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THE DETERMINATION OF BOUNDARY COEFFICIENTS FROM FAR FIELD MEASUREMENTS

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Dedicated to Chuck Groetsch for his fundamental contributions to the field of inverse problems.

ABSTRACT. We consider the problem of determining either the surface impedance $\lambda = \lambda(x)$ or surface conductivity $\eta = \eta(x)$ from far field data corresponding to time-harmonic incident plane waves scattered by a coated infinite cylinder. We show that λ and η are uniquely determined from the far field data and provide a numerical algorithm for determining these quantities.

1. Introduction. Inverse problems connected with the detection of decoys play a special role in inverse electromagnetic scattering theory since for such problems the shape of the scattering object is typically known a priori. For example, in order to distinguish between a real missile and a decoy coated with metallic paint the shapes are the same and known and the target identification problem is based on distinguishing between a perfect conductor and a dielectric coated with a thin highly conducting layer. Assuming that the frequency is chosen such that the thickness of the coating is less than the skin depth, the problem then becomes one of determining the surface conductivity, i.e., the problem of determining a coefficient in a boundary condition. In other decoy problems the hostile object can be a perfect conductor coated by a thin dielectric layer, i.e., in this case the surface impedance serves as target signature. Problems associated with the detection of decoys are further complicated by the fact that the far-field data is measured over a limited aperture and the directions of the incident plane waves used to interrogate the target are also restricted to a limited aperture.

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In this paper we will consider two inverse scattering problems which are associated with the detection of decoys. The first problem is to determine the surface impedance $\lambda = \lambda(x)$ of a scattering object of known shape from limited aperture far field data and the second is to determine the surface conductivity $\eta = \eta(x)$ using the same type of data. (For simplicity we restrict ourselves to the case of TM polarized waves scattered by an infinite cylinder.) We first show that under appropriate assumptions λ and η are both uniquely determined from limited aperture far field data. In the case of the surface impedance λ this is a new proof of a known result (Theorem 8.11 of [**3, 4**]), but using weaker hypotheses on λ and the boundary, whereas for η the proof is both new and a significant improvement on previous results (Theorem 8.26 of [**4, 7**]). We then show how λ and η can be determined by solving a linear integral equation of the first kind (if λ and η are constant we obtain a formula for λ and η and avoid the need to solve an integral equation) and provide some numerical examples showing the practicality of our inversion algorithm for moderate values of λ or η. We note that in both the uniqueness theorems and the reconstruction algorithms we allow for the case of partially coated obstacles. For related results on the determination of boundary coefficients from far field data, we refer the reader to [**1, 10, 13, 16**].

2. The surface impedance. We consider the scattering of an electromagnetic time harmonic plane wave by a perfectly conducting infinite cylinder that is (partially) coated by an inhomogeneous dielectric material. This leads to a mixed boundary value problem for the Helmholtz equation [**6**]. In particular, let $D \subset \mathbf{R}^2$ be an open, bounded region with Lipschitz boundary ∂D such that $\mathbf{R}^2 \setminus \overline{D}$ is connected. We assume that the boundary ∂D has a Lipschitz dissection $\partial D = \partial D_D \cup \Pi \cup \partial D_I$, where ∂D_D and ∂D_I are disjoint, relatively open subsets of ∂D, having Π as their common boundary in ∂D. Furthermore, boundary conditions of Dirichlet and impedance type with the surface impedance a bounded measurable function $\lambda \in L_\infty(\partial D_I)$ are specified on ∂D_D and ∂D_I, respectively. We assume that the surface impedance is positive and uniformly bounded, i.e., $\lambda(x) \geq \lambda_0 > 0$ for almost all $x \in \partial D_I$. Let ν denote the unit outward normal vector defined almost everywhere on $\partial D_D \cup \partial D_I$. The total field $u = u^s + e^{ikx\cdot d}$ given as the sum of the unknown scattered wave and incident plane wave satisfies

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbf{R}^2 \setminus \overline{D}, \tag{1}$$

$$u = 0 \quad \text{on } \partial D_D, \tag{2}$$

$$\frac{\partial u}{\partial \nu} + i\lambda(x) u = 0 \quad \text{on } \partial D_I, \tag{3}$$

where $k > 0$ is the wave number and d is a unit vector in the direction of propagation of the incident plane waves. Moreover the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{4}$$

uniformly in $\widehat{x} = x/|x|$ with $r = |x|$.

The well-posedness of the exterior mixed boundary value problem is established in [**6**] (in [**6**] λ was assumed to be constant, but all the results remain valid if $\lambda = \lambda(x) \in L_\infty(\partial D_I)$). In particular it is shown that the direct scattering problem (1)–(4) has a unique solution $u \in H^1_{\text{loc}}(\mathbf{R}^2 \setminus \overline{D})$.

It is easy to see [4, 11] that the scattered field has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\widehat{x}, d) + O(r^{-3/2}),$$

where u_∞ is the *far field pattern* of the scattered wave. The far field pattern defines the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\widehat{x}) := \int_\Omega u_\infty(\widehat{x}, d) g(d)\, ds(d), \quad g \in L^2(\Omega) \tag{5}$$

where $\Omega := \{x \in \mathbf{R}^2 : |x| = 1\}$ is the unit circle. A Herglotz wave function with kernel $g \in L^2(\Omega)$ is an entire solution of the Helmholtz equation defined by

$$v_g^i(x) = \int_\Omega e^{ikx\cdot d} g(d)\, ds(d), \quad x \in \mathbf{R}^2. \tag{6}$$

In this paper we assume that D is known. Then the *inverse problem* we consider here is to determine the surface impedance λ as a function in $L_\infty(\partial D_I)$ from a knowledge of $u_\infty(\widehat{x}, d)$ for $\widehat{x} \in \Omega_0 \subseteq \Omega$ and $d \in \Omega_1 \subseteq \Omega$.

First we give a uniqueness proof for $\lambda \in L_\infty(\partial D_I)$. To this end we prove the following equality.

Theorem 2.1. *Let v_g^i be the Herglotz wave function with kernel g and $v_g := v_g^s + v_g^i$ the solution of (1)–(4) with $e^{ikx\cdot d}$ replaced by v_g^i. Then*

$$\int_{\partial D} \lambda(x)|v_g|^2\, ds = -k\|Fg\|^2 + \sqrt{8\pi k}\, Im\left(e^{i\pi/4}(Fg,\, g)\right) \tag{7}$$

where F is the far field operator and $(\cdot,\,\cdot)$ is the inner product over $L^2(\Omega)$.

Proof. We follow the proof in [**12**]. We first note that if v^s and w^s are two radiating solutions of the Helmholtz equation with far field patterns v_∞ and w_∞ then from Green's theorem we have that

$$\int_{\partial D} \left(v^s \frac{\partial \overline{w^s}}{\partial \nu} - \overline{w^s}\frac{\partial v^s}{\partial \nu}\right) ds = -2ik \int_\Omega v_\infty \overline{w_\infty}\, ds. \tag{8}$$

Furthermore, if $v^s \in H^1_{\mathrm{loc}}(\mathbf{R}^2 \setminus \overline{D})$ is a radiating solution to the Helmholtz equation with far field pattern v_∞ then

$$v_\infty(\widehat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \left(v^s(y) \frac{\partial e^{-ik\widehat{x}\cdot y}}{\partial \nu_y} - e^{-ik\widehat{x}\cdot y}\frac{\partial v^s(y)}{\partial \nu}\right) ds(y),$$

and hence if w_h^i is a Herglotz wave function with kernel wave function with kernel h then an interchange of orders of integration shows that

$$\int_{\partial D} \left(v^s \frac{\partial \overline{w_h^i}}{\partial \nu} - \overline{w_h^i}\frac{\partial v^s}{\partial \nu}\right) ds = \sqrt{8\pi k}\, e^{-i\pi/4} \int_\Omega \overline{h(d)} v_\infty(d)\, ds. \tag{9}$$

Now let $v_g = v_g^s + v_g^i$ be the solution of (1)–(4) with kernel $e^{ikx\cdot d}$ replaced by v_g^i. Then, since $v_g = 0$ on ∂D_D and using (8) and (9), we have that

$$
\begin{aligned}
2i \int_{\partial D} & \lambda(x)|v_g|^2 ds(x) \\
&= \int_{\partial D} \left(v_g \frac{\partial \overline{v_g}}{\partial \nu} - \overline{v_g} \frac{\partial v_g}{\partial \nu} \right) ds \\
&= \int_{\partial D} \left(v_g^s \frac{\partial \overline{v_g^s}}{\partial \nu} - \overline{v_g^s} \frac{\partial v_g^s}{\partial \nu} \right) ds + \int_{\partial D} \left(v_g^s \frac{\partial \overline{v_g^i}}{\partial \nu} - \overline{v_g^i} \frac{\partial v_g^s}{\partial \nu} \right) ds \\
&\quad + \int_{\partial D} \left(v_g^i \frac{\partial \overline{v_g^s}}{\partial \nu} - \overline{v_g^s} \frac{\partial v_g^i}{\partial \nu} \right) \\
&= -2ik \int_\Omega |v_{g,\infty}|^2 \, ds + \sqrt{8\pi k}\, e^{-i\pi/4} \\
&\quad \times \int_\Omega \overline{g}\, v_{g,\infty} \, ds - \sqrt{8\pi k}\, e^{i\pi/4} \int_\Omega g\, \overline{v_{g,\infty}} \, ds \\
&= -2ik\|Fg\|^2 + \sqrt{8\pi k}\, e^{-i\pi/4}(Fg,\, g) - \sqrt{8\pi k}\, e^{i\pi/4}(g,\, Fg),
\end{aligned}
$$

which proves the theorem. □

Theorem 2.2. *Let*

$$
\mathcal{W} := \left\{ f \in L^2(\partial D_I) : \begin{array}{c} f = v_g|_{\partial D_I} \text{ for some } g \in L^2(\Omega) \\ \text{where } v_g \text{is as in Theorem 2.1} \end{array} \right\}.
$$

Then $\mathcal{W}$ is dense in $L^2(\partial D_I)$.

Proof. It suffices to show that if $\varphi \in L^2(\partial D_I)$ satisfies

$$
\int_{\partial D_I} v_g \varphi \, ds = 0 \tag{10}
$$

for all $f = v_g|_{\partial D_I} \in \mathcal{W}$ then $\varphi = 0$. Suppose (10) is true for all such v_g and let $w \in H^1_{\mathrm{loc}}(\mathbf{R}^2 \setminus \overline{D})$ be the solution [**6**] of

$$
\begin{aligned}
\Delta w + k^2 w = 0 &\quad \text{in } \mathbf{R}^2 \setminus \overline{D} \\
w = 0 &\quad \text{on } \partial D_D \\
\frac{\partial w}{\partial \nu} + i\lambda(x) w = \varphi &\quad \text{on } \partial D_I \\
\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial r} - ikw \right) = 0.&
\end{aligned}
$$

Then since $v_g = 0$ on ∂D_D, using the boundary conditions and the integral representation formula, we have that

$$\begin{aligned}
0 &= \int_{\partial D} v_g \varphi \, ds = \int_{\partial D} v_g \left(\frac{\partial w}{\partial \nu} + i\lambda w \right) ds \\
&= \int_{\partial D} \left[v_g^s \frac{\partial w}{\partial \nu} + i\lambda v_g^s w + v_g^i \frac{\partial w}{\partial \nu} + i\lambda v_g^i w \right] ds \\
&= \int_{\partial D} \left[v_g^s \frac{\partial w}{\partial \nu} + w \left(-\frac{\partial v_g^s}{\partial \nu} - \frac{\partial v_g^i}{\partial \nu} - i\lambda v_g^i \right) \right] ds \\
&\quad + \int_{\partial D} \left(v_g^i \frac{\partial w}{\partial \nu} + i\lambda v_g^i w \right) ds \\
&= \int_{\partial D} \left(v_g^s \frac{\partial w}{\partial \nu} - \frac{\partial v_g^s}{\partial \nu} w \right) ds + \int_{\partial D} \left(v_g^i \frac{\partial w}{\partial \nu} - \frac{\partial v_g^i}{\partial \nu} w \right) ds \\
&= \int_{\partial D} \left(v_g^i \frac{\partial w}{\partial \nu} - \frac{\partial v_g^i}{\partial \nu} w \right) ds \quad \text{for all } g \in L^2(\Omega).
\end{aligned}$$

Substituting (6) into the last line of this equation and interchanging the order of integration now implies that

$$\int_{\partial D} \left(e^{ikx\cdot d} \frac{\partial w}{\partial \nu}(x) - \frac{\partial e^{ikx\cdot d}}{\partial \nu} w(x) \right) ds(x) = 0 \quad \text{for } d \in \Omega,$$

and hence $w_\infty(d) = 0$ which implies $w = 0$ by Rellich's lemma. Hence $\varphi = 0$.

Theorem 2.3. *Let u_∞^j be the far field pattern for* (1)–(4) *corresponding to $\lambda = \lambda_j$, $u = u^j$, $j = 1, 2$, and let Ω_0 and Ω_1 be open nonempty subsets of the unit circle Ω. Then if $u_\infty^1(\widehat{x}, d) = u_\infty^2(\widehat{x}, d)$ for $\widehat{x} \in \Omega_0$, $d \in \Omega_1$, $\lambda_1(x) = \lambda_2(x)$ almost everywhere on $\partial D_{I_1} = \partial D_{I_2}$.*

Proof. Since $u_\infty(\widehat{x}, d)$ is an analytic function of $\widehat{x}$ and d on $\Omega \times \Omega$, the hypotheses of the theorem imply that $u_\infty^1(\widehat{x}, d) = u_\infty^2(\widehat{x}, d)$ for $\widehat{x}$, $d \in \Omega$. Hence by Rellich's lemma $u^1 = u^2$ in $\mathbf{R}^2 \setminus \overline{D}$ for all $d \in \Omega$ which implies that $v_g = v_g^1 = v_g^2$ on ∂D, where v_g^j is the solution of (1)–(4) with $e^{ikx\cdot d}$ replaced by v_g^i and $\lambda = \lambda_j$. By Holmgren's theorem and the boundary condition satisfied by u^j, $\partial D_{I_1} = \partial D_{I_2} = \partial D_I$. From

Theorem 2.1 we have that

$$\int_{\partial D_I} [\lambda_1(x) - \lambda_2(x)]\, |v_g|^2 \, dx = 0.$$

Viewing the $L_\infty(\partial D_I)$ function $\lambda_1 - \lambda_2$ as a self-adjoint operator on $L^2(\partial D_I)$ we have by Theorem 2.2 that $\lambda_1(x) = \lambda_2(x)$ almost everywhere on ∂D_I (cf. [**15**, Theorem 9.2-2]) which proves the uniqueness for both the support of the coating and the surface impedance.

Next we show how to reconstruct the surface impedance λ. To this end we consider the *far field equation*

$$\int_{\Omega_1} u_\infty(\widehat{x}, d) g(d)\, ds(d) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\widehat{x}\cdot z} \qquad (11)$$
$$g \in L^2(\Omega_1), \quad \widehat{x} \in \Omega_0, \qquad z \in D.$$

In [**3**] combined with the result of [**2**] it is shown that for $z \in D$ and for every $\varepsilon > 0$ there exists a $g_\varepsilon^z \in L^2(\Omega)$ with support in $\overline{\Omega}_1$ such that

$$\left\| \int_{\Omega_1} u_\infty(\widehat{x}, d) g_\varepsilon^z(d)\, ds(d) - \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\widehat{x}\cdot z} \right\|_{L^2(\Omega_0)} \leq \varepsilon, \qquad (12)$$

and the corresponding Herglotz function $v^i_{g_\varepsilon^z}$ is such that

$$\lim_{\varepsilon \to 0} \| v^i_{g_\varepsilon^z} - u_z \|_{H^1(D)} = 0$$

where $u_z \in H^1(D)$ is the unique solution of the interior boundary value problem

$$\Delta u_z + k^2 u_z = 0 \quad \text{in } D \qquad (13)$$

$$u_z = -\Phi(\cdot, z) \quad \text{on } \partial D_D \qquad (14)$$

$$\frac{\partial u_z}{\partial \nu} + i\lambda(x) u_z = -\frac{\partial \Phi(\cdot, z)}{\partial \nu} - i\lambda(x)\Phi(\cdot, z) \quad \text{on } \partial D_I, \qquad (15)$$

$\Phi(x, z) := (i/4) H_0^{(1)}(k|x - z|)$ is the fundamental solution to the Helmholtz equation and $H_0^{(1)}$ is a Hankel function of the first kind of order zero. Using the interior regularity results for elliptic equations it is easy to see that, for $z \in D$, $\lim_{\varepsilon \to 0} v^i_{g^z_\varepsilon}(z) = u_z(z)$.

The following lemmas were proven [**3**].

Lemma 2.4. *For every point $z \in D$ we have that*

$$\int_{\partial D_I} \lambda(x) \left| u_z(x) + \Phi(x,z) \right|^2 ds(x) = -1/4 - Im \, (u_z(z)) \tag{16}$$

where u_z is defined by (13)–(15).

Lemma 2.5. *Let $B_r \subset D$ be a ball of radius r, and denote by*

$$\mathcal{V} := \left\{ f \in L^2(\partial D_I) : \begin{array}{l} f = (u_z + \Phi(\cdot, z))|_{\partial D_I}, \\ z \in B_r \ \textit{and} \ u_z \ \textit{the solution of} \ (13)\text{–}(15) \end{array} \right\}.$$

Then $\mathcal{V}$ is complete in $L^2(\partial D_I)$.

Equation (16) can be seen as an integral equation of the first kind for λ. Since $u_z + \Phi$ vanishes on ∂D_D we can replace the region of integration ∂D_I by ∂D. Using Lemma 2.5, it is easy to see in the same way as in the proof of Theorem 2.3 that the left hand side of this equation is an injective compact integral operator with positive kernel defined on $L_\infty(\partial D_I)$. Using Tikhonov regularization techniques it is possible to determine λ by finding the regularized solution of (16) in $L^2(\partial D)$ with noisy kernel and noisy right hand side. In particular, if the surface impedance is a positive constant $\lambda > 0$ we obtain that

$$\lambda = \frac{-1/4 - \mathrm{Im}\ (u_z(z))}{\|u_z + \Phi(\cdot\,; z)\|^2_{L^2(\partial D)}}, \quad z \in D, \tag{17}$$

where we have used the fact that $u_z(\cdot) + \Phi(\cdot, z) = 0$ on ∂D_D.

3. The surface conductivity. We now consider the scattering of an electromagnetic time harmonic plane wave by an inhomogeneous dielectic infinite cylinder, that is, (partially) coated by a thin highly conducting layer. In particular, consider the scattering of a time-harmonic plane wave by a partially coated infinitely long cylinder with axis in the z-direction and assume that the incident field is propagating in a direction perpendicular to the cylinder such that the electric field is parallel to the z-axis. This leads to a mixed transmission problem for the Helmholtz equation [**8**]. In particular let D be as in Section 2 and let $n(x)$, $x \in D$, denote the index of refraction of the dielectric obstacle. We assume that $n(x) > n_0 > 0$ and is piecewise continuous in $\overline{D}$. The physical properties of the thin highly conducting coating are represented by the surface conductivity η. We assume that the surface conductivity $\eta \in L_\infty(\partial D_2)$ is positive and uniformly bounded, i.e., $\eta(x) \geq \eta_0 > 0$ for almost all $x \in \partial D_2$. Let ν again denote the unit outward normal vector defined almost everywhere on $\partial D_1 \cup \partial D_2$. Then the total field $u = u^s + e^{ikx\cdot d}$ outside D and the interior field w inside D satisfy the mixed transmission boundary value problem

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbf{R}^2 \setminus \overline{D}, \tag{18}$$

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in } D \tag{19}$$

$$w - u = 0 \qquad \text{on } \partial D \tag{20}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial D_1 \tag{21}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu} = i\eta(x)u \quad \text{on } \partial D_2 \tag{22}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial v} - iku^s\right) = 0 \tag{23}$$

where again $k > 0$ is the wave number and $r = |x|$. In [**8**] it is shown that there exists a unique solution $w \in H^1(D)$ and $u \in H^1_{\text{loc}}(\mathbf{R}^2 \setminus \overline{D})$ to the mixed transmission problem (18)–(23). The scattered field u^s again has the asymptotic behavior [**4, 11**]

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\widehat{x}, d) + O(r^{-3/2})$$

as $r \to \infty$ where $u_\infty(\widehat{x}, d)$ is defined for $\widehat{x}, d \in \Omega$ is the far field pattern.

In this section we again assume that the support of the inhomogeneity D is known. Then the *inverse problem* we consider here is to determine the surface conductivity η as a function in $L_\infty(\partial D_2)$ from a knowledge of $u_\infty(\widehat{x}, d)$ for $\widehat{x} \in \Omega_0 \subseteq \Omega$ and $d \in \Omega_1 \subseteq \Omega$. Note that we do not assume that $n(x)$ is known and we will not reconstruct it.

We first want to prove a uniqueness theorem for $\eta \in L_\infty(\partial D_2)$. Let us denote by $\widetilde{\eta} \in L_\infty(\partial D)$ the extension by zero to the whole boundary of the surface conductivity η.

Theorem 3.1. *Let v_g^i be the Herglotz wave function with kernel g and v_g and w_g be the solution of (18)–(23) with $e^{ikx\cdot d}$ replaced by v_g^i. Then*

$$\int_{\partial D} \widetilde{\eta}(x)|v_g|^2\, ds = -k\|Fg\|^2 + \sqrt{8\pi k}\ Im\left(e^{i\pi/4}(Fg,\, g)\right) \tag{24}$$

where F is the far field operator $(\cdot,\, \cdot)$ is the inner product over $L^2(\Omega)$.

Proof. The proof is similar to the proof of Theorem 2.1. In particular from Theorem 2.1 we have the relations (8) and (9). Now let v_g and w_g be solutions of (18)–(23) with $e^{ikx\cdot d}$ replaced by v_g^i. Then from the second Green's identity by using the equations for v_g and w_g and the transmission condition together with the fact that n is real we have that

$$2i\int_{\partial D} \widetilde{\eta}(x)|v_g|^2 ds(x) = \int_{\partial D}\left(v_g\frac{\partial \overline{v_g}}{\partial \nu} - \overline{v_g}\frac{\partial v_g}{\partial \nu}\right) ds, \tag{25}$$

and the theorem now follows in the same way as in the proof of Theorem 2.1.

We call k a Dirichlet eigenvalue for D if

$$\begin{aligned} \Delta v + k^2 n(x) v &= 0 \quad \text{in } D \\ v &= 0 \quad \text{on } \partial D \end{aligned}$$

has non trivial solutions.

Theorem 3.2. *Assume that k is not a Dirichlet eigenvalue for D, and define*

$$\mathcal{Z} := \left\{ f \in L^2(\partial D) : \begin{array}{c} f = v_g|_{\partial D} \text{ for some } g \in L^2(\Omega) \\ \text{where } v_g \text{ is as in Theorem 3.1} \end{array} \right\}.$$

Then $\mathcal{Z}$ is dense in $L^2(\partial D)$.

Proof. It suffices to show that if $\varphi \in L^2(\partial D)$ satisfies

$$\int_{\partial D} v_g \varphi \, ds = 0 \tag{26}$$

for all $f = v_g|_{\partial D} \in \mathcal{W}$ then $\varphi = 0$. To this end, let $v_g = v_g^i + v_g^s \in H^1_{\mathrm{loc}}(\mathbf{R}^2 \setminus \overline{D})$ and $w_g \in H^1(D)$ be the solutions of (18)–(23) with $e^{ikx\cdot d}$ replaced by v_g^i, and suppose (26) is true for all such v_g. We construct $u \in H^1_{\mathrm{loc}}(\mathbf{R}^2 \setminus \overline{D})$ and $w \in H^1(D)$ as the unique solution [**8**] of

$$\begin{aligned}
\Delta u + k^2 u &= 0 && \text{in } \mathbf{R}^2 \setminus \overline{D}\\
\Delta w + k^2 n(x) w &= 0 && \text{in } D\\
w - u &= 0 && \text{on } \partial D\\
\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu} &= i\widetilde{\eta}(x) u + \varphi && \text{on } \partial D\\
\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) &= 0
\end{aligned}$$

Then, using the boundary conditions and Green's second identity, we have that

$$\begin{aligned}
0 &= \int_{\partial D} v_g \varphi \, ds = \int_{\partial D} v_g \left(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu} - i\tilde{\eta} u\right) ds\\
&= \int_{\partial D} w_g \frac{\partial w}{\partial \nu}\, ds - \int_{\partial D} (v_g^s + v_g^i)\frac{\partial u}{\partial \nu}\, ds - \int_{\partial D} i\widetilde{\eta} v_g u \, ds\\
&= \int_{\partial D} w_g \frac{\partial w}{\partial \nu}\, ds - \int_{\partial D} (v_g^s + v_g^i)\frac{\partial u}{\partial \nu}\, ds\\
&\quad - \int_{\partial D} \left(\frac{\partial w_g}{\partial \nu} - \frac{\partial (v_g^s + v_g^i)}{\partial \nu}\right) u \, ds\\
&= \int_{\partial D} \left(w_g \frac{\partial w}{\partial \nu} - \frac{\partial w_g}{\partial \nu} w\right) ds - \int_{\partial D} \left(v_g^s \frac{\partial u}{\partial \nu} - \frac{\partial v_g^s}{\partial \nu} u\right) ds\\
&\quad - \int_{\partial D} \left(v_g^i \frac{\partial u}{\partial \nu} - \frac{\partial v_g^i}{\partial \nu} u\right) ds\\
&= \int_{\partial D} \left(v_g^i \frac{\partial u}{\partial \nu} - \frac{\partial v_g^i}{\partial \nu} u\right) ds \quad \text{for all } g \in L^2(\Omega).
\end{aligned}$$

This implies that

$$\int_{\partial D} \left(e^{ikx\cdot d} \frac{\partial u}{\partial \nu} - \frac{\partial e^{ikx\cdot d}}{\partial \nu} u\right) ds(x) = 0, \quad \text{for } d \in \Omega,$$

and hence $u_\infty(d) = 0$ which implies $u = 0$ and since k is not a Dirichlet eigenvalue $w = 0$ and therefore $\varphi = 0$. This ends the proof. $\square$

Theorem 3.3. *Assume that k is not a Dirichlet eigenvalue for D. Let u^j_∞ be the far field pattern for* (18)–(23) *corresponding to $\widetilde{\eta} := \widetilde{\eta}_j$, $u = u^j$, $w = w^j$, $j = 1, 2$, and let Ω_0 and Ω_1 be open nonempty subsets of the unit circle Ω. Then if $u^1_\infty(\widehat{x}, d) = u^2_\infty(\widehat{x}, d)$ for $\widehat{x} \in \Omega_0$ and $d \in \Omega_1$, $\widetilde{\eta}_1 = \widetilde{\eta}_2$ as functions in $L_\infty(\partial D)$. In particular the support of coatings $\partial D^1_2 = \partial D^2_2$ coincide and $\eta_1(x) = \eta_2(x)$ almost everywhere on $\partial D^1_2 = \partial D^2_2$.*

Proof. As in the proof of Theorem 2.3, we have that $u^1_\infty(\widehat{x}, d) = u^2_\infty(\widehat{x}, d)$ for $\widehat{x}, d \in \Omega$ and hence by Rellich's lemma then $u^1 = u^2$ in $\mathbf{R}^2 \setminus \overline{D}$ for all $d \in \Omega$ which implies that $v_g = v^1_g = v^2_g$ on ∂D. From Theorem 3.1 we have that

$$\int_{\partial D} [\widetilde{\eta}_1(x) - \widetilde{\eta}_2(x)]\, |v_g|^2 \, dx = 0.$$

Viewing the $L_\infty(\partial D)$ function $\widetilde{\eta}_1 - \widetilde{\eta}_2$ as a self-adjoint operator on $L^2(\partial D)$ we have by Theorem 3.2 that $\eta_1 = \eta_2$ as functions in $L_\infty(\partial D)$ (cf. [**15**, Theorem 9.2-2]). In particular, this means that the support of coatings $\partial D^1_2 = \partial D^2_2$ coincide and $\eta_1(x) = \eta_2(x)$ almost everywhere on $\partial D^1_2 = \partial D^2_2$, which proves the theorem.

We proceed next with the reconstruction of η. To this end we consider again the far field equation

$$\int_{\Omega_1} u_\infty(\widehat{x}, d) g(d)\, ds(d) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\widehat{x}\cdot z} \qquad (27)$$
$$g \in L^2(\Omega_1), \quad \widehat{x} \in \Omega_0, \quad z \in D.$$

For $z \in D$, it is easy to see (cf. [**11**, Theorem 8.9]) that the far field equation has a solution $g^z \in L^2(\Omega)$ if and only if the *interior* *transmission problem*

$$\begin{aligned} \Delta v_z + k^2 v_z &= 0 && \text{in } D \\ \Delta w_z + k^2 n(x) w_z &= 0 && \text{in } D \\ w_z - v_z &= \Phi(\cdot, z) && \text{on } \partial D \\ \frac{\partial w_z}{\partial \nu} - \frac{\partial v_z}{\partial \nu} &= \frac{\partial \Phi(\cdot, z)}{\partial \nu} && \text{on } \partial D_1 \end{aligned} \tag{28}$$

$$\frac{\partial w_z}{\partial \nu} - \frac{\partial v_z}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} + i\eta(v_z + \Phi(\cdot, z)) \qquad \text{on } \partial D_2 \tag{29}$$

has a solution v_z, w_z such that $v_z := v_{g^z}$ is a Herglotz wave function with kernel g^z. We call k a *transmission eigenvalue* if the homogeneous interior transmission problem, i.e., (28)–(29) with $\Phi(\cdot, z) = 0$, has a non trivial solution. In [**5**] it is shown that, provided k is not a transmission eigenvalue, the interior transmission problem has a unique solution $w_z \in L^2(D)$ and $v_z \in L^2(D)$ such that $w_z - v_z \in H^2(D)$. In addition it can be shown that the set of transmission eigenvalue is at most discrete. From this result, Theorem 3.3 of [**5**] and the approximation property of the Herglotz wave functions with compactly supported kernels proven in subsection 1.3 of [**2**], one can show, provided that k is not a transmission eigenvalue, that for $z \in D$ and for every $\varepsilon > 0$ there exists a $g_\varepsilon^z \in L^2(\Omega)$ with support in $\overline{\Omega}_1$ such that

$$\left\| \int_{\Omega_1} u_\infty(\widehat{x}, d) g_\varepsilon^z(d)\, ds(d) - \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\widehat{x}\cdot z} \right\|_{L^2(\Omega_0)} \le \varepsilon, \tag{30}$$

and the corresponding Herglotz function $v^i_{g_\varepsilon^z}$ is such that

$$\lim_{\varepsilon \to 0} \| v^i_{g_\varepsilon^z} - v_z \|_{L^2(D)} = 0$$

where v_z, w_z is the unique solution of the interior transmission problem (28)–(29).

In order to connect the approximate solution to the far field equation with the solution of the interior transmission problem and the forward scattering problem we remark that the direct scattering problem is well defined for incident fields in

$$H_{\text{inc}}(D) := \{u \in L^2(D) : \Delta u + k^2 u = 0 \text{ in the distributional sense}\}.$$

To see this, for an incident field $u^i \in H_{\mathrm{inc}}(D)$ we can rewrite (18)–(23) in terms of the scattered field u^s in $\mathbf{R}^2 \setminus \overline{D}$ and $u^s := w - u^i$ in D in the form

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbf{R}^2 \setminus \overline{D}, \tag{31}$$

$$\Delta u^s + k^2 n u^s = (1-n)u^i \quad \text{in} \tag{32}$$

$$u^s_- - u^s_+ = 0 \quad \text{on } \partial D \tag{33}$$

$$\frac{\partial u^s_-}{\partial \nu} - \frac{\partial u^s_+}{\partial \nu} = 0 \quad \text{on } \partial D_1 \tag{34}$$

$$\frac{\partial u^s_-}{\partial \nu} - \frac{\partial u^s_+}{\partial \nu} = i\eta(x) u^s_+ \quad \text{on } \partial D_2 \tag{35}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial v} - iku^s\right) = 0 \tag{36}$$

where u^s_+ and u^s_- denote the limiting values of u^s as x tends to ∂D from outside D and inside D, respectively. A standard variational approach shows the existence of a unique solution to this transmission problem in $H^1(D) \times H^1_{\mathrm{loc}}(D)$ that depends continuously on the data $u^i \in H_{\mathrm{inc}}(D)$ in the $L^2(D)$ norm. For $z \in D$, if w_z, v_z is a solution of the interior transmission problem (28)–(29), then $u^s := \Phi(\cdot, z)$ in $\mathbf{R}^2 \setminus \overline{D}$ and $u^s = v_z - w_z$ in D satisfy the direct scattering problem (31)–(36) with incident field $u^i := w_z$. Now if $v^s_{g^z_\varepsilon}$ is the scattered field corresponding to $v^i_{g^z_\varepsilon}$, then $v^s_{g^z_\varepsilon} \to u^s = v_z - w_z$ in $H^1(D)$ and $v^s_{g^z_\varepsilon} \to \Phi(\cdot, z)$ in $H^1_{\mathrm{loc}}(\mathbf{R}^2 \setminus \overline{D})$ as $\varepsilon \to 0$. Using this fact (see also [**9**]) it is possible to justify the use of Green's formula applied to w_z and v_z and thus obtain the following result:

Lemma 3.4. *Assume that k is not a transmission eigenvalue. Then for every point $z \in D$ we have that*

$$\int_{\partial D_2} \eta(x) \left|v_z(x) + \Phi(x,z)\right|^2 \, ds(x) = -1/4 - \mathrm{Im}\,\left(v_z(z)\right), \tag{37}$$

where v_z, w_z is the solution of the interior transmission problem (28)–(29).

Proof. Set $V_z = v_z + \Phi(\cdot, z)$. Then, using the fact that n is real,

$$\begin{aligned} 0 &= \int_{\partial D} \left(w_z \frac{\partial \overline{w}_z}{\partial \nu} - \overline{w}_z \frac{\partial w_z}{\partial \nu} \right) ds \\ &= \int_{\partial D} \left(V_z \frac{\partial \overline{V_z}}{\partial \nu} - \overline{V_z} \frac{\partial V_z}{\partial \nu} \right) ds - 2i \int_{\partial D_2} \eta(x) V_z \overline{V_z}\, ds. \end{aligned}$$

But

$$\begin{aligned} \int_{\partial D} & \left(V_z \frac{\partial \overline{V_z}}{\partial \nu} - \overline{V_z} \frac{\partial V_z}{\partial \nu} \right) ds \\ &= \int_{\partial D} \left(\Phi(\cdot, z) \frac{\partial \overline{\Phi(\cdot, z)}}{\partial \nu} - \overline{\Phi(\cdot, z)} \frac{\Phi(\cdot, z)}{\partial \nu} \right) ds \\ &\quad + \int_{\partial D} \left(v_z \frac{\partial \overline{\Phi(\cdot, z)}}{\partial \nu} - \overline{\Phi(\cdot, z)} \frac{\partial v_z}{\partial \nu} \right) ds \\ &\quad + \int_{\partial D} \left(\Phi(\cdot, z) \frac{\partial \overline{v}_z}{\partial \nu} - \overline{v}_z \frac{\partial \Phi(\cdot, z)}{\partial \nu} \right) ds \\ &= -2ik \int_{\Omega} \Phi_\infty(\cdot, z) \overline{\Phi_\infty(\cdot, z)}\, ds + \overline{v}_z(z) - v_z(z) \\ &= -\frac{i}{2} - 2i \mathrm{Im}\,(v_z(z)), \end{aligned}$$

i.e.,

$$\int_{\partial D_2} \eta(x) \left| v_z(x) + \Phi(x, z) \right|^2 \, ds(x) = -1/4 - \mathrm{Im}\, \left(v_z(z) \right).$$

Lemma 3.5. *Assume that k is neither a Dirichlet or a transmission eigenvalue for D. Let $B_r \subset D$ be a ball of radius r, and let*

$$\mathcal{X} := \left\{ f \in L^2(\partial D_2) : \begin{array}{l} f = (v_z + \Phi(\cdot, z))|_{\partial D_2}, \\ z \in B_r \textit{ and } v_z, w_z \textit{ the solution of } (28)\text{–}(29) \end{array} \right\}.$$

Then $\mathcal{X}$ is complete in $L^2(\partial D_2)$.

Proof. Using Green's second formula, it is shown in [**5**, Section 3] that if v_z, w_z is a solution to the interior transmission problem (28)–(29)

then $v_z|_{\partial D}$ is well defined in $L^2(\partial D)$. Now let $V_z := v_z + \Phi(\cdot, z)$ and τ be a function in $L^2(\partial D_2)$ such that for every $z \in B_r$

$$\int_{\partial D_2} V_z \, \tau \, ds = 0.$$

Construct $v \in L^2(D)$ and $w \in L^2(D)$ such that $v - w \in H^2(D)$ as the unique solution of the interior transmission problem

$$\begin{aligned}
\Delta v + k^2 \, v &= 0 && \text{in } D \\
\Delta w + k^2 n(x) w &= 0 && \text{in } D \\
w - v &= 0 && \text{on } \partial D \\
\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} &= 0 && \text{on } \partial D_1 \\
\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} &= i\eta v + \tau && \text{on } \partial D_2.
\end{aligned} \tag{38}$$

The existence of such a solution is shown in [**5**, Section 3]. Then we have that

$$\begin{aligned}
0 = \int_{\partial D_2} V_z \, \tau \, ds &= \int_{\partial D_2} V_z \left(\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} - i\eta v \right) ds \\
&= \int_{\partial D} V_z \frac{\partial w}{\partial \nu} \, ds - \int_{\partial D} V_z \frac{\partial v}{\partial \nu} \, ds - i \int_{\partial D_2} \eta \, v V_z \, ds.
\end{aligned} \tag{39}$$

Next from the equations for w_z and w, the divergence theorem and the transmission conditions we have that

$$\begin{aligned}
\int_{\partial D} V_z \frac{\partial w}{\partial \nu} \, ds = \int_{\partial D} w_z \frac{\partial w}{\partial \nu} \, ds &= \int_{\partial D} \frac{\partial w_z}{\partial \nu} \, w \, ds \\
&= \int_{\partial D} \frac{\partial V_z}{\partial \nu} \, v \, ds + i \int_{\partial D_2} \eta \, v V_z \, ds,
\end{aligned} \tag{40}$$

and substituting (40) into (39) yields

$$\begin{aligned}
0 = \int_{\partial D} \left(\frac{\partial v}{\partial \nu} \, V_z - \frac{\partial V_z}{\partial \nu} \, v \right) ds &= \int_{\partial D} \left(\frac{\partial v}{\partial \nu} \, v_z - \frac{\partial v_z}{\partial \nu} \, v \right) ds \\
+ \int_{\partial D} \left(\frac{\partial v}{\partial \nu} \, \Phi(\cdot, z) - \frac{\partial \Phi(\cdot, z)}{\partial \nu} \, v \right) ds &= v(z) \quad \text{for all } z \in B_r.
\end{aligned} \tag{41}$$

The unique continuation principle for the Helmholtz equation now implies that $v = 0$ in D. Hence if k is not a Dirichlet eigenvalue for D then $w \equiv 0$ and therefore $\varphi = 0$ which proves the lemma.

We recall that v_z can be approximated by $v^i_{g^z_\varepsilon}$. Equation (37) can now be seen as an integral equation of the first kind for η where by setting $\eta = \widetilde{\eta}$ we can assume that the region of integration is ∂D instead of ∂D_2. Using Lemma 3.5, it is easy to see in the same way as in the proof Theorem 3.3 that the left hand side of this equation is an injective compact integral operator with positive kernel defined on $L_\infty(\partial D_2)$. Using Tikhonov regularization techniques it is possible to determine η by finding the regularized solution of (37) in $L^2(\partial D)$ with noisy kernel and noisy right hand side.

In particular, if the dielectric obstacle is fully coated (i.e., $\partial D = \partial D_2$) and the surface conductivity is a positive constant $\eta > 0$ we obtain that

$$\eta = \frac{-1/4 - \operatorname{Im}\,(v_z(z))}{\|v_z + \Phi(\cdot\,;z)\|^2_{L^2(\partial D)}} \qquad z \in D \tag{42}$$

4. Numerical results. In this section we shall present some numerical results, using synthetic far field data, that illustrate the theory given in previous sections. Following the method of our previous papers [**6, 7**] we shall use far field patterns computed via a cubic finite element scheme that uses the perfectly matched layer to terminate the computational domain.

For a given scatterer (all our examples will be computed using the triangular scatterer in Figure 1) we generate an approximate far field pattern for N equally spaced incident angles in the aperture Ω_1 and record the far field pattern at N equally spaced points in Ω_0. There is no reason to require the same number of points in both apertures, and indeed our code allows an arbitrary choice of points (not even equally spaced) but we do not investigate this aspect here. We always make the choice that the incident and measurement apertures are given by $\Omega_1 = -\Omega_0$. This corresponds to backscattering (i.e., the measurement and source points are in the same portion of physical space). The data thus gives an $N \times N$ approximation to the matrix of A of multistatic far field measurements where $A_{\ell,m} = u_\infty(\widehat{x}_\ell, -\widehat{x}_m)$ and $\widehat{x}_\ell$, $1 \leq \ell \leq N$, is a set of equally spaced points in Ω_0. The entries of this matrix are further corrupted by random noise as described in [**6, 7**] with the same noise level as in those papers. In particular, to each entry $A_{l,m}$ we obtain "noisy" entries $(A_\varepsilon)_{l,m}$ using

$$(A_\varepsilon)_{l,m} = A_{l,m}\,(1 + \varepsilon(\xi_{1,m,n} + i\,\xi_{2,m,n}))$$

where $\varepsilon = 0.01$ and $\xi_{1,m,n}$ and $\xi_{2,m,n}$ are given by a random number generator uniformly distributed in the range $[-1,\,1]$.

We shall assume throughout this paper that D is known (i.e., orientation, position and shape) and that the goal is to compute the impedance or conductivity. Using the data matrix A in the same way as in the previously mentioned papers, we can compute a discrete approximation to a regularized solution of (11) and (27) using Tikhonov regularization and Morozov's discrepancy principle for any auxilliary source point $z \in D$ [**4, 11**]. This then provides a discrete kernel for the Herglotz wave function

$$v_g(x) = \int_{\Omega_1} \mathrm{e}^{ikx\cdot d} g(d)\, ds(d)$$

where the integral is replaced by mid-point quadrature. Assuming that the discrete kernel approximating g is computed sufficiently accurately, the resulting discrete Herglotz wave function will then be an approximation to the function u_z in the integral equation (16) or the function v_z in (37). Indeed these integral equations are identical once u_z or v_z are known, so we will now discuss only (16).

Since we know D we can choose $z \in D$ and then, as described in the previous paragraph, we can compute an approximation to u_z. The integral in (16) is then approximated using quadrature on ∂D. Here we use the finite element grid used for the forward problem and then use the trapezoidal rule with five points on each subinterval and as discussed after Lemma 2.5, without loss of generality we replace $\partial D_I = \partial D$. A linear system for the values of λ at the quadrature points is obtained by taking P points $z \in D$. Arbitrarily we choose these P points to be uniformly distributed on a circle of radius r_0 about the origin (our example scatterer contains the origin). The resulting discrete approximation to (16) is highly ill-conditioned so we again use Tikhonov regularization via the $L^2(\partial D)$ norm of $d\lambda/ds$ where s is the arc length along ∂D (implemented via finite differences). This

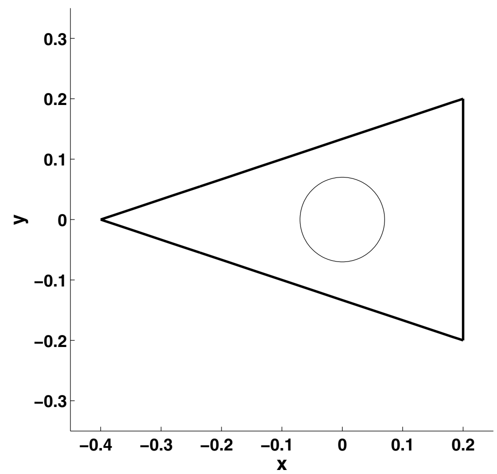

FIGURE 1. The shape of the scatterer used in these experiments. The same triangle is used for both conductivity and impedance problems. In the first case (impedance) the scatterer is impenetrable so the region inside the triangle is not meshed. For the second case conductivity, the triangle is penetrable and the inside is meshed for the forward problem. The circle shown inside the triangle is the position of auxiliary sources z used in the computation of the conductivity or impedance functions.

regularization is particularly well suited to computing a constant λ. We choose the regularization parameter giving the solution which best approximates the exact solution in the discrete L^2-norm over a discrete set of parameter values: $10^{-1}, 10^{-2}, \dots, 10^{-10}$.

Our example domain D is the triangle shown in Figure 1. This is very unlikely to be a Herglotz domain and hence represents a reasonable test of our algorithms.

Full aperture reconstructions. We first consider full aperture data in which $\Omega_0 = \Omega_1 = \Omega$. Our first graphs in Figure 2 show the

TABLE 1. Table of parameter values for the results in this section.

N	51 (except Fig. 5 when $N = 25$)	r_0	0.07
k	0.2, 2 or 5 (depending on the figure)	P	100

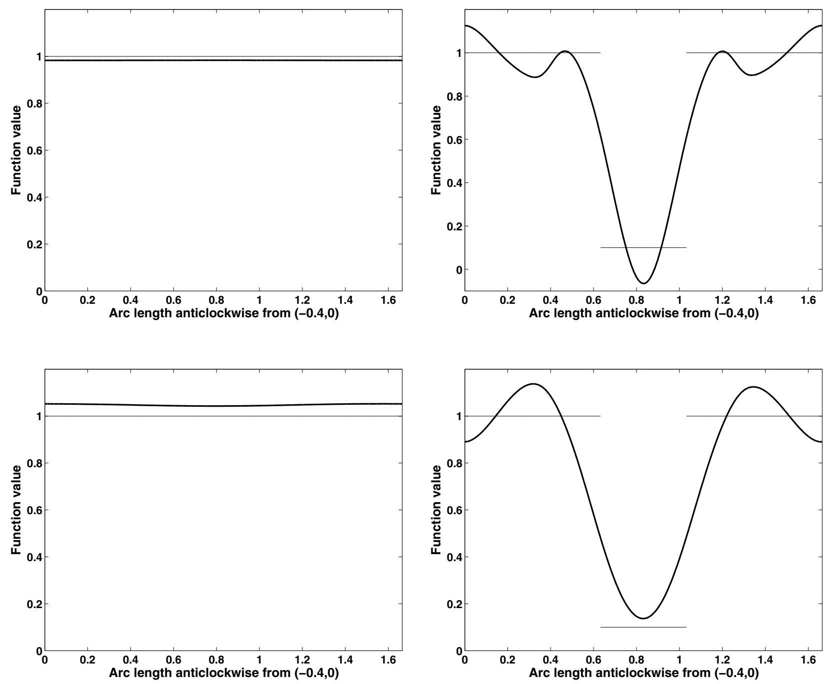

FIGURE 2. Reconstructions of the constant functions $\lambda = 1$ (top left), $\eta = 1$ (bottom left), and piecewise constant λ and η (top right and bottom right, respectively) using full aperture data when $k = 2$. The results are shown as a function of arc length measured anti-clockwise from the leftmost point of the triangle and the exact solution is shown as a thin line.

results of reconstructing a constant function and a piecewise constant function on ∂D when $k = 2$. Other data is given in Table 1. For the constant function, $\lambda = 1$ or $\eta = 1$, while for the piecewise constant function $\lambda = 1$ on the long sides of the triangle and $\lambda = 0.2$ on the short side (the same function is also used for η). The results clearly indicate the variation of the piecewise constant function, although the reconstructions are smooth (as is to be expected from regularization) and have a large error (for example an $L^2(\partial D)$ relative error of 18% in the case of the reconstruction of the piecewise constant λ).

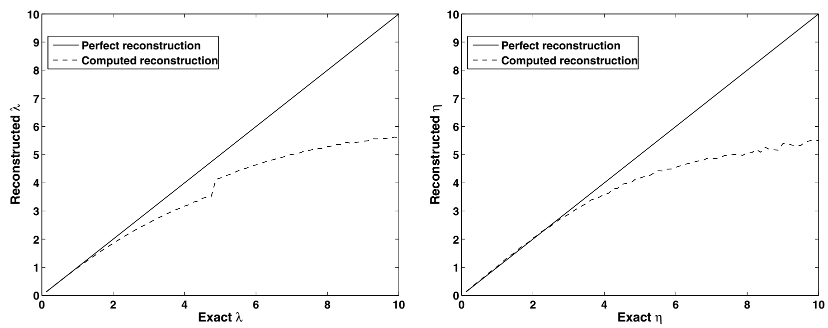

FIGURE 3. Reconstruction of various constant impedance λ (left panel) and η (right panel) when $k = 2$ using full aperture data. As λ or η increase the method loses accuracy. This implies that it is difficult or perhaps impossible to distinguish between a perfect conductor and an impedance or conductive boundary.

Our goal is to distinguish different coatings by variations in η or λ, so in Figure 3 we examine the performance of the algorithm for a constant parameter $\lambda = C$ (or $\eta = C$ as appropriate) as C varies from 0.125 to 10. For both the impedance and conductivity parameters the accuracy of the reconstruction deteriorates as C increases. This implies that it would be difficult to differentiate between, for example, $\lambda = 5$ and $\lambda = 500$ (the latter being very close to a perfect conductor). The method can differentiate between objects with low impedance or conductivity (the range is slightly larger for the conductivity η than for the impedance λ).

Limited aperture reconstructions. In this section we only attempt to reconstruct the constant impedance $\lambda = 1$ or the constant conductivity $\eta = 1$. We investigate the sensitivity of the reconstruction to the size and orientation of the measurement aperture Ω_0. We first examine three limited apertures:

1. The aperture subtends $\pi/8$ radians and extends from polar angle $\theta = 0$ to $\theta = \pi/8$.

2. The aperture subtends $\pi/4$ radians and extends from polar angle $\theta = -\pi/8$ to $\theta = \pi/8$.

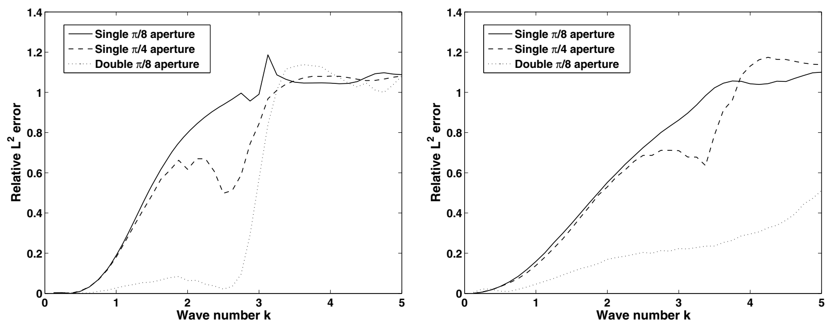

FIGURE 4. The error in limited aperture reconstructions as a function of wave number k for various apertures (impedance: left panel, conductive: right panel). We use three apertures: a single aperture subtending $\pi/8$ between polar angle $\theta = 0$ and $\theta = \pi/8$, a single aperture subtending $\pi/4$ between $\theta = -\pi/8$ to $\theta = \pi/8$, and a pair of $\pi/8$ apertures between $\theta = 0$ and $\theta = \pi/8$ and between $\theta = \pi/2$ and $\theta = 5\pi/8$. The single larger aperture and the smaller aperture behave similarly in that accuracy deteriorates similarly as k increases. The pair of apertures allows the use of a higher wave number.

3. The aperture consists of two disconnected components, the first extends from polar angle $\theta = 0$ to $\theta = \pi/8$ and the second is from $\theta = \pi/2$ to $\theta = 5\pi/8$. This has the same total aperture as the second case above.

Results are shown in Figure 4. For low wave number, all the apertures provide the same high accuracy of reconstruction. As k increases the error in the reconstruction for both the connected apertures increases at the same rate despite their difference in size. (From these limited experiments we feel it is premature to offer an explanation as to why the error appears to increase as k increases). For the disconnected aperture the error remains substantially lower than the other apertures for a wider range of wave numbers. The reason for the better performance of the disconnected aperture is not clear.

The accuracy of the reconstruction depends both on the aperture size and also the orientation of the aperture relative to the triangle. We investigate this next. In Figure 5 we show the error in the reconstruction for a different aperture size (the angle subtended by the aperture) and starting angle measured counter clockwise from the x axis. For this experiment we use $N = 25$ incident fields to decrease overall computer time. When $k = 5$ we see that for the reconstruction

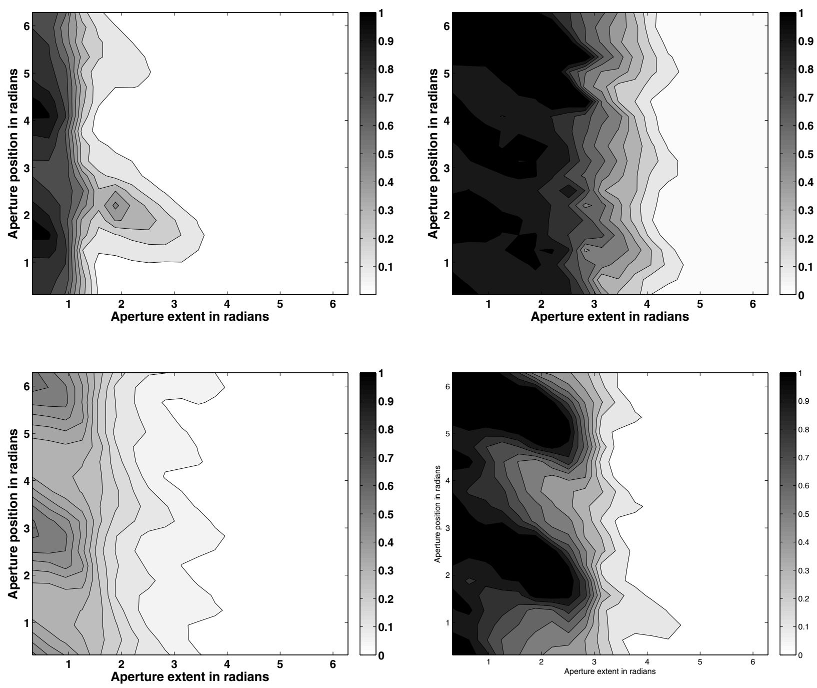

FIGURE 5. Contour maps of reconstruction error as a function of aperture extent (i.e. the angle subtended by the aperture) and the start angle of the aperture. The top panels are for impedance ($k = 2$ left and $k = 5$ right) and the lower panels are for conductivity. At the higher wave number the error is poor for any aperture smaller than roughly 4 radians. For most aperture positions a smaller aperture can be used when k decreases as is to be expected.

of λ the aperture needs to be over π in extent in order to give less than 10 % error regardless of orientation. When $k = 2$ a smaller aperture can be used (below 2 radians) in most directions. Similar results are seen for the conductivity η although this example is less sensitive to the aperture position than for the impedance.

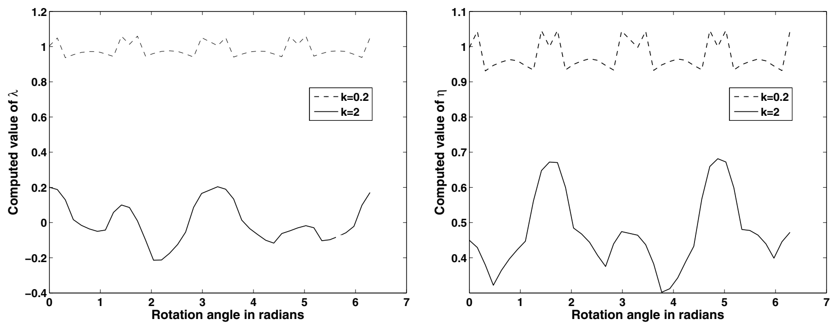

FIGURE 6. A test to show the sensitivity of the reconstruction of $\lambda = 1$ (left panel) or $\eta = 1$ (right panel) to incorrect orientation of the target. Data for a rotated triangle (rotated about the origin) is used assuming the triangle is in its original orientation. For low wave number the reconstruction is not very sensitive to orientation (the four fold symmetry is likely due to the damping layer used in the forward problem). For higher wave numbers the reconstruction is more variable and also overall less accurate.

Throughout the paper we have assumed that D is known (i.e. its position, shape and orientation). As we might expect, at resonance frequencies the orientation of the target is vital for obtaining a reliable reconstruction. In Figure 6 we show the results of using the far field pattern generated by a rotated target (about the origin) on a limited aperture from $\theta = 0$ to $\theta = \pi/8$. But we have performed the inversion assuming the target is in its standard orientation. When $k = 2$ the reconstruction of λ or η, which is already poor due to the limited aperture, varies strongly with orientation. Of course for low frequency (for example $k = 0.2$) the reconstruction is much better and roughly independent of orientation (the fourfold symmetry in the reconstruction is most likely caused by errors in the far field pattern due to the PML which is less efficient for low frequency.

5. Conclusion. In the paper we have given uniqueness theorems and reconstruction algorithms for determining the surface impedance λ or surface conductivity η for an obstacle from noisy far field data. Although our algorithm works well for moderate values of λ and η, numerical examples show that, as λ or η increase, our reconstruction algorithms become less accurate. We believe that this is due to the fact that both the kernel and right hand side of the integral equation satisfied by λ or η become very small as λ or η become large. Hence, in order to be effective in distinguishing between coated decoys and perfect conductors (i.e., λ or η equal to infinity) more work needs to be done on improving the accuracy of our algorithms for large values of λ or η (possibly through the use of techniques from asymptotic analysis).

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