Received: 7 July 2017
DOI: 10.1002/mma.4664

RESEARCH ARTICLE

WILEY

A boundary integral equation for the transmission eigenvalue problem for Maxwell equation

F. Cakoni[1] | O. Ivanyshyn Yaman[2] | R. Kress[3] | F. Le Louër[4]

[1]Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA

[2]Department of Mathematics, Izmir Institute of Technology, Urla, Izmir, 35430, Turkey

[3]Institut für Numerische und Angewandte Mathematik, Universität Göttingen, Göttingen, Germany

[4]Sorbonne University, Université de technologie de Compiègne, Laboratory of Applied Mathematics of Compiègne, 60203 Compiègne cedex, France

Correspondence
Rainer Kress, Institut für Numerische und Angewandte Mathematik, Universität Göttingen, Göttingen, Germany.
Email: kress@math.uni-goettingen.de

Communicated by: M. Dauge

Funding information
AFOSR, Grant/Award Number: FA9550-13-1-0199 ; NSF, Grant/Award Number: DMS-1602802, ; Simons Foundation, Grant/Award Number: 392261 ; Türkiye Bilimsel ve Teknolojik Ara?tirma Kurumu (Scientific and Technological Research Council of Turkey), Grant/Award Number: 116F299

MSC Classification: 35R30; 35J25; 35P25; 35P05

We propose a new integral equation formulation to characterize and compute transmission eigenvalues in electromagnetic scattering. As opposed to the approach that was recently developed by Cakoni, Haddar and Meng (2015) which relies on a two-by-two system of boundary integral equations, our analysis is based on only one integral equation in terms of the electric-to-magnetic boundary trace operator that results in a simplification of the theory and in a considerable reduction of computational costs. We establish Fredholm properties of the integral operators and their analytic dependence on the wave number. Further, we use the numerical algorithm for analytic nonlinear eigenvalue problems that was recently proposed by Beyn (2012) for the numerical computation of the transmission eigenvalues via this new integral equation.

KEYWORDS

boundary integral equations, inhomogeneous media, inverse scattering, transmission eigenvalues

1 | INTRODUCTION

The transmission eigenvalue problem for Maxwell's equation arises in scattering theory for time-harmonic electromagnetic waves in inhomogeneous media. If n denotes the refractive index of an inhomogeneous medium with support $D \in \mathbb{R}^3$ in electromagnetic scattering, the transmission eigenvalue problem is formulated as finding $k \in \mathbb{C}$ for which the homogeneous problem

$$\operatorname{curl}\operatorname{curl} E - k^2 nE = 0 \quad \text{in } D, \tag{1}$$

Dedicated to Erich Martensen on the occasion of his 90th birthday

$$\operatorname{curl}\operatorname{curl} E_0 - k^2 E_0 = 0 \quad \text{in } D, \tag{2}$$

$$\nu \times E = \nu \times E_0 \quad \text{on } \partial D, \tag{3}$$

$$\nu \times \operatorname{curl} E = \nu \times \operatorname{curl} E_0 \quad \text{on } \partial D \tag{4}$$

has nontrivial solutions $E, E_0 \in L^2(D)$. Here, we assume that D is bounded and has a connected complement $\mathbb{R}^3 \backslash D$ with sufficiently smooth boundary ∂D and ν denotes the outward unit normal vector.

Transmission eigenvalues can be seen as the extension of the concept of resonant frequencies for impenetrable objects to the case of penetrable media. They are related to nonscattering frequencies. If $E = E^i + E^s$ is the total field for scattering of an incident field E^i such that $E^s = 0$ everywhere, then E and $E_0 = E^i$ solve (1)-(4). Conversely, if (1)-(4) has a nontrivial solution E, E_0, and E_0 can be extended outside D as a solution to curlcurl $E_0 - k^2 E_0 = 0$, and this extension E_0 is considered as the incident field, then the corresponding scattered field is $E^s = 0$.

The transmission eigenvalue problem is a nonlinear and non-self-adjoint eigenvalue problem that is not covered by the standard theory of eigenvalue problems for elliptic equations, and as such in recent years, its analysis has been an attractive subject of investigation. For existence of transmission eigenvalues for the Maxwell equations, we refer to Cakoni et al[1,2] and to numerical computations via finite element methods to Monk and Sun.[3]

In this paper, we extend the boundary integral equation approach that we developed in Cakoni and Kress[4] for the transmission eigenvalue problem for the Helmholtz equation in the case of a constant refractive index. Boundary integral equation methods were first used in the context of transmission eigenvalues for the Helmholtz problem by Cossonnière and Haddar.[5,6] In their work, they used Green's representation formula to derive a system of two linear boundary integral equations that are equivalent to the transmission eigenvalue problem and depend nonlinearly on the eigenvalue parameter k. Using parts of the analysis in Cossonnière and Haddar,[5,6] we were able to develop a new formulation that leads to only one linear boundary integral equation in terms of a Dirichlet-to-Neumann operator and used it also for numerical computations of transmission eigenvalues (see Cakoni and Kress[4]). The analysis in Cossonnière and Haddar[5,6] was extended by Cakoni et al[7] to the transmission eigenvalue problem for the Maxwell equations leading again to a system of two linear integral equations. It is the purpose of this paper to extend our approach from Cakoni and Kress[4] to again obtain a formulation with only one integral equation.

Assuming that $n > 0$ with $n \neq 1$ is constant, the main idea is to derive an integral equation from a characterization of the transmission eigenvalues in terms of the electric-to-magnetic boundary trace operator $\mathcal{A}_{k,n}$. For any field E defined in D, throughout the paper, we denote by

$$\gamma E := \nu \times (E \times \nu)$$

the tangential trace of E on ∂D. Then, $\mathcal{A}_{k,n}$ is defined by

$$\mathcal{A}_{k,n} : c \mapsto \gamma \operatorname{curl} E, \tag{5}$$

where E is the unique solution to

$$\operatorname{curl}\operatorname{curl} E - k^2 n E = 0 \quad \text{in } D, \tag{6}$$
$$\nu \times E = c \quad \text{on } \partial D, \tag{7}$$

assuming that k^2 is not an eigenvalue for this problem. We further assume that k^2 is also not an eigenvalue for the case when $n = 1$ and, for ease of notation, set $k_n := k\sqrt{n}$ and write $\mathcal{A}_k = \mathcal{A}_{k,1}$ and $\mathcal{A}_{k_n} = \mathcal{A}_{k,n}$. Then, k is a transmission eigenvalue if and only if the kernel of the operator

$$A(k) := \mathcal{A}_k - \mathcal{A}_{k_n} \tag{8}$$

is nontrivial. To get rid of the restriction on k^2 not to be an eigenvalue for (6)-(7), we will find it necessary to modify the boundary condition (7) into a nonlocal impedance condition to be specified later on. Before we do that, we need to discuss the appropriate trace spaces for solutions of the transmission eigenvalue problem and provide further preparations.

2 | OPERATORS AND TRACE SPACES

To represent the electric-to-magnetic boundary trace operator in terms of integral operators, we need to introduce the single-layer potential $\mathcal{S}_k$ defined by

$$(\mathcal{S}_k \psi)(x) := 2 \int_{\partial D} \psi(y) \Phi_k(x, y) \, ds(y), \quad x \in \mathbb{R}^3 \backslash \partial D, \tag{9}$$

in terms of the fundamental solution

$$\Phi_k(x,y) = \frac{1}{4\pi}\,\frac{e^{ik|x-y|}}{|x-y|}\,, \quad x \neq y. \tag{10}$$

The factor 2 in the definition of $\mathcal{S}_k$ later on avoids the occurrence of a factor $1/2$ in our representations of the operator $A(k)$. It is known (see, eg, McLean[8, Theorem 7.2]) that if ∂D is $C^{2,1}$-smooth, the linear operator $\mathcal{S}_k : H^{s-\frac{1}{2}}(\partial D) \to H^{s+1}(D)$ is bounded for $-1 \leq s \leq 2$. We define the restriction of $\mathcal{S}_k$ and of its normal derivative to the boundary ∂D by

$$(S_k\psi)(x) := 2\int_{\partial D} \psi(y)\Phi(x,y)\; ds(y), \quad x \in \partial D, \tag{11}$$

$$(K_k'\psi)(x) := 2\int_{\partial D} \psi(y)\frac{\partial}{\partial \nu_x}\;\Phi(x,y)\; ds(y), \quad x \in \partial D. \tag{12}$$

Then, by the trace theorem,

$$S_k : H^{s-\frac{1}{2}}(\partial D) \to H^{s+\frac{1}{2}}(\partial D), \tag{13}$$

$$K_k' : H^{s-\frac{1}{2}}(\partial D) \to H^{s-\frac{1}{2}}(\partial D) \tag{14}$$

are bounded for $-1 \leq s \leq 2$.

To discuss the two basic electromagnetic boundary integral operators, we define the space

$$H_{\mathrm{t}}^s(\partial D) := \left\{ a \in H^s(\partial D) : \nu \cdot a = 0 \right\}$$

of tangential fields, and as in Colton and Kress,[9, Chapter 6] we introduce the operators

$$(M_k a)(x) := 2\int_D \nu(x) \times \mathrm{curl}_x \left\{ \Phi_k(x,y)a(y) \right\} ds(y), \quad x \in \partial D \tag{15}$$

and

$$(N_k a)(x) := 2\gamma \mathrm{curl}\;\mathrm{curl}\int_D \Phi_k(x,y)a(y)\; ds(y), \quad x \in \partial D. \tag{16}$$

The definition of N_k differs slightly from the one in Colton and Kress.[9] The operator M_k is a pseudo-differential operator of order -1, it maps $H_{\mathrm{t}}^{s-\frac{1}{2}}(\partial D)$ compactly into itself, and N_k is a pseudo-differential operator of order 1, it maps $H_{\mathrm{t}}^{s-\frac{1}{2}}(\partial D)$ into $H_{\mathrm{t}}^{s-\frac{3}{2}}(\partial D)$ for $0 \leq s \leq 2$ (see Colton and Kress[9]).

The solutions to the transmission eigenvalue problem belong to $L^2_{\mathrm{curl}^2}(D)$, where

$$L^2_{\mathrm{curl}^2}(D) := \left\{ E \in L^2(D) : \mathrm{curl}\;\mathrm{curl}\; E \in L^2(D), \mathrm{div}\, E = 0 \right\}$$

is equipped with the norm

$$\|E\|^2_{L^2_{\mathrm{curl}^2}(D)} = \|E\|^2_{L^2(D)} + \|\mathrm{curl}\;\mathrm{curl}\; E\|^2_{L^2(D)}.$$

Therefore, we need to discuss both their traces

$$\gamma E \in H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D) \quad \text{and} \quad \gamma\;\mathrm{curl}\; E \in H_{\mathrm{t}}^{-\frac{3}{2}}(\partial D).$$

For $E \in L^2_{\mathrm{curl}^2}(D)$, in view of the second vector Green integral theorem (see Colton and Kress[9, p. 190]), its trace $\gamma E \in H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D)$ is defined by duality

$$\langle \gamma E, f \rangle_{H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D), H_{\mathrm{t}}^{\frac{1}{2}}(\partial D)} = \int_D (E \cdot \mathrm{curl}\;\mathrm{curl}\, F - F \cdot \mathrm{curl}\;\mathrm{curl}\; E)\, dx,$$

where $F \in H^2(D)$ is such that $\operatorname{div} F = 0$ in D and $\nu \times \gamma F = 0$ and γ curl $F = -f$ on ∂D. Similarly, the trace γ curl $E \in H_t^{-\frac{3}{2}}(\partial D)$ is defined

$$\langle \gamma \text{ curl } E, f\rangle_{H_t^{-\frac{3}{2}}(\partial D), H_t^{\frac{3}{2}}(\partial D)} = \int_D (E \cdot \text{curl curl } F - F \cdot \text{curl curl } E)\, dx,$$

where $F \in H^2(D)$ is chosen such that $\operatorname{div} F = 0$ in D and $\nu \times \gamma F = f$ and γ curl $F = 0$ on ∂D.

For these definitions of the traces, we require a lifting result due to Haddar,[10] which ensures that for given fields $a \in H_t^{\frac{3}{2}}(\partial D)$ and $b \in H_t^{\frac{1}{2}}(\partial D)$, there exist $F \in H^2(D)$ such that $\operatorname{div} F = 0$ in D and $\nu \times \gamma F = a$ and γ curl $F = b$ on ∂D. In Haddar,[10] the existence of $G \in H^2(D)$ satisfying $\nu \times \gamma G = a$ and γ curl $G = b$ on ∂D is established. By the Helmholtz decomposition, we have $G = \text{curl}\, A + \text{grad}\, u$ with $A, u \in H^3(D)$. Then, with the unique harmonic function $v \in H^3(D)$ satisfying $v = u$ on ∂D, the field $F = \text{curl}\, A + \text{grad}\, v$ has the required properties.

Obviously, $\mathcal{S}_k a$ satisfies the vector Helmholtz equation in D, and the operator curl $\mathcal{S}_k : H^{-\frac{1}{2}}(\partial D) \to L^2_{\text{curl}^2}(D)$ is bounded. By a duality argument, it is possible to extend the vector jump relations on ∂D to the case of densities $a \in H_t^{-\frac{1}{2}}(\partial D)$.

Lemma 2.1. *The field $E = \text{curl } \mathcal{S}_k a$ with density $a \in H_t^{-\frac{1}{2}}(\partial D)$ has boundary traces*

$$\nu \times \gamma E_{\pm} = M_k a \mp a \in H_t^{-\frac{1}{2}}(\partial D) \tag{17}$$

and

$$\gamma \text{ curl } E = N_k a \in H^{-\frac{3}{2}}(\partial D). \tag{18}$$

3 | VECTOR POTENTIAL THEORY REVISITED

Before we proceed with the closer investigation of the transmission eigenvalue problem, we provide a regularity property from vector potential theory that we will be use in our further analysis. This section follows Martensen[11, p. 252] and puts some classical facts on harmonic vector fields into a contemporary Sobolev space framework. R.K. was particularly pleased to work this out since he became familiar with the topic during his early times in mathematics almost 50 years ago as a PhD student and postdoctoral researcher with Erich Martensen in Darmstadt.

For $m = 0, 1, \ldots$, we introduce the spaces of divergence free vector fields

$$H^{m,0}(D) := \left\{ F \in H^m(D) : \operatorname{div} F = 0 \right\}.$$

Then, for $m = 0, 1$, given $F \in H^{m,0}(D)$, we define a harmonic function $u \in H^{m+1}(D)$ as the single-layer potential

$$u(x) := \int_{\partial D} \Phi_0(x, y)\ \nu(y) \cdot F(y)\ ds(y), \quad x \in D, \tag{19}$$

and a harmonic function $v \in H^{m+1}_{\text{loc}}(\mathbb{R}^3 \backslash \bar{D})$ as the unique solution to the exterior Neumann problem

$$\frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} \quad \text{on } \partial D \tag{20}$$

vanishing at infinity. Then, we consider the field

$$A(x) := \int_D \Phi_0(x, y) F(y)\ dy + \int_{\partial D} \Phi_0(x, y)\ [u(y) - v(y)]\ \nu(y)\ ds(y) \tag{21}$$

for $x \in D$ and the operator $V : F \mapsto A$.

Lemma 3.1. *The operator V is bounded from $H^{m,0}(D)$ into $H^{m+2,0}(D)$.*

Proof. The mapping of F into the volume potential in (21) is bounded from $H^m(D)$ into $H^{m+2}(D)$. By the trace theorem, the mapping of F into the potential u is bounded from $H^m(D)$ into $H^{m+1}(D)$, and the mapping of u into the boundary trace of v is bounded from $H^{m+1}(D)$ into $H^{m+\frac{1}{2}}(\partial D)$. Finally, the mapping from $(u|_{\partial D} - v|_{\partial D})$ to the single-layer potential in (21) is bounded from $H^{m+\frac{1}{2}}(\partial D)$ into H^{m+2}. Hence, the boundedness of V from $H^m(D)$ into $H^{m+2}(D)$ follows.

It remains to show that $\operatorname{div} A = 0$ in D. Taking the divergence in (21), interchanging differentiation and integration and using $\operatorname{div} F = 0$ yields

$$\operatorname{div} A(x) = -\int_D \operatorname{div}_y[\Phi_0(x,y)F(y)]\,dy - \int_{\partial D}[u(y)-v(y)]\,\frac{\partial\Phi_0(x,y)}{\partial\nu(y)}\,ds(y)$$

for $x \in D$. By the Gauss divergence theorem and (19), it follows

$$\operatorname{div} A(x) = -u(x) - \int_{\partial D}[u(y)-v(y)]\,\frac{\partial\Phi_0(x,y)}{\partial\nu(y)}\,ds(y), \quad x \in D.$$

Finally, the boundary condition (20) and Green's integral formula for harmonic functions imply $\operatorname{div} A = 0$ in D. □

Assume now that we are given fields $F \in H^{m,0}(D)$ and $g \in H_t^{m+\frac{3}{2}}(\partial D)$. Then, the field $H_0 = \operatorname{curl} V(F) \in H^{m+1,0}(D)$ satisfies

$$\operatorname{curl} H_0 = F, \quad \operatorname{div} H_0 = 0 \quad \text{in } D$$

because of $\operatorname{curl} H_0 = \operatorname{curl}\operatorname{curl} A = -\Delta A + \operatorname{grad}\operatorname{div} A = F$. Now, we consider the Neumann problem for a harmonic function $w \in H^{m+2}(D)$ with boundary condition

$$\frac{\partial w}{\partial \nu} = \nu \cdot H_0 + \operatorname{Div} g \quad \text{on } \partial D \tag{22}$$

and normalization $\int_{\partial D} w\,ds = 0$. Note that the right hand side in (22) belongs to $H^{m+\frac{1}{2}}(\partial D)$, and in view of the Gauss divergence theorem, it satisfies the solvability condition

$$\int_{\partial D}[\nu \cdot H_0 + \operatorname{Div} g]\,ds = 0.$$

Then,

$$H := H_0 - \operatorname{grad} w$$

satisfies

$$\operatorname{curl} H = F, \quad \operatorname{div} H = 0 \quad \text{in } D$$

and the boundary condition

$$\nu \cdot H = -\operatorname{Div} g \quad \text{on } \partial D. \tag{23}$$

The mapping $(H_0, g) \mapsto w$ is bounded from $H^{m+1}(D) \times H_t^{m+\frac{3}{2}}(\partial D)$ into $H^{m+2}(D)$. Therefore, in turn, the mapping $(F, g) \mapsto H$ is bounded from $H^m(D) \times H_t^{m+\frac{3}{2}}(\partial D)$ into $H^{m+1}(D)$.

The field $E_0 = \operatorname{curl} V(H) \in H^{m+2,0}(D)$ in turn now satisfies

$$\operatorname{curl} E_0 = H, \quad \operatorname{div} E_0 = 0 \quad \text{in } D.$$

The mapping $H \to E_0$ is bounded from $H^{m+1,0}(D)$ into $H^{m+2,0}(D)$. We consider the Dirichlet boundary value problem for a harmonic field E_1, that is, $\operatorname{curl} E_1 = 0$ and $\operatorname{div} E_1 = 0$ in D with tangential component

$$\nu \times E_1 = \nu \times E_0 - g \quad \text{on } \partial D.$$

From (23) and curl $E_0 = 0$, we observe that

$$\operatorname{Div}\,[\nu \times E_0 - g] = 0. \tag{24}$$

We seek the solution in the form

$$E_1(x) = \operatorname{curl} \int_{\partial D} \Phi_0(x, y) a(y)\, ds(y), \quad x \in D,$$

with density $a \in H_{\mathrm{t}}^{m+\frac{3}{2}}(\partial D)$. The boundary condition is satisfied provided a solves the integral equation

$$M_0 a - a = 2\nu \times E_0 - 2g. \tag{25}$$

For a simply connected domain D, the operator $M_0 - I$ is injective in the Hölder space $C_{\mathrm{t}}^{0,\alpha}(\partial D)$ (see Colton and Kress[12, Theorem 5.4]). Using the Fredholm alternative in dual systems, it can be seen that $M_0 - I$ is also injective in $H_{\mathrm{t}}^{m+\frac{3}{2}}(\partial D)$ (see Kress[13, Theorem 4.20]). By the Riesz theory, injectivity of $M_0 - I$ now implies bijectivity with a bounded inverse $(M_0 - I)^{-1}$. We refrain from presenting the technical details required to show solvability of (25) by using the quotient space with respect to the kernel of $M - I$ in the case of multiply connected domains D.

Taking the surface divergence of (25) leads to the homogeneous equation (see Colton and Kress[9, Theorem 6.17])

$$K_0' \operatorname{Div}\, a + \operatorname{Div}\, a = 0.$$

From this, it follows that the single-layer potential with density Div$a$ is constant in D (see Kress[13, Theorem 6.21]). Therefore, we have

$$\operatorname{curl} E_1(x) = \operatorname{grad} \operatorname{div} \int_{\partial D} \Phi_0(x, \cdot) a\, ds = \operatorname{grad} \int_{\partial D} \Phi_0(x, \cdot) \operatorname{Div}\, a\, ds = 0$$

for $x \in D$. Because of the boundedness of $(M_0 - I)^{-1}$ on $H_{\mathrm{t}}^{m+\frac{3}{2}}(\partial D)$, the mapping taking (E_0, g) into E_1 is bounded from $H^{m+2,0}(D) \times H_{\mathrm{t}}^{m+\frac{3}{2}}(\partial D)$ into $H^{m+2,0}(D)$.

Then,

$$E := E_0 - E_1 \in H^{m+2,0}(D) \tag{26}$$

satisfies

$$\operatorname{curl} \operatorname{curl} E = F, \quad \operatorname{div} E = 0 \quad \text{in } D \tag{27}$$

and the boundary condition

$$\nu \times E = g \quad \text{on } \partial D. \tag{28}$$

The total mapping $(F, g) \mapsto E$ from $H^{m,0}(D) \times H_{\mathrm{t}}^{m+\frac{3}{2}}(\partial D)$ into $H^{m+2,0}(D)$ is bounded. Using the vector Green integral theorem, it can be shown that the solution $E \in H^{m+2,0}(D)$ to (27)-(28) is unique.

We can summarize this into the following theorem (compare Theorem 1.3 in Taylor[14, p. 305]).

Theorem 3.2. *The unique solution $E \in H^{m,0}(D)$ of curl curl $E = F$ satisfying $\nu \times E = g$ on ∂D with $F \in H^{m,0}(D)$ and $g \in H_{\mathrm{t}}^{m+\frac{3}{2}}(\partial D)$ belongs to $H^{m+2,0}(D)$, and the linear mapping taking (F, g) into E is bounded from $H^{m,0}(D) \times H_{\mathrm{t}}^{m+\frac{3}{2}}(\partial D)$ into $H^{m+2,0}(D)$ for $m = 0, 1$.*

We require a further regularity result that we base on the vector Green formula for divergence free vector fields

$$\begin{aligned} E(x) = &\int_D \Phi_0(x, \cdot) \operatorname{curl} \operatorname{curl} E\, dy + \operatorname{grad} \int_{\partial D} \Phi_0(x, \cdot)\, \nu \cdot E\, ds - \operatorname{curl} \int_{\partial D} \Phi_0(x, \cdot)\, \nu \times E\, ds \\ &- \int_{\partial D} \Phi_0(x, \cdot)\, \nu \times \operatorname{curl} E\, ds \end{aligned} \tag{29}$$

for $x \in D$. This representation formula can be viewed as a limit of the Stratton-Chu formula for Maxwell's equations and a proof for smooth functions can be found, for example, in Kress.[15] It can be seen to be valid for $E \in L^2_{\mathrm{curl}^2}(D)$ with the boundary values interpreted in the sense of the traces $\gamma E \in H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D)$ and γ curl $E \in H_{\mathrm{t}}^{-\frac{3}{2}}(\partial D)$ as introduced in the previous section. Since for divergence free $E \in L^2_{\mathrm{curl}^2}(D)$, we have $\Delta E \in L^2(D)$, the trace $\nu \cdot E \in H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D)$ also is well defined (see Cakoni and Kress[4]). We will use (29) to establish the following regularity properties.

Theorem 3.3. *For each $E \in L^2_{curl^2}(D)$, we can estimate*

$$\|\gamma \mathrm{curl}\, E\|_{H^{-\frac{3}{2}}(\partial D)} \leq C\left[\|\nu \times \gamma E\|_{H^{-\frac{1}{2}}(\partial D)} + \|\mathrm{curl}\ \mathrm{curl}\ E\|_{L^2(D)}\right] \tag{30}$$

for some $C > 0$ depending only on D.

Proof. We introduce the volume potential

$$W(x) := 2\int_D \Phi_0(x,\cdot)\mathrm{curl}\ \mathrm{curl}\ E\ dy, \quad x \in D,$$

and, with the aid of Lemma 2.1, take the trace of the curl in (29) to obtain the integral equation

$$a + M_0 a = \gamma\ \mathrm{curl}\ W - \nu \times N_0\ (\nu \times E) \tag{31}$$

for $a := \nu \times \gamma\, \mathrm{curl}\, E$. Analogous to $M_0 - I$, for a simply connected domain D, the operator $M_0 + I$ is injective in the Hölder space $C_{\mathrm{t}}^{0,\alpha}(\partial D)$. Arguing as in the proof of the previous theorem, we obtain solvability of (31) in $H_{\mathrm{t}}^{-\frac{3}{2}}(\partial D)$ with a bounded inverse $(I + M_0)^{-1}$. Then, in view of the boundedness of the volume potential from $L^2(D)$ into $H^2(D)$ and the boundedness of N_0 from $H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D)$ into $H_{\mathrm{t}}^{-\frac{3}{2}}(\partial D)$, estimating the right-hand side of (31), we complete the proof. □

Lemma 3.4. *Each $E \in L^2_{curl^2}(D)$ with trace $\gamma E \in H_{\mathrm{t}}^{\frac{3}{2}}(\partial D)$ on ∂D is in $H^{2,0}(D)$.*

Proof. For given $E \in L^2_{\mathrm{curl}^2}(D)$, denote by $\tilde{E} \in H^{2,0}(D)$, the unique solution of curl curl $\tilde{E}$ = curl curl E in D with $\nu \times \gamma\tilde{E} = \nu \times E$ on ∂D from Theorem 3.2 and consider the difference $E_0 = E - \tilde{E}$. Then, by the definition of the trace γE_0, we have that $\int_D E_0 \cdot \mathrm{curl}\ \mathrm{curl}\, F\ dx = 0$ for any $F \in H^{2,0}(D)$ with $\gamma F = 0$ on ∂D. Inserting the unique solution $F \in H^{2,0}(D)$ of curl curl $F = \overline{E}_0$ with $\gamma F = 0$ on ∂D gives $E_0 = 0$, that is, $E = \tilde{E} \in H^{2,0}(D)$. □

4 | A NONLOCAL IMPEDANCE CONDITION

Now, we are ready to consider the impedance problem

$$\mathrm{curl}\ \mathrm{curl}\ E - k^2 n E = 0 \quad \text{in } D \tag{32}$$

with the nonlocal boundary condition

$$\nu \times E - i\eta\gamma S_0^3 \gamma\ \mathrm{curl}\ E = c \quad \text{on } \partial D \tag{33}$$

for $E \in L^2_{\mathrm{curl}^2}(D)$ where $c \in H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D)$ is given. Here, S_0 is the single-layer operator for the Laplace case, and $\eta \geq 0$ is a parameter that we introduce to include the perfect conductor boundary condition as the special case $\eta = 0$. The nonlocal boundary condition has no physical meaning but helps us to circumvent some regularity issues. For this reason, the analysis presented below remains valid after replacing the compact operator S_0 by any positive definite pseudo-differential operator of order -1.

Theorem 4.1. *The impedance boundary value problem (32)-(33) has a unique solution.*

Proof. To show uniqueness, assume that E solves the impedance problem for $c = 0$. From the homogeneous boundary condition (33) and the fact that $\gamma S_0^3 : H_{\mathrm{t}}^{-\frac{3}{2}}(\partial D) \to H_{\mathrm{t}}^{\frac{3}{2}}(\partial D)$, we observe that

$$\nu \times E = i\eta\gamma\, S_0^3 \gamma \operatorname{curl} E \in H_{\mathrm{t}}^{\frac{3}{2}}(\partial D).$$

Therefore, by Theorem 3.2 with $m = 0$ in combination with Lemma 3.4, we have $E \in H^2(D)$. Then from the first vector Green integral theorem (see Colton and Kress[9, p. 190]), using the positive definiteness of S_0, we obtain that

$$\int_D \left\{ |\operatorname{curl} E|^2 - k^2 |E|^2 \right\} dx = i \int_{\partial D} \gamma S_0^{\frac{3}{2}} \gamma \operatorname{curl} E \cdot \gamma S_0^{\frac{3}{2}} \gamma \operatorname{curl} \overline{E}\, ds,$$

whence $\gamma \operatorname{curl} E = 0$ on ∂D follows. Then, the boundary condition (33) implies $\nu \times E = 0$ on ∂D and from the Stratton-Chu formula (see Colton and Kress[9]), we deduce that $E = 0$ in D.

We seek the solution in the form

$$E = \operatorname{curl} \mathcal{S}_k a \tag{34}$$

with density $a \in H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D)$. Clearly, E satisfies the differential equation (32) and belongs to $L^2_{\mathrm{curl}^2}(D)$. In view of (17) and (18), the boundary condition (33) is satisfied provided a solves the integral equation

$$(-I + M_k - i\eta\gamma S_0^3 N_k) a = 2c. \tag{35}$$

Injectivity of

$$Q_k := -I + M_k - i\eta\gamma S_0^3 N_k$$

is a consequence of the uniqueness for the impedance problem, which implies that E defined by (34) for a solution a of $Q_k a = 0$ vanishes $E = 0$ in D. Then, (18) yields $\nu \times E_+ = 0$ on ∂D and from this uniqueness for the perfect conductor exterior boundary value problem implies that $E = 0$ in $\mathbb{R}^3 \setminus D$. Finally, from this and (17), we obtain $a = 0$. As noted above, $M_k : H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D) \to H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D)$ is compact and $N_k : H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D) \to H_{\mathrm{t}}^{-\frac{3}{2}}(\partial D)$ is bounded. As consequence of the latter, $\gamma S_0^3 N_k : H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D) \to H_{\mathrm{t}}^{\frac{3}{2}}(\partial D)$ is bounded and therefore $\gamma S_0^3 N_k : H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D) \to H_{\mathrm{t}}^{-\frac{1}{2}}(\partial D)$ is compact. By the Riesz theory, injectivity of Q_k implies bijectivity and boundedness of Q_k^{-1}. □

We will also need the impedance boundary value problem for purely imaginary $k = i$ and $\eta = i$. In this case, we have the boundary condition

$$\nu \times E + \gamma S_0^3 \gamma \operatorname{curl} E = c \quad \text{on } \partial D \tag{36}$$

and obtain uniqueness and existence along the same lines.

The solution to the nonlocal impedance problem defines an impedance-to-magnetic boundary trace operator by

$$\mathcal{B}_{k,n} : c \mapsto \gamma \operatorname{curl} E \tag{37}$$

where E is the unique solution to (32)-(33). We set $k_n := k\sqrt{n}$ and write $\mathcal{B}_k = \mathcal{B}_{k,1}$ and $\mathcal{B}_{k_n} = \mathcal{B}_{k,n}$. Obviously, k is a transmission eigenvalue if and only if the kernel of the operator

$$B(k,\eta) := \mathcal{B}_k - \mathcal{B}_{k_n} \tag{38}$$

is nontrivial. From the proof of Theorem 4.1, we observe that

$$\mathcal{B}_k = N_k Q_k^{-1}$$

and the difference of this operators corresponding to k and k_n is given by

$$B(k,\eta) = N_k Q_k^{-1} - N_{k_n} Q_{k_n}^{-1}.$$

5 | THE TRANSMISSION EIGENVALUE PROBLEM

We now are ready for our investigation of the transmission eigenvalue problem.

Lemma 5.1. *The linear operators*

$$a \mapsto \operatorname{curl} \mathcal{S}_k Q_k^{-1} a - \operatorname{curl} \mathcal{S}_{k_n} Q_{k_n}^{-1} a \tag{39}$$

from $H_t^{-\frac{1}{2}}(\partial D)$ *into* $H^2(D)$ *and* $B(k,\eta) : H_t^{-\frac{1}{2}}(\partial D) \to H_t^{\frac{1}{2}}(\partial D)$ *are bounded.*

Proof. By definition, $B(k)a$ is the tangential trace on the boundary ∂D of the curl of

$$E := \operatorname{curl} \mathcal{S}_k Q_k^{-1} a - \operatorname{curl} \mathcal{S}_{k_n} Q_{k_n}^{-1} a, \quad a \in H_t^{-\frac{1}{2}}(\partial D).$$

By Lemma 2.1, we have $\gamma \operatorname{curl} E = N_k Q_k^{-1} a - N_{k_n} Q_{k_n}^{-1} a$ with the mapping $a \to \gamma \operatorname{curl} E$ bounded form $H_t^{-\frac{1}{2}}(\partial D)$ into $H_t^{-\frac{3}{2}}(\partial D)$. From

$$\operatorname{curl}\operatorname{curl} E = k^2 \operatorname{curl} \mathcal{S}_k Q_k^{-1} a - k_n^2 \operatorname{curl} \mathcal{S}_{k_n} Q_{k_n}^{-1} a \tag{40}$$

we observe that $E \in L^2_{\mathrm{curl}^2}(D)$ and, clearly, E satisfies the boundary condition

$$\nu \times E - i\eta\gamma S_0^3 \gamma \operatorname{curl} E = 0 \quad \text{on } \partial D, \tag{41}$$

whence $\nu \times \gamma E \in H_t^{\frac{3}{2}}(\partial D)$ follows with the mapping $a \to \nu \times E$ bounded from $H_t^{-\frac{1}{2}}(\partial D)$ into $H_t^{\frac{3}{2}}(\partial D)$. Therefore, from Lemma 3.4 and Theorem 3.2 with $m = 0$, the statement of the theorem follows. □

Theorem 5.2. *Let* $\kappa > 0$ *and* $\kappa_n := \kappa\sqrt{n}$. *Then,*

$$(\kappa_n^2 - \kappa^2) B(i\kappa, i) : H_t^{-\frac{1}{2}}(\partial D) \to H_t^{\frac{1}{2}}(\partial D)$$

is coercive, ie,

$$(\kappa_n^2 - \kappa^2)\langle B(i\kappa, i)a, a\rangle_{H_t^{\frac{1}{2}}(\partial D), H_t^{-\frac{1}{2}}(\partial D)} \geq C\|a\|^2_{H_t^{-\frac{1}{2}}(\partial D)}$$

for all $a \in H_t^{-\frac{1}{2}}(\partial D)$ *and some* $C > 0$.

Proof. For $E \in H^2(D)$ with $\operatorname{div} E = 0$, we transform

$$\begin{aligned}
&\overline{E} \cdot (\operatorname{curl}\operatorname{curl} + \kappa^2)(\operatorname{curl}\operatorname{curl} + \kappa_n^2)E - \operatorname{curl}\operatorname{curl} \overline{E} \cdot \operatorname{curl}\operatorname{curl} E - (\kappa^2 + \kappa_n^2) \operatorname{curl} \overline{E} \cdot \operatorname{curl} E - \kappa^2\kappa_n^2\, \overline{E} \cdot E \\
&= -\overline{E} \cdot \Delta \operatorname{curl}\operatorname{curl} E + \Delta\overline{E} \cdot \operatorname{curl}\operatorname{curl} E - (\kappa^2 + \kappa_n^2) \operatorname{div}[\overline{E} \times \operatorname{curl} E].
\end{aligned}$$

From this, by the vector Green theorem (see Colton and Kress[9, p.190]) and the Gauss divergence theorem, we obtain

$$\begin{aligned}
&\int_D \overline{E} \cdot (\operatorname{curl}\operatorname{curl} + \kappa^2)(\operatorname{curl}\operatorname{curl} + \kappa_n^2)E\, dx - \int_D \left[|\operatorname{curl}\operatorname{curl} E|^2 + (\kappa^2 + \kappa_n^2)\, |\operatorname{curl} E|^2 + \kappa^2\kappa_n^2\, |E|^2\right] dx \\
&= \int_{\partial D} \left\{ \nu \times \operatorname{curl}\overline{E} \cdot \operatorname{curl}\operatorname{curl} E - \nu \times \overline{E} \cdot [\operatorname{curl} \Delta E + (\kappa^2 + \kappa_n^2) \operatorname{curl} E] \right\} ds.
\end{aligned} \tag{42}$$

Now, for $a \in H_t^{-\frac{1}{2}}(\partial D)$, we define

$$E := \operatorname{curl} \mathcal{S}_{i\kappa} Q_{i\kappa}^{-1} a - \operatorname{curl} \mathcal{S}_{i\kappa_n} Q_{i\kappa_n}^{-1} a.$$

By the preceding Lemma 5.1 we have that $E \in H^2(D)$. Then, we find

$$(\operatorname{curl}\operatorname{curl} + \kappa^2)(\operatorname{curl}\operatorname{curl} + \kappa_n^2)E = 0 \tag{43}$$

and

$$\operatorname{curl}\operatorname{curl} E = -\kappa^2 \operatorname{curl}\mathcal{S}_{i\kappa} Q_{i\kappa}^{-1} a + \kappa_n^2 \operatorname{curl}\mathcal{S}_{i\kappa_n} Q_{i\kappa_n}^{-1} a. \tag{44}$$

Therefore, we have the boundary conditions

$$\nu \times E + \gamma S_0^3 \gamma \operatorname{curl} E = 0, \quad \gamma \operatorname{curl} E = B(i\kappa, i)a \quad \text{on } \partial D \tag{45}$$

and

$$\nu \times \operatorname{curl}\operatorname{curl} E + \gamma S_0^3 \gamma \operatorname{curl}\operatorname{curl}\operatorname{curl} E = -(\kappa^2 - \kappa_n^2)a \quad \text{on } \partial D. \tag{46}$$

We use them to obtain that

$$\begin{aligned}
&\nu \times \operatorname{curl} \overline{E} \cdot \operatorname{curl}\operatorname{curl} E - \nu \times \overline{E} \cdot [\operatorname{curl} \Delta E + (\kappa^2 + \kappa_n^2) \operatorname{curl} E \\
&= (\kappa^2 - \kappa_n^2) a \cdot \overline{B(i\kappa, i)a} + \gamma \operatorname{curl} \overline{E} \cdot \gamma S_0^3 \gamma \operatorname{curl}\operatorname{curl}\operatorname{curl} E \\
&\quad - \gamma S_0^3 \gamma \operatorname{curl} \overline{E} \cdot \gamma \operatorname{curl}\operatorname{curl}\operatorname{curl} E + (\kappa^2 + \kappa_n^2)\, \gamma S_0^3 \gamma \operatorname{curl} \overline{E} \cdot \gamma \operatorname{curl} E
\end{aligned}$$

on ∂D. From this and (42) and (45), using the self adjointness of S_0, it follows that

$$\begin{aligned}
&- \int_D \left[|\operatorname{curl}\operatorname{curl} E|^2 + (\kappa^2 + \kappa_n^2)\, |\operatorname{curl} E|^2 + \kappa^2 \kappa_n^2\, |E|^2 \right] dx \\
&= \int_{\partial D} \left[(\kappa^2 - \kappa_n^2)\, a \overline{B(i\kappa, i)a} + (\kappa^2 + \kappa_n^2)\, \gamma S_0^{\frac{3}{2}} \gamma \operatorname{curl} \overline{E} \cdot \gamma S_0^{\frac{3}{2}} \gamma \operatorname{curl} E \right] ds,
\end{aligned}$$

whence

$$(\kappa_n^2 - \kappa^2) \int_{\partial D} a \overline{B(i\kappa, i)a}\, ds \geq \int_D \left[|\operatorname{curl}\operatorname{curl} E|^2 + \kappa^2 \kappa_n^2\, |E|^2 \right] dx \tag{47}$$

follows.

Straightforward computation show that

$$[\operatorname{curl}\operatorname{curl}]^2 E = F(E)$$

where

$$F(E) := (\kappa^2 + \kappa_n^2) \operatorname{curl}\operatorname{curl} E - \kappa^2 \kappa_n^2 E.$$

From this and (46), applying Theorem 3.3 to $\operatorname{curl}\operatorname{curl} E$ and using the trace theorem and the boundedness of S_0, we obtain

$$\|a\|_{H^{-\frac{1}{2}}(\partial D)} \leq c \int_D \left[|\operatorname{curl}\operatorname{curl} E|^2 + \kappa^2 \kappa_n^2\, |E|^2 \right] dx. \tag{48}$$

Combining (47) and (48) completes the proof. □

Theorem 5.3. *The operator*

$$B(k, \eta) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} B(i|k|, i) : H_t^{-\frac{1}{2}}(\partial D) \to H_t^{-\frac{1}{2}}(\partial D)$$

is compact.

Proof. For $a \in H_t^{-\frac{1}{2}}(\partial D)$, we define

$$E_0 := \operatorname{curl} \mathcal{S}_k Q_k^{-1} a - \operatorname{curl} \mathcal{S}_{k_n} Q_{k_n}^{-1} a$$

and

$$E_i := \operatorname{curl} \mathcal{S}_{i|k|} Q_{i|k|}^{-1} a - \operatorname{curl} \mathcal{S}_{i|k_n|} Q_{i|k_n|}^{-1} a$$

and let

$$E := E_0 + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} E_i.$$

Then, $E \in H^2(D)$ by Lemma 5.1 and E satisfies the boundary conditions (see Equation 45)

$$\nu \times E + \gamma S_0^3 \gamma \operatorname{curl} E = (i+1)\gamma S_0^3 \gamma \operatorname{curl} E_0 \tag{49}$$

and

$$\nu \times \operatorname{curl}\operatorname{curl} E + \gamma S_0^3 \gamma \operatorname{curl}\operatorname{curl}\operatorname{curl} E = (i+1)\gamma S_0^3 \gamma \operatorname{curl}\operatorname{curl}\operatorname{curl} E_0 \tag{50}$$

on ∂D. Furthermore, it is straightforward to check that

$$[\operatorname{curl}\operatorname{curl}]^2 E = F(E_0, E_i), \tag{51}$$

where

$$F(E_0, E_i) := -k^2 k_n^2 E_0 - (k^2 + k_n^2) \operatorname{curl}\operatorname{curl} E_0 - \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} \left[|k|^2 |k_n|^2 E_i - (|k|^2 + |k_n|^2) \operatorname{curl}\operatorname{curl} E_i \right] \tag{52}$$

belongs to $L^2(D)$ by Lemma 5.1 and the map $a \mapsto F$ is bounded from $H_t^{-\frac{1}{2}}(\partial D)$ into $L^2(D)$.

From (44), we see that the mappings taking a into $\gamma S_0^3 \gamma \operatorname{curl}^3 E$ and into $\gamma S_0^3 \gamma \operatorname{curl}^3 E_0$ are bounded from $H_t^{-\frac{1}{2}}(\partial D)$ into $H_t^{\frac{3}{2}}(\partial D)$. Therefore, applying Theorem 3.2 with $m = 0$, together with Lemma 3.4, for $\operatorname{curl}\operatorname{curl} E$, we obtain that $\operatorname{curl}\operatorname{curl} E \in H^2(D)$ with the mapping $a \mapsto \operatorname{curl}\operatorname{curl} E$ bounded from $H_t^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$. By Lemma 5.1, the mappings taking a into $\gamma S_0^3 \gamma \operatorname{curl} E$ and into $\gamma S_0^3 \gamma \operatorname{curl} E_0$ are bounded from $H_t^{-\frac{1}{2}}(\partial D)$ into $H_t^{\frac{5}{2}}(\partial D)$. Therefore, applying Theorem 3.2 now for E (with $m = 1$ and $\operatorname{curl}\operatorname{curl} E \in H^1(D)$) then in turn shows that $E \in H^3(D)$ with the mapping $a \mapsto E$ being bounded from $H_t^{-\frac{1}{2}}(\partial D)$ into $H^3(D)$. Therefore, the mapping $a \mapsto \gamma \operatorname{curl} E$ is bounded from $H_t^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{3}{2}}(\partial D)$. Now, noting that

$$\gamma \operatorname{curl} E = B(k, \eta) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} B(i|k|, i),$$

the statement of the theorem follows from the compact embedding of $H_t^{\frac{3}{2}}(\partial D)$ into $H_t^{\frac{1}{2}}(\partial D)$. □

In summary, Theorems 5.2 and 5.3 imply the following result, from which in particular, we can reestablish the well known discreteness (see for example, Cakoni et al[1]) of the transmission eigenvalues for the special case of a constant refractive index. Analyticity in the theorem follows from the analyticity of the kernels of the integral operators with respect to the wave number k.

Theorem 5.4. *$B(k; \eta) : H_t^{-\frac{1}{2}}(\partial D) \to H_t^{-\frac{1}{2}}(\partial D)$ is a Fredholm operator with index zero and analytic in $\{k \in \mathbb{C} : Re(k) > 0$ and $Im(k) \geq 0\}$.*

Let us denote by $\mathbb{E}$ the set of all positive k such that k^2 or k_n^2 is a Maxwell eigenvalue for a perfect conductor D. Then, $A(k)$ is defined for $k \in \mathbb{C} \backslash \mathbb{E}$. Then, setting $\eta = 0$, Theorem 5.4 contains the following result as a special case.

Corollary 5.5. *$A(k) : H_t^{-\frac{1}{2}}(\partial D) \to H_t^{-\frac{1}{2}}(\partial D)$ is a Fredholm operator with index zero and analytic in $\mathbb{C}\backslash\mathbb{E}$.*

6 | NUMERICAL EXAMPLES

In this final section, we will illustrate the use of our integral equation formulation for the numerical computation of electromagnetic interior transmission eigenvalues. Analytically derived transmission eigenvalues for a ball due to Monk and Sun[3] and numerical results by Kleefeld[16] based on a boundary element collocation method for a two-by-two boundary integral equation formulation are used as benchmarks.

In our case, the nonlinear operator $B(k,\eta)$ given by (38) is discretized using the spectrally accurate method that was proposed by Ganesh and Hawkins[17] for solving the magnetic field integral equation and extended to the electric field integral equation by Le Louër.[18] We then apply Beyn algorithm[19] for the solution of nonlinear eigenvalue problems for large-sized matrices that are analytic with respect to the eigenvalue parameter.

The spectral method is based on a spherical parametrization of the simply connected boundary ∂D so that the integral operators are transported onto the unit sphere $\mathbb{S}^2$ by means of the bicontinuous invertible Piola transform of the boundary parametrization (see Ivanyshyn et al[20, Equation (2.8)]). The operator N_k differs slightly from the operator C_k in Le Louër[18] but can be retrieved through the properties of the Piola transform (see Ivanyshyn et al[20, Equation (3.2)]). For any $n_{sph} \in \mathbb{N}$, the parametrized versions of the integral operators M_k and N_k both defined on $H_t^{-\frac{1}{2}}(\mathbb{S}^2)$ are then projected onto the space of dimension $m = 2(n_{sph}+1)^2 - 2$ spanned by the orthonormal tangential vector spherical harmonics

$$\frac{1}{\sqrt{\ell(\ell+1)}}\nabla_{\mathbb{S}^2} Y_{\ell,j} \quad \text{and} \quad \frac{1}{\sqrt{\ell(\ell+1)}}\mathrm{Curl}_{\mathbb{S}^2} Y_{\ell,j}, \quad 1 \le \ell \le n_{sph}, \ |j| \le \ell,$$

of degree less than or equal to n_{sph} (see Colton and Kress[9, Equation (6.60)]). Here, in the numerical examples, we chose $n_{sph} = 20$, that is, $m = 882$.

We recall that the single-layer operator S_0 used in the analysis of the two previous sections can be replaced by any positive definite compact operator of order -1. As alternative, we chose the operator given by

$$\Lambda = \mathrm{Curl}_{\mathbb{S}^2}(-\Delta_{\mathbb{S}^2})^{-\frac{3}{2}}\mathrm{Curl}_{\mathbb{S}^2} - \nabla_{\mathbb{S}^2}(-\Delta_{\mathbb{S}^2})^{-\frac{3}{2}}\mathrm{Div}_{\mathbb{S}^2}$$

in terms of the surface curl and divergence. Its discretization is given by the diagonal matrix

$$\Lambda_m = \begin{pmatrix} D_m & 0 \\ 0 & D_m \end{pmatrix} \ \text{with} \ D_m = \mathrm{diag}\left(\underbrace{\frac{1}{\sqrt{\ell(\ell+1)}}, \ldots, \frac{1}{\sqrt{\ell(\ell+1)}}}_{(2\ell+1)\ \text{times}}\right)_{1\le\ell\le n_{sph}},$$

and, as compared to S_0, it reduces the computational costs of the $m \times m$ matrix setup of the operator $B(k,\eta)$ in Beyn's algorithm.

For the contour integral in Beyn's method (see Cakoni and Kress[4] and Beyn[19]), we chose ellipses

$$\partial\Omega = \left\{ \frac{k_{max}+k_{min}}{2} + \frac{k_{max}-k_{min}}{2}\cos t + i\beta \sin t \, : \, t \in [0, 2\pi] \right\}$$

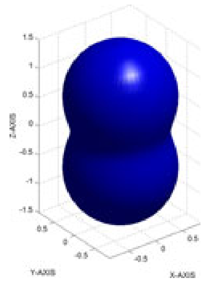
and used 128 quadrature points in the composite trapezoidal rule. Here, (k_{min}, k_{max}) is the interval in which we are searching for the transmission eigenvalues and $\beta = 0.01$ corresponds to the minor axis of the ellipse that is chosen rather small. The tolerance in Beyn's algorithm is chosen 10^{-12}. We consider only the case $n > 1$, the transmission eigenvalues for $n < 1$ can be computed directly by the relation $k(1/n) = \sqrt{n}\, k(n)$ (see also Cakoni and Kress[4]).

In the first example, we tested the proposed method for domains with known real interior transmission eigenvalues and consider the unit ball and a peanut shaped domain with the parametrization

$$z(\theta, \phi) = \frac{3\sqrt{\cos^2\theta + 0.25\sin^2\theta}}{2} e(\theta, \phi)$$

where

$$e(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$



As refractive index, we chose both $n = 4$ and $n = 16$. From Table 1, it can be seen that the eigenvalues and their multiplicities as given in the brackets coincide with those in Monk and Sun[3] for $n = 16$ and in Kleefeld[16] for $n = 4$.

In the next example, we consider transmission eigenvalues for ellipsoids

$$z(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, C\cos\theta)$$

for various choices for the axis C (see Figure 1).

According to Theorem 2.5 in Monk and Sun,[3] the first transmission eigenvalue $k_{1,D}$ for a domain D lies in the interval

$$\frac{k_{1,n}}{r_2} \leq k_{1,D} \leq \frac{k_{1,n}}{r_1}, \tag{53}$$

where k_1 is the smallest transmission eigenvalue for the unit ball and a given refractive index n; r_1 is the radius of the largest ball B_{r_1} such that $B_{r_1} \subset D$, and r_2 is the radius of the smallest ball B_{r_2} such that $D \subset B_{r_2}$. Hence, the transmission eigenvalues for a unit ball computed in Table 1 can be used for choosing the interval (k_{min}, k_{max}) in Beyn's algorithm.

TABLE 1 Benchmark interior transmission eigenvalues

n = 16	n = 4	
Unit Ball	**Unit Ball**	**Peanut**
1.1654 [3]	3.1415 [3]	2.9966 [2]
1.4608 [3]	3.4928 [5]	3.0393 [2]
1.4751 [5]	3.5928 [3]	3.3624 [1]
1.7640 [5]	3.6924 [5]	3.4172 [1]
1.7774 [7]	3.9026 [7]	3.5150 [1]

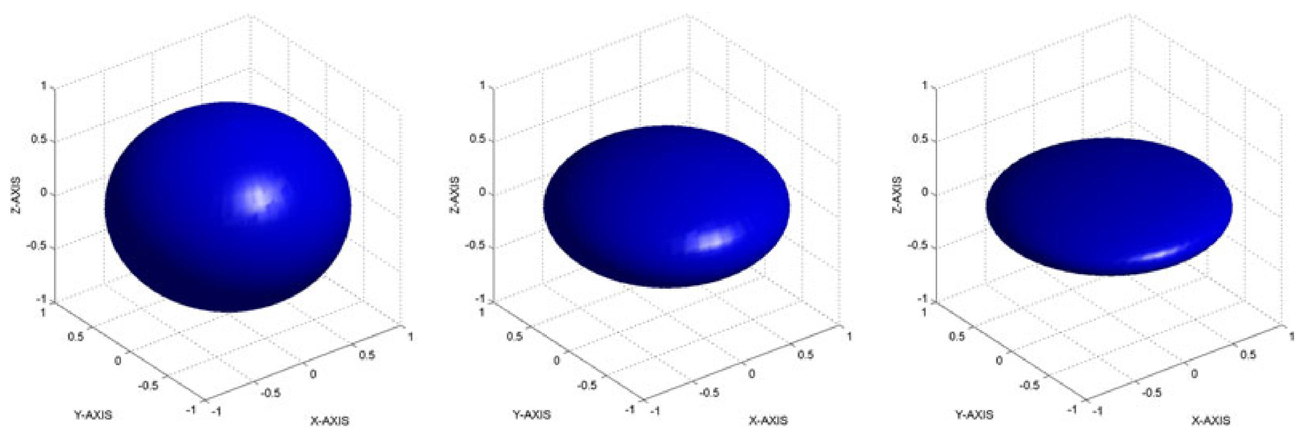


FIGURE 1 Ellipsoids [Colour figure can be viewed at wileyonlinelibrary.com]

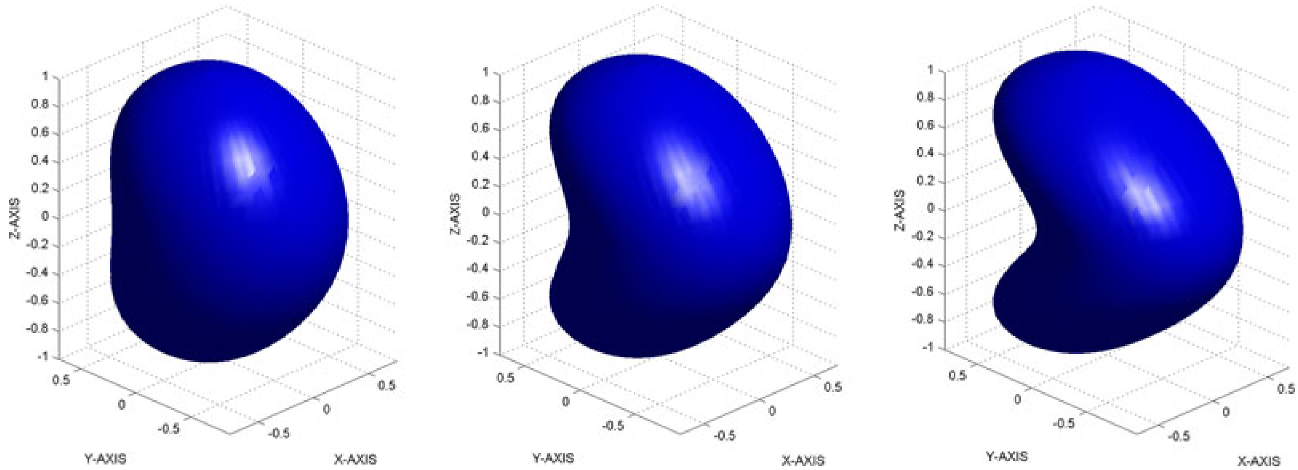


FIGURE 2 Bean-shaped domains [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 2 Transmission eigenvalues for ellipsoids

n = 4			**n = 16**		
$C = 0.8$	$C = 0.5$	$C = 0.3$	$C = 0.8$	$C = 0.5$	$C = 0.3$
3.3364 [1]	4.2885 [1]	5.8344 [1]	1.2340 [1]	1.5009 [1]	2.1395 [1]
3.4080 [2]	4.3422 [1]	5.8593 [2]	1.2989 [2]	1.7311 [2]	2.2301 [2]
3.7002 [2]	4.3836 [2]	6.0630 [2]	1.5222 [2]	1.8272 [2]	2.4517 [2]
3.7553 [1]	4.4479 [2]	6.2544 [2]	1.5472 [1]	1.9483 [1]	2.7021 [2]
3.7687 [2]	4.5405 [2]	6.5506 [2]	1.5778 [2]	2.0042 [2]	2.8619 [2]

TABLE 3 Transmission eigenvalues for bean-shaped domain

n = 4			**n = 16**		
$\epsilon = 0.1$	$\epsilon = 0.2$	$\epsilon = 0.3$	$\epsilon = 0.1$	$\epsilon = 0.2$	$\epsilon = 0.3$
3.6204 [1]	3.7326 [1]	4.2940 [1]	1.4010 [1]	1.4726 [1]	1.5729 [1]
3.6483 [1]	3.8266 [1]	4.3705 [1]	1.4106 [1]	1.5022 [1]	1.5843 [1]
3.8692 [1]	3.9356 [1]	4.9770 [1]	1.4407 [1]	1.5030 [1]	1.6219 [1]
3.9110 [1]	3.9407 [1]	5.1771 [1]	1.6206 [1]	1.6591 [1]	1.7295 [1]
4.0046 [1]	4.1786 [1]	5.7341 [1]	1.6300 [1]	1.6997 [1]	1.7948 [1]

In the last example, we present transmission eigenvalues for bean-shaped domains with parametrization

$$z(\theta, \phi) = (a(\theta) \sin\theta \cos\phi, a(\theta) \sin\theta \sin\phi - \epsilon \cos(\pi \cos\theta), \cos\theta),$$

where $a(\theta) = \sqrt{0.64(1 - \epsilon \cos(\pi \cos\theta))}$ and $\epsilon > 0$ (see Figure 2).

Tables 2 and 3 illustrate the dependence of the eigenvalues on the refractive index and the geometry, ie, a smaller refractive index or squeezing the domain both lead to an increase of the transmission eigenvalues in agreement with (53).

The imaginary part of all computed eigenvalues is less than 10^{-8} except for the bean-shaped domain with $\epsilon = 0.3$, where the imaginary part is less than 10^{-6}. This is due to the concave shape of the domain, the accuracy can be increased by increasing the discretization parameters n_{sph}.

Analogous to the two dimensional case,[4] the computational cost of the proposed method is about only one-half of that for the two-by-two systems presented in Cakoni et al[7] and Kleefeld.[16] This is significant because of the large dimension of the matrices and the vector valued equations.

ACKNOWLEDGEMENTS

The research of F.C. is supported in parts by AFOSR grant FA9550-13-1-0199, NSF Grant DMS-1602802, and Simons Foundation Award 392261. F.C. gratefully acknowledges the hospitality of the University of Göttingen during her visit with R.K. when part of this work was initiated. The research of O.I.Y. is partially supported by TÜBITAK under grant 116F299.

ORCID

R. Kress http://orcid.org/0000-0002-8052-3500

REFERENCES

1. Cakoni F, Gintides D, Haddar H. The existence of an infinite discrete set of transmission eigenvalues. *SIAM J Math Anal*. 2010;42:237-255.
2. Cakoni F, Kirsch A. On the interior transmission eigenvalue problem. *Int J Comput Sci Math*. 2010;3:142-167.
3. Monk P, Sun J. Finite element methods for Maxwell's transmission problem. *SIAM J Sci Comp*. 2012;34:B247-B244.
4. Cakoni F, Kress R. A boundary integral equation method for the transmission eigenvalue problem. *Appl Anal*. 2017;96:23-38.
5. Cossonnière A. Valeurs propres de transmission et leur utilisation dans l'identification d'inclusions à partir de mesures électromagnétiques. *PhD Thesis*: Université Toulouse; 2011.
6. Cossonnière A, Haddar H. Surface integral formulation of the interior transmission problem. *J Integral Equ Appl*. 2013;25:341-376.
7. Cakoni F, Haddar H, Meng S. Boundary integral equations for the transmission eigenvalue problem for Maxwell's equations. *J Integral Equ Appl*. 2015;27:375-406.
8. McLean W. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge: Cambridge University Press; 2000.
9. Colton D, Kress R. *Inverse Acoustic and Electromagnetic Scattering Theory*. 3rd ed. New York: Springer; 2013.
10. Haddar H. The interior transmission problem for anisotropic Maxwell's equations and its applications to inverse problem. *Math Meth Appl Sci*. 2004;17:2111-2129.
11. Martensen E. *Potentialtheorie*. Stuttgart: Teubner; 1968.
12. Colton D, Kress R. *Integral Equation Methods in Scattering Theory*. Philadelphia: SIAM; 2013.
13. Kress R. *Linear Integral Equations*. 3rd ed. New York: Springer; 2013.
14. Taylor ME. *Partial Differential Equations*. New York: Springer; 1996.
15. Kress R. Die Behandlung zweier Randwertprobleme für die vektorielle Poissongleichung nach einer Integralgleichungsmethode. *Arch Ration Mech Anal*. 1970;39:206-226.
16. Kleefeld A. Numerical methods for acoustic and electromagnetic scattering: transmission boundary-value problems, interior transmission eigenvalues, and the factorization method. *Habilitation Thesis*: Brandenburg University of Technology Cottbus; 2015.
17. Ganesh M, Hawkins SC. A high-order tangential basis algorithm for electromagnetic scattering by curved surfaces. *J Comput Phys*. 2008;227:4543-4562.
18. Le Louër F. Spectrally accurate numerical solution of hypersingular boundary integral equations for three-dimensional electromagnetic wave scattering problems. *J Comput Phys*. 2014;275:662-666.
19. Beyn WJ. An integral method for solving non-linear eigenvalue problems. *Linear Algebra Appl*. 2012;436:3839-3863.
20. Ivanyshyn Yaman O, Le Louër F. Material derivatives of boundary integral operators in electromagnetism and application to inverse scattering problems. *Inverse Prob*. 2016;32:095003. 24pp

How to cite this article: Cakoni F, Ivanyshyn Yaman O, Kress R, Le Louër F. A boundary integral equation for the transmission eigenvalue problem for Maxwell equation. *Math Meth Appl Sci*. 2018;41:1316–1330. https://doi.org/10.1002/mma.4664