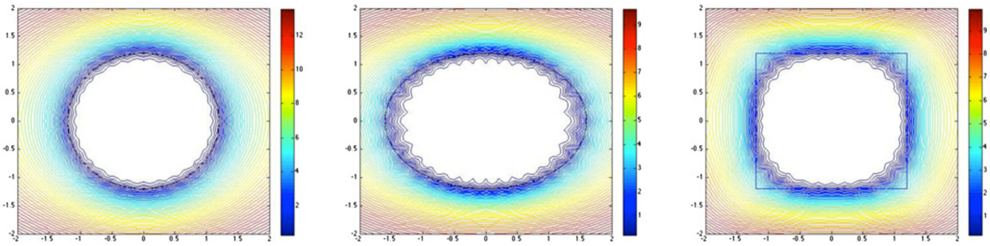
$$z \in D_1 \backslash \overline{D} \quad \textit{if and only if} \quad \sum_j \frac{|\left(\Phi_z, \phi_j\right)|^2}{|\lambda_j|} < \infty$$

where $\Phi_z := \Phi(\cdot, z)|_{\partial C}$, with $\Phi(\cdot, z)$ being the fundamental solution of the Helmholtz equation given by (1).

From practical point of view in order to determine the support of D from interior sources and measurements it suffices to sample only within the region D_1.

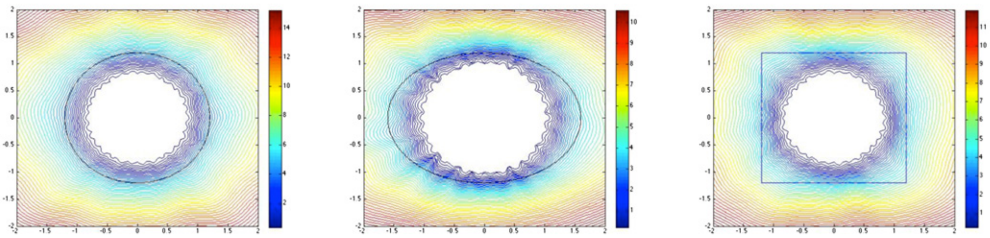
5. Numerical examples

In this section we provide some preliminary numerical results to show the viability of the factorization method to determine the support of a cavity surrounded by anisotropic inhomogeneous media. For a given anisotropic medium and artificial point sources on the given manifold ∂C, we can compute the near field data using a finite element method combined with PML on the artificial boundary. Having the simulated data $u^s(x, y)$, $x, y \in \partial C$, we compute a discretized version of the near field operator and of $N_\#$, and then apply the criterion described



(a) (b) (c)

Figure 2. Panels (a), (b) and (c) show the reconstruction of a circle with radius 1.2, of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with noise free data. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.8. Here $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$ and the true geometry of the cavity is indicated by the solid line.



(a) (b) (c)

Figure 3. Panels (a), (b) and (c) show the reconstruction of a circle with radius 1.2, of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with 1% noise. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.8. Here $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$ and the true geometry is indicated by the solid line. The sampling points z are in $[-2, 2]^2$.

in theorem 4.6 to reconstruct the interior of the cavity D. In particular, we compute the eigensystem $(\phi_j, \lambda_j)_{j=1..M}$ of the symmetric matrix that approximate $N_\#$ and then use the discrete version of the Picard's criteria. To visualize the cavity we plot the contour lines of

$$W(z) := \left[\sum_{j=1}^{M} \frac{| < \Phi_z, \phi_j > |^2}{|\lambda_j|} \right]^{-1}$$

for z varying in a region large enough to contain the D. The cavity is the region where $W(z)$ takes values close to zero. For more details in the implementation of the factorization method see [10].

In this paper, we present the reconstruction of a circle, an ellipse and a square in the two-dimensional case. The exact geometry and the reconstructions are shown in the figures below. In all the examples presented here the region D_1 is the disk of radius 2. In the examples presented in figures 2 and 3, C is the disk of radius 0.8 (30 incident point sources and 30 corresponding measurements equally distributed on ∂C), the anisotropic medium has the constitutive parameters $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$, and the wave number is $k = 5$. Reconstructions are given for noise free data and 1% white noise added. The sampling point z moves in a grid covering the square $[-2, 2]^2$.

In order to study the sensitivity of reconstructions on the size of the measurement manifold ∂C, we show reconstructions for the configuration of the examples in figure 2 where now ∂C

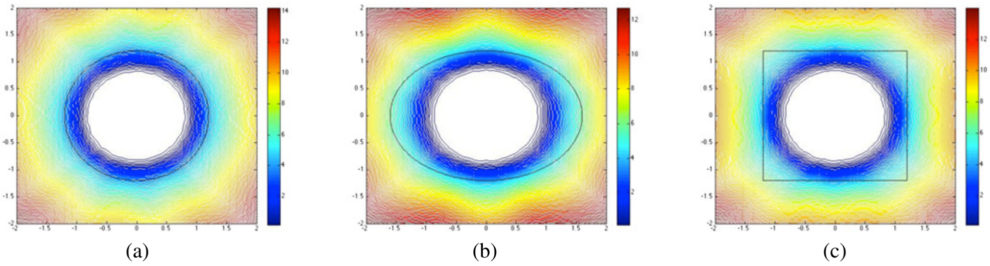


Figure 4. Panels (a), (b) and (c) show the reconstruction of a circle with radius 1.2, of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with noise free data. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.4. Here $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$ and the true geometry is indicated by the solid line. The sampling points z are in $[-2, 2]^2$.

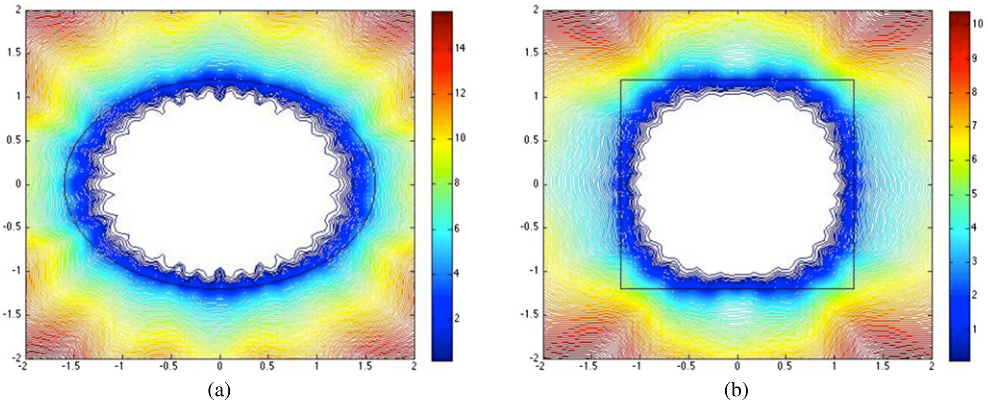
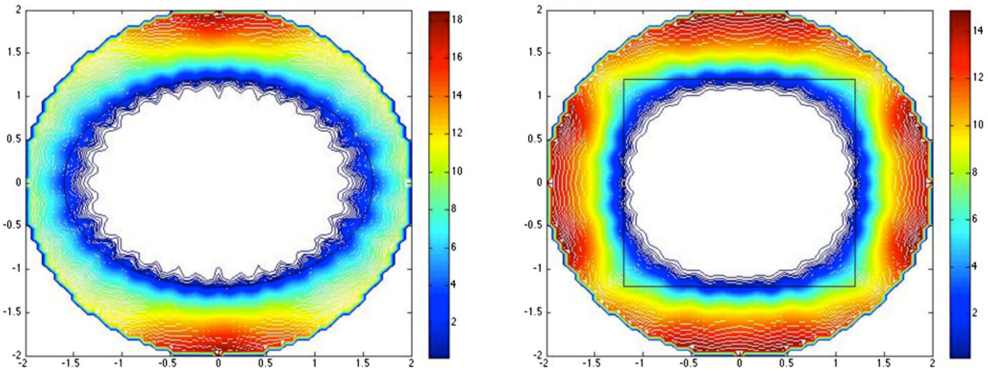


Figure 5. Panels (a) and (b) show the reconstruction of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with noise free data. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.8. Here $A = [0.6, 0; 0, 0.8]$, $n = 0.8$ and the true geometry of the cavity is indicated by the solid line. The sampling points z are in $[-2, 2]^2$.

is the circle of radius 0.4. The results presented in figure 4 confirm that the reconstructions become worse as C gets smaller although the number of sources and receivers remains the same. We also consider the anisotropic media with matrix A satisfying (loosely speaking) $A - I < 0$, namely $A = [0.6, 0; 0, 0.8]$ for the ellipse and square and the reconstructions are presented in figure 5. Finally as explained in theorem 4.7 it is possible to avoid the (real) exterior transmission eigenvalues (which in particular cases are proven to exists cf [5]) if the sampling point z remains only inside D_1, i.e. in the inhomogeneous layer and the cavity. The examples presented in figure 6 with sampling region $D_1 \setminus \overline{C}$ for the ellipse and the square confirm that this confinement of sampling region does not affect the quality of reconstructions.

6. Remarks on non-physical incident sources

Our justification of the factorization method works for incident waves being complex conjugate of point sources, which are non-physical. However, it is well known that these non-physical sources can be approximated arbitrarily close by linear combination of physical point sources



(a) (b)

Figure 6. Panels (a) and (b) show the reconstruction of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with noise free data. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.8. Here $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$ and the true geometry of the cavity is indicated by the solid line. The sampling points z are in $D_1 \setminus \overline{C}$.

(these fact is also discussed in [9]). It is interesting that for non-penetrable cavities the factorization method can be justified for physical incident waves. Our analysis can be carried through for the problem when D_1 is contained in a large ball B_R with homogeneous medium in $B_R \setminus \overline{D_1}$ and zero Dirichlet or Neumann conditions on ∂B_R. In this case it is possible to take the incident wave to be the Dirichlet or Neumann Green's function for the homogeneous media, which is real valued, and then everything in the paper works exactly in the same way. We should also mention that formal implementation of the factorization method with physical point sources provides reasonable reconstructions.

Acknowledgments

The research of SM is supported in part by the NSF grant DMS-1106972 and the DRI of INRIA (Associate Team ISIP). The research of FC is supported in part by the Air Force Office of Scientific Research Grant FA9550-13-1-0199 and NSF grant DMS-1106972. SM greatly acknowledge the hospitality of the DeFI Team during his visit at Ecole Polytechnique.

References

[1] Cakoni F, Colton D and Meng S 2014 The inverse scattering problem for a penetrable cavity with internal measurements *AMS Contemp. Math.* **615** 71–88

[2] Cakoni F and Colton D 2014 *A Qualitative Approach to Inverse Scattering Theory* (Berlin: Springer)

[3] Colton D and Kress R 2013 *Inverse Acoustic and Electromagnetic Scattering Theory* 3rd edn (New York: Springer)

[4] Hu Y, Cakoni F and Liu J The inverse problem for a partially coated cavity with interior measurements *Appl. Anal.* at press

[5] Colton D, Leung Y J and Meng S The inverse spectral problem for exterior transmission eigenvalues (submitted)

[6] Ikehata M 2012 An inverse acoustic scattering problem inside a cavity with dynamical back-scattering data *Inverse Problems* **28** 095016

[7] Ikehata M and Itou H 2012 On reconstruction of a cavity in a linearized viscoelastic body from infinitely many transient boundary data *Inverse Problems* **28** 125003

[8] Jakubik P and Potthast R 2008 Testing the integrity of some cavity-the Cauchy problem and the range test *Appl. Numer. Math.* **58** 899–914
[9] Kirsch A 2007 An integral equation for Maxwell's equations in a layered medium with an application to the factorization method *J. Integral Eqns and Appl.* **19** 333–57
[10] Kirsch A and Gringerg N 2008 *The Factorization Method for Inverse Problems* (Oxford: Oxford University Press)
[11] Kress R 1999 *Linear Integral Equations* 2nd edn (New York: Springer)
[12] Lechleiter A 2009 The factorization method is independent of transmission eigenvalues *Inverse Problems Imaging* **3** 123–38
[13] Liu X 2014 The factorization method for cavities *Inverse Problems* **30** 015006
[14] McLean W 2000 *Strongly Elliptic Systems and Boundary Integral Equations* (Cambridge: Cambridge University Press)
[15] Qin H H and Cakoni F 2011 Nonlinear integral equations for shape reconstruction in the inverse interior scattering problem *Inverse Problems* **27** 035005
[16] Nédélec J C 2001 *Acoustic and Electromagnetic Equations Integral Representations for Harmonic Problems (Applied Mathematical Sciences* vol 144*)* (New York: Springer)
[17] Qin H H and Colton D 2012 The inverse scattering problem for cavities with impedance boundary condition *J. Adv. Comput. Math.* **36** 157–74
[18] Qin H H and Colton D 2012 The inverse scattering problem for cavities *J. Appl. Numer. Math.* **62** 699–708
[19] Zeng F, Cakoni F and Sun J 2012 An inverse electromagnetic scattering problem for cavity *Inverse Problems* **27** 125002