

---

## On the interior transmission eigenvalue problem

---

Fioralba Cakoni

Department of Mathematical Sciences,  
University of Delaware,  
Newark, Delaware 197167, USA  
E-mail: cakoni@math.udel.edu

Andreas Kirsch\*

Department of Mathematics,  
Karlsruhe Institute of Technology (KIT),  
76131 Karlsruhe, Germany  
E-mail: andreas.kirsch@kit.edu

\*Corresponding author

**Abstract:** We consider the transmission eigenvalue problem corresponding to the scattering problem for anisotropic media for both the scalar Helmholtz equation and Maxwell's equations in the case when the contrast in the scattering media occurs in two independent functions. We prove the existence of an infinite discrete set of transmission eigenvalues provided that the two contrasts are of opposite signs. In this case we provide bounds for the first transmission eigenvalue in terms of the ratio of refractive indices. In the case of the same sign contrasts for the scalar case we show the existence of a finite number of transmission eigenvalues under restrictive assumptions on the strength of the scattering media.

**Keywords:** interior transmission problem; transmission eigenvalues; inhomogeneous medium; inverse scattering.

**Reference** to this paper should be made as follows: Cakoni, F. and Kirsch, A. (2010) 'On the interior transmission eigenvalue problem', *Int. J. Computing Science and Mathematics*, Vol. 3, Nos. 1/2, pp.142–167.

**Biographical notes:** F. Cakoni received her PhD Degree in 1996 from the University of Tirana (Albania) and University of Patras (Greece). From 1998 to 2000 she was Alexander Von Humboldt Researcher at the University of Stuttgart (Germany). Since 2000 she has been on the Faculty of the Department of Mathematical Sciences at the University of Delaware, where she became Professor in 2010. Her research focuses on integral equations and variational methods applied to direct and inverse problems in acoustic, electromagnetic and elastic scattering theory, and boundary value problems for partial differential equations. She is the author of the monograph *Qualitative Methods in Inverse Scattering Theory* (Springer, jointly with D. Colton).

A. Kirsch received his PhD Degree in 1978 and the habilitation Degree in 1984 both from the Department of Mathematics at the

University of Göttingen (Germany). From 1986 to 1988 he worked as a Professor at the Geophysics Institute of the University of Göttingen before he accepted a call to a professorship for Applied Mathematics at the University of Erlangen (Germany). Since 1996 he holds a Professorship for Mathematics at the Karlsruhe Institute of Technology (KIT, Germany), the former University of Karlsruhe. His research focuses on integral equations, general inverse problems and, in particular, on direct and inverse scattering problems for acoustic, elastic, and electromagnetic waves. He is author of the monographs *An Introduction to the Mathematical Theory of Inverse Problems* (Springer), *Optimization Methods in Electromagnetic Radiation* (Springer, jointly with T.S. Angell), and *The Factorisation Method for Inverse Problems* (Oxford University Press, jointly with N. Grinberg).

---

## 1 Introduction

The transmission eigenvalue problem is a new class of non-selfadjoint eigenvalue problems that first appeared in inverse scattering theory for inhomogeneous medium. It is a boundary value problem for a set of equations defined in a bounded domain coinciding with the support of the scattering object. In connection with scattering theory, the solution of the transmission eigenvalue problem can be viewed as finding an incident wave such that for a given inhomogeneous medium the scattered field is zero. It can be shown that this can in principle occur for only at most a discrete set of values of the wave number, so-called transmission eigenvalues. In addition, the relationship between the eigenvalue one of the scattering matrix corresponding to the scattering by an inhomogeneous medium and transmission eigenvalues is well known (Colton and Kress, 1998) (this is referred to as the inside-outside duality in some physics literature (Dietz et al., 1995), see also Introduction in Kirsch (2009)). We refer the reader to Cakoni and Colton (2006), Colton et al. (1989), Colton and Kress (1998), Colton and Monk (1988), Kirsch (2007), Kirsch and Grinberg (2008), and Rynne and Sleeman (1992) for more discussion on the relevance of the interior transmission problem to the scattering theory for inhomogeneous medium (for a comprehensive discussion of the interior transmission problem up to the data see the survey paper (Colton et al., 2007)).

The study of transmission eigenvalue problems has recently become an attractive research topic. Although the interior transmission problem was introduced in 1988–1989 and in Colton et al. (1989); Colton and Monk (1988), and until recently the only known result on transmission eigenvalues was the fact that they form at most a discrete set with positive infinity as the only possible accumulation point. The first result about the existence of transmission eigenvalues was published in Päivärinta and Sylvester (2008) for the case of the reduced wave equation in an isotropic inhomogeneous medium where it was shown that there exist a finite number of transmission eigenvalues provided that the index of refraction is large enough. This paper was soon followed by Cakoni and Haddar (2009) and Kirsch (2009) where the same result was proven for anisotropic media and Maxwell's equations. Subsequently the difficult case of a medium with cavities,

i.e., regions with zero contrast, was investigated in Cakoni et al. (2010a). Further progress on the question of the existence of transmission eigenvalues was made in Cakoni and Gintides (2010) where the assumption on the size of the index of refraction was removed. The story was completed in Cakoni et al. (2010b) where the existence of an infinite discrete set of transmission eigenvalues was proven for all the above cases. However, except for some partial results obtained in Cakoni et al. (2002) and Cakoni and Haddar (2001), the investigation of the transmission eigenvalue problem up to now is limited to the case of inhomogeneous medium with one contrast function which is a restrictive model of the scattering of acoustic, electromagnetic or elastic waves by inhomogeneities. The goal of this paper is to extend the investigation of transmission eigenvalue problem for the case of scattering media with two independent refractive indices. In this case, the reduction of transmission eigenvalue problem to a nonlinear eigenvalue problem for a fourth order differential operator is no longer applicable. In this paper, we follow the approach developed in Kirsch (2009).

The plan of our paper is as follows. In Section 2 we formulate the transmission eigenvalue problem for the case of the scalar Helmholtz equation and show the existence of an infinite set of transmission eigenvalues for spherically stratified media with two radial contrasts. Then we proceed with the investigation of the eigenvalue problem for the general inhomogeneous anisotropic media, in which case for technical reasons we distinguish between two possible subclasses. In this regard, in Section 2.1 we discuss the case of a medium with two contrasts of different signs, where we prove the existence of an infinite discrete set of transmission eigenvalues that accumulate only at positive infinity. Furthermore, we provide bounds for the first transmission eigenvalue involving the geometry of the domain and the ratio of two refractive indices. The case of media with contrasts of the same sign is discussed in Section 2.2 where the existence of a finite set of transmission eigenvalues is shown under some assumptions on the refractive indices. Finally, in Section 3 we extend our approach to the case of Maxwell's equations assuming anisotropic electric permittivity but constant magnetic permeability different from that of the background medium. The case of Maxwell's equations with both refractive indices functions of spatial variable is still open.

## 2 The scalar Helmholtz equation

We assume that  $D$  is a bounded connected domain of  $\mathbb{R}^3$ , with Lipschitz boundary  $\partial D$  and denote by  $\nu$  the outward unit normal defined almost everywhere on  $\partial D$  (to fix our ideas we present the analysis in  $\mathbb{R}^3$ , but everything holds true in  $\mathbb{R}^2$  as well). Let  $Q \in L^\infty(D, \mathbb{C}^{3 \times 3})$  be a matrix valued function such that  $Q(x)$  is Hermitean for almost all  $x \in D$ . Furthermore, we assume that there exists  $\alpha > 0$  such that  $\xi \cdot (I + Q)\bar{\xi} \geq \alpha|\xi|^2$  all  $\xi \in \mathbb{C}^3$  and almost everywhere in  $D$ . We also consider a scalar real valued function  $p \in L^\infty(D)$  such that  $1 + p \geq \beta$  on  $D$  for some constant  $\beta > 0$ . The transmission eigenvalue problem reads: find  $(u, w) \in H^1(D) \times H^1(D)$  that satisfies

$$\nabla \cdot [(I + Q)\nabla u] + \lambda(1 + p)u = 0 \quad \text{and} \quad \Delta w + \lambda w = 0 \quad \text{in } D, \quad (2.1)$$

$$u = w \quad \text{and} \quad \nu \cdot (I + Q)\nabla u = \nu \cdot \nabla w \quad \text{on } \partial D. \quad (2.2)$$

The variational form of (2.1)–(2.2) is the following coupled pair of variational equations

$$\iint_D [\nabla w \cdot \nabla \bar{\psi} - \lambda w \bar{\psi}] dx = 0 \quad \text{for all } \psi \in H_0^1(D), \quad (2.3)$$

$$\iint_D [(I + Q)\nabla u \cdot \nabla \bar{\psi} - \lambda(1 + p)u\bar{\psi}] dx = \iint_D [\nabla w \cdot \nabla \bar{\psi} - \lambda w \bar{\psi}] dx \quad (2.4)$$

for all  $\psi \in H^1(D)$ .

**Definition 2.1:** Values of  $\lambda > 0$  for which the transmission eigenvalue problem (2.1)–(2.2) has a nontrivial solution  $(u, w) \in H^1(D) \times H^1(D)$  with  $(u, w) \neq (0, 0)$  are called transmission eigenvalues. The corresponding nonzero solution  $(u, w)$  is called transmission eigenfunction.

**Example 2.2** (The spherically symmetric case): In the case when  $D := B_R$  is a ball of radius  $R$  centred at the origin and both contrasts  $Q := q(r)I$  and  $p := p(r)$  depend only on the radial variable we can directly show that there exists an infinite set of transmission eigenvalues. We assume that  $q \in C^2(\overline{B_R})$  and  $p \in C^2(\overline{B_R})$  are real valued such that  $1 + q > 0$  and  $1 + p > 0$  in  $\overline{B_R}$ . Obviously, if both  $p = 0$  and  $q = 0$  every  $\lambda > 0$  is a transmission eigenvalue (i.e., this corresponds to the case when there is no inhomogeneity and therefore no waves are scattered). To avoid such a situation we assume that

$$\delta := \frac{1}{R} \int_0^R \left( \frac{1 + p(\rho)}{1 + q(\rho)} \right)^{\frac{1}{2}} d\rho \neq 1. \quad (2.5)$$

We restrict our attention to solutions of (2.1)–(2.2) that depends only on  $r = |x|$ . Then clearly  $w$  must be of the form

$$w(x) = a_0 j_0(\sqrt{\lambda}r)$$

where  $j_0$  is the spherical Bessel function of order zero and  $a_0$  is a constant. Next, making the substitution  $u(x) = [1 + q(r)]^{-1/2}U(x)$  we see that the first equation in (2.1) takes the following form

$$\Delta U + \left( \lambda \frac{1 + p(r)}{1 + q(r)} - m(r) \right) U = 0$$

where

$$m(r) = \frac{1}{\sqrt{1 + q(r)}} \Delta \sqrt{1 + q(r)}.$$

Hence, setting

$$u(x) = \frac{b_0}{[1 + q(r)]^{\frac{1}{2}}} \frac{y(r)}{r}$$

where  $b_0$  is a constant, straightforward calculations show that if  $y$  is a solution of

$$y'' + \left( \lambda \frac{1+p(r)}{1+q(r)} - m(r) \right) y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

then  $u$  satisfies the first equation in (2.1). Let us denote by

$$n(r) := \frac{1+p(r)}{1+q(r)}.$$

Following Colton and Kress (1998) (see also (Colton et al., 2007)), in order to solve the above initial value problem for  $y$  we use the Liouville transformation

$$z(\xi) := [n(r)]^{\frac{1}{4}} y(r) \quad \text{where} \quad \xi(r) := \int_0^r [n(\rho)]^{\frac{1}{2}} d\rho$$

which yields the following initial value problem for  $z(\xi)$

$$z'' + [\lambda - p(\xi)]z = 0, \quad z(0) = 0, \quad z'(0) = [n(0)]^{-\frac{1}{4}} \tag{2.6}$$

where

$$p(\xi) := \frac{n''(r)}{4[n(r)]^2} - \frac{5}{16} \frac{[n'(r)]^2}{[n(r)]^3} + \frac{m(r)}{n(r)}.$$

Now exactly in the same way as in Colton and Kress (1998, p. 228) by writing (2.6) as a Volterra integral equation and using the methods of successive approximations, we obtain the following asymptotic behaviour for  $y$

$$y(r) = \frac{1}{\sqrt{\lambda}[n(0)n(r)]^{1/4}} \sin \left( \sqrt{\lambda} \int_0^r [n(\rho)]^{1/2} d\rho \right) + \mathcal{O} \left( \frac{1}{\lambda} \right) \quad \text{and}$$

$$y'(r) = \left[ \frac{n(r)}{n(0)} \right]^{1/4} \cos \left( \sqrt{\lambda} \int_0^r [n(\rho)]^{1/2} d\rho \right) + \mathcal{O} \left( \frac{1}{\sqrt{\lambda}} \right)$$

uniformly on  $[0, R]$ . Applying the boundary conditions (2.2) on  $\partial B_R$ , i.e.,

$$\frac{b_0}{[1+q(R)]^{1/2}} \frac{y(R)}{R} = a_0 j_0(\sqrt{\lambda}R),$$

$$b_0(1+q(R)) \frac{d}{dr} \left( \frac{1}{[1+q(r)]^{1/2}} \frac{y(r)}{r} \right) \Big|_{r=R} = a_0 \frac{d}{dr} j_0(\sqrt{\lambda}r) \Big|_{r=R},$$

we see that a nontrivial solution to (2.1)–(2.2) exists if and only if

$$W(\lambda) = \det \begin{pmatrix} \frac{1}{[1+q(R)]^{1/2}} \frac{y(R)}{R} & j_0(\sqrt{\lambda}R) \\ [1+q(R)] \frac{d}{dr} \left( \frac{1}{[1+q(r)]^{1/2}} \frac{y(r)}{r} \right) \Big|_{r=R} & \sqrt{\lambda} j_0'(\sqrt{\lambda}R) \end{pmatrix} = 0. \tag{2.7}$$

Since  $j_0(\sqrt{\lambda}) = \sin \sqrt{\lambda}r/\sqrt{\lambda}r$ , from the above asymptotic behaviour of  $y(r)$  we have that

$$W(\lambda) = \frac{1}{\sqrt{\lambda}R^2} \left[ A \sin(\delta\sqrt{\lambda}R) \cos(\sqrt{\lambda}R) - B \cos(\delta\sqrt{\lambda}R) \sin(\sqrt{\lambda}R) \right] + \mathcal{O}\left(\frac{1}{\lambda}\right) \quad (2.8)$$

where  $\delta$  is given by (2.5) and

$$A = \frac{1}{[1+q(R)]^{1/2}} \frac{1}{[n(0)n(R)]^{1/4}}, \quad B = [1+q(R)]^{1/2} \left[ \frac{n(R)}{n(0)} \right]^{1/4}.$$

Since  $\delta \neq 1$  the first term in (2.8) is a periodic function if  $\delta$  is rational and almost-periodic (see Katznelson, 1968, Section VI. 5) if  $\delta$  is irrational, therefore taking positive and negative values. This means that for large enough  $\lambda$ ,  $W(\lambda)$  has infinitely many zeros which proves the existence of infinitely many transmission eigenvalues.

In the following we need to consider a particular case of the above spherically stratified media where  $Q := q_0I$  and  $p := p_0$  are both constant such that  $1 + q_0 > 0$  and  $1 + p_0 > 0$ . In this case the interior transmission eigenvalue problem reads as

$$\Delta u + \lambda \frac{1+p_0}{1+q_0} u = 0 \quad \text{in } B_R, \quad (2.9)$$

$$\Delta w + \lambda w = 0 \quad \text{in } B_R, \quad (2.10)$$

$$u = w \quad \text{on } \partial B_R, \quad (2.11)$$

$$(1+q_0) \frac{\partial u}{\partial r} = \frac{\partial w}{\partial r} \quad \text{on } \partial B_R, \quad (2.12)$$

where  $r = |x|$ . To solve (2.9)–(2.12) we make the ansatz

$$w(r, \hat{x}) = a_n j_n(\sqrt{\lambda}r) Y_n^m(\hat{x}), \quad u(r, \hat{x}) = b_n j_n \left( \sqrt{\lambda \frac{1+p_0}{1+q_0}} r \right) Y_n^m(\hat{x})$$

where  $j_n$  are spherical Bessel functions of order  $n$ ,  $Y_n^m$  are the spherical harmonics and  $\hat{x} = x/r$ . Then the corresponding transmission eigenvalues are zeros of the following determinants

$$W(\lambda) = \det \begin{pmatrix} j_n(\sqrt{\lambda}R) & j_n \left( \sqrt{\lambda \frac{1+p_0}{1+q_0}} R \right) \\ \sqrt{\lambda} j_n'(\sqrt{\lambda}R) & \sqrt{\lambda(1+p_0)(1+q_0)} j_n' \left( \sqrt{\lambda \frac{1+p_0}{1+q_0}} R \right) \end{pmatrix}. \quad (2.13)$$

□

Our main concern in this paper is to prove the existence of transmission eigenvalues for the general case. To this end we set  $v = w - u$ . Then  $v \in H_0^1(D)$  and if  $(u, w)$  satisfies (2.1)–(2.2), subtracting the equation for  $u$  in (2.1) from the equation for  $w$  in (2.1) we obtain that  $v$  satisfies

$$\begin{aligned} \nabla \cdot [(I+Q)\nabla v] + \lambda(1+p)v &= \nabla \cdot [Q\nabla w] + \lambda p w \quad \text{in } D, \\ \nu \cdot (I+Q)\nabla v &= \nu \cdot Q\nabla w \quad \text{on } \partial D. \end{aligned}$$

The weak formulation of the above problem is  $v \in H_0^1(D)$  such that

$$\begin{aligned} & \iint_D [Q \nabla w \cdot \nabla \bar{\psi} - \lambda p w \bar{\psi}] dx \\ &= \iint_D [(I + Q) \nabla v \cdot \nabla \bar{\psi} - \lambda(1 + p)v \bar{\psi}] dx \end{aligned} \quad (2.14)$$

for all  $\psi \in H^1(D)$ . For a given  $v \in H_0^1(D)$  we can define  $w := w_v$  by considering the Neumann boundary value problem (2.14) for  $w$ . In order to analyse the above Neuman problem for  $w$ , in the following we use the following bounds of  $Q$  and  $p$ :

$$q^* = \sup_{x \in D} \sup_{\xi \in \mathbb{C}^3, |\xi|=1} (\bar{\xi} \cdot Q(x)\xi), \quad q_* = \inf_{x \in D} \inf_{\xi \in \mathbb{C}^3, |\xi|=1} (\bar{\xi} \cdot Q(x)\xi), \quad (2.15)$$

$$p^* = \sup_{x \in D} p(x) \quad \text{and} \quad p_* = \inf_{x \in D} p(x). \quad (2.16)$$

Note that from the assumption at the beginning of this section we have that  $1 + q_* > 0$  and  $1 + p_* > 0$ . For reasons that will become clear later, we need to distinguish between two cases, namely  $Q$  and  $p$  have opposite sign and  $Q$  and  $p$  have the same sign.

### 2.1 The case with contrast $Q$ and $p$ of opposite sign

In this section we consider the case when  $Q$  and  $p$  have opposite sign, more precisely either  $q_* > 0$  and  $p^* < 0$ , or  $q^* < 0$  and  $p_* > 0$ . For the corresponding scattering problem this means that either the contrast in the scattering medium represented by the matrix  $Q$  is positive and the contrast represented by  $p$  is negative, or the other way around (note that the corresponding parameters of the background media here are assumed to be equal to one).

**Lemma 2.3:** *Assume that either  $q_* > 0$  and  $p^* < 0$ , or  $q^* < 0$  and  $p_* > 0$ . Then there exists  $\delta > 0$  such that for every  $v \in H_0^1(D)$  and  $\lambda \in \mathbb{C}$  with  $\text{Re}\lambda > -\delta$  there exists a unique solution  $w := w_v \in H^1(D)$  of (2.14). The operator  $A_\lambda : H_0^1(D) \rightarrow H^1(D)$ , defined by  $v \mapsto w_v$ , is bounded and depends analytically on  $\lambda \in \{z \in \mathbb{C} : \text{Re}z > -\delta\}$ .*

*Proof:* First we note a difference between the cases  $\lambda = 0$  and  $\lambda \neq 0$ . Setting  $\psi = 1$  in (2.14) we obtain as a necessary condition that

$$\lambda \iint_D p w dx = \lambda \iint_D (1 + p) v dx,$$

i.e.,  $\int_D p w dx = \int_D (1 + p) v dx$  in the case  $\lambda \neq 0$ . In the case  $\lambda = 0$ , however, the solution  $w$  is only unique up to a constant which we choose such that this equality holds as well.

Therefore, we make an ansatz for the solution in the form  $w = \tilde{w} + c$  where  $c$  is constant and  $\tilde{w} \in \tilde{H}^1(D)$  where

$$\tilde{H}^1(D) = \left\{ \psi \in H^1(D) : \iint_D p \psi dx = 0 \right\} \quad (2.17)$$

equipped with the  $H^1(D)$  norm. Let us denote by  $\mu > 0$  the constant which satisfies

$$\mu = \inf_{\psi \in \tilde{H}^1(D)} \frac{\|\nabla\psi\|_{L^2(D)}^2}{\|\psi\|_{L^2(D)}^2}. \quad (2.18)$$

By standard arguments (of Poincaré type) one shows that  $\mu$  is positive. The definition of  $\mu$  yields

$$\frac{\mu}{\mu + 1} \|\psi\|_{H^1(D)}^2 \leq \|\nabla\psi\|_{L^2(D)}^2 \leq \|\psi\|_{H^1(D)}^2 \quad (2.19)$$

for all  $\psi \in \tilde{H}^1(D)$ , i.e.,  $\|\nabla\psi\|_{L^2(D)}$  is an equivalent norm in  $\tilde{H}^1(D)$ .

Substituting  $w = \tilde{w} + c$  into  $\int_D pw \, dx = \int_D (1+p)v \, dx$  and using  $\int_D p\tilde{w} \, dx = 0$  yields

$$c = \frac{1}{\int_D p \, dx} \int_D (1+p)v \, dx. \quad (2.20)$$

In particular,  $c$  is independent of  $\lambda$ . Substituting the form  $w = \tilde{w} + c$  into (2.14) yields

$$\int_D [Q\nabla\tilde{w} \cdot \nabla\bar{\psi} - \lambda p\tilde{w}\bar{\psi}] \, dx = \int_D [(I+Q)\nabla v \cdot \nabla\bar{\psi} - \lambda(1+p)v\bar{\psi}] \, dx \quad (2.21)$$

for all  $\psi \in \tilde{H}^1(D)$ . Let  $\sigma = 1$  if  $q_* > 0$  and  $p^* < 0$  holds, and  $\sigma = -1$  if  $q^* < 0$  and  $p_* > 0$  holds. Furthermore, let us denote by  $\mathcal{A}_\lambda(\tilde{w}, \psi)$  the left hand side of (2.21) multiplied by  $\sigma$ . Hence we have that

$$\begin{aligned} \operatorname{Re} \mathcal{A}_\lambda(\psi, \psi) &= \sigma \int_D [Q\nabla\psi \cdot \nabla\bar{\psi} - (\Re\lambda)p|\psi|^2] \, dx \\ &\geq \min(|q^*|, |q_*|) \|\nabla\psi\|_{L^2(D)}^2 - \delta \max(|p^*|, |p_*|) \|\psi\|_{L^2(D)}^2 \\ &\geq \left[ \frac{\mu}{\mu + 1} \min(|q^*|, |q_*|) - \delta \max(|p^*|, |p_*|) \right] \|\psi\|_{H^1(D)}^2 \end{aligned} \quad (2.22)$$

for all  $\psi \in \tilde{H}^1(D)$  where we have used (2.19). Therefore,  $\mathcal{A}_\lambda(\cdot, \cdot)$  is coercive for sufficiently small  $\delta > 0$  with lower bound which is independent of  $\lambda$ . Consequently, there exists a unique solution  $\tilde{w} \in \tilde{H}^1(D)$  of (2.21) which depends continuously on  $v$ . Furthermore,  $w = \tilde{w} + c$  satisfies (2.14) because of the definition of  $c$ . Therefore, we conclude that the bounded linear operator  $A_\lambda : H_0^1(D) \rightarrow H^1(D)$  which maps  $v$  to the unique solution  $w$  of (2.14) is well defined and depends analytically on  $\lambda$ .  $\square$

We set again  $w_v = A_\lambda v$  and denote by  $L_\lambda v \in H_0^1(D)$  the unique Riesz representation of the bounded conjugate-linear functional

$$\psi \longrightarrow \int_D [\nabla w_v \cdot \nabla\bar{\psi} - \lambda w_v \bar{\psi}] \, dx \quad \text{for } \psi \in H_0^1(D),$$



i.e.,

$$(L_\lambda v, \psi)_{H^1(D)} = \iint_D [\nabla w_v \cdot \nabla \bar{\psi} - \lambda w_v \bar{\psi}] dx \quad \text{for } \psi \in H_0^1(D). \quad (2.23)$$

Then also  $L_\lambda$  depends analytically on  $\lambda \in \{z \in \mathbb{C} : \Re z > -\delta\}$ . Now we are able to connect a transmission eigenfunction, i.e., a nontrivial solution  $(u, w)$  of (2.1)–(2.2), to the kernel of the operator  $L_\lambda$ .

**Theorem 2.4:**

- (a) *Let  $(u, w) \in H^1(D) \times H^1(D)$  be a transmission eigenfunction corresponding to some (real)  $\lambda > 0$ . Then  $v = w - u \in H_0^1(D)$  solves  $L_\lambda v = 0$ .*
- (b) *Let  $v \in H_0^1(D)$  satisfy  $L_\lambda v = 0$  for some (real)  $\lambda > 0$ . Furthermore, let  $w = w_v = A_\lambda v \in H^1(D)$  be as in the construction of  $A_\lambda$  in Lemma 2.3, i.e., the solution of (2.14). Then  $(u, w) \in H^1(D) \times H^1(D)$  is a transmission eigenfunction where  $u = w - v$ .*

*Proof:* (a) Formula (2.3) implies that  $(L_\lambda v, \psi)_{H^1(D)} = 0$  for all  $\psi \in H_0^1(D)$  which means that  $L_\lambda v = 0$ .

(b) Next let  $L_\lambda v = 0$ , i.e.,

$$\iint_D [\nabla w_v \cdot \nabla \bar{\psi} - \lambda w_v \bar{\psi}] dx = 0 \quad \text{for } \psi \in H_0^1(D)$$

which means that  $w = w_v$  solves the Helmholtz equation in  $D$ . With  $u := w - v$  the Cauchy data of  $w$  and  $u$  coincides. Finally, the equation (2.4) for  $u$  follows from (2.14).  $\square$

**Theorem 2.5:**

- (a) *The operator  $L_\lambda : H_0^1(D) \rightarrow H_0^1(D)$  is selfadjoint for all  $\lambda \in \mathbb{R}_{\geq 0}$ .*
- (b) *Let  $\sigma = 1$  if  $q_* > 0$  and  $p_* < 0$ , and  $\sigma = -1$  if  $q_* < 0$  and  $p_* > 0$ . Then  $\sigma L_0 : H_0^1(D) \rightarrow H_0^1(D)$  is coercive, i.e.,  $(\sigma L_0 v, v)_{H^1(D)} \geq c \|v\|_{H^1(D)}^2$  for all  $v \in H_0^1(D)$  and  $c > 0$  independent of  $v$ .*
- (c)  *$L_\lambda - L_0$  is compact in  $H_0^1(D)$ .*
- (d) *There exists at most a countable number of real  $\lambda > 0$  for which  $L_\lambda$  fails to be one-to-one, i.e., the set of transmission eigenvalues is discrete, and infinity is the only possible accumulation point.*

*Proof:* (a) First we show that  $L_\lambda$  is selfadjoint for all  $\lambda \in \mathbb{R}_{\geq 0}$ . To this end for every  $v_1, v_2 \in H_0^1(D)$  let  $w_1 := w_{v_1}$  and  $w_2 := w_{v_2}$  be the corresponding solution of (2.14). Then we have that

$$(L_\lambda v_1, v_2)_{H^1(D)} = \iint_D [\nabla w_1 \cdot \nabla \bar{v}_2 - \lambda w_1 \bar{v}_2] dx$$

$$\begin{aligned}
 &= \iint_D [(I + Q) \nabla w_1 \cdot \nabla \bar{v}_2 - \lambda(1 + p) w_1 \bar{v}_2] dx \\
 &\quad - \iint_D [Q \nabla w_1 \cdot \nabla \bar{v}_2 - \lambda p w_1 \bar{v}_2] dx.
 \end{aligned} \tag{2.24}$$

Using now (2.14) twice, first for  $v = v_2$  and corresponding  $w = w_2$  and  $\psi = w_1$ , and then for  $v = v_1$ , corresponding  $w = w_1$  and  $\psi = v_2$ , yields

$$\begin{aligned}
 (L_\lambda v_1, v_2)_{H^1(D)} &= \iint_D [Q \nabla w_1 \cdot \nabla \bar{w}_2 - \lambda p w_1 \bar{w}_2] dx \\
 &\quad - \iint_D [(I + Q) \nabla v_1 \cdot \nabla \bar{v}_2 - \lambda(1 + p) v_1 \bar{v}_2] dx
 \end{aligned} \tag{2.25}$$

which is a selfadjoint expression for  $v_1$  and  $v_2$ .

(b) Next we show that  $\sigma L_0 : H_0^1(D) \rightarrow H_0^1(D)$  is a coercive operator. Using the definition of  $L_0$  in (2.23) and the fact that  $w = w_v = v + u$  we have

$$(L_0 v, v)_{H^1(D)} = \iint_D \nabla w \cdot \nabla \bar{v} dx = \iint_D |\nabla v|^2 dx + \iint_D \nabla u \cdot \nabla \bar{v} dx. \tag{2.26}$$

From (2.14) for  $\lambda = 0$  and  $\psi = u$  we have that

$$\iint_D \nabla u \cdot \nabla \bar{v} dx = \iint_D Q \nabla u \cdot \nabla \bar{u} dx. \tag{2.27}$$

If  $q_* > 0$  then we have  $\iint_D Q \nabla u \cdot \nabla \bar{u} dx \geq q_* \|\nabla u\|_{L^2(D)}^2 \geq 0$  and hence

$$(L_0 v, v)_{H^1(D)} \geq \iint_D |\nabla v|^2 dx.$$

From Poincaré's inequality in  $H_0^1(D)$  we have that  $\|\nabla v\|_{L^2(D)}$  is an equivalent norm in  $H_0^1(D)$  and this proves the coercivity of  $L_0$ .

Now we assume  $q_* < 0$ . From (2.25) with  $v_1 = v_2 = v$  and  $\lambda = 0$  we have

$$\begin{aligned}
 -(L_0 v, v)_{H^1(D)} &= - \iint_D Q \nabla w \cdot \nabla \bar{w} dx + \iint_D (I + Q) \nabla v \cdot \nabla \bar{v} dx \\
 &\geq (1 + q_*) \iint_D |\nabla v|^2 dx
 \end{aligned} \tag{2.28}$$

which proves again the coercivity of  $-L_0$  since  $1 + q_* > 0$ .

Part (c) of the theorem follows from the compact embedding of  $H_0^1(D)$  into  $L^2(D)$ . We omit the proof here and include the proof of a similar result for the more complicated case of Maxwell's equations in Section 3.

(d) Since  $(\sigma L_0)^{-1}$  exists and  $\lambda \mapsto L_\lambda$  is analytic on  $\{z \in \mathbb{C} : \operatorname{Re} z > -\delta\}$ , this follows directly from the analytic Fredholm theory, see Colton and Kress (1998).  $\square$

Now we are ready to prove the existence of infinitely many (real) transmission eigenvalues, i.e., the existence of a sequence of  $\lambda_j \in \mathbb{R}$ ,  $j \in \mathbb{N}$ , and corresponding  $v_j \in H_0^1(D)$  such that  $v_j \neq 0$  and  $L_{\lambda_j} v_j = 0$ . From now on we restrict ourselves to real and positive  $\lambda$ . Note that since  $\sigma L_0 : H_0^1(D) \rightarrow H_0^1(D)$  is a positive definite operator the kernel of  $L_\lambda$  coincides with the kernel of  $I + (\sigma L_0)^{-1/2} C_\lambda (\sigma L_0)^{-1/2}$  where  $C_\lambda = \sigma(L_\lambda - L_0)$  is compact. Letting  $T_\lambda := -(\sigma L_0)^{-1/2} C_\lambda (\sigma L_0)^{-1/2}$ , it is known that the compact and selfadjoint operator  $T_\lambda$  has an infinite sequence of eigenvalues  $\mu_j(\lambda)$ ,  $j \in \mathbb{N}$  such that  $\mu_j(\lambda) \rightarrow +\infty$  as  $j \rightarrow +\infty$  that can be ordered in increasing order. Furthermore, they satisfy the max-min principle

$$\mu_j(\lambda) = \sup_{W \subset \mathcal{W}_j} \inf_{u \in W \setminus \{0\}} \frac{(T_\lambda u, u)_{H^1(D)}}{\|u\|_{H^1(D)}} \tag{2.29}$$

where  $\mathcal{W}_j$  denotes the set of all  $j$ -dimensional subspaces  $W$  of  $H_0^1(D)$ . From the max-min principle we conclude that  $\mu_j$ ,  $j \in \mathbb{N}$ , are continuous with respect to  $\lambda$  on  $[0, \infty)$ . Furthermore, by the above discussion  $\lambda$  is a transmission eigenvalue if and only if  $\mu_j(\lambda) = 1$  for some  $j \in \mathbb{N}$ .

**Remark:** The multiplicity of a transmission eigenvalue is finite. Indeed if  $\lambda$  is a transmission eigenvalue then, since  $\sigma L_0$  is a positive definite operator we have that 1 is an eigenvalue of the compact selfadjoint operator  $T_\lambda := -(\sigma L_0)^{-1/2} C_\lambda (\sigma L_0)^{-1/2}$ . This means that the kernel of  $I + (\sigma L_0)^{-1/2} C_\lambda (\sigma L_0)^{-1/2}$  is finite dimensional and so is the kernel of  $L_\lambda$ .

Making use of the above discussion, the proof of the existence of transmission eigenvalues is now based on the following theorem (see e.g. Päivärinta and Sylvester (2008) for the proof):

**Theorem 2.6:** *Let  $L_\lambda : H_0^1(D) \rightarrow H_0^1(D)$  be defined as in (2.23) and let  $\sigma = 1$  if  $q_* > 0$  and  $p^* < 0$  holds, and  $\sigma = -1$  if  $q^* < 0$  and  $p_* > 0$  holds. Assume that:*

- 1 *there is a  $\lambda_0 \geq 0$  such that  $\sigma L_{\lambda_0}$  is positive on  $H_0^1(D)$  and*
- 2 *there is a  $\lambda_1 > \lambda_0$  such that  $\sigma L_{\lambda_1}$  is non positive on some  $m$ -dimensional subspace  $W_m$  of  $H_0^1(D)$ .*

*Then there are  $m$  transmission eigenvalues in  $[\lambda_0, \lambda_1]$  counting their multiplicity.*

Using now Theorem 2.6 and adapting the ideas developed in Cakoni et al. (2010b) and Cakoni and Gintides (2010), we are ready to prove the main theorem of this section. We recall the notations of  $q^*$ ,  $q_*$ ,  $p^*$ , and  $p_*$  from (2.15) and (2.16).

**Theorem 2.7:** *Suppose that the matrix function  $Q$  and the function  $p$  are such that either  $q_* > 0$  and  $p^* < 0$ , or  $q^* < 0$  and  $p_* > 0$ . Then there exists an infinite sequence of transmission eigenvalues  $\lambda_j$  with  $+\infty$  as their only accumulation point*

*Proof:* Let us first assume that  $q_* > 0$  and  $p^* < 0$  (i.e.,  $\sigma = 1$  in Theorem 2.6). First, we recall that the assumption (1) of Theorem 2.6 is satisfied with  $\lambda_0 = 0$  i.e.,

$(L_0 v, v)_{H^1(D)} > 0$  for all  $v \in H_0^1(D)$  with  $v \neq 0$ . Next, by definition of  $L_\lambda$  and the fact that  $w = u + v$  have

$$\begin{aligned} (L_\lambda v, v)_{H^1(D)} &= \iint_D [\nabla w \cdot \nabla \bar{v} - \lambda w \bar{v}] dx \\ &= \iint_D [\nabla u \cdot \nabla \bar{v} - \lambda u \bar{v} + |\nabla v|^2 - \lambda |v|^2] dx. \end{aligned} \quad (2.30)$$

We also have that  $u$  satisfies

$$\iint_D [Q \nabla u \cdot \nabla \bar{\psi} - \lambda p u \bar{\psi}] dx = \iint_D [\nabla v \cdot \nabla \bar{\psi} - \lambda v \bar{\psi}] dx \quad (2.31)$$

for all  $\psi \in H^1(D)$ . Now taking  $\psi = u$  in (2.31) and plugging the result into (2.30) yields

$$(L_\lambda v, v)_{H^1(D)} = \iint_D [Q \nabla u \cdot \nabla \bar{u} - \lambda p |u|^2 + |\nabla v|^2 - \lambda |v|^2] dx. \quad (2.32)$$

Let now  $B_r \subset D$  be an arbitrary ball of radius  $r$  included in  $D$  and let  $\hat{\lambda}$  be a transmission eigenvalue corresponding to the ball  $B_r$  with constant contrasts  $q_*$  and  $p^*$ . Let  $\hat{w}, \hat{u}$  be the non-zero solutions to the corresponding homogenous interior transmission problem, i.e the solution of (2.9)–(2.12) with  $q_0 := q_*$ ,  $p_0 := p^*$  and  $R = r$ , and set  $\hat{v} := \hat{w} - \hat{u} \in H_0^1(B_r)$ . We denote the corresponding operator by  $\hat{L}_\lambda$ . Of course, by construction we have that (2.32) holds for this situation as well, i.e., since  $\hat{L}_\lambda \hat{v} = 0$ ,

$$0 = (\hat{L}_\lambda \hat{v}, \hat{v})_{H^1(B_r)} = \iint_{B_r} [q_* |\nabla \hat{u}|^2 - \hat{\lambda} p^* |\hat{u}|^2 + |\nabla \hat{v}|^2 - \hat{\lambda} |\hat{v}|^2] dx. \quad (2.33)$$

Next we denote by  $\tilde{v} \in H_0^1(D)$  the extension of  $\hat{v} \in H_0^1(B_r)$  by zero to the whole of  $D$  and let  $\tilde{w} := w_{\tilde{v}}$  be the corresponding solution to (2.14) and  $\tilde{u} := \tilde{w} - \tilde{v}$ . In particular  $\tilde{u} \in H^1(D)$  satisfies

$$\begin{aligned} \iint_D [Q \nabla \tilde{u} \cdot \nabla \bar{\psi} - \hat{\lambda} p \tilde{u} \bar{\psi}] dx &= \iint_D [\nabla \tilde{v} \cdot \nabla \bar{\psi} - \hat{\lambda} \tilde{v} \bar{\psi}] dx \\ &= \iint_{B_r} [\nabla \hat{v} \cdot \nabla \bar{\psi} - \hat{\lambda} \hat{v} \bar{\psi}] dx \\ &= \iint_{B_r} [q_* \nabla \hat{u} \cdot \nabla \bar{\psi} - \hat{\lambda} p^* \hat{u} \bar{\psi}] dx \end{aligned} \quad (2.34)$$

for all  $\psi \in H^1(D)$ . Therefore, for  $\psi = \tilde{u}$  we have by the Cauchy-Schwarz inequality,

$$\begin{aligned} \iint_D [Q \nabla \tilde{u} \cdot \nabla \bar{\tilde{u}} - \hat{\lambda} p |\tilde{u}|^2] dx &= \iint_{B_r} [q_* \nabla \hat{u} \cdot \nabla \bar{\tilde{u}} + \hat{\lambda} p^* |\hat{u} \tilde{u}|] dx \\ &\leq \left[ \iint_{B_r} [q_* |\nabla \hat{u}|^2 + \hat{\lambda} p^* |\hat{u}|^2] dx \right]^{1/2} \left[ \iint_{B_r} [q_* |\nabla \tilde{u}|^2 + \hat{\lambda} p^* |\tilde{u}|^2] dx \right]^{1/2} \\ &\leq \left[ \iint_{B_r} [q_* |\nabla \hat{u}|^2 - \hat{\lambda} p^* |\hat{u}|^2] dx \right]^{1/2} \left[ \iint_D [Q \nabla \tilde{u} \cdot \nabla \bar{\tilde{u}} - \hat{\lambda} p |\tilde{u}|^2] dx \right]^{1/2} \end{aligned}$$

since  $|p| = -p \geq -p^* = |p^*|$  and thus

$$\iint_D [Q \nabla \tilde{u} \cdot \nabla \bar{u} - \hat{\lambda} p |\tilde{u}|^2] dx \leq \iint_{B_r} [q_* |\nabla \hat{u}|^2 - \hat{\lambda} p^* |\hat{u}|^2] dx.$$

Substituting this into (2.32) for  $\lambda = \hat{\lambda}$  and  $v = \tilde{v}$  yields

$$\begin{aligned} (L_{\hat{\lambda}} \tilde{v}, \tilde{v})_{H^1(D)} &= \iint_D [Q \nabla \tilde{u} \cdot \nabla \bar{u} - \hat{\lambda} p |\tilde{u}|^2 + |\nabla \tilde{v}|^2 - \hat{\lambda} |\tilde{v}|^2] dx \\ &\leq \iint_{B_r} [q_* |\nabla \hat{u}|^2 - \hat{\lambda} p^* |\hat{u}|^2 + |\nabla \hat{v}|^2 - \hat{\lambda} |\hat{v}|^2] dx = 0 \end{aligned} \quad (2.35)$$

by (2.33). Hence from Theorem 2.6 we have that there is a transmission eigenvalue  $\lambda$  in  $(0, \hat{\lambda}]$ . Now we fix an arbitrary  $m \in \mathbb{N}$  and take  $\epsilon > 0$  small enough such that  $D$  contains  $m$  disjoint balls  $B_\epsilon^1, B_\epsilon^2, \dots, B_\epsilon^m$  of radius  $\epsilon$ . Let  $\hat{\lambda}_\epsilon$  be a transmission eigenvalue for each of these balls with  $q_*$  and  $p^*$ . Let  $\hat{w}_\epsilon^j, \hat{u}_\epsilon^j$  be the non-zero solutions of the corresponding homogeneous interior transmission problem and  $\tilde{v}_\epsilon^j := \hat{w}_\epsilon^j - \hat{u}_\epsilon^j \in H_0^1(B_\epsilon^j)$ ,  $j = 1, \dots, m$ . Let now  $\tilde{v}_\epsilon^j \in H_0^1(D)$  be the extension by zero to the whole of  $D$  of  $\hat{v}_\epsilon^j \in H_0^1(B_\epsilon^j)$ . Note that  $\{\tilde{v}_\epsilon^1, \tilde{v}_\epsilon^2, \dots, \tilde{v}_\epsilon^m\}$  are linearly independent and orthogonal in  $H_0^1(D)$  since they have disjoint supports. From the argument above we have that  $L_{\hat{\lambda}_\epsilon}$  is non positive on the  $m$  dimensional subspace of  $H_0^1(D)$  spanned by  $\{\tilde{v}_\epsilon^1, \tilde{v}_\epsilon^2, \dots, \tilde{v}_\epsilon^m\}$ . Hence there exist  $m$  transmission eigenvalues in  $(0, \hat{\lambda}_\epsilon]$  counting their multiplicity. Since  $m$  was arbitrary this part of the theorem is proved.

The case of  $q^* < 0$  and  $p_* > 0$  can be treated in the same way if we consider  $-L_\lambda$  and  $q_0 := q^*$ ,  $p_0 := p_*$  in place of  $L_\lambda$  and  $q_0 := q_*$ ,  $p_0 := p^*$ .  $\square$

We can obtain a better lower bound for transmission eigenvalues. To this end we first assume that  $q_* > 0$  and  $p^* < 0$  and consider (2.32) again, i.e.,

$$(L_\lambda v, v)_{H^1(D)} = \iint_D [Q \nabla u \cdot \nabla \bar{u} - \lambda p |u|^2 + |\nabla v|^2 - \lambda |v|^2] dx.$$

The first term is estimated by

$$\iint_D [Q \nabla u \cdot \nabla \bar{u} - \lambda p |u|^2] dx \geq \min(q_*, \lambda |p^*|) \|u\|_{H^1(D)}^2 \geq 0$$

and, since  $v \in H_0^1(D)$ , we have that  $\|\nabla v\|_{L^2(D)}^2 \geq \Lambda_1(D) \|v\|_{L^2(D)}^2$  where  $\Lambda_1(D)$  is the first Dirichlet eigenvalue of  $-\Delta$  in  $D$ . Therefore,  $(L_\lambda v, v)_{H^1(D)} > 0$  as long as  $\lambda < \Lambda_1(D)$ . Thus, we can conclude that all transmission eigenvalues  $\lambda$  are such that  $\lambda \geq \Lambda_1(D)$ .

Next we consider  $q^* < 0$  and  $p_* > 0$  and from (2.25) since  $w = u + v$

$$\begin{aligned} -(L_\lambda v, v)_{H^1(D)} &= \iint_D [(-Q)(\nabla u + \nabla v) \cdot (\nabla \bar{u} + \nabla \bar{v}) + \lambda p |u + v|^2] dx \\ &\quad + \iint_D [(I + Q) \nabla v \cdot \nabla \bar{v} - \lambda(1 + p) |v|^2] dx. \end{aligned}$$

In this case

$$\begin{aligned} & \iint_D [(-Q)(\nabla u + \nabla v) \cdot (\nabla \bar{u} + \nabla \bar{v}) + \lambda p |u + v|^2] dx \\ & \geq \min(|q^*|, \lambda p_*) \|u + v\|_{H^1(D)}^2 \geq 0 \end{aligned}$$

whereas

$$\iint_D [(I + Q)\nabla v \cdot \nabla \bar{v} - \lambda(1 + p)|v|^2] dx \geq [(1 + q_*)\Lambda_1(D) - \lambda(1 + p^*)] \|v\|_{L^2}^2.$$

Hence  $0 < \lambda < \frac{1+q_*}{1+p^*} \Lambda_1(D)$  are no transmission eigenvalues. Therefore all transmission eigenvalues satisfy  $\lambda \geq \frac{1+q_*}{1+p^*} \Lambda_1(D)$ , where again  $\Lambda_1(D)$  is the first Dirichlet eigenvalue of  $-\Delta$  in  $D$ . From the above discussion and the proof of Theorem 2.7 we have the following lower and upper bounds for the first transmission eigenvalue.

**Theorem 2.8:** *Let  $B_R \subset D$  be the largest ball contained in  $D$ . Let  $\lambda_1(D, Q, p)$  be the first transmission eigenvalue corresponding to (2.1)–(2.2).*

(1) *If  $q_* > 0$  and  $p^* < 0$  then*

$$\Lambda_1(D) \leq \lambda_1(D, Q, p) \leq \lambda_1(B_R, q_*, p^*)$$

*where  $\lambda_1(B_R, q_*, p^*)$  is the first transmission eigenvalue corresponding to the ball  $B_R$  with contrast  $q_0 := q_*$  and  $p_0 := p^*$  in (2.9)–(2.12).*

(2) *If  $q^* < 0$  and  $p_* > 0$  then*

$$\frac{1 + q_*}{1 + p^*} \Lambda_1(D) \leq \lambda_1(D, Q, p) \leq \lambda_1(B_R, q^*, p_*)$$

*where  $\lambda_1(B_R, q^*, p_*)$  is the first transmission eigenvalue corresponding to the ball  $B_R$  with contrast  $q_0 := q^*$  and  $p_0 := p_*$  in (2.9)–(2.12).*

## 2.2 The case with contrast $Q$ and $p$ of the same sign

Now we turn our attention to the case when  $Q$  and  $p$  have the same sign. The interior transmission problem for this case has been studied in Cakoni et al. (2002) and Cakoni and Haddar (2001). In particular there it is shown that transmission eigenvalues form at most a discrete set with  $+\infty$  as the only possible accumulation point. Here our main concern is to show the existence of transmission eigenvalues. To this regard we limit ourselves to the case when both contrasts  $Q$  and  $p$  are positive, i.e.,  $q_* > 0$  and  $p_* > 0$ . We follow the same procedure as in Section 2.1. In particular, for a given  $v \in H_0^1(D)$  we need to solve the Neumann problem for  $w := w_v$  given by (2.14) in the weak formulation. Unfortunately, this problem is not solvable for all  $\lambda$  which forces us to put restrictions on  $\lambda$ ,  $Q$  and  $p$ . Therefore, in the following we prove the existence of at least one transmission eigenvalue under restrictive assumptions on  $Q$  and  $p$ . Let  $B_r \subset D$  be a ball of radius

$r$  included in  $D$  and set  $\hat{\lambda} = \lambda_1(q_*/2, B_r)$  to be the first transmission eigenvalue of (2.9)–(2.12) with  $R = r$ ,  $q_0 := q_*/2$ , and  $p_0 := 0$ . Furthermore we require that  $p^* > 0$  is small enough such that

$$p^* < \frac{\mu}{2\hat{\lambda}}q_* \tag{2.36}$$

with  $\hat{\lambda} = \lambda_1(q_*/2, B_r)$  and  $\mu$  from (2.18). We can now prove a result analogue to Lemma 2.3.

**Lemma 2.9:** *For every  $v \in H_0^1(D)$  and  $\lambda \in \mathbb{C}$  with  $\text{Re}\lambda < \hat{\lambda}$  there exists a unique solution  $w := w_v \in H^1(D)$  of (2.14). The operator  $A_\lambda : H_0^1(D) \rightarrow H^1(D)$ , defined by  $v \mapsto w_v$ , is bounded and depends analytically on  $\lambda \in \{z \in \mathbb{C} : \text{Re}z < \hat{\lambda}\}$ .*

*Proof:* We proceed exactly in the same way as in Lemma 2.3 to look for the solution in the form  $w = \tilde{w} + c$  where  $\tilde{w} \in \tilde{H}^1(D)$  solves (2.21) and the constant  $c$  is given by (2.20). Denoting the left hand side of (2.21) again by  $\mathcal{A}_\lambda(\tilde{w}, \psi)$  and using (2.18) and (2.36) we have that

$$\begin{aligned} \text{Re}\mathcal{A}_\lambda(\psi, \psi) &= \iint_D [Q\nabla\psi \cdot \nabla\bar{\psi} - (\text{Re}\lambda)p|\psi|^2] dx \\ &\geq q_*\|\nabla\psi\|_{L^2(D)}^2 - \frac{p^*\hat{\lambda}}{\mu}\|\nabla\psi\|_{L^2(D)}^2 \\ &\geq \left[ q_* - \frac{p^*\hat{\lambda}}{\mu} \right] \|\nabla\psi\|_{L^2(D)}^2 \geq \frac{q_*}{2} \frac{\mu}{\mu+1} \|\psi\|_{H^1(D)}^2 \end{aligned} \tag{2.37}$$

for all  $\psi \in \tilde{H}^1(D)$ . Therefore, we have that  $\mathcal{A}_\lambda$  is coercive and there exists a unique solution  $\tilde{w} \in \tilde{H}^1(D)$  which depends continuously on  $v$ . The rest of the proof continues the same way as in Lemma 2.3.  $\square$

Now we can define the operator  $L_\lambda : H_0^1(D) \rightarrow H_0^1(D)$  for  $\lambda \in \{z \in \mathbb{C} : \text{Re}z < \hat{\lambda}\}$  by (2.23). Obviously, Theorem 2.4 is valid for  $L_\lambda$  in the current case as long as  $\lambda \in [0, \hat{\lambda})$ . Furthermore, exactly in same way as Theorem 2.5 we can prove the following two theorems.

**Theorem 2.10:** *Let again  $\hat{\lambda} = \lambda_1(q_*/2, B_r)$  be the first transmission eigenvalue of (2.9)–(2.12) with  $R = r$ ,  $q_0 := q_*/2$ , and  $p_0 := 0$ . Then  $L_\lambda$  is selfadjoint for all  $\lambda \in [0, \hat{\lambda})$  and is of the form  $L_0 - C_\lambda$  where  $C_\lambda$  is compact and  $L_0$  is coercive on  $H_0^1(D)$ . In particular,  $(L_0v, v)_{H^1(D)} \geq \|\nabla v\|_{L^2(D)}^2 \geq c\|v\|_{H^1(D)}^2$  for all  $v \in H_0^1(D)$ .*

We can now use Theorem 2.6 to prove that there exists at least one transmission eigenvalue  $\lambda \in (0, \hat{\lambda})$  under the assumptions stated at the beginning of this section. We recall that  $\hat{\lambda} = \lambda_1(q_*/2, B_r)$  denotes the first transmission eigenvalue corresponding to a ball  $B_r \subset D$  of radius  $r$  contained in  $D$  with contrasts  $q_0 = q_*/2$  and  $p_0 = 0$ . Let  $\hat{w}, \hat{u}$  be the non-zero solutions to the corresponding interior transmission eigenvalue problem. i.e the solution of (2.9)–(2.12) with  $R = r$ ,

$q_0 := q_*/2$ ,  $p_0 := 0$ , and set  $\hat{v} := \hat{w} - \hat{u} \in H_0^1(B_r)$ . Let  $\tilde{v}$  be the extension by zero of  $\hat{v}$  in the whole of  $D$  and  $\tilde{u}$  defined again by (2.31), i.e.,

$$\iint_D [Q\nabla\tilde{u} \cdot \nabla\bar{\psi} - \hat{\lambda}p\tilde{u}\bar{\psi}] dx = \iint_D [\nabla\tilde{v} \cdot \nabla\bar{\psi} - \hat{\lambda}\tilde{v}\bar{\psi}] dx$$

for all  $\psi \in H^1(D)$ . Also we recall (2.32) in the form

$$(L_{\hat{\lambda}}\tilde{v}, \tilde{v})_{H^1(D)} = \iint_D [Q\nabla\tilde{u} \cdot \nabla\bar{\tilde{u}} - \hat{\lambda}p|\tilde{u}|^2 + |\nabla\tilde{v}| - \hat{\lambda}|\tilde{v}|^2] dx \quad (2.38)$$

and have to estimate the first two terms in this expression. Analogously to (2.34) for  $\psi = \tilde{u}$ , we have that

$$\begin{aligned} & \iint_D [Q\nabla\tilde{u} \cdot \nabla\bar{\tilde{u}} - \hat{\lambda}p|\tilde{u}|^2] dx \\ &= \iint_{B_r} [\nabla\hat{v} \cdot \nabla\bar{\tilde{u}} - \hat{\lambda}\hat{v}\bar{\tilde{u}}] dx = \frac{q_*}{2} \iint_{B_r} \nabla\hat{u} \cdot \nabla\bar{\tilde{u}} dx \\ &\leq \left[ \frac{q_*}{2} \iint_{B_r} |\nabla\hat{u}|^2 dx \right]^{1/2} \left[ \frac{q_*}{2} \iint_{B_r} |\nabla\tilde{u}|^2 dx \right]^{1/2} \\ &= \left[ \frac{q_*}{2} \iint_{B_r} |\nabla\hat{u}|^2 dx \right]^{1/2} \left[ \iint_{B_r} [q_*|\nabla\tilde{u}|^2 - \frac{q_*}{2}|\nabla\tilde{u}|^2] dx \right]^{1/2} \\ &\leq \left[ \frac{q_*}{2} \iint_{B_r} |\nabla\hat{u}|^2 dx \right]^{1/2} \left[ \iint_{B_r} [Q\nabla\tilde{u} \cdot \nabla\bar{\tilde{u}} - \hat{\lambda}p|\tilde{u}|^2] dx \right]^{1/2} \end{aligned}$$

where we have used the estimate (2.36) for  $p^*$ . Therefore,

$$\iint_D [Q\nabla\tilde{u} \cdot \nabla\bar{\tilde{u}} - \hat{\lambda}p|\tilde{u}|^2] dx \leq \frac{q_*}{2} \iint_{B_r} |\nabla\hat{u}|^2 dx.$$

Substituting this into (2.38) yields

$$(L_{\hat{\lambda}}\tilde{v}, \tilde{v})_{H^1(D)} \leq \iint_{B_r} \left[ \frac{q_*}{2} |\nabla\hat{u}|^2 + |\nabla\hat{v}| - \hat{\lambda}|\hat{v}|^2 \right] dx = (\hat{L}_{\hat{\lambda}}\hat{v}, \hat{v})_{H^1(B_r)} = 0$$

which proves that there is a transmission eigenvalue in  $(0, \hat{\lambda}]$ .

**Remark 2:** If  $p^*$  is small enough such that (2.36) is satisfied for an  $r > 0$  that in  $D$  we can fit  $m$  balls of radius  $r$ , then in the same way as in the proof of Theorem 2.7 we can show that there are  $m$  transmission eigenvalues in  $(0, \hat{\lambda}]$  counting their multiplicity.

We finish this section by noticing that for a fixed  $Q$  the largest upper bound for  $p^*$  is  $\frac{\mu q_*}{2\lambda_1(q_*/2, B_R)}$  for the largest ball  $B_R$  included in  $D$ . Furthermore note that the smaller the contrast  $Q$  is the larger the contrast  $p$  is allowed in our approach.



### 3 Maxwell's equations

We make again the assumption that  $p \in L^\infty(D)$  is real valued and non-negative<sup>1</sup> and  $Q \in L^\infty(D, \mathbb{C}^{3 \times 3})$  is matrix-valued such that  $Q(x)$  is Hermitian for almost all  $x \in D$ . Furthermore, we assume that there exists  $0 < q_* < q^* < 1$  such that<sup>2</sup>  $q_*|z|^2 \leq z \cdot Q(x)\bar{z} \leq q^*|z|^2$  for all  $z \in \mathbb{C}^3$  and almost all  $x \in D$ . Again,  $D$  is a bounded and connected domain with Lipschitz boundary. We consider the scattering of time-harmonic electromagnetic waves in media where  $\varepsilon = \varepsilon(x)$  and  $\mu$  are given by  $\varepsilon(x) = \varepsilon_0(1 - Q(x))^{-1}$  and  $\mu = \mu_0(1 + p)$ .

We assume that the reader is familiar with the standard spaces in this context. The space  $H(\text{curl}, D)$  is defined as the completion of  $C^\infty(D, \mathbb{C}^3)$  with respect to the norm

$$\|u\|_{H(\text{curl}, D)} = \sqrt{(u, u)_{H(\text{curl}, D)}}$$

where

$$(u, v)_{H(\text{curl}, D)} = \iint_D [\text{curl } u \cdot \text{curl } \bar{v} + u \cdot \bar{v}] dx.$$

The subspace of vanishing tangential traces is denoted by  $H_0(\text{curl}, D)$ , i.e.,

$$H_0(\text{curl}, D) = \{u \in H(\text{curl}, D) : \nu \times u = 0 \text{ on } \partial D\}.$$

The trace is well defined, see e.g. Monk (2003).

**Definition 3.1:** Let  $\sigma \in \{+1, -1\}$ . The number  $\lambda > 0$  is called an *interior transmission eigenvalue* with respect to  $I + \sigma Q$  and  $1 - \sigma p$  if there exists real-valued  $(u, w) \in H(\text{curl}, D) \times H(\text{curl}, D)$  with  $(u, w) \neq (0, 0)$  such that

$$\text{curl } \text{curl } w - \lambda w = 0 \quad \text{in } D \tag{3.39}$$

and

$$\text{curl}((I + \sigma Q) \text{curl } u) - \lambda(1 - \sigma p)u = 0 \quad \text{in } D, \tag{3.40}$$

and the Cauchy data of  $u$  and  $v$  coincide, i.e.,

$$\nu \times u = \nu \times w \text{ on } \partial D \quad \text{and} \quad \nu \times ((I + \sigma Q) \text{curl } u) = \nu \times \text{curl } w \text{ on } \partial D. \tag{3.41}$$

The variational forms are

$$\iint_D [\text{curl } w \cdot \text{curl } \bar{\psi} - \lambda w \cdot \bar{\psi}] dx = 0 \quad \text{for all } \psi \in H_0(\text{curl}, D), \tag{3.42}$$

$$\begin{aligned} & \iint_D [(I + \sigma Q) \text{curl } u \cdot \text{curl } \bar{\psi} - \lambda(1 - \sigma p)u \cdot \bar{\psi}] dx \\ &= \iint_D [\text{curl } w \cdot \text{curl } \bar{\psi} - \lambda w \cdot \bar{\psi}] dx \end{aligned} \tag{3.43}$$

for all  $\psi \in H(\text{curl}, D)$ .

As in the scalar case we have to discuss the case of  $D$  being a ball and  $Q$  is a scalar constant.

**Example 3.2:** Let  $D$  be a ball of radius  $R > 0$  centred at the origin and let  $Q(x) = qI$  and  $p \geq 0$  be scalar and constant. Then the transmission eigenvalue problem has the form

$$\operatorname{curl} \operatorname{curl} w - \lambda w = 0 \quad \text{and} \quad \operatorname{curl} \operatorname{curl} u - \lambda \frac{1 - \sigma p}{1 + \sigma q} u = 0 \quad \text{in } D, \quad (3.44)$$

and

$$\nu \times u = \nu \times w \quad \text{and} \quad (1 + \sigma q)\nu \times \operatorname{curl} u = \nu \times \operatorname{curl} w \quad \text{on } \partial D. \quad (3.45)$$

For abbreviation we set  $\rho = \sqrt{(1 - \sigma p)/(1 + \sigma q)}$  and  $\eta = 1 + \sigma q$  and  $k = \sqrt{\lambda}$ . Let  $Y_n = Y_n(\hat{x})$  where  $|\hat{x}| = 1$  be a (non-trivial and real valued) spherical harmonic of order  $n \in \mathbb{N}$ ,  $n \geq 1$ . We make the ansatz for  $w$  and  $u$  in the form

$$w(x) = \alpha \operatorname{curl} [j_n(kr)Y_n(\hat{x})x], \quad u(x) = \beta \operatorname{curl} [j_n(k\rho r)Y_n(\hat{x})x]$$

for some  $\alpha, \beta \in \mathbb{R}$  to be determined. Here,  $r, \hat{x}$  are the spherical polar coordinates of  $x$ , i.e.,  $x = r\hat{x}$ . Setting  $\phi(x) = j_n(kr)Y_n(\hat{x})$  we note that (see Colton and Kress, 1998)  $w(x) = \alpha \operatorname{curl} [\phi(x)x] = \alpha \nabla \phi(x) \times x$  satisfies  $\operatorname{curl}^2 w - k^2 w = 0$  and, analogously,  $\operatorname{curl}^2 u - k^2 \rho^2 u = 0$  in  $D$ . We compute the boundary data as

$$\nu(x) \times w(x) = \alpha \hat{x} \times (\nabla \phi(x) \times x) = \alpha R j_n(kR) \operatorname{Grad} Y_n(\hat{x}), \quad |x| = r = R,$$

with tangential gradient  $\operatorname{Grad} Y_n$  of  $Y_n$  and

$$\begin{aligned} \nu(x) \times \operatorname{curl} w(x) &= \alpha \hat{x} \times \nabla \left( \frac{\partial \phi}{\partial r}(x) + \phi(x) \right) \\ &= \alpha (k j_n'(kR) + j_n(kR)) \hat{x} \times \operatorname{Grad} Y_n(\hat{x}), \quad |x| = r = R, \end{aligned}$$

and analogously for  $u$ . The functions  $u$  and  $w$  satisfy the transmission conditions (3.45) if, and only if,  $\alpha$  and  $\beta$  satisfy the linear system

$$\begin{pmatrix} j_n(kR) & -j_n(k\rho R) \\ k j_n'(kR) + j_n(kR) & -\eta [k \rho j_n'(k\rho R) + j_n(k\rho R)] \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the matrix can be studied (as a function of  $k = \sqrt{\lambda}$ ) in the same way as in Example 2.2 (compare (2.13)) and yields the existence of infinitely many eigenvalues which eigenvalues which converge to infinity.  $\square$

We define again the difference  $v = w - u$  and observe that  $v \in H_0(\operatorname{curl}, D)$  satisfies the equation

$$\sigma [\operatorname{curl} (Q \operatorname{curl} w) + \lambda p w] = \operatorname{curl} ((I + \sigma Q) \operatorname{curl} v) - \lambda(1 - \sigma p)v \quad \text{in } D, \quad (3.46)$$

$$\sigma \nu \times (Q \operatorname{curl} w) = \nu \times ((I + \sigma Q) \operatorname{curl} v) \quad \text{on } \partial D, \quad (3.47)$$

i.e., in variational form

$$\begin{aligned} & \sigma \iint_D [Q \operatorname{curl} w \cdot \operatorname{curl} \bar{\psi} + \lambda p w \cdot \bar{\psi}] dx \\ &= \iint_D [(I + \sigma Q) \operatorname{curl} v \cdot \operatorname{curl} \bar{\psi} - \lambda(1 - \sigma p)v \cdot \bar{\psi}] dx \end{aligned} \quad (3.48)$$

for all  $\psi \in H(\operatorname{curl}, D)$ . It is the aim to define the operator  $L_\lambda$  in the same way as in the previous section. Therefore, we have to study first the solution operator  $A_\lambda : H_0(\operatorname{curl}, D) \rightarrow H(\operatorname{curl}, D)$  which maps  $v$  into  $w$ . We note that, by substituting  $\psi = \nabla \rho$  for some  $\rho \in H^1(D)$  into (3.48) we have that

$$\iint_D [\sigma p w + (1 - \sigma p)v] \cdot \nabla \bar{\rho} dx = 0 \quad \text{for all } \rho \in H^1(D) \quad (3.49)$$

provided  $\lambda \neq 0$ . We require this equation also for  $\lambda = 0$ .

**Theorem 3.3:** *Let  $p \in L^\infty(D)$  non-negative. Then there exists  $\delta > 0$  such that for every  $\lambda \in \{z \in \mathbb{C} : \operatorname{Re} z > -\delta\}$  and every  $v \in H_0(\operatorname{curl}, D)$  there exists a unique  $w = w(\lambda, v) \in H(\operatorname{curl}, D)$  with (3.48) and (3.49). The solution operator  $A_\lambda : v \mapsto w$  is bounded from  $H_0(\operatorname{curl}, D)$  into  $H(\operatorname{curl}, D)$  and depends analytically on  $\lambda$ .*

*Proof:* We make use of the Helmholtz decomposition  $H(\operatorname{curl}, D) = Y \oplus \nabla H^1(D)$  where

$$Y = \left\{ w \in H(\operatorname{curl}, D) : \iint_D p w \cdot \nabla \rho dx = 0 \text{ for all } \rho \in H^1(D) \right\}.$$

To prove existence for given  $v \in H_0(\operatorname{curl}, D)$  we make the ansatz  $w = \tilde{w} + \nabla \phi$  where  $\phi \in H^1(D)$  solves

$$\sigma \iint_D p \nabla \phi \cdot \nabla \bar{\rho} dx = - \iint_D (1 - \sigma p)v \cdot \nabla \bar{\rho} dx \quad \text{for all } \rho \in H^1(D) \quad (3.50)$$

and  $\tilde{w} \in Y$  solves (3.48) for all  $\psi \in Y$ . The solution  $\phi \in H^1(D)$  exists, is independent of  $\lambda$  and depends continuously on  $v$  since it is defined by an ordinary Neumann problem.

To study the existence of  $\tilde{w} \in Y$  we introduce the sesquilinear form

$$\mathcal{A}_\lambda(w, \psi) := \sigma \iint_D [Q \operatorname{curl} w \cdot \operatorname{curl} \bar{\psi} + \lambda p w \cdot \bar{\psi}] dx, \quad w, \psi \in H(\operatorname{curl}, D),$$

and consider it on  $Y \times Y$ . First we note that  $\mathcal{A}_0$  is coercive on  $Y \times Y$ , i.e., there exists  $\mu > 0$  with

$$\sigma \iint_D Q \operatorname{curl} \psi \cdot \operatorname{curl} \bar{\psi} dx \geq \mu \|\psi\|_{H(\operatorname{curl}, D)}^2 \quad \text{for all } \psi \in Y,$$

(see Monk (2003) for a proof). For  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}\lambda > -\delta$  we estimate

$$\begin{aligned} \operatorname{Re}\mathcal{A}_\lambda(\psi, \psi) &= \sigma \iint_D [Q \operatorname{curl} \psi \cdot \operatorname{curl} \bar{\psi} + (\operatorname{Re}\lambda)p|\psi|^2] dx \\ &\geq (\mu - \delta\|p\|_{L^\infty(D)}) \|\psi\|_{H(\operatorname{curl}, D)}^2 \quad \text{for all } \psi \in Y, \end{aligned}$$

and this is coercive for sufficiently small  $\delta > 0$ . Since the right hand side of (3.48) is bounded the theorem of Lax and Milgram yields existence, uniqueness and continuous dependence of a solution  $\tilde{w} \in Y$  of (3.48), restricted to  $\psi \in Y$ . With  $w = \tilde{w} + \nabla\phi$  we conclude that, for  $\psi \in Y$ ,

$$\begin{aligned} \mathcal{A}_\lambda(w, \psi) &= \mathcal{A}_\lambda(\tilde{w}, \psi) + \mathcal{A}_\lambda(\nabla\phi, \psi) \\ &= \iint_D [\operatorname{curl} v \cdot (I + \sigma Q) \operatorname{curl} \bar{\psi} - \lambda(1 - \sigma p)v \cdot \bar{\psi}] dx \\ &\quad + \lambda\sigma \iint_D p\nabla\phi \cdot \bar{\psi} dx \\ &= \iint_D [\operatorname{curl} v \cdot (I + \sigma Q) \operatorname{curl} \bar{\psi} - \lambda(1 - \sigma p)v \cdot \bar{\psi}] dx \end{aligned}$$

since  $\psi \in Y$ . For  $\psi = \nabla\rho$  we have that

$$\begin{aligned} \mathcal{A}_\lambda(w, \nabla\rho) &= \lambda\sigma \iint_D p(\tilde{w} + \nabla\phi) \cdot \nabla\bar{\rho} dx = \lambda\sigma \iint_D p\nabla\phi \cdot \nabla\bar{\rho} dx \\ &= -\lambda \iint_D (1 - \sigma p)v \cdot \nabla\bar{\rho} dx \end{aligned}$$

by the definition (3.50) of  $\phi$ . Therefore,  $w$  solves (3.48) for all  $\psi \in H(\operatorname{curl}, D)$ . Equation (3.49) is also satisfied by the definition of  $\phi$ . This proves existence. For proving uniqueness let  $w \in H(\operatorname{curl}, D)$  be a solution of (3.48) and (3.49) for  $v = 0$ . From (3.49) we conclude that  $w \in Y$ . Substituting  $\psi = w$  in (3.48) and using the coercivity yields  $w = 0$ . This ends the proof.  $\square$

This theorem assures the solvability of (3.46) and (3.47) for  $w$  for given  $v \in H(\operatorname{curl}, D)$ . It remains to require that  $w$  solves the equation  $\operatorname{curl} \operatorname{curl} w - \lambda w = 0$ . Therefore, we define the mapping  $L_\lambda$  from  $H_0(\operatorname{curl}, D)$  into itself by the property that  $L_\lambda v$  is the Riesz representation of the conjugate-linear and bounded functional

$$\psi \mapsto \iint_D [\operatorname{curl} w_v \cdot \operatorname{curl} \bar{\psi} - \lambda w_v \cdot \bar{\psi}] dx, \quad \psi \in H_0(\operatorname{curl}, D),$$

where  $w_v = A_\lambda v$ , i.e.,

$$\begin{aligned} (L_\lambda v, \psi)_{H(\operatorname{curl}, D)} &= \iint_D [\operatorname{curl} w_v \cdot \operatorname{curl} \bar{\psi} - \lambda w_v \cdot \bar{\psi}] dx \\ &\quad \text{for all } \psi \in H_0(\operatorname{curl}, D). \end{aligned} \tag{3.51}$$

The proof of the following result which corresponds to Theorem 2.4 is obvious.

**Theorem 3.4:**

- a *Let  $(u, w)$  be a transmission eigenfunction corresponding to  $\lambda$ . Then  $v = w - u \in H_0(\text{curl}, D)$  solves  $L_\lambda v = 0$ .*
- b *Let  $v \in H_0(\text{curl}, D)$ ,  $v \neq 0$ , satisfy  $L_\lambda v = 0$ . Furthermore, let  $w = A_\lambda v \in H(\text{curl}, D)$  be the solution of (3.48) and (3.49). Then  $(u, w)$  is an eigenfunction where  $u = w - v$ .*

So far, we allowed  $p \in L^\infty(D)$  to be space dependent and non-negative. From now on we make the assumption that  $p > 0$  is constant and positive (and  $p < 1$  if the case of  $\sigma = 1$  is considered). The case  $p = 0$  can be treated analogously, see Kirsch (2009). Then we note that  $\text{div}u = \text{div}w = 0$  in the variational sense for any pair of eigenfunctions corresponding to an eigenvalue  $\lambda > 0$ . Therefore,  $v \in X_0$  and  $w \in X$  where

$$X_0 = \left\{ v \in H_0(\text{curl}, D) : \iint_D v \cdot \nabla \rho \, dx = 0 \text{ for all } \rho \in H_0^1(D) \right\},$$

$$X = \left\{ v \in H(\text{curl}, D) : \iint_D v \cdot \nabla \rho \, dx = 0 \text{ for all } \rho \in H_0^1(D) \right\},$$

i.e., we have the Helmholtz decompositions  $H(\text{curl}, D) = X \oplus \nabla H_0^1(D)$  and  $H_0(\text{curl}, D) = X_0 \oplus \nabla H_0^1(D)$ . We note from (3.48) and (3.49) that  $A_\lambda$  maps  $X_0$  into  $X$  and  $A_\lambda \nabla \phi = -\frac{1-\sigma p}{\sigma p} \nabla \phi$  for all  $\phi \in H_0^1(D)$  and all  $\lambda \in \mathbb{C}$  with  $\text{Re}\lambda > -\delta$ . Analogously,  $L_\lambda$  maps  $X_0$  into itself and  $L_\lambda \nabla \phi = \lambda \frac{1-\sigma p}{\sigma p} \nabla \phi$  for all  $\phi \in H_0^1(D)$  and all  $\lambda \in \mathbb{C}$  with  $\text{Re}\lambda > -\delta$ . From this, an elementary calculation yields the estimate

$$\begin{aligned} & \sigma(L_\lambda(v_0 + \nabla \phi), v_0 + \nabla \phi)_{H(\text{curl}, D)} \\ &= \sigma(L_\lambda v_0, v_0)_{H(\text{curl}, D)} + \lambda \frac{1-\sigma p}{p} \|\nabla \phi\|_{L^2(D)}^2 \leq \sigma(L_\lambda v_0, v_0)_{H(\text{curl}, D)} \end{aligned} \quad (3.52)$$

for all  $v_0 \in X_0$  and  $\phi \in H_0^1(D)$ .

Again, the transmission eigenvalues are just the parameters  $\lambda$  for which  $L_\lambda$  fails to be injective. Now we continue with the investigation of  $L_\lambda$ . Analogously to Theorem 2.5 we show:

**Theorem 3.5:**

- (a)  *$\sigma L_0$  is selfadjoint and coercive on  $X_0$ , in particular*

$$\sigma(L_0 v, v)_{H(\text{curl}, D)} \geq c \|v\|_{H(\text{curl}, D)}^2 \text{ for all } v \in X_0$$

where  $c > 0$  is independent of  $v$ .

- (b)  *$L_\lambda$  depends analytically on  $\lambda \in \{z \in \mathbb{C} : \text{Re}z > -\delta\}$  with  $\delta > 0$  from Theorem 3.3.*
- (c)  *$L_\lambda - L_0$  is selfadjoint and compact in  $X_0$  for all  $\lambda \in \mathbb{R}_{\geq 0}$ .*

*Proof:* (a) First we show that  $L_\lambda$  is selfadjoint for all  $\lambda \geq 0$ . For  $v_1, v_2 \in X_0$  and corresponding  $w_1, w_2 \in X$  we have:

$$\begin{aligned} (L_\lambda v_1, v_2)_{H(\text{curl}, D)} &= \iint_D [\text{curl } w_1 \cdot \text{curl } \bar{v}_2 - \lambda w_1 \cdot \bar{v}_2] dx \\ &= \iint_D [(I + \sigma Q) \text{curl } w_1 \cdot \text{curl } \bar{v}_2 - \lambda(1 - \sigma p) w_1 \cdot \bar{v}_2] dx \\ &\quad - \sigma \iint_D [Q \text{curl } w_1 \cdot \text{curl } \bar{v}_2 + \lambda p w_1 \cdot \bar{v}_2] dx. \end{aligned}$$

Now we use (3.48) twice, first for  $v = v_2$ , corresponding  $w = w_2$ , and  $\psi = w_1$ , and then for  $v = v_1$ , corresponding  $w = w_1$ , and  $\psi = v_2$ . This yields

$$\begin{aligned} (L_\lambda v_1, v_2)_{H(\text{curl}, D)} &= \sigma \iint_D [\text{curl } \bar{w}_2 \cdot Q \text{curl } w_1 + \lambda p \bar{w}_2 \cdot w_1] dx \\ &\quad - \iint_D [\text{curl } v_1 \cdot (I + \sigma Q) \text{curl } \bar{v}_2 - \lambda(1 - \sigma p) v_1 \cdot \bar{v}_2] dx, \end{aligned}$$

and this is a selfadjoint expression in  $v_1$  and  $v_2$ .

Now we show that  $\sigma L_0$  is coercive.

Let first  $\sigma = +1$ . Using the definition of  $L_0$  and  $w_v = v + u$  we conclude

$$\begin{aligned} (L_0 v, v)_{H(\text{curl}, D)} &= \iint_D \text{curl } w_v \cdot \text{curl } \bar{v} dx \\ &= \iint_D |\text{curl } v|^2 dx + \iint_D \text{curl } u \cdot \text{curl } \bar{v} dx. \end{aligned}$$

Now we use (3.48) for  $\sigma = +1$  and  $\lambda = 0$  to derive the variational form of  $u \in X$  in the form

$$\iint_D Q \text{curl } u \cdot \text{curl } \bar{\psi} dx = \iint_D \text{curl } v \cdot \text{curl } \bar{\psi} dx$$

for all  $\psi \in X$ . For  $\psi = u$  this yields

$$\begin{aligned} (L_0 v, v)_{H(\text{curl}, D)} &= \iint_D |\text{curl } v|^2 dx + \iint_D Q \text{curl } u \cdot \text{curl } \bar{u} dx \\ &\geq \iint_D |\text{curl } v|^2 dx \end{aligned}$$

which yields coercivity of  $L_0$  since  $v \mapsto \|\text{curl } v\|_{L^2(D)}$  is an equivalent norm in  $X_0$ . Let now  $\sigma = -1$ . Then we use (3.48) for  $\sigma = -1$  and  $\lambda = 0$  twice (as above) and write

$$\begin{aligned} -(L_0 v, v)_{H(\text{curl}, D)} &= - \iint_D (I - Q) \text{curl } w_v \cdot \text{curl } \bar{v} dx - \iint_D Q \text{curl } w_v \cdot \text{curl } \bar{v} dx \\ &= \iint_D Q \text{curl } w_v \cdot \text{curl } \bar{w}_v dx + \iint_D (I - Q) \text{curl } v \cdot \text{curl } \bar{v} dx \\ &\geq (1 - q_+) \iint_D |\text{curl } v|^2 dx \end{aligned}$$

which yields again coercivity of  $-L_0$ .

- (b) This follows directly from the continuity of  $\lambda \mapsto A_\lambda$  and the definition  $L_\lambda$ .
- (c) Let  $v^{(j)} \in X_0$  converge weakly to zero in  $X_0$ . The space  $X_0$  is compactly imbedded in  $L^2(D, \mathbb{C}^3)$ , see, e.g. Corollary 3.51 of Monk (2003). Therefore,  $v^{(j)}$  converges to zero in  $L^2(D, \mathbb{C}^3)$ . Denote the corresponding solutions of (3.48) by  $w_\lambda^{(j)} = A_\lambda v^{(j)} \in X$ . By the continuity of the operator  $A_\lambda$  we conclude that  $w_\lambda^{(j)}$  converges weakly to zero in  $X$ . From (3.49) we conclude that  $(1 - \sigma p)v^{(j)} + \sigma p w_\lambda^{(j)} \in \tilde{Y}$  where

$$\tilde{Y} = \left\{ u \in H(\text{curl}, D) : \iint_D u \cdot \nabla \rho \, dx = 0 \text{ for all } \rho \in H^1(D) \right\}$$

denotes the space of divergence-free fields with vanishing normal components on  $\partial D$ . Also this subspace  $\tilde{Y}$  is compactly imbedded in  $L^2(D, \mathbb{C}^3)$ . Therefore,  $(1 - \sigma p)v^{(j)} + \sigma p w_\lambda^{(j)}$  converges to zero in  $L^2(D, \mathbb{C}^3)$  and thus also  $w_\lambda^{(j)}$ . We note that

$$\begin{aligned} & ((L_\lambda - L_0)v^{(j)}, \psi)_{H(\text{curl}, D)} \\ &= \iint_D [\text{curl}(w_\lambda^{(j)} - w_0^{(j)}) \cdot \text{curl} \bar{\psi} - \lambda w_\lambda^{(j)} \cdot \bar{\psi}] \, dx \end{aligned} \quad (3.53)$$

for all  $\psi \in X_0$ . The difference  $\tilde{w}^{(j)} = w_\lambda^{(j)} - w_0^{(j)} \in X$  satisfies the variational equation

$$\begin{aligned} & \sigma \iint_D [Q \text{curl} \tilde{w}^{(j)} \cdot \text{curl} \bar{\psi} + \lambda p w_\lambda^{(j)} \cdot \bar{\psi}] \, dx \\ &= -\lambda(1 - \sigma p) \iint_D v^{(j)} \cdot \bar{\psi} \, dx \quad \text{for all } \psi \in X. \end{aligned}$$

We set  $\psi = \tilde{w}^{(j)}$  and estimate

$$\begin{aligned} \iint_D Q \text{curl} \tilde{w}^{(j)} \cdot \text{curl} \tilde{w}^{(j)} \, dx &= \lambda \left| \iint_D [(1 - \sigma p)v^{(j)} + \sigma p w_\lambda^{(j)}] \cdot \overline{\tilde{w}^{(j)}} \, dx \right| \\ &\leq \lambda \|(1 - \sigma p)v^{(j)} + \sigma p w_\lambda^{(j)}\|_{L^2(D)} \|\tilde{w}^{(j)}\|_{L^2(D)}. \end{aligned}$$

Since the right hand side converges to zero and  $Q$  is positive definite we conclude that also  $\|\text{curl} \tilde{w}^{(j)}\|_{L^2(D)^2}$  converges to zero and thus  $\tilde{w}^{(j)} \rightarrow 0$  in  $H(\text{curl}, D)$ . Finally, from (3.53) we conclude that

$$\begin{aligned} \|(L_\lambda - L_0)v^{(j)}\|_{H(\text{curl}, D)} &= \sup_{\|\psi\|_{H(\text{curl}, D)}=1} ((L_\lambda - L_0)v^{(j)}, \psi)_{H(\text{curl}, D)} \\ &\leq \|\tilde{w}^{(j)}\|_{H(\text{curl}, D)} + \lambda \|w_\lambda^{(j)}\|_{L^2(D)} \rightarrow 0. \end{aligned}$$

This ends the proof.  $\square$

Now we continue in the same spirit as in Theorem 2.7.

**Theorem 3.6:** *Suppose that the matrix function  $Q \in L^\infty(D, \mathbb{C}^{3 \times 3})$  and the constant  $p$  satisfy the conditions from the beginning of this section. Then there exists an infinite sequence of transmission eigenvalues  $\lambda_j$  with  $+\infty$  as their only accumulation point.*

*Proof:* Again, assumption (1) of Theorem 2.6 with  $H_0^1(D)$  replaced by  $X_0$  is satisfied with  $\lambda_0 = 0$  i.e.,  $\sigma(L_0 v, v)_{H(\text{curl}, D)} > 0$  for all  $v \in X_0$  with  $v \neq 0$  by Theorem 3.5. Next, by definition of  $L_\lambda$  and the fact that  $w = u + v$  have for  $\lambda \in \mathbb{R}_{\geq 0}$  and  $v \in H_0(\text{curl}, D)$

$$\begin{aligned} (L_\lambda v, v)_{H(\text{curl}, D)} &= \iint_D [\text{curl } w \cdot \text{curl } \bar{v} - \lambda w \cdot \bar{v}] dx \\ &= \iint_D [\text{curl } u \cdot \text{curl } \bar{v} - \lambda u \cdot \bar{v} + |\text{curl } v|^2 - \lambda |v|^2] dx. \end{aligned} \quad (3.54)$$

From (3.48) we also have that  $u$  satisfies

$$\sigma \iint_D [Q \text{curl } u \cdot \text{curl } \bar{\psi} + \lambda p u \cdot \bar{\psi}] dx = \iint_D [\text{curl } v \cdot \text{curl } \bar{\psi} - \lambda v \cdot \bar{\psi}] dx \quad (3.55)$$

for all  $\psi \in H(\text{curl}, D)$ . Now taking  $\psi = u$  in (3.55) and plugging the result into (3.54) yields

$$\sigma(L_\lambda v, v)_{H(\text{curl}, D)} = \iint_D [Q \text{curl } u \cdot \text{curl } \bar{u} + \lambda p |u|^2 + \sigma |\text{curl } v|^2 - \lambda \sigma |v|^2] dx. \quad (3.56)$$

Let now  $B_r \subset D$  be an arbitrary ball of radius  $r$  included in  $D$  and let  $\hat{\lambda}$  be a transmission eigenvalue corresponding to the ball  $B_r$  with contrasts  $q_*$  and  $p$ . Let  $\hat{w}$ ,  $\hat{u}$  be the non-zero solutions to the corresponding homogenous interior transmission problem, i.e the solution of (3.44)–(3.45) with  $q_0 := q_*$ ,  $p_0 := p$  and  $R = r$ , and set  $\hat{v} := \hat{w} - \hat{u} \in H_0(\text{curl}, B_r)$ . By construction we have that (3.56) holds for this situation as well, i.e., since  $\hat{L}_{\hat{\lambda}} \hat{v} = 0$

$$\begin{aligned} 0 &= \sigma(\hat{L}_{\hat{\lambda}} \hat{v}, \hat{v})_{H(\text{curl}, B_r)} \\ &= \iint_{B_r} [q_* |\text{curl } \hat{u}|^2 + \hat{\lambda} p |\hat{u}|^2 + \sigma |\text{curl } \hat{v}|^2 - \hat{\lambda} \sigma |\hat{v}|^2] dx. \end{aligned} \quad (3.57)$$

Next we denote by  $\tilde{v}$  the extension of  $\hat{v} \in H_0(\text{curl}, B_r)$  by zero to the whole of  $D$  and note that  $\tilde{v} \in H_0(\text{curl}, D)$  (see, e.g. Monk, 2003). Furthermore, let  $\tilde{w} := w_{\tilde{v}}$  be the corresponding solution to (3.48) and  $\tilde{u} := \tilde{w} - \tilde{v}$ . In particular  $\tilde{u}$  satisfies

$$\begin{aligned} &\sigma \iint_D [Q \text{curl } \tilde{u} \cdot \text{curl } \bar{\psi} + \hat{\lambda} p \tilde{u} \cdot \bar{\psi}] dx \\ &= \iint_D [\text{curl } \tilde{v} \cdot \text{curl } \bar{\psi} - \hat{\lambda} \tilde{v} \cdot \bar{\psi}] dx = \iint_{B_r} [\text{curl } \hat{v} \cdot \text{curl } \bar{\psi} - \hat{\lambda} \hat{v} \cdot \bar{\psi}] dx \\ &= \sigma \iint_{B_r} [q_* \text{curl } \hat{u} \cdot \text{curl } \bar{\psi} + \hat{\lambda} p \hat{u} \cdot \bar{\psi}] dx \end{aligned} \quad (3.58)$$



for all  $\psi \in H(\text{curl}, D)$ . Therefore, for  $\psi = \tilde{u}$  we have by the Cauchy-Schwarz inequality,

$$\begin{aligned} & \iint_D [Q \text{curl } \tilde{u} \cdot \text{curl } \tilde{u} + \hat{\lambda} p |\tilde{u}|^2] dx \\ &= \iint_{B_r} [q_* \text{curl } \hat{u} \cdot \text{curl } \tilde{u} + \hat{\lambda} p \hat{u} \cdot \tilde{u}] dx \\ &\leq \left[ \iint_{B_r} [q_* |\text{curl } \hat{u}|^2 + \hat{\lambda} p |\hat{u}|^2] dx \right]^{1/2} \left[ \iint_{B_r} [q_* |\text{curl } \tilde{u}|^2 + \hat{\lambda} p |\tilde{u}|^2] dx \right]^{1/2} \\ &\leq \left[ \iint_{B_r} [q_* |\text{curl } \hat{u}|^2 + \hat{\lambda} p |\hat{u}|^2] dx \right]^{1/2} \left[ \iint_D [Q \text{curl } \tilde{u} \cdot \text{curl } \tilde{u} + \hat{\lambda} p |\tilde{u}|^2] dx \right]^{1/2} \end{aligned}$$

and thus

$$\iint_D [Q \text{curl } \tilde{u} \cdot \text{curl } \tilde{u} + \hat{\lambda} p |\tilde{u}|^2] dx \leq \iint_{B_r} [q_* |\text{curl } \hat{u}|^2 + \hat{\lambda} p |\hat{u}|^2] dx.$$

Substituting this into (3.56) yields

$$\begin{aligned} \sigma(L_{\hat{\lambda}} \tilde{v}, \tilde{v})_{H(\text{curl}, D)} &= \iint_D [Q \text{curl } \tilde{u} \cdot \text{curl } \tilde{u} + \hat{\lambda} p |\tilde{u}|^2 + \sigma |\text{curl } \tilde{v}|^2 - \hat{\lambda} \sigma |\tilde{v}|^2] dx \\ &\leq \iint_{B_r} [q_* |\text{curl } \hat{u}|^2 + \hat{\lambda} p |\hat{u}|^2 + \sigma |\text{curl } \hat{v}|^2 - \hat{\lambda} \sigma |\hat{v}|^2] dx = 0 \end{aligned}$$

by (3.57). We note that, in general,  $\tilde{v} \in H_0(\text{curl}, D)$  fails to be in  $X_0$ . However, if we define  $v_0 \in X_0$  to be the orthogonal projection of  $\tilde{v}$  in  $X_0$  then, by (3.52),

$$\sigma(L_{\hat{\lambda}} v_0, v_0)_{H(\text{curl}, D)} \leq \sigma(L_{\hat{\lambda}} \tilde{v}, \tilde{v})_{H(\text{curl}, D)} \leq 0.$$

Now we argue as in the proof of Theorem 2.7. □

## References

- Cakoni, F. and Colton, D. (2006) *Qualitative Methods in Inverse Scattering Theory: An Introduction*, Springer, Berlin.
- Cakoni, F., Colton, D. and Haddar, H. (2002) ‘The linear sampling method for anisotropic media’, *J. Comp. Appl. Math.* Vol. 146, pp.285-299.
- Cakoni, F., Colton, D. and Haddar, H. (2001a) ‘The interior transmission problem for regions with cavities’, *SIAM J. Math. Analysis*, Vol. 42, pp.145–162.
- Cakoni, F. and Gintides, D. (2010) ‘New results on transmission eigenvalues’, *Inverse Problems and Imaging*, Vol. 4, pp.39–48.
- Cakoni, F., Gintides, D. and Haddar, H. (2010b) ‘The existence of an infinite discrete set of transmission eigenvalues’, *SIAM J. Math. Anal.*, Vol. 42, pp.237–255.
- Cakoni, F. and Haddar, H. (2001) *The Linear Sampling Method for Anisotropic Media: Part 2*, Preprints 2001/26, MSRI Berkeley, California.
- Cakoni, F. and Haddar, H. (2009) ‘On the existence of transmission eigenvalues in an inhomogeneous medium’, *Applicable Analysis*, Vol. 88, pp.475–493.

- Colton, D., Kirsch, A. and Päivärinta, L. (1989) 'Far field patterns for acoustic waves in an inhomogeneous medium', *SIAM Jour. Math. Anal.*, Vol. 20, pp.1472–1483.
- Colton, D. and Kress, R. (1998) *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd ed., Vol. 13 in Applied Mathematical Sciences, Springer, New York.
- Colton, D. and Monk, P. (1988) 'The inverse scattering problem for acoustic waves in an inhomogeneous medium', *Quart. Jour. Mech. Applied Math*, Vol. 41, pp.97–125.
- Colton, D., Päivärinta, L. and Sylvester, J. (2007) 'The interior transmission problem', *Inverse Problems and Imaging* Vol. 1, pp.13–28.
- Dietz, B., Eckmann, J.P., Pillet, C.A., Smilansky, U. and Ussishkin, I. (1995) 'Inside-outside duality for planar billiards: A numerical study', *Physical Review E*, pp.4222–4234.
- Katznelson, Y. (1968) *An Introduction to Harmonic Analysis*, John Wiley and Sons, Inc. New York.
- Kirsch, A. (2007) 'An integral equation approach and the interior transmission problem for Maxwell's equations', *Inverse Problems and Imaging*, Vol. 1, pp.159–179.
- Kirsch, A. (2009) 'On the existence of transmission eigenvalues', *Inverse Problems and Imaging* Vol. 3, pp.155–172.
- Kirsch, A. and Grinberg, N. (2008) *The Factorization Method for Inverse Problems*, Oxford University Press, Oxford.
- Monk, P. (2003) *Finite Element Methods for Maxwell's Equations*, Oxford University Press.
- Päivärinta, L. and Sylvester, J. (2008) 'Transmission eigenvalues', *SIAM J. Math. Anal.*, Vol. 40, pp.738–753.
- Reed, M. and Simon, B. (1980) *Functional Analysis*, Academic Press, San Diego, London.
- Rynne, B.P. and Sleeman, B.D. (1992) 'The interior transmission problem and inverse scattering from inhomogeneous media', *SIAM J. Math. Anal.*, Vol. 22, pp.1755–1762.

## Notes

<sup>1</sup>We need to assume that  $p(x) \leq p^* < 1$  if  $\sigma = 1$  in Definition 3.1.

<sup>2</sup>The assumption  $q^* < 1$  is only necessary in the case  $\sigma = -1$  in Definition 3.1.