Inverse Problems and Imaging doi:
Volume 0, No. 0, 0, 0

TRANSMISSION EIGENVALUES FOR INHOMOGENEOUS MEDIA CONTAINING OBSTACLES

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Abstract. We consider the interior transmission problem corresponding to the inverse scattering by an inhomogeneous (possibly anisotropic) media in which an impenetrable obstacle with Dirichlet boundary conditions is embedded. Our main focus is to understand the associated eigenvalue problem, more specifically to prove that the transmission eigenvalues form a discrete set and show that they exist. The presence of Dirichlet obstacle brings new difficulties to already complicated situation dealing with a non-selfadjoint eigenvalue problem. In this paper, we employ a variety of variational techniques under various assumptions on the index of refraction as well as the size of the Dirichlet obstacle.

1. Introduction.

In the recent years, the interior transmission eigenvalue problem has become an important area of research in inverse scattering theory. This interest is motivated by the fact that transmission eigenvalues carry information about the material properties of the scattering object and these eigenvalues can in principle be determined from the scattering data [7]. For a connection of the interior transmission problem with the scattering problem we refer the reader to [3], [13], [14] and [19]. Following the first proof of the existence of transmission eigenvalues in [21] and then in [10], a flux of results on the study of transmission eigenvalues and their application in obtaining estimates on material properties of inhomogeneous scattering media has emerged in the literature [4], [5], [8], [11], [16], [17], [18], (and the references therein). All these studies have considered the case when the contrast in the scattering medium does not change sign. In [6] for the scalar case and in [15] for the case of Maxwell's equations, the transmission eigenvalues have been studied

2000 *Mathematics Subject Classification.* Primary: 35R30, 35Q60, 35J40, 78A25.
Key words and phrases. Interior transmission problem, transmission eigenvalues, inhomogeneous medium, inverse scattering problem.
The research of F. Cakoni was supported in part by the U.S. Air Force Office of Scientific Research Grant FA-9550-11-1-0189 and the NSF Grant DMS-1106972. The support of the Associate Team ISIP between DeFi-INRIA at Ecole Polytechnique and the University of Delaware is gratefully acknowledged.

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for inhomogeneous media with voids, i.e. subregions where the index of refraction is the same as of the background media. Recently some progress has been made in the study of transmission eigenvalue problem for media with contrast that can change sign [1], [22]. In particular, there it is proven that the transmission eigenvalues for a discrete (possibly empty) set provided that the sign condition on the contrast is required only at the boundary of the inhomogeneity.

In this paper we investigate the interior transmission problem and corresponding transmission eigenvalues for inhomogeneous media that contains a perfect conductor inside, for both isotropic and anisotropic case. In the context of electromagnetic scattering, this problem corresponds to the scattering by an inhomogeneous media with space varying electric permittivity and magnetic permeability which contains inside a perfect conductor. From practical point of view the importance of this problem lies in the possibility of using transmission eigenvalues to detect anomalies inside inhomogeneous media in non-destructive testing. This type of problem is considered in [20] where the authors recover the obstacle embedded in an inhomogeneous media. In Section 2 with start our investigation with the isotropic Helmholtz equation and prove that there exists a discrete infinite set of real transmission eigenvalues, provided that the real-valued index of refraction $n := n(x)$ in the medium satisfies $0 < n < 1$ where one is the background index of refraction. Our approach does not work if $n > 1$. Then we continue in Section 3 with the anisotropic Helmholtz equation assuming that the contrast in the scattering medium appears in the main operator (which can be a matrix valued function) as well as in the lower term. If A and n denote the refractive indices in the main operator and lower terms, respectively, based on the T-coercivity developed in [1] and [2] we are able to prove the discreteness of transmission eigenvalues for $A - I > 0$ and any $n > 0$ or $I - A > 0$ and $1 - n > 0$. Our results on the existence of transmission eigenvalues for the anisotropic case are more restrictive. More specifically, adapting the approach developed in [12], for the case of $A - I > 0$ and $0 < n < 1$ or $n > 1$ and small enough, we can show the existence of finitely many transmission eigenvalues assuming that the area of the interior Dirichlet inclusion is small enough.

2. **The scalar isotropic case.**

We start our discussion by considering the case of the interior transmission problem for an isotropic inhomogeneous medium with a Dirichlet obstacle inside. Let $D \subset \mathbb{R}^d$, $d = 2, 3$ be a simply connected and bounded region with piece-wise smooth boundary $\Gamma := \partial D$. Inside D, we consider a region $D_0 \subset D$ possibly be multiply connected with piece-wise smooth boundary $\Sigma := \partial D_0$ such that $\mathbb{R}^d \backslash \overline{D}_0$ is connected. We assume that D_0 is an impenetrable obstacle satisfying the Dirichlet boundary condition, whereas $D \backslash \overline{D}_0$ is an inhomogenous medium with index of refraction n where $n \in L^\infty(D \backslash \overline{D}_0)$ is such that $n \geq c > 0$. Let ν denote the unit outward normal to Γ and Σ.

The interior transmission problem corresponding to the scattering problem for the scatterer D reads

$$\text{(ITPH)} \qquad \begin{cases} \Delta w + k^2 n w = 0 & \text{in } D \backslash \overline{D}_0 \\ \Delta v + k^2 v = 0 & \text{in } D \\ w - v = g & \text{on } \Gamma \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = h & \text{on } \Gamma \\ w = 0 & \text{on } \Sigma. \end{cases}$$

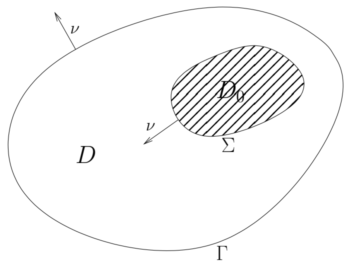

Figure 1. Geometry and notations.

Due to the fact that the function w is only defined in $D\backslash\overline{D}_0$, the first difficulty that we meet is to correctly define a solution to this problem in appropriate function spaces. Indeed, the difference u between w and v can only be considered in the set $D\backslash\overline{D}_0$ and we do not have enough information about u and in particular about its normal derivative $\dfrac{\partial u}{\partial \nu}$ on the boundary Σ to conclude the H^2-regularity for u. In particular, u is not necessarily in $H^2(D\backslash\overline{D}_0)$ and the only thing we can say is that $\Delta u \in L^2(D\backslash\overline{D}_0)$. Thus we introduce the Hilbert space

$$H^1_\Delta(D\backslash\overline{D}_0) := \left\{u \in H^1(D\backslash\overline{D}_0) \text{ such that } \Delta u \in L^2(D\backslash\overline{D}_0)\right\}$$

and we define a weak solution to (ITP4.1) as follows:

Definition 2.1. *For given $g \in H^{3/2}(\Gamma)$ and $h \in H^{1/2}(\Gamma)$, a weak solution to* (ITPH) *is a pair of functions $w \in L^2(D\backslash\overline{D}_0)$ and $v \in L^2(D)$ satisfying the first two equations of* (ITPH) *in the distributional sense such that $w = 0$ on Σ and $u = w - v \in H^1_\Delta(D\backslash\overline{D}_0)$ satisfies the boundary conditions on Γ, $u = g$ and $\dfrac{\partial u}{\partial \nu} = h$.*

2.1. **Variational formulation.** In order to analyze (ITPH) we first write the problem as a forth order partial differential equation. To this end, let us assume that $1/|n-1| \in L^\infty(D\backslash\overline{D}_0)$ and let w and v be a weak solution to (ITPH). Then $u := w - v$ satisfies

$$\Delta u + k^2 n u = -k^2(n-1)v \quad \text{in } D\backslash\overline{D}_0. \tag{1}$$

Dividing both sides of (1) by $(n-1)$ and applying the operator $(\Delta + k^2)$ we get a fourth order equation for u in $D\backslash\overline{D}_0$

$$(\Delta + k^2)\frac{1}{n-1}(\Delta + k^2 n)u = 0 \quad \text{in } D\backslash\overline{D}_0 \tag{2}$$

together with the boundary conditions on Γ

$$u = g \quad ; \quad \frac{\partial u}{\partial \nu} = h \quad \text{on } \Gamma \tag{3}$$

and on Σ, we have that

$$u = -v \quad \text{on } \Sigma. \tag{4}$$

Furthermore v satisfies Helmholtz equation in D_0

$$\Delta v + k^2 v = 0 \text{ in } D_0 \tag{5}$$

with continuity of the Cauchy data across Σ that can be written using (1) as

$$\left(\frac{1}{k^2(n-1)}(\Delta+k^2n)u\right)^+ = v^- \text{ and } \frac{\partial}{\partial\nu}\left(\frac{1}{k^2(n-1)}(\Delta+k^2n)u\right)^+ = \frac{\partial v^-}{\partial\nu}. \tag{6}$$

Conversely, it is easily verified that a solution $u \in H^1_\Delta(D\backslash\overline{D}_0)$ and $v \in L^2(D_0)$ of (2)-(6) defines a weak solution w and v to (ITPH) by

$$v := \frac{-1}{k^2(n-1)}(\Delta+k^2n)u \text{ in } D\backslash\overline{D}_0 \text{ and } w := u+v \text{ in } D\backslash\overline{D}_0.$$

Thus (2)-(6) and the interior transmission problem are equivalent. Now, we are ready to write the interior the interior transmission problem in a variational formulation. Indeed for a solution (v,w) of (ITPH) we define u in D by $u = w - v$ in $D\backslash\overline{D}_0$ and $u = -v$ in D_0. Then clearly u is in $H^1(D) \cap H^1_\Delta(D\backslash\overline{D}_0)$, satisfies (2)-(3),

$$u^+ = u^- \text{ on } \Sigma,$$

$$\left(\frac{-1}{k^2(n-1)}(\Delta u+k^2nu)\right)^+ = -u^+ \text{ and } \frac{\partial}{\partial\nu}\left(\frac{-1}{k^2(n-1)}(\Delta u+k^2nu)\right)^+ = -\frac{\partial u^-}{\partial\nu} \text{ on } \Sigma$$

and

$$\Delta u + k^2 u = 0 \text{ in } D_0.$$

Taking a test function φ such that $\varphi = 0$ and $\dfrac{\partial\varphi}{\partial\nu} = 0$ on Γ, multiplying (2) by φ and integrating by parts and using the boundary conditions, we obtain

$$\begin{aligned}
0 &= \int_{D\backslash\overline{D}_0}(\Delta+k^2)\frac{1}{n-1}(\Delta u+k^2nu)\overline{\varphi}dx \\
&= \int_{D\backslash\overline{D}_0}(\Delta+k^2)\frac{1}{n-1}(\Delta u+k^2u)\overline{\varphi}dx + k^2\int_{D\backslash\overline{D}_0}(\Delta u+k^2u)\overline{\varphi}dx \\
&= \int_{D\backslash\overline{D}_0}\frac{1}{n-1}(\Delta u+k^2u)(\Delta\overline{\varphi}+k^2\overline{\varphi})dx + k^2\int_{D\backslash\overline{D}_0}(\Delta u+k^2u)\overline{\varphi}dx \\
&\quad + \int_\Sigma\left(\frac{1}{n-1}(\Delta u+k^2u)\right)^+\frac{\partial\varphi^+}{\partial\nu}ds - \int_\Sigma\frac{\partial}{\partial\nu}\left(\frac{1}{n-1}(\Delta u+k^2u)\right)^+\overline{\varphi}^+ds \\
&= \int_{D\backslash\overline{D}_0}\frac{1}{n-1}(\Delta u+k^2u)(\Delta\overline{\varphi}+k^2\overline{\varphi})dx + k^2\int_{D\backslash\overline{D}_0}(\Delta u+k^2u)\overline{\varphi}dx \\
&\quad + k^2\int_\Sigma\frac{\partial u^+}{\partial\nu}\overline{\varphi}^+ds - k^2\int_\Sigma\frac{\partial u^-}{\partial\nu}\overline{\varphi}^-ds \\
&= \int_{D\backslash\overline{D}_0}\frac{1}{n-1}(\Delta u+k^2u)(\Delta\overline{\varphi}+k^2\overline{\varphi})dx + k^4\int_{D\backslash\overline{D}_0}u\overline{\varphi}dx - k^2\int_{D\backslash\overline{D}_0}\nabla u\cdot\nabla\overline{\varphi}dx \\
&\quad + k^4\int_{D_0}u\overline{\varphi}dx - k^2\int_{D_0}\nabla u\cdot\nabla\overline{\varphi}dx \\
&= \int_{D\backslash\overline{D}_0}\frac{1}{n-1}(\Delta u+k^2u)(\Delta\overline{\varphi}+k^2\overline{\varphi})dx + k^4\int_D u\overline{\varphi}dx - k^2\int_D\nabla u\cdot\nabla\overline{\varphi}dx.
\end{aligned}$$

Now, let θ be a lifting function in $H^2(D)$ such that $\theta = g$ and $\dfrac{\partial\theta}{\partial\nu} = h$ on Γ. Then $u_0 := u - \theta \in H^1_0(D) \cap H^1_\Delta(D\backslash\overline{D}_0)$ and the natural variational space for the above variational problem is the Hilbert space given by

$$W := \left\{ u \in H_0^1(D) \cap H_\Delta^1(D\backslash\overline{D}_0) \,\text{such that}\, \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \right\}$$

equipped with the norm

$$||u||_W^2 = ||u||_{H^1(D)}^2 + ||\Delta u||_{L^(D\backslash\overline{D}_0)}^2.$$

Therefore, the variational formulation of the interior transmission problem becomes: find $u_0 \in W$ such that

$$(7) \quad \int_{D\backslash\overline{D}_0} \frac{1}{n-1}(\Delta u_0 + k^2 u_0)(\Delta\overline{\varphi} + k^2\overline{\varphi})dx + k^4 \int_D u_0\overline{\varphi}dx - k^2 \int_D \nabla u_0 \cdot \nabla\overline{\varphi}dx$$
$$= -\int_{D\backslash\overline{D}_0} \frac{1}{n-1}(\Delta\theta + k^2\theta)(\Delta\overline{\varphi} + k^2\overline{\varphi})dx - k^4 \int_D \theta\overline{\varphi}dx + k^2 \int_D \nabla\theta \cdot \nabla\overline{\varphi}dx$$

for all $\varphi \in W$. By taking appropriate test functions it is easy to see that a solution of the variational problem (7) defines a week solution to (2)-(6) and therefore to the interior transmission problem.

Remark 2.1. One can remark that on the contrary to the previously studied cases [10], since u is less regular, only the first order term on the left hand side of (7) defines a compact operator whereas the last term does not. Furthermore for n greater than one, the operator defined by the following bilinear from

$$\tilde{\mathcal{A}}_k(u,\varphi) := \int_{D\backslash\overline{D}_0} \frac{1}{n-1}(\Delta u + k^2 u)(\Delta\overline{\varphi} + k^2\overline{\varphi})dx - k^2 \int_D \nabla u \cdot \nabla\overline{\varphi}dx$$

has no chance to be coercive because of the negative sign in front of the last term of the operator. For this reason, using this variational formulation, we are only able to treat the problem for n less than one, since in this case we can show that $-\tilde{\mathcal{A}}_k$ is indeed coercive.

Next, we denote by $n_* = \inf_{D\backslash\overline{D}_0} n(x)$ and $n^* = \sup_{D\backslash\overline{D}_0} n(x)$ and from now on we assume that $n_* < n(x) < n^* < 1$.

Let us define the following sesquilinear forms

$$\mathcal{A}_k(u,\varphi) := \int_{D\backslash\overline{D}_0} \frac{1}{1-n}(\Delta u + k^2 u)(\Delta\overline{\varphi} + k^2\overline{\varphi})dx + k^4 \int_D u\overline{\varphi}dx + k^2 \int_D \nabla u \cdot \nabla\overline{\varphi}dx$$

and

$$\mathcal{B}(u,\varphi) := 2\int_D u\overline{\varphi}dx$$

and the bounded linear functional

$$\ell(\varphi) := -\int_{D\backslash\overline{D}_0} \frac{1}{n-1}(\Delta\theta + k^2\theta)(\Delta\overline{\varphi} + k^2\overline{\varphi})dx - k^4 \int_D \theta\overline{\varphi}dx + k^2 \int_D \nabla\theta \cdot \nabla\overline{\varphi}dx$$

Then the interior transmission problem in the variational form now consists of finding $u_0 \in W$ such that

$$\mathcal{A}_k(u_0,\varphi) - k^4\mathcal{B}(u_0,\varphi) = \ell(\varphi) \quad \text{for all } \varphi \in W.$$

Using the Riesz representation theorem we define two bounded linear operators $A_k : W \to W$ and $B : W \to W$ by

$$(A_k u, \varphi)_W := \mathcal{A}_k(u,\varphi) \text{ and } (Bu,\varphi)_W := \mathcal{B}(u,\varphi).$$

Theorem 2.1. *Assume that* $n_* < n(x) < n^* < 1$. *Then*

(i) *The operator $B : W \to W$ is compact.*
(ii) *The operator $A_k : W \to W$ is coercive.*

Proof. (i) The compactly embedding of $H^1(D)$ into $L^2(D)$ implies that B is compact operator on W.

(ii) Now we show that A_k is coercive. Setting $\gamma = \dfrac{1}{1-n_*}$ and using the equality

$$\gamma X^2 - 2\gamma XY + (1+\gamma)Y^2 = \varepsilon\left(Y - \frac{\gamma}{\varepsilon}X\right)^2 + \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)X^2 + (1+\gamma-\varepsilon)Y^2, \tag{8}$$

for $X = ||\Delta u||^2_{D\setminus\overline{D}_0}$ and $Y = k^2||u||_{D\setminus\overline{D}_0}$, where for a generic region $\mathcal{O} \in \mathbb{R}^d$, $\|\cdot\|_{\mathcal{O}}$ denotes the $L^2(\mathcal{O})$, we have

$$\begin{aligned}
(A_k u, u)_W &= \int_{D\setminus\overline{D}_0} \frac{1}{1-n}|\Delta u + k^2 u|^2 dx + k^4||u||^2_{D\setminus\overline{D}_0} + k^2||\nabla u||^2_D + k^4||u||^2_{D_0} \\
&\geq \gamma||\Delta u||^2_{D\setminus\overline{D}_0} - 2k^2\gamma||\Delta u||_{D\setminus\overline{D}_0}||u||_{D\setminus\overline{D}_0} + k^4(1+\gamma)||u||^2_{D\setminus\overline{D}_0} \\
&\quad + k^2||\nabla u||^2_D + k^4||u||^2_{D_0} \\
&\geq \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)||\Delta u||^2_{D\setminus\overline{D}_0} + k^4(\gamma+1-\varepsilon)||u||^2_{D\setminus\overline{D}_0} + k^2||\nabla u||^2_D + k^4||u||^2_{D_0}
\end{aligned}$$

where $\gamma < \varepsilon < \gamma + 1$. For such an ε, we conclude that there exists a constant $C > 0$ such that

$$(A_k u, u) \geq C||u||^2_W$$

for all $u \in W$ which proves that $A_k : W \to W$ is coercive.

□

The above theorem shows that the operator $A_k - k^4B$ is Fredholm with index zero, whence a solution exists if the uniqueness holds. In the following with be concerned with the injectivity $A_k - k^4B$ which leads to the study of the transmission eigenvalues which are in fact the of main interest in this paper.

2.2. **Transmission eigenvalues.** The interior transmission eigenvalue problem in the considered case is

$$\text{(TEP)} \qquad \begin{cases} \Delta w + k^2 n w = 0 & \text{in } D\setminus\overline{D}_0 \\ \Delta v + k^2 v = 0 & \text{in } D \\ w - v = 0 & \text{on } \Gamma \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma \\ w = 0 & \text{on } \Sigma. \end{cases}$$

As already known from the literature [4], [21], [14] this eigenvalue problem is non self-adjoint end therefor it may have complex transmission eigenvalues. However for this study we are limited to the case of real eigenvalues corresponding to (TEP).

Definition 2.2. *The values of $k > 0$ for which (TEP) has a nontrivial solution are called the* transmission eigenvalues.

In term of the operators defined above $k > 0$ is a transmission eigenvalue if the kernel of the operator $A_k - k^4B$ is nontrivial. In the following we are concerned with the existence and discreteness of transmission eigenvalues.

Theorem 2.2. *Assume that $n_* < n(x) < n^* < 1$. Then the set of transmission eigenvalues is discrete and $+\infty$ is the only possible accumulation point.*

Proof. To prove the discreteness of transmission eigenvalues we use the analytic Fredholm theory [13]. We have seen earlier that thanks to the coercivity of $\mathcal{A}_k(\cdot,\cdot)$, A_k^{-1} exists as a bounded operator on W. Thus, the transmission eigenvalues are the values of $k > 0$ for which $I - k^4 A_k^{-1} B$ has a nontrivial kernel. Furthermore, the operator A_k is obviously analytic with respect to $k \in \mathbb{C}$ and hence the mapping $k \mapsto A_k^{-1}$ is analytic in a neighbourhood of the real axis. To apply the analytic Fredholm theorem, it remains to show that $I - k^4 A_k^{-1} B$ or $A_k - k^4 B$ is injective for at least one k. To this end, we recall the Poincaré inequality which is valid for all $u \in H_0^1(D)$

$$||u||_D^2 \leq \frac{1}{\lambda_0(D)} ||\nabla u||_D^2$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D. Then, for all $u \in W$ we have that

$$\begin{aligned}\mathcal{A}_k(u,u) - k^4 \mathcal{B}(u,u) &= \int_{D\setminus\overline{D}_0} \frac{1}{1-n} |\Delta u + k^2 u|^2 dx - k^4 ||u||_D^2 + k^2 ||\nabla u||_D^2 \\ &\geq k^2 \left(||\nabla u||_D^2 - k^2 ||u||_D^2 \right) \\ &\geq k^2 ||\nabla u||_D^2 \left(1 - \frac{k^2}{\lambda_0(D)} \right).\end{aligned}$$

We deduce that $\mathcal{A}_k(u,u) - k^4 \mathcal{B}(u,u) > 0$ for all $k > 0$ such that $k^2 < \lambda_0(D)$ and hence $A_k - k^4 B$ is injective for such k. Hence, the analytical Fredholm theory implies that the set of transmission eigenvalues is discrete and from the analyticity with $+\infty$ and the only possible accumulation point. □

Remark 2.2. From the previous theorem, we deduce a lower bound for the first transmission eigenvalue. Indeed, if $k > 0$ is a transmission eigenvalue, then

$$k \geq \lambda_0(D).$$

Next we want to prove the existence of transmission eigenvalues following [10]. If we consider the generalized eigenvalue problem

$$A_k - \lambda(k) B u = 0 \qquad u \in W$$

which is known to have an infinite sequence of eigenvalues $\lambda_j(k)$, $j \in \mathbb{N}$, then the transmission eigenvalues are the solutions $\lambda_j(k) = k^4$ The proof of the existence of transmission eigenvalues makes use of the following theorem shown in [11]

Theorem 2.3. *Let $k \longmapsto A_k$ be a continuous mapping from $]0,\infty[$ to the set of self-adjoint and positive definite bounded linear operators on W and let B be a self-adjoint and non negative compact bounded linear operator on W. We assume that there exists two positive constant $k_0 > 0$ and $k_1 > 0$ such that*

1. *$A_{k_0} - k_1^4 B$ is positive on W,*
2. *$A_{k_1} - k_1^4 B$ is non positive on a m dimensional subspace of W.*

Then each of the equations $\lambda_j(k) = k^4$ for $j = 1, \ldots, m$, has at least one solution in $[k_0, k_1]$ where $\lambda_j(k)$ is the j^{th} eigenvalue (counting multiplicity) of A_k with respect to B, i.e. $\ker(A_k - \lambda_j(k)B) \neq \{0\}$.

Theorem 2.4. *Assume that $n_* < n(x) < n^* < 1$. There exist an infinite discrete set of transmission eigenvalues.*

Proof. We have already seen that for $k_0 < \lambda_0(D)$, then $A_{k_0} - k_1^4 B$ is positive in W. Now let us find k_1 such that $A_{k_1} - k_1^4 B$ is non positive in a subspace of W. Let B_r^j, $j = 1...M(r)$, be $M(r)$ balls of radius r included in $D\backslash\overline{D}_0$.

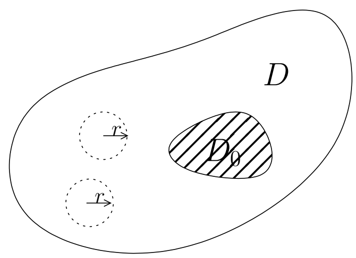

FIGURE 2. Balls of radius r included in $D\backslash\overline{D}_0$.

We denote by k_1 the first transmission eigenvalue corresponding to the interior transmission problem for B_r^j for all $j = 1..M(r)$ with index of refraction n^* which is know to exist [13], and let $u_j \in H_0^2(B_r^j)$, $1 \leq j \leq M(r)$, be the corresponding eigenvector which satisfy

$$\int_{B_r^j} \frac{1}{1-n^*}(\Delta u_j + k_1^2 n^* u_j)(\Delta\overline{\varphi} + k_1^2\overline{\varphi})dx = 0$$

for all $\varphi \in H_0^2(B_r^j)$. We denote by $\tilde{u}_j \in H_0^2(D)$ the extension of u_j by zero to the whole of D and we define a $M(r)$-dimensional subspace of W by $\mathcal{V} :=$ Vect $\{\tilde{u}_j, 1 \leq j \leq M(r)\}$. Since for $j \neq m$, $\tilde{u}_j$ and $\tilde{u}_m$ have disjoint support, for $u = \sum_{j=1}^{M(r)} \alpha_j \tilde{u}_j \in \mathcal{V}$, we have

$$\begin{aligned}
&\mathcal{A}_{k_1}(u,u) - k_1^4\mathcal{B}(u,u) \\
&\quad = \sum_{j=1}^{M(r)} |\alpha_j|^2 \left(\int_{D\backslash\overline{D}_0} \frac{1}{1-n}|\Delta\tilde{u}_j + k_1^2\tilde{u}_j|^2 dx - k_1^4 \int_D |\tilde{u}_j|^2 dx + k_1^2 \int_D |\nabla\tilde{u}_j|^2 dx \right) \\
&\quad = \sum_{j=1}^{M(r)} |\alpha_j|^2 \left(\int_{B_r^j} \frac{1}{1-n}|(\Delta u_j + k_1^2 u_j)|^2 dx - k_1^4 \int_{B_r^j} |u_j|^2 dx + k_1^2 \int_{B_r^j} |\nabla u_j|^2 dx \right) \\
&\quad \leq \sum_{j=1}^{M(r)} |\alpha_j|^2 \left(\frac{1}{1-n^*} \int_{B_r^j} |(\Delta u_j + k_1^2 u_j)|^2 dx - k_1^4 \int_{B_r^j} |u|^2 dx + k_1^2 \int_{B_r^j} |\nabla u_j|^2 dx \right) \\
&\quad = \sum_{j=1}^{M(r)} |\alpha_j|^2 \left(\int_{B_r^j} \frac{1}{1-n^*}(\Delta u_j + k_1^2 n^* u_j)(\Delta\overline{u}_j + k_1^2\overline{u}_j)dx \right) = 0.
\end{aligned}$$

Thus, we conclude that there exist $M(r)$ transmission eigenvalues in $]\lambda_0(D), k_1]$. Letting $r \to 0$, we have that $M(r) \to \infty$ and thus we can now deduce that there exists an infinite set of transmission eigenvalues.

□

We close this section with a monotonicity result for the first transmission eigenvalue with respect to the size of D_0, which can be useful in non-destructive testing.

We denote by $k_1(D_0, n)$ the first transmission eigenvalue corresponding to (ITPH) with a perfect conductor D_0 and index of refraction n inside $D\backslash\overline{D}_0$.

Theorem 2.5. *Let $D_0 \subset D_0'$ and $n < 1$. Then*

$$k_1(D_0', n) \leq k_1(D_0, n).$$

Proof. Let $\tilde{u} \in W$ be the eigenvector corresponding to $k_1(D_0, n)$. Then $\tilde{u}$ satisfies

$$\int_{D\backslash\overline{D}_0} \frac{1}{1-n}|\Delta\tilde{u}+k_1(D_0,n)^2\tilde{u}|^2dx - k_1(D_0,n)^4\int_D |\tilde{u}|^2dx + k_1(D_0,n)^2\int_D |\nabla\tilde{u}|^2dx = 0.$$

Since $D\backslash D_0' \subset D\backslash\overline{D}_0$, we have $\tilde{u} \in W(D_0) \subset W(D_0')$ and

$$\begin{aligned}\mathcal{A}_{k_1(D_0,n)}(\tilde{u},\tilde{u}) - k_1(D_0,n)^4\mathcal{B}(\tilde{u},\tilde{u}) &= \int_{D\backslash\overline{D}_0'} \frac{1}{1-n}|\Delta\tilde{u} + k_1(D_0,n)^2\tilde{u}|^2dx \\ &\quad - k_1(D_0,n)^4\int_D |\tilde{u}|^2dx + k_1(D_0,n)^2\int_D |\nabla\tilde{u}|^2dx \\ &\leq \int_{D\backslash\overline{D}_0} \frac{1}{1-n}|\Delta\tilde{u} + k_1(D_0,n)^2\tilde{u}|^2dx - k_1(D_0,n)^4\int_D |\tilde{u}|^2dx \\ &\quad + k_1(D_0,n)^2\int_D |\nabla\tilde{u}|^2dx = 0.\end{aligned}$$

Hence $(A_{k_1(D_0,n)} - k_1(D_0,n)^4B)\tilde{u} < 0$, where $A_{k_1(D_0,n)}$ and B are the operators corresponding to $D\backslash\overline{D}_0'$ and thus can deduce that $k_1(D_0', n) \leq k_1(D_0, n)$. □

Remark 2.3. The Fredholm property of the interior transmission problem and the discreteness of transmission eigenvalues can be proven also for complex valued index of refraction n such that $1 > \Re(n) \geq c > 0$ and $\Im(n) \geq 0$. It merely suffices to take the real part of $\mathcal{A}(\cdot,\cdot)$ when proving the coercivity property in part (ii) Theorem 2.1. However, it is easy to show by taking the $\Im(\mathcal{A}_k(u,u) - k^4\mathcal{B}(u,u))$ that there are no transmission eigenvalues if $\Im(n) > 0$ almost everywhere in $D\backslash\overline{D}_0$.

3. **The anisotropic case.** In this section, we consider that the medium inside $D\backslash\overline{D}_0$ is anisotropic. In particular, let A be a $d \times d$, $d = 2, 3$ matrix-real valued function whose entries are in $L^\infty(D\backslash\overline{D}_0)$ such that A is symmetric and $(\overline{\xi}\cdot A(x)\xi) \geq c > 0$, $(\overline{\xi}\cdot A(x)\xi) \geq c' > 0$, for all $\xi \in \mathbb{C}^d$. Again, we take $n \in L^\infty(D\backslash\overline{D}_0)$ to be a real valued function such that $n \geq c > 0$. We focus here only in the study of interior transmission eigenvalue problem which in this case reads: find $v \in H^1(D)$ and $w \in H^1(D\backslash\overline{D}_0)$ such that

$$\text{(TEPA)} \qquad \begin{cases} \nabla\cdot A\nabla w + k^2 n w = 0 & \text{in } D\backslash\overline{D}_0 \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \Gamma \\ \nu\cdot A\nabla w = \nu\cdot\nabla v & \text{on } \Gamma \\ w = 0 & \text{on } \Sigma. \end{cases}$$

As it will become clear later on, if one is interested in the solvability of the interior transmission problem with nonzero boundary data, our analysis proves the Fredholm structure of the problem. Again we focus on real values of k and define transmission eigenvalues as follows:

Definition 3.1. *The values of $k > 0$ for which (TEPA) has a nontrivial solution are called* transmission eigenvalues.

Due to the nature of the problem we employ different techniques for proving the discreetness and the existence of transmission eigenvalues. We star with the discreteness question.

In the following, we denote by

$$\gamma^* := \sup_{D\setminus\overline{D}_0} \sup_{||\xi||=1} (\overline{\xi} \cdot A(x)\xi) \quad \text{and} \quad \gamma_* := \inf_{D\setminus\overline{D}_0} \inf_{||\xi||=1} (\overline{\xi} \cdot A(x)\xi).$$

3.1. **The discretness of transmission eigenvalues.** To find a variational formulation for the system (TEPA), we multiply the first and second equations by w' and v' respectively, where v' and w' are two test functions such that $w' = 0$ on Σ and integrate by parts to obtain

$$\int_{D\setminus\overline{D}_0} A\nabla w \cdot \nabla \overline{w}' dx - k^2 \int_{D\setminus\overline{D}_0} n w \overline{w}' dx - \int_\Gamma \overline{w}' \frac{\partial w}{\partial \nu_A} ds = 0 \tag{9}$$

and

$$- \int_D \nabla v \cdot \nabla \overline{v}' dx + k^2 \int_D v \overline{v}' dx + \int_\Gamma \overline{v}' \frac{\partial v}{\partial \nu} ds = 0. \tag{10}$$

Adding both (9) and (10) and using the boundary conditions, we have that

$$\int_{D\setminus\overline{D}_0} A\nabla w \cdot \nabla \overline{w}' dx - \int_D \nabla v \cdot \nabla \overline{v}' dx + k^2 \int_D v \overline{v}' dx - k^2 \int_{D\setminus\overline{D}_0} n w \overline{w}' dx = 0$$

Setting

$$\mathbb{H} := \left\{ (v, w) \in H^1(D) \times H^1(D\setminus\overline{D}_0) / w = 0 \text{ on } \Sigma, \text{ such that } v = w \text{ on } \Gamma \right\},$$

the variational formulation of (TEPA) becomes: find (v, w) in $\mathbb{H}$ such that for all (v', w') in $\mathbb{H}$,

$$a_k((v, w), (v', w')) = 0 \tag{11}$$

where

$$\begin{aligned} a_k((v, w), (v', w')) &= \int_{D\setminus\overline{D}_0} A\nabla w \cdot \nabla \overline{w}' dx - \int_D \nabla v \cdot \nabla \overline{v}' dx \\ &+ k^2 \int_D v \overline{v}' dx - k^2 \int_{D\setminus\overline{D}_0} n w \overline{w}' dx. \end{aligned}$$

One can easily verify that finding a solution to (11) is equivalent to finding a solution to (TEPA).

Obviously, due to the negative sign in front of the term $\int_D \nabla v \cdot \nabla \overline{v}' dx$, it is not possible to show directly that the variational formulationleads to a Fredholm type. To get around this difficulty, we use the concept of T-coercivity which has been initially used for the study of metamaterials in [2] and [1]. To this end let us recall the T-coercivity concept.

Definition 3.2. *Let T be a bijective bounded linear operator on a Hilbert space V. A bilinear form $b(\cdot, \cdot)$ is T-coercive on $V \times V$ if*

$$\exists \gamma > 0, \forall v \in V, |b(v, Tv)| \geq \gamma ||v||_V^2.$$

The proof of the following theorem can be found in [2].

Theorem 3.1. *Let $\ell(\cdot)$ be a continuous linear form on V and let $a(\cdot,\cdot)$ be a continuous bilinear form on $V \times V$. Assume that a can be splitted as $a(\cdot,\cdot) = b(\cdot,\cdot) + c(\cdot,\cdot)$ where the bilinear forms $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are both continuous and linear on $V \times V$, and that the bounded linear operator $C \in \mathcal{L}(V)$ associated with $c(\cdot,\cdot)$ is compact. Assume moreover that there exists a bijective bounded linear $T \in \mathcal{L}(V)$ such that $b(\cdot,\cdot)$ is T-coercive on $V \times V$. Then the variational problem of finding $u \in V$ such that*

$$\forall v \in V, a(u,v) = \ell(v) \tag{12}$$

has a solution if and only if the uniqueness holds (i.e. the only solution of (12) *with $\ell = 0$ is $u = 0$).*

3.1.1. *The case of $(A - I)$ positive.* In this section, we assume that $1 < \gamma_* < \gamma^*$. Our goal is now to apply Theorem 3.1 to (11), and the key is to be able to construct an appropriate bijection $T \in \mathcal{L}(\mathbb{H})$. An obvious first idea would be to consider the linear operator of the form $T(v,w) := (-v,w)$ in order to to change the sign of $\displaystyle\int_D \nabla v \cdot \nabla \overline{v}' dx$ in the variational formulation (11). Unfortunately, $(-v,w)$ is not in $\mathbb{H}$ since $-v \neq w$ on Γ. Thus, we need to modify this operator so that it satisfies all the properties of $\mathbb{H}$. To this end, we introduce the step function χ such that $\chi = 1$ in $D \backslash \overline{D}_0$ and $\chi = 0$ in D_0. We now define the bijective bounded linear operator $T : \mathbb{H} \to \mathbb{H}$ $(T^2 = I)$ by

$$T: \begin{array}{ccc} \mathbb{H} & \to & \mathbb{H} \\ (v,w) & \mapsto & (-v + 2\chi w, w). \end{array}$$

Since $w = 0$ on Σ, the function $-v + 2\chi w$ is continuous across Σ which implies that the function $-v + 2\chi w$ is in $H^1(D)$ and consequently the operator T is well defined on $\mathbb{H}$. Now, with the help of T we can define a new bilinear form

$$\begin{aligned} \tilde{a}_k((v,w),(v',w')) &= a_k((v,w),T(v',\overline{w}')) \\ &= \int_{D\backslash \overline{D}_0} A\nabla w \cdot \nabla \overline{w}' dx + \int_D \nabla v \cdot \nabla \overline{v}' dx - k^2 \int_D v\overline{v}' dx \\ &\quad - k^2 \int_{D\backslash \overline{D}_0} nw\overline{w}' dx - 2\int_D \nabla v \cdot \nabla(\chi \overline{w}') dx + 2k^2 \int_D v\chi\overline{w'} dx \end{aligned}$$

and we show in the following that it satisfies the Fredholm property.

Lemma 3.2. *The bilinear form $\tilde{a}_k(\cdot,\cdot) : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ satisfies the Fredholm property.*

Proof. We can write $\tilde{a}_k((v,w),(v',w')) = b((v,w),(v',w')) + c_k((v,w),(v',w'))$ where

$$\begin{aligned} b((v,w),(v',w')) = &\int_{D\backslash \overline{D}_0} A\nabla w \cdot \nabla \overline{w}' dx + \int_D \nabla v \cdot \nabla \overline{v}' dx \\ &- 2\int_{D\backslash \overline{D}_0} \nabla v \cdot \nabla \overline{w}' dx + \int_D v\overline{v}' dx + \int_{D\backslash \overline{D}_0} w\overline{w}' dx \end{aligned}$$

and

$$c_k((v,w),(v',w')) = -(k^2+1)\int_D v\overline{v}' dx - \int_{D\backslash \overline{D}_0} (k^2 n + 1) w\overline{w}' dx + 2k^2 \int_{D\backslash \overline{D}_0} v\overline{w}' dx.$$

From Riesz's representation theorem, we define the bounded linear operator C_k from $\mathbb{H}$ into $\mathbb{H}$ by

$$c_k((v,w),(v',w')) = (C_k(v,w),(v',w')).$$

The compact embedding of $H^1(D)$ into $L^2(D)$ implies that C_k is a compact operator for all $k>0$. We now show that $b(\cdot,\cdot)$ is coercive.

$$
\begin{aligned}
b((v,w),(v,w)) = & \int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}dx + \int_D |\nabla v|^2dx + ||v||^2_D + ||w||^2_{D\setminus\overline{D}_0}\\
& - 2\int_{D\setminus\overline{D}_0} \nabla v\cdot\nabla\overline{w}dx\\
& \geq \gamma_*||\nabla w||^2_{D\setminus\overline{D}_0} + ||\nabla v||^2_D + ||v||^2_D + ||w||^2_{D\setminus\overline{D}_0} - 2\int_{D\setminus\overline{D}_0} \nabla v\cdot\nabla\overline{w}dx.
\end{aligned}
$$

Using the following inequality

$$
\begin{aligned}
\left|-2\int_{D\setminus\overline{D}_0} \nabla v\cdot\nabla\overline{w}dx\right| & \leq \int_{D\setminus\overline{D}_0} |\nabla v\cdot\nabla\overline{w}|dx\\
& \leq \frac{1}{\eta}||\nabla v||^2_{D\setminus\overline{D}_0} + \eta||\nabla w||^2_{D\setminus\overline{D}_0}
\end{aligned}
$$

with $\eta>0$, we then obtain

$$
\begin{aligned}
b((v,w),(v,w)) & \geq (\gamma_*-\eta)||\nabla w||^2_{D\setminus\overline{D}_0} + \left(1-\frac{1}{\eta}\right)||\nabla v||^2_D + ||v||^2_D + ||w||^2_{D\setminus\overline{D}_0}\\
& \geq C\left(||v||^2_{H^1(D)} + ||w||^2_{H^1(D\setminus\overline{D}_0)}\right)
\end{aligned}
$$

with $C>0$ if $1<\eta<\gamma_*$. We can finally conclude from (a slightly modified version of)Theorem 3.1 that $\tilde{a}_k(\cdot,\cdot)$ satisfies the Fredholm property. □

From the above theorem the bounded linear operator $B:\mathbb{H}\to\mathbb{H}$ defined by mean of Riesz's representation theorem as

$$
b((v,w),(v',w')) = (B(v,w),(v',w'))
$$

is invertible

Remark 3.1. Note that the operator $C_k:\mathbb{H}\to\mathbb{H}$ depends analytically on $k\in\mathbb{C}$. Also note that the operator B does not depend on k. Thus the eigenvalue problem becomes $(I+B^{-1}C_k)(v,w)=0$ where $B^{-1}C_k:\mathbb{H}\to\mathbb{H}$ is compact and the mapping $k\to B^{-1}C_k$ is analytic in $\mathbb{C}$.

Theorem 3.3. *Assume that* $1<\gamma_*<\gamma^*<\infty$ *and* $0<n_*\leq n(x)\leq n^*<\infty$ *where where* $\gamma^* := \sup_{D\setminus\overline{D}_0}\sup_{||\xi||=1}(\overline{\xi}\cdot A(x)\xi)$, $\gamma_* := \inf_{D\setminus\overline{D}_0}\inf_{||\xi||=1}(\overline{\xi}\cdot A(x)\xi)$, $n_* = \inf_{D\setminus\overline{D}_0} n(x)$ *and* $n^* = \sup_{D\setminus\overline{D}_0} n(x)$. *Then the set of transmission eigenvalues is discrete.*

Proof. To apply the analytic Fredholm theory, from Remark 3.1 it remains to show that there exists a $k \in \mathbb{C}$ for which $B + C_k$ is injective. We set $k = \mathrm{i}\kappa$.

$$\begin{aligned}
\tilde{a}_{\mathrm{i}\kappa}((v,w),(v,w)) = & \int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}dx + \int_D |\nabla v|^2 dx + \kappa^2\int_D |v|^2dx + \kappa^2\int_{D\setminus\overline{D}_0} n|w|^2dx \\
& - 2\int_{D\setminus\overline{D}_0} \nabla v\cdot\nabla\overline{w}dx - 2\kappa^2\int_{D\setminus\overline{D}_0} v\overline{w}dx \\
\geq & \gamma_*||\nabla w||^2_{D\setminus\overline{D}_0} + ||\nabla v||^2_D + \kappa^2||v||^2_D + \kappa^2 n_*||w||^2_{D\setminus\overline{D}_0} \\
& - \frac{1}{\eta}||\nabla v||^2_D - \eta||\nabla w||^2_{D\setminus\overline{D}_0} - \frac{\kappa^2}{\alpha}||v||^2_D - \kappa^2\alpha||w||^2_{D\setminus\overline{D}_0} \\
\geq & (\gamma_* - \eta)||\nabla w||^2_{D\setminus\overline{D}_0} + \left(1-\frac{1}{\eta}\right)||\nabla v||^2_D + \kappa^2\left(1-\frac{1}{\alpha}\right)||v||^2_D \\
& + \kappa^2\left(n_* - \alpha\right)||w||^2_{D\setminus\overline{D}_0}
\end{aligned}$$

where $n_* = \inf_{D\setminus\overline{D}_0} n(x)$. Furthermore, $w \in H^1(D\setminus\overline{D}_0)$ and it vanishes on the boundary Σ which implies the Poincaré inequality

$$||w||^2_{D\setminus\overline{D}_0} \leq \lambda||\nabla w||^2_{D\setminus\overline{D}_0},$$

and consequently

$$\begin{aligned}
\tilde{a}_{\mathrm{i}\kappa}((v,w),(v,w)) \geq & \left((\gamma_* - \eta) - \kappa^2\lambda|n_* - \alpha|\right)||\nabla w||^2_{D\setminus\overline{D}_0} \\
& + \kappa^2\left(1-\frac{1}{\alpha}\right)||v||^2_D + \left(1-\frac{1}{\eta}\right)||\nabla v||^2_D.
\end{aligned}$$

Then, for κ^2 small enough, $1 < \eta < \gamma_*$ and $\alpha > 1$, we deduce that $\tilde{a}_{\mathrm{i}\kappa}$ is coercive and $B + C_{i\kappa}$ is injective. The analytic Fredholm theory now ensures the discreteness of the set of transmission eigenvalues. □

Note that the discreteness of transmission eigenvalues for the case of $A - I > 0$ is proven without any sign requirement on the contrast $n - 1$.

3.1.2. *The case of* $(I - A)$ *positive.* In this section, we assume that $0 < \gamma_* < \gamma^* < 1$ We again use the T-coercivity to show discreteness of transmission eigenvalues. As it will become clear later on, for this case we can prove the discretnes under the additional assumption that $n < 1$ only.

We recall that (v,w) is a solution to the interior transmission problem (TEPA) if and only if $u \in \mathbb{H}$ is the solution of the variational problem (11). Now, we use the cutoff function $\chi \in C^\infty(D)$ satisfying $0 \leq \chi \leq 1$ in $D\setminus\overline{D}_0$ and $\mathrm{supp}(\chi) \cap D_0 = \emptyset$. Similarly to the approach in Section 3.1.1, we define a bijective bounded linear operator T from $\mathbb{H}$ to $\mathbb{H}$ by

$$\begin{aligned}
T: \quad \mathbb{H} \quad & \rightarrow \quad \mathbb{H} \\
(v,w) \quad & \mapsto \quad (-v, w - 2\chi v).
\end{aligned}$$

Again we consider the new bilinear form $\tilde{a}_k$ given by

$$\begin{aligned}\tilde{a}_k((v,w),(v',w')) &= a_k((v,w),T(v',\overline{w}'))\\ &= \int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}'dx + \int_D \nabla v\cdot\nabla\overline{v}'dx - k^2\int_D v\overline{v}'dx\\ &- k^2\int_{D\setminus\overline{D}_0} nw\overline{w}'dx - 2\int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla(\chi\overline{v}')dx + 2k^2\int_{D\setminus\overline{D}_0} nw\chi\overline{v}'dx.\end{aligned}$$

Lemma 3.4. *The bilinear form $\tilde{a}_k(\cdot,\cdot)$ satisfies the Fredholm property.*

Proof. We can write $\tilde{a}_k((v,w),(v',w')) = b((v,w),(v',w'))+c_k((v,w),(v',w'))$ where

$$\begin{aligned}b((v,w),(v',w')) = &\int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}'dx + \int_D \nabla v\cdot\nabla\overline{v}'dx\\ &- 2\int_{D\setminus\overline{D}_0}\chi A\nabla w\cdot\nabla\overline{v}'dx + \int_D v\overline{v}'dx + \int_{D\setminus\overline{D}_0} w\overline{w}'dx\end{aligned}$$

and

$$\begin{aligned}c_k((v,w),(v',w')) = &-(k^2+1)\int_D v\overline{v}'dx - \int_{D\setminus\overline{D}_0}(k^2n+1)w\overline{w}'dx\\ &- 2\int_{D\setminus\overline{D}_0}\overline{v}'A\nabla w\cdot\nabla\chi dx + 2k^2\int_{D\setminus\overline{D}_0} nw\chi\overline{v}'dx.\end{aligned}$$

From Riesz's representation theorem, we define the bounded operator C_k from $\mathbb{H}$ into $\mathbb{H}$ by

$$c_k((v,w),(v',w')) = (C_k(v,w),(v',w'))_{\mathbb{H}}.$$

The compact embedding of $H^1(D)$ into $L^2(D)$ implies that C_k is a compact operator for all $k>0$. Next we show that $b(\cdot,\cdot)$ is coercive. To this end, let (v,w) be in $\mathbb{H}$.

$$\begin{aligned}b((v,w),(v,w)) &= \int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}dx + \int_D|\nabla v|^2dx + ||v||_D^2 + ||w||^2_{D\setminus\overline{D}_0}\\ &- 2\int_{D\setminus\overline{D}_0}\chi A\nabla w\cdot\nabla\overline{v}dx\\ &\geq \frac{1}{\gamma^*}||A\nabla w||^2_{D\setminus\overline{D}_0} + ||\nabla v||_D^2 + ||v||_D^2 + ||w||^2_{D\setminus\overline{D}_0}\\ &- 2\int_{D\setminus\overline{D}_0}\chi A\nabla w\cdot\nabla\overline{v}dx.\end{aligned}$$

Using the following inequality

$$\begin{aligned}\left|-2\int_{D\setminus\overline{D}_0}\chi A\nabla w\cdot\nabla\overline{v}dx\right| &\leq \int_{\mathrm{supp}(\chi)}|A\nabla w\cdot\nabla\overline{v}|dx\\ &\leq \eta||\nabla v||_D^2 + \frac{1}{\eta}||A\nabla w||^2_{D\setminus\overline{D}_0}\end{aligned}$$

with $\eta > 0$ to be chosen later. Then

$$\begin{aligned}
b((v,w),(v,w)) &\geq \frac{1}{\gamma^*}||A\nabla w||^2_{D\setminus\overline{D}_0} + ||\nabla v||^2_D + ||v||^2_D + ||w||^2_{D\setminus\overline{D}_0} \\
&\quad - \eta||\nabla v||^2_{D\setminus\overline{D}_0} - \frac{1}{\eta}||A\nabla w||^2_{D\setminus\overline{D}_0} \\
&\geq \left(\frac{1}{\gamma^*} - \frac{1}{\eta}\right)||\nabla w||^2_{D\setminus\overline{D}_0} + (1-\eta)||\nabla v||^2_D + ||v||^2_D + ||w||^2_{D\setminus\overline{D}_0} \\
&\geq C\left(||v||^2_{H^1(D)} + ||w||^2_{H^1(D\setminus\overline{D}_0)}\right)
\end{aligned}$$

with $C > 0$ if $\gamma^* < \eta < 1$. We can conclude that $\tilde{a}_k$ satisfies the Fredholm property. □

Again we define the invertible bounded linear operator $B : \mathbb{H} \to \mathbb{H}$ associated with the coersive bilinear form $b(\cdot,\cdot)$ as follows $b((v,w),(v',w')) = (B(v,w),(v',w'))_{\mathbb{H}}$. The the transmission eigenvalue problem is equivalent to

$$(B + C_k)u = 0 \qquad \text{or} \qquad (I + B^{-1}C_k)u = 0 \qquad \text{in } \mathbb{H}. \tag{13}$$

Furthermore the mapping $k \to C_k$ is analytic in $\mathbb{C}$.

Remark 3.2. *One can remark that the Fredholm property of $\tilde{a}_k(\cdot,\cdot)$ holds true for any $n \geq c > 0$. The restriction on the sign of $n-1$ appears in the next theorem, and is needed to show that there exists at least one k for which $B + C_k$ is injective.*

Theorem 3.5. *Assume that $0 < \gamma_* < \gamma^* < 1$ and $0 < n_* \leq n(x) \leq n^* < 1$ where $\gamma^* := \sup_{D\setminus\overline{D}_0}\sup_{||\xi||=1}(\overline{\xi}\cdot A(x)\xi)$, $\gamma_* := \inf_{D\setminus\overline{D}_0}\inf_{||\xi||=1}(\overline{\xi}\cdot A(x)\xi)$, $n_* = \inf_{D\setminus\overline{D}_0} n(x)$ and $n^* = \sup_{D\setminus\overline{D}_0} n(x)$. Then the set of transmission eigenvalues is discrete.*

Proof. To apply the analytic Fredholm theory to (13), it remains to show that there exists a k for which $B + C_k$ is injective. To this end

$$\begin{aligned}
\tilde{a}_{i\kappa}((v,w),(v,w)) &= \int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}dx + \int_D |\nabla v|^2dx + \kappa^2\int_D |v|^2dx \\
&\quad + \kappa^2\int_{D\setminus\overline{D}_0} n|w|^2dx - 2\int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla(\chi\overline{v})dx - 2\kappa^2\int_{\mathrm{supp}(\chi)} nw\overline{v} \\
&\geq \frac{1}{\gamma^*}||A\nabla w||^2_{D\setminus\overline{D}_0} + ||\nabla v||^2_D + \kappa^2||v||^2_D + \frac{\kappa^2}{n^*}||nw||^2_{D\setminus\overline{D}_0} - \frac{1}{\eta}||A\nabla w||^2_{D\setminus\overline{D}_0} \\
&\quad - \eta||\nabla v||^2_D - \frac{1}{\alpha}||A\nabla w||^2_{D\setminus\overline{D}_0} - \alpha C||v||^2_D - \frac{\kappa^2}{\beta}||nw||^2_{D\setminus\overline{D}_0} - \kappa^2\beta||v||^2_D \\
&\quad \left(\frac{1}{\gamma^*} - \frac{1}{\eta} - \frac{1}{\alpha}\right)||A\nabla w||^2_{D\setminus\overline{D}_0} + \left(\kappa^2(1-\beta) - \alpha C\right)||v||^2_D \\
&\quad + (1-\eta)||\nabla v||^2_D + \kappa^2\left(\frac{1}{n^*} - \frac{1}{\beta}\right)||nw||^2_{D\setminus\overline{D}_0}
\end{aligned}$$

where $C = ||\nabla\chi||^2$.

Let $\gamma^* < \eta < 1$, $n^* < \beta < 1$ and α be such that $\frac{1}{\gamma^*} - \frac{1}{\eta} - \frac{1}{\alpha} > 0$. Then for κ large enough we have that $\kappa^2(1-\beta) - \alpha C > 0$, and thus $\tilde{a}_{i\kappa}$ is coercive which means $B + C_{i\kappa}$ is injective. Then the analytic Fredholm theory now ensures the discreteness of the set of transmission eigenvalues. □

3.2. **The existence of transmission eigenvalues.** The T-coercivity approach does not provide any framework for proving the existence of transmission eigenvalues. For this question we adapt the approach introduced in [12], [18] to treat the case $A - I >)$ and $n > 1$ or $n < 1$. Unfortunately, due to the presence of the Dirichlet obstacle D_0 this approach provides only the existence of a finite set of transmission eigenvalues provided that the area of D_0 is small enough. In the case when $n > 1$ we also require n to be small enough. The existence of transmission eigenvalues for $I - A > 0$ is still open.

Throughout this section we assume that $1 < \gamma_* < \gamma^* < +\infty$ where $\gamma^* := \sup_{D\backslash\overline{D}_0} \sup_{||\xi||=1}(\overline{\xi} \cdot A(x)\xi)$ and $\gamma_* := \inf_{D\backslash\overline{D}_0} \inf_{||\xi||=1}(\overline{\xi} \cdot A(x)\xi)$. Recall that $n_* = \inf_{D\backslash\overline{D}_0} n(x)$ and $n^* = \sup_{D\backslash\overline{D}_0} n(x)$.

If we consider the new variable $u := w - v$ in $D\backslash\overline{D}_0$, then u is in $H^1(D\backslash\overline{D}_0)$, $u = 0$ on Γ and v satisfies the mixed boundary problem depending on u in $D\backslash\overline{D}_0$

$$\begin{cases} \nabla \cdot (I - A)\nabla v + k^2(1-n)v = \nabla \cdot A\nabla u + k^2 nu & \text{in } D\backslash\overline{D}_0, \\ \nu \cdot (A - I)\nabla v = \nu \cdot A\nabla u & \text{on } \Gamma, \\ -v = u & \text{on } \Sigma. \end{cases} \tag{14}$$

We define

$$H^1_\Gamma(D\backslash\overline{D}_0) := \left\{ u \in H^1(D\backslash\overline{D}_0) \text{ such that } u = 0 \text{ on } \Gamma \right\}$$

and

$$H^1_\Sigma(D\backslash\overline{D}_0) := \left\{ u \in H^1(D\backslash\overline{D}_0) \text{ such that } u = 0 \text{ on } \Sigma \right\}.$$

The next step is to solve the mixed boundary value problem (14) for v as a function of u. To this end, for a fixed $u \in H^1_\Gamma(D\backslash\overline{D}_0)$, we define the lifting function $\theta \in H^1(D\backslash\overline{D}_0)$ such that $\theta = -u$ on Σ. Setting $v_0 := v - \theta$, the variational formulation of (14) as a problem for v_0 now becomes: find $v_0 \in H^1_\Sigma(D\backslash\overline{D}_0)$ such that

$$\begin{aligned} &\int_{D\backslash\overline{D}_0} \big((A - I)\nabla v_0 \cdot \nabla\overline{\varphi} - k^2(n-1)v_0\overline{\varphi}\big)\,dx \\ &= -\int_{D\backslash\overline{D}_0} \big(A\nabla u \cdot \nabla\overline{\varphi} - k^2 nu\overline{\varphi}\big)\,dx - \int_{D\backslash\overline{D}_0} \big((A - I)\nabla\theta \cdot \nabla\overline{\varphi} - k^2(n-1)\theta\overline{\varphi}\big)\,dx \end{aligned} \tag{15}$$

for all $\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)$.

First, we want to show that problem (15) is well-posed using Lax-Milgram theorem. Since the right-hand side is obviously a continuous function of φ in $H^1_\Sigma(D\backslash\overline{D}_0)$, it only remains to show that the left-hand side is coercive. In the next theorem, we see that the latter is always true for $n < 1$ or for $n > 1$ small enough. Setting

$$\mu := \inf_{\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)} \frac{||\nabla\varphi||^2_{D\backslash\overline{D}_0}}{||\varphi||^2_{D\backslash\overline{D}_0}},$$

we have that for all $\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)$,

$$\frac{\mu}{\mu+1}||\varphi||^2_{H^1(D\backslash\overline{D}_0)} \leq ||\nabla\varphi||^2_{D\backslash\overline{D}_0}.$$

Note that $\mu > 0$ coincides with the first eigenvalue of $-\Delta$ in $D\backslash\overline{D}_0$ with mixed Neumann-Dirichlet boundary conditions.

Let B_r be a ball of radius r included in $D\backslash\overline{D}_0$ and let $\hat{k}' > 0$ be the first transmission eigenvalue of the interior transmission problem for B_r with $A = \frac{\gamma_*}{2}I$ and $n = 1$:

$$\begin{cases} \nabla \cdot \frac{\gamma_*}{2}\nabla w + k^2 w = 0 & \text{in } B_r \\ \Delta v + k^2 v = 0 & \text{in } B_r \\ w = v & \text{on } \partial B_r \\ \nu \cdot \frac{\gamma_*}{2}\nabla w = \nu \cdot \nabla v & \text{on } \partial B_r \end{cases} \tag{16}$$

The existence of such $\hat{k}' > 0$ is proven in [12], [13]. In the case when $n-1$ is positive, i.e $n_* > 1$, we further assume that

$$n^* - 1 \leq \frac{\gamma_* \mu}{2\hat{k}'^2}. \tag{17}$$

Lemma 3.6. *For every u in $H^1_\Gamma(D\backslash\overline{D}_0)$ and $k \geq 0$ satisfying $k \leq \hat{k}'$ if $n > 1$, there exists a unique solution $v_0 \in H^1_\Sigma(D\backslash\overline{D}_0)$ of* (15) *and consequently a unique $v_u := v_0 + \theta \in H^1(D\backslash\overline{D}_0)$ of* (14).

Proof. We denote

$$B_k(v,\varphi) := \int_{D\backslash\overline{D}_0} \big((A - I)\nabla v \cdot \nabla\overline{\varphi} - k^2(n-1)v\overline{\varphi}\big)\, dx$$

First assume that $1 - n > 0$. Then

$$\begin{aligned} B_k(v,v) &\geq (\gamma_* - 1)||\nabla v||^2_{D\backslash\overline{D}_0} \\ &\geq (\gamma_* - 1)\frac{\mu}{\mu+1}||v||^2_{H^1(D\backslash\overline{D}_0)}. \end{aligned}$$

Thus B_k is coercive for $k \geq 0$ if $n - 1 < 0$. From Lax-Milgram theorem, we deduce that there exists a unique solution v_0 of (15) depending continuously on u.

Now assume that $n - 1 > 0$ and more precisely that n satisfies (17)

$$\begin{aligned} B_k(v,v) &\geq (\gamma_* - 1)||\nabla v||^2_{D\backslash\overline{D}_0} - (k^2)(n^* - 1)||v||^2_{D\backslash\overline{D}_0} \\ &\geq \left((\gamma_* - 1)^2 - \frac{\hat{k}'^2(n^* - 1)}{\mu}\right)||\nabla v||^2_{D\backslash\overline{D}_0} \\ &\geq \left(\frac{\gamma_*}{2} - 1\right)\frac{\mu}{\mu+1}||v||^2_{H^1(D\backslash\overline{D}_0)}. \end{aligned}$$

In this case B_k is coercive for $0 \leq k \leq \hat{k}'$ if $n - 1 > 0$ and the result again follows from the Lax-Milgram theorem. ☐

Hence we can now define a linear bounded operator A_k by

$$\begin{aligned} A_k :\quad H^1_\Gamma(D\backslash\overline{D}_0) &\rightarrow H^1(D\backslash\overline{D}_0) \\ u &\mapsto v_u := v_0 + \theta. \end{aligned}$$

for $k \geq 0$ if $n - 1 < 0$ and $0 \leq k \leq \hat{k}'$ if $n - 1 > 0$.

Assume now that k^2 is not a Dirichlet eigenvalue for $-\Delta$ is D_0, and let v be the unique solution in $H^1(D_0)$ to

$$\begin{cases} \Delta v + k^2 v = 0 & \text{in } D_0 \\ v = \varphi & \text{on } \Sigma \end{cases} \tag{18}$$

for some $\varphi \in H^{1/2}(\Sigma)$, In this case, we define the Dirichlet to Neumann operator T_k by

$$T_k : \quad H^{1/2}(\Sigma) \quad \to \quad H^{-1/2}(\Sigma)$$
$$\varphi \quad \mapsto \quad \frac{\partial v}{\partial \nu}$$

where v is solution to (18).

Using the Riesz representation theorem, we can define the operator

$$L_k : H^1_\Gamma(D\backslash\overline{D}_0) \to H^1_\Gamma(D\backslash\overline{D}_0)$$

by

$$\langle L_k u, \varphi\rangle_{H^1(D\backslash\overline{D}_0)} = \int_{D\backslash\overline{D}_0} \left(-\nabla v_u \cdot \nabla\overline{\varphi} + k^2 v_u \overline{\varphi}\right) dx - \int_\Sigma T_k v_u \overline{\varphi} ds$$

for all $\varphi \in H^1_\Gamma(D\backslash\overline{D}_0)$, where last integral is understood in the sense of $H^{-1/2}(\Sigma)$, $H^{1/2}(\Sigma)$ duality.

It is obvious that the mapping $k \to L_k$ is continuous in the domain of definition, i.e. for $k \geq 0$ if $n-1<0$ and $0 \leq k \leq \hat{k}'$ if $n-1>0$ such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_0. The next theorem introduces an equivalent formulation to (TEPA).

Theorem 3.7. *Assume that $k \geq 0$ if $n-1<0$ and $0 \leq k \leq \hat{k}'$ if $n-1>0$, such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_0.*

(i) *Let (w, v) be a solution of* (TEPA) *for some $k > 0$. Then $u := w - v \in H^1_\Gamma(D\backslash\overline{D}_0)$ solves $L_k u = 0$.*

(ii) *Let $u \in H^1_\Gamma(D\backslash\overline{D}_0)$ such that $L_k u = 0$. If $v := A_k u$, the pair $w := (u+v, v)$ is solution to* (TEPA).

Proof. (i) If (w, v) is a solution of (TEPA), then, $v = A_k u$ where $u := w - v$ and solves the Helmholtz equation in D. In particular, v solves Helmholtz equation in $D\backslash\overline{D}_0$ and $\dfrac{\partial v}{\partial \nu} = T_k v$ on Σ. Then, for all $\varphi \in H^1_\Gamma(D\backslash\overline{D}_0)$,

$$\begin{aligned} 0 &= \int_{D\backslash\overline{D}_0} (\Delta v + k^2 v)\overline{\varphi} dx \\ &= \int_{D\backslash\overline{D}_0} \left(-\nabla v \cdot \nabla\overline{\varphi} + k^2 v \overline{\varphi}\right) dx - \int_\Sigma \frac{\partial v}{\partial \nu}\overline{\varphi} ds = \langle L_k u, \varphi\rangle_{H^1(D\backslash\overline{D}_0)} . \end{aligned}$$

Then $L_k u = 0$.

(ii) Let $u \in H^1_\Gamma(D\backslash\overline{D}_0)$ such that $L_k u = 0$. We define $v := A_k u$ in $D\backslash\overline{D}_0$ and in D_0, v is defined as the solution to

$$\begin{cases} \Delta v + k^2 v = 0 & \text{in } D_0 \\ v = A_k u & \text{on } \Sigma. \end{cases}$$

Then, v is in $H^1(D)$ and since $L_k u = 0$, v satisfies $\Delta v + k^2 v = 0$ in D. Besides, $v = A_k u$ in $D\backslash\overline{D}_0$ implies that the pair $w := (u+v, v)$ is solution to (TEPA). □

The following theorem states some properties of the operator L_k.

Theorem 3.8. *Assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_0, and $k \geq 0$ if $n-1<0$ and $0 \leq k < \hat{k}'$ if $n-1>0$.*

(i) *The operator $L_k : H^1_\Gamma(D\backslash\overline{D}_0) \to H^1_\Gamma(D\backslash\overline{D}_0)$ is self-adjoint.*

(ii) $L_k - L_0 : H^1_\Gamma(D\backslash\overline{D}_0) \to H^1_\Gamma(D\backslash\overline{D}_0)$ *is compact.*
(iii) *The operator* $L_0 : H^1_\Gamma(D\backslash\overline{D}_0) \to H^1_\Gamma(D\backslash\overline{D}_0)$ *is coercive.*

Proof. (i) Let u_1, $u_2 \in H^1_\Gamma(D\backslash\overline{D}_0)$ and $v_1 = A_k u_1$, $v_2 = A_k u_2$. Thus

$$\langle L_k u_1, u_2\rangle_{H^1(D\backslash\overline{D}_0)} = -\int_{D\backslash\overline{D}_0} \big((I-A)\nabla v_1 \cdot \nabla \bar{u_2} - k^2(1-n)v_1\bar{u_2}\big)\,dx - \int_{D\backslash\overline{D}_0} \big(A\nabla v_1 \cdot \nabla \bar{u_2} - k^2 n v_1 \bar{u_2}\big)\,dx - \int_\Sigma T_k(v_1)\bar{u_2}ds. \tag{19}$$

From the equality (15), we have for $i = 1, 2$ and all $\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)$

$$\int_{D\backslash\overline{D}_0} \big(A\nabla u_i \cdot \nabla\bar{\varphi} - k^2 n u_i \bar{\varphi}\big)\,dx = \int_{D\backslash\overline{D}_0} \big((I-A)\nabla v_i \cdot \nabla\bar{\varphi} - k^2(1-n)v_i\bar{\varphi}\big)\,dx.$$

Taking $i = 2$ with $\varphi = v_1$ and $i = 1$ with $\varphi = u_2$ in the above, the expression (19) for L_k becomes

$$\begin{aligned}\langle L_k u_1, u_2\rangle_{H^1(D\backslash\overline{D}_0)} &= \int_{D\backslash\overline{D}_0} \big((A-I)\nabla v_2 \cdot \nabla\bar{v_1} - k^2(n-1)v_2\bar{v_1}\big)\,dx \\ &\quad - \int_{D\backslash\overline{D}_0} \big(A\nabla u_1 \cdot \nabla\bar{u_2} - k^2 n u_1 \bar{u_2}\big)\,dx + \int_\Sigma T_k(v_1)\bar{v_2}ds \\ &= \int_{D\backslash\overline{D}_0} \big((A-I)\nabla v_2 \cdot \nabla\bar{v_1} - k^2(n-1)v_2\bar{v_1}\big)\,dx \\ &\quad - \int_{D\backslash\overline{D}_0} \big(A\nabla u_1 \cdot \nabla\bar{u_2} - k^2 n u_1 \bar{u_2}\big)\,dx + \int_{D_0} \big(\nabla v_1 \cdot \nabla v_2 - k^2 v_1 \bar{v_2}\big)\,dx\end{aligned}$$

which is a symmetric expression for u_1 and u_2.

(ii) The compactness of $L_k - L_0$ is obtained from the compact embedding of $H^1(D\backslash\overline{D}_0)$ into $L^2(D\backslash\overline{D}_0)$. Indeed, let (u_j) be a sequence of $H^1_\Gamma(D\backslash\overline{D}_0)$ weakly converging to zero in $H^1_\Gamma(D\backslash\overline{D}_0)$. Since $H^1_\Gamma(D\backslash\overline{D}_0)$ is compactly embedded in $L^2(D\backslash\overline{D}_0)$, we deduce that the sequence (u_j) strongly converges to zero in $L^2(D\backslash\overline{D}_0)$. Let us denote $v_k^j := A_k u_j \in H^1(D\backslash\overline{D}_0)$ and $v_0^j := A_0 u_j \in H^1(D\backslash\overline{D}_0)$. Since the operators A_k and A_0 are continuous from $H^1_\Gamma(D\backslash\overline{D}_0)$ into $H^1(D\backslash\overline{D}_0)$, we deduce that v_k^j and v_0^j weakly converge to zero in $H^1(D\backslash\overline{D}_0)$ and consequently, strongly converge to zero in $L^2(D\backslash\overline{D}_0)$. Furthermore, from (15), v_k^j and v_0^j satisfy for all $\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)$,

$$\int_{D\backslash\overline{D}_0} \Big((A-I)\nabla v_k^j \cdot \nabla\overline{\varphi} - k^2(n-1)v_k^j\overline{\varphi}\Big)\,dx = -\int_{D\backslash\overline{D}_0} \big(A\nabla u_j \cdot \nabla\overline{\varphi} - k^2 n u_j \overline{\varphi}\big)\,dx$$

and

$$\int_{D\backslash\overline{D}_0} (A-I)\nabla v_0^j \cdot \nabla\overline{\varphi}dx = -\int_{D\backslash\overline{D}_0} A\nabla u_j \cdot \nabla\overline{\varphi}dx.$$

Letting $\tilde{v}_j := v_0^j - v_k^j$, and taking the difference between the two previous equations yield

$$\int_{D\backslash\overline{D}_0} \Big((A-I)\nabla\tilde{v}_j \cdot \nabla\overline{\varphi} + k^2(n-1)v_k^j\overline{\varphi}\Big)\,dx = -k^2\int_{D\backslash\overline{D}_0} n u_j \overline{\varphi}dx. \tag{20}$$

Now, for $\varphi = \tilde{v}_j$ in (20), applying Cauchy-Schwarz inequality, we obtain

$$\left|\int_{D\setminus\overline{D}_0}(A-I)\nabla\tilde{v}_j\cdot\nabla\overline{\tilde{v}_j}dx\right| = k^2\left|\int_{D\setminus\overline{D}_0}\left((1-n)v_k^j+nu_j\right)\overline{\tilde{v}_j}dxdx\right|$$
$$\leq k^2||(1-n)v_k^j+nu_j||_{D\setminus\overline{D}_0}||\tilde{v}_j||_{D\setminus\overline{D}_0}.$$

Since $||(1-n)v_k^j+nu_j||_{D\setminus\overline{D}_0}$ is bounded and $||\tilde{v}_j||_{D\setminus\overline{D}_0}$ tends to zero, from the fact that $A-I$ is positive definite, we deduce that $\nabla\tilde{v}_j$ converges to zero in $L^2(D\setminus\overline{D}_0)$ and consequently $\tilde{v}_j$ converges to zero in $H^1(D\setminus\overline{D}_0)$.

Now, since for all $\varphi \in H^1_\Sigma(D\setminus\overline{D}_0)$,

$$\langle(L_k-L_0)u_j,\varphi\rangle_{H^1(D\setminus\overline{D}_0)} = \int_{D\setminus\overline{D}_0}\nabla\tilde{v}_j\cdot\nabla\overline{\varphi}dx+k^2\int_{D\setminus\overline{D}_0}v_k^j\overline{\varphi}dx+\int_\Sigma(T_0v_0^j-T_kv_k^j)\overline{\varphi}ds,$$

we have that

$$||(L_k-L_0)u_j||_{H^1(D\setminus\overline{D}_0)} = \sup_{||\varphi||_{H^1(D\setminus\overline{D}_0)}=1}\langle(L_k-L_0)u_j,\varphi\rangle_{H^1(D\setminus\overline{D}_0)}$$
$$\leq ||\nabla\tilde{v}_j||_{D\setminus\overline{D}_0}+k^2||v_k^j||_{D\setminus\overline{D}_0}+||\tilde{v}_j||_{H^{1/2}(\Sigma)}.$$

The right-hand side tends to zero and consequently $(L_k-L_0)u_j$ strongly tends to zero in $H^1(D\setminus\overline{D}_0)$. Then, L_k-L_0 is compact.

(iii) Now we show that L_0 is coersive. To this end for $u \in H^1_\Gamma(D\setminus\overline{D}_0)$ we have that

$$\langle L_0u,u\rangle_{H^1(D\setminus\overline{D}_0)} = -\int_{D\setminus\overline{D}_0}\nabla v_u\cdot\nabla\bar{u}dx-\int_\Sigma\frac{\partial v_u}{\partial\nu}\bar{u}ds$$
$$= -\int_{D\setminus\overline{D}_0}\nabla v_u\cdot\nabla\bar{u}dx+\int_\Sigma\frac{\partial v_u}{\partial\nu}\bar{v}_uds$$
$$= -\int_{D\setminus\overline{D}_0}\nabla w_u\cdot\nabla\bar{u}dx+\int_{D\setminus\overline{D}_0}|\nabla u|^2dx+\int_{D_0}|\nabla v_u|^2dx.$$

Replacing v_u by w_u-u in (15) for $k=0$ and $\varphi = w_u$, we obtain

$$\int_{D\setminus\overline{D}_0}\nabla u\cdot\nabla\bar{w}_udx = \int_{D\setminus\overline{D}_0}(I-A)\nabla w_u\cdot\nabla\bar{w}_udx$$

Therefore

$$\langle L_0u,u\rangle = \int_{D\setminus\overline{D}_0}(A-I)\nabla w_u\cdot\nabla\bar{w}_udx+\int_{D\setminus\overline{D}_0}|\nabla u|^2dx+\int_{D_0}|\nabla v_u|^2dx. \quad (21)$$

Since $(A-I)$ is positive definite, we deduce that L_0 is coercive, which ends the proof of the theorem

□

Note that the mapping $k\to L_k$ is continuous in its domain of definition, i.e. for $k\geq 0$ if $n-1<0$ and $0\leq k\leq\hat{k}'$ if $n-1>0$, such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_0. The proof of existence of transmission eigenvalues is based on the following theorem which is a modified version of Theorem 2.3 [12].

Theorem 3.9. *Let $L_k: H^1_\Gamma(D\setminus\overline{D}_0)\to H^1_\Gamma(D\setminus\overline{D}_0)$ be as defined above. If*

(a) *there exists k_0 such that L_{k_0} is positive on $H^1_\Gamma(D\setminus\overline{D}_0)$, and*
(b) *there exists k_1 such that L_{k_1} is non positive on some m-dimensional subspace of $H^1_\Gamma(D\setminus\overline{D}_0)$.*

Then there exists m transmission eigenvalues in $[k_0, k_1]$ counting with their multiplicity provided that the entire interval $[k_0, k_1]$ belongs to the domain of definition of the mapping $k \to L_k$.

Theorem 3.10. *Assume that $A - I > 0$ and that either $n_* < n < n^* < 1$ or $1 < n_* < n < n^* \leq 1 + \frac{\gamma_* \mu}{2\hat{k}'^2}$. Then there exists at least one transmission eigenvalue provided that the area of D_0 is small enough.*

Proof. We have shown in Theorem 3.8 that L_0 is coercive, thus the assumption (a) of Theorem 3.9 is satisfied for $k_0 = 0$.

First assume that $n < 1$. Let B_r be the largest ball included in $D \backslash \overline{D}_0$ of radius r and let us denote by $\hat{k}$ the first transmission eigenvalue of the interior transmission problem in B_r with $A = \gamma_* I$ and $n = n^*$, i.e.

$$\begin{cases} \nabla \cdot \gamma_* \nabla w + k^2 n^* w = 0 & \text{in } B_r \\ \Delta v + k^2 v = 0 & \text{in } B_r \\ w = v & \text{on } \partial B_r \\ \nu \cdot \gamma_* \nabla w = \nu \cdot \nabla v & \text{on } \partial B_r. \end{cases} \tag{22}$$

Assume now that the area of D_0 is small enough such that the first Dirichlet eigenvalue for $-\Delta$ in D_0 is greater than $\hat{k}$ (this is possible since due to the Faber-Krahn inequality the first Dirichlet eigenvalue for $-\Delta$ in D_0 is greater than $C/\text{area}D_0$) Thus the operator L_k is well defined for all $k \in [0, \hat{k}]$. denote by $\hat{w}$ and $\hat{v}$ the corresponding eigenvectors and we set $\hat{u} := \hat{w} - \hat{v} \in H^1_0(B_r)$. We shall show that we can find $u \in H^1_\Gamma(D \backslash \overline{D}_0)$ such that $\langle L_{\hat{k}} u, u \rangle \leq 0$ so that the assumption (b) of Theorem 3.9 is satisfied.

From the equation satisfied by $\hat{v}$ in B_r and using the fact that $\hat{u} = 0$ on ∂B_r and $\hat{v} = \hat{w} - \hat{u}$, we first have

$$0 = \int_{B_r} \left(\Delta \hat{v} + k^2 \hat{v} \right) \overline{\hat{u}} dx = \int_{B_r} \left(\nabla \hat{v} \cdot \nabla \overline{\hat{u}} - \hat{k}^2 \hat{v} \overline{\hat{u}} \right) dx \tag{23}$$

$$= \int_{B_r} \left(\nabla \hat{w} \cdot \nabla \overline{\hat{u}} - \hat{k}^2 \hat{w} \overline{\hat{u}} - |\nabla \hat{u}|^2 + \hat{k}^2 |\hat{u}|^2 \right) dx. \tag{24}$$

On the other hand, replacing $\hat{v}$ by $\hat{w} - \hat{u}$ in the variational formulation satisfied by $\hat{v}$ and $\hat{w}$ we have

$$\int_{B_r} \left(\nabla \hat{u} \cdot \nabla \overline{\varphi} - \hat{k}^2 \hat{u} \overline{\varphi} \right) dx = \int_{B_r} \left((1 - \gamma_*) \nabla \hat{w} \cdot \nabla \overline{\varphi} - \hat{k}^2 (1 - n^*) \hat{w} \overline{\varphi} \right) dx$$

for all $\varphi \in H^1(B_r)$. In particular for $\varphi = \hat{w}$, we obtain

$$\int_{B_r} \left(\nabla \hat{w} \cdot \nabla \overline{\hat{u}} - \hat{k}^2 \hat{w} \overline{\hat{u}} \right) dx = \int_{B_r} \left((1 - \gamma_*) |\nabla \hat{w}|^2 - \hat{k}^2 (1 - n^*) |\hat{w}|^2 \right) dx. \tag{25}$$

From (24) and (25), we finally get the equality

$$\int_{B_r} \left((1 - \gamma_*) |\nabla \hat{w}|^2 - \hat{k}^2 (1 - n^*) |\hat{w}|^2 - |\nabla \hat{u}|^2 + \hat{k}^2 \hat{u}|^2 \right) dx = 0. \tag{26}$$

Now we denote by $\tilde{u}$ the extension of $\hat{u}$ by zero to all of $D \backslash \overline{D}_0$. Since $\tilde{u} \in H^1_\Gamma(D \backslash \overline{D}_0)$, we can define $\tilde{v} := v_{\tilde{u}}$ the corresponding solution to

$$\begin{cases} \nabla \cdot (I - A) \nabla v + \hat{k}^2 (1 - n) v = \nabla \cdot A \nabla \tilde{u} + \hat{k} n \tilde{u} & \text{in } D \backslash \overline{D}_0 \\ \nu \cdot (I - A) \nabla v = \nu \cdot A \nabla \tilde{u} & \text{on } \Gamma \\ v = -\tilde{u} = 0 & \text{on } \Sigma \end{cases}$$

and we set $\tilde{w} := \tilde{u} + \tilde{v} \in H^1_\Sigma(D\backslash\overline{D}_0)$. We first remark that replacing $\tilde{v}$ by $\tilde{w} - \tilde{u}$ in (15) and for $\varphi = \tilde{u}$, yields

$$\int_{D\backslash\overline{D}_0} \Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{u}} - \hat{k}^2(n-1)\tilde{w}\overline{\tilde{u}}\Big)\,dx = -\int_{D\backslash\overline{D}_0}\Big(|\nabla\tilde{u}|^2 - \hat{k}^2|\tilde{u}|^2\Big)\,dx.$$

Consequently, replacing $\tilde{v}$ by $\tilde{w} - \tilde{u}$ in the expression of $L_{\hat{k}}$ and using the definition of $\tilde{u}$, we obtain

$$\begin{aligned}\langle L_{\hat{k}}\tilde{u},\tilde{u}\rangle_{H^1(D\backslash\overline{D}_0)} &= -\int_{D\backslash\overline{D}_0}\Big(\nabla\tilde{v}\cdot\nabla\overline{\tilde{u}} - \hat{k}^2\tilde{v}\tilde{u}\Big)\,dx\\ &= -\int_{D\backslash\overline{D}_0}\Big(\nabla\tilde{w}\cdot\nabla\overline{\tilde{u}} - \hat{k}^2\tilde{w}\tilde{u} - |\nabla\tilde{u}|^2 + \hat{k}^2|\tilde{u}|^2\Big)\,dx\\ &= \int_{D\backslash\overline{D}_0}\Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n-1)|\tilde{w}|^2 + |\nabla\tilde{u}|^2 - \hat{k}^2|\tilde{u}|^2\Big)\,dx\\ &= \int_{D\backslash\overline{D}_0}\Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n-1)|\tilde{w}|^2\Big)\,dx + \int_{B_r}\Big(|\nabla\hat{u}|^2 - \hat{k}^2|\hat{u}|^2\Big)\,dx.\end{aligned}$$

Now, considering again (15) with $\tilde{v} = \tilde{w} - \tilde{u}$ and using the definition of $\tilde{u}$, for all $\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)$, we have

$$\begin{aligned}&\int_{D\backslash\overline{D}_0}\Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\varphi} - \hat{k}^2(n-1)\tilde{w}\overline{\varphi}\Big)\,dx = -\int_{D\backslash\overline{D}_0}\Big(\nabla\tilde{u}\cdot\nabla\overline{\varphi} - \hat{k}^2\tilde{u}\overline{\varphi}\Big)\,dx\\ &= -\int_{B_r}\Big(\nabla\hat{u}\cdot\nabla\overline{\varphi} - \hat{k}^2\hat{u}\overline{\varphi}\Big)\,dx = \int_{B_r}\Big((\gamma_*-1)\nabla\hat{w}\cdot\nabla\overline{\varphi} - \hat{k}^2(n^*-1)\hat{w}\overline{\varphi}\Big)\,dx.\end{aligned}$$

In particular, for $\varphi = \tilde{w} \in H^1_\Sigma(D\backslash\overline{D}_0)$ we obtain

$$\begin{aligned}\int_{D\backslash\overline{D}_0}&\Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n-1)|\tilde{w}|^2\Big)\,dx\\ &= \int_{B_r}\Big((\gamma_*-1)\nabla\hat{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n^*-1)\hat{w}\overline{\tilde{w}}\Big)\,dx\end{aligned}\tag{27}$$

The Cauchy-Schwarz inequality applied to the right-hand side of (27) gives

$$\begin{aligned}&\int_{D\backslash\overline{D}_0}\Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n-1)|\tilde{w}|^2\Big)\,dx = \int_{B_r}\Big((\gamma_*-1)\nabla\hat{w}\cdot\nabla\overline{\tilde{w}} + \hat{k}^2(1-n^*)\hat{w}\overline{\tilde{w}}\Big)\,dx\\ &\le \left(\int_{B_r}\Big((\gamma_*-1)|\nabla\hat{w}|^2 + \hat{k}^2(1-n^*)|\hat{w}|^2\Big)\,dx\right)^{1/2}\left(\int_{B_r}\Big((\gamma_*-1)|\nabla\tilde{w}|^2 + \hat{k}^2(1-n^*)|\tilde{w}|^2\Big)\,dx\right)^{1/2}\\ &\le \left(\int_{B_r}\Big((\gamma_*-1)|\nabla\hat{w}|^2 + \hat{k}^2(1-n^*)|\hat{w}|^2\Big)\,dx\right)^{1/2}\left(\int_{D\backslash\overline{D}_0}\Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n-1)|\tilde{w}|^2\Big)\,dx\right)^{1/2}\end{aligned}$$

and finally

$$\begin{aligned}\int_{D\backslash\overline{D}_0}&\Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n-1)|\tilde{w}|^2\Big)\,dx\\ &\le \int_{B_r}\Big((\gamma_*-1)|\nabla\hat{w}|^2 + \hat{k}^2(1-n^*)|\hat{w}|^2\Big)\,dx.\end{aligned}\tag{28}$$

Therefore, from (28) and (26), we obtain that

$$\begin{aligned}\left\langle L_{\hat{k}}\tilde{u},\tilde{u}\right\rangle_{H^1(D\setminus\overline{D}_0)} &= \int_{D\setminus\overline{D}_0}\left((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}}-\hat{k}^2(n-1)|\tilde{w}|^2\right)dx\\ &\quad+\int_{B_r}\left(|\nabla\hat{u}|^2-\hat{k}^2|\hat{u}|^2\right)dx\\ &\le\int_{B_r}\left((\gamma_*-1)|\nabla\hat{w}|^2+\hat{k}^2(1-n^*)|\hat{w}|^2+|\nabla\hat{u}|^2-\hat{k}^2|\hat{u}|^2\right)dx=0.\end{aligned}$$

We can conclude that there exists a transmission eigenvalue in $(0,\hat{k}]$.

Now assume that $1<n_*<n<n^*\le 1+\frac{\gamma_*\mu}{2\hat{k}'^2}$. Again, we assume that the area of D_0 is small enough such that the first Dirichlet eigenvalue for $-\Delta$ in D_0 is greater than $\hat{k}'$. (We recall that $\hat{k}'$ is the first transmission eigenvalue of the interior transmission problem for B_r with $A=\frac{\gamma_*}{2}$ and $n=1$ given in (16).) We denote by $\hat{w}$ and $\hat{v}$ the eigenvectors corresponding to $\hat{k}'$ and set $\hat{u}:=\hat{w}-\hat{v}\in H_0^1(B_r)$. From the equation satisfied by $\hat{v}$ and using the fact that $\hat{u}=0$ on ∂B_r and $\hat{v}=\hat{w}-\hat{u}$ we first have

$$0=\int_{B_r}\left(\nabla\hat{v}\cdot\nabla\overline{\hat{u}}-\hat{k}'^2\hat{v}\overline{\hat{u}}\right)dx=\int_{B_r}\left(\nabla\hat{w}\cdot\nabla\overline{\hat{u}}-\hat{k}'^2\hat{w}\overline{\hat{u}}-|\nabla\hat{u}|^2+\hat{k}'^2|\hat{u}|^2\right)dx.$$

On the other hand, replacing $\hat{v}$ by $\hat{w}-\hat{u}$ in the variational formulation satisfied by $\hat{v}$ and $\hat{w}$ we have

$$\int_{B_r}\left(\nabla\hat{u}\cdot\nabla\overline{\varphi}-\hat{k}'^2\hat{u}\overline{\varphi}\right)dx=\int_{B_r}\left(1-\frac{\gamma_*}{2}\right)\nabla\hat{w}\cdot\nabla\overline{\varphi}dx$$

for all $\varphi\in H^1(B_r)$. In particular for $\varphi=\hat{w}$, we obtain

$$\int_{B_r}\left(\nabla\hat{w}\cdot\nabla\overline{\hat{u}}-\hat{k}'^2\hat{w}\overline{\hat{u}}\right)dx=\int_{B_r}\left(1-\frac{\gamma_*}{2}\right)|\nabla\hat{w}|^2dx. \tag{29}$$

Combining the above equations, we finally obtain

$$\int_{B_r}\left(\left(1-\frac{\gamma_*}{2}\right)|\nabla\hat{w}|^2-|\nabla\hat{u}|^2+\hat{k}'^2\hat{u}|^2\right)dx=0. \tag{30}$$

Now we denote by $\tilde{u}$ the extension of $\hat{u}$ by zero to all of $D\setminus\overline{D}_0$. Since $\tilde{u}\in H^1_\Sigma(D\setminus\overline{D}_0)$, we can define $\tilde{v}:=v_{\tilde{u}}$ the corresponding solution to

$$\begin{cases}\nabla\cdot(I-A)\nabla v+\hat{k}'^2(1-n)v=\nabla\cdot A\nabla\tilde{u}+\hat{k}'n\tilde{u} & \text{in } D\setminus\overline{D}_0\\ \nu\cdot(I-A)\nabla v=\nu\cdot A\nabla\tilde{u} & \text{on } \Gamma\\ v=-\tilde{u}=0 & \text{on } \Sigma\end{cases} \tag{31}$$

and we set $\tilde{w}:=\tilde{u}+\tilde{v}$. We first remark that replacing $\tilde{v}$ by $\tilde{w}-\tilde{u}$ in (15), we have

$$\begin{aligned}\int_{D\setminus\overline{D}_0}\left((A-I)\nabla\tilde{w}\cdot\nabla\overline{\varphi}-\hat{k}'^2(n-1)\tilde{w}\overline{\varphi}\right)dx &= -\int_{D\setminus\overline{D}_0}\left(\nabla\tilde{u}\cdot\nabla\overline{\varphi}-\hat{k}'^2\tilde{u}\overline{\varphi}\right)dx\\ &= -\int_{B_r}\left(\nabla\hat{u}\cdot\nabla\overline{\varphi}-\hat{k}'^2\hat{u}\overline{\varphi}\right)dx\\ &= \int_{B_r}\left(\frac{\gamma_*}{2}-1\right)\nabla\hat{w}\cdot\nabla\overline{\varphi}dx\end{aligned}$$

In particular, for $\varphi = \tilde{w} \in H^1_\Sigma(D\backslash\overline{D}_0)$, we obtain

$$\int_{D\backslash\overline{D}_0} \left((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}'^2(n-1)|\tilde{w}|^2\right)dx = \int_{B_r}\left(\frac{\gamma_*}{2}-1\right)\nabla\hat{w}\cdot\nabla\overline{\tilde{w}}dx. \tag{32}$$

The Cauchy-Schwarz inequality applied to the right-hand side of (32) gives

$$\begin{aligned}
&\int_{D\backslash\overline{D}_0} \left((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}'^2(n-1)|\tilde{w}|^2\right)dx = \int_{B_r}\left(\frac{\gamma_*}{2}-1\right)\nabla\hat{w}\cdot\nabla\overline{\tilde{w}}dx\\
&\leq \left(\int_{B_r}\left(\frac{\gamma_*}{2}-1\right)|\nabla\hat{w}|^2dx\right)^{1/2}\left(\int_{B_r}\left(\frac{\gamma_*}{2}-1\right)|\nabla\tilde{w}|^2dx\right)^{1/2}\\
&= \left(\int_{B_r}\left(\frac{\gamma_*}{2}-1\right)|\nabla\hat{w}|^2dx\right)^{1/2}\left(\int_{B_r}\left((\gamma_*-1)|\nabla\tilde{w}|^2 - \frac{\gamma_*}{2}|\nabla\tilde{w}|^2\right)dx\right)^{1/2}\\
&\leq \left(\int_{B_r}\left(\frac{\gamma_*}{2}-1\right)|\nabla\hat{w}|^2dx\right)^{1/2}\left(\int_{D\backslash\overline{D}_0}\left((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}'^2(n-1)|\tilde{w}|^2\right)dx\right)^{1/2}
\end{aligned}$$

and finally

$$\int_{D\backslash\overline{D}_0} \left((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}'^2(n-1)|\tilde{w}|^2\right)dx \leq \int_{B_r}\left(\frac{\gamma_*}{2}-1\right)|\nabla\hat{w}|^2dx. \tag{33}$$

Therefore, we obtain

$$\begin{aligned}
\left\langle L_{\hat{k}'}\tilde{u},\tilde{u}\right\rangle_{H^1(D\backslash\overline{D}_0)} &= -\int_{D\backslash\overline{D}_0}\left(\nabla\tilde{v}\cdot\nabla\overline{\tilde{u}} - \hat{k}'^2\tilde{v}\overline{\tilde{u}}\right)dx\\
&= -\int_{D\backslash\overline{D}_0}\left(\nabla\tilde{w}\cdot\nabla\overline{\tilde{u}} - \hat{k}'^2\tilde{w}\overline{\tilde{u}} - |\nabla\tilde{u}|^2 + \hat{k}'^2|\tilde{u}|^2\right)dx\\
&= \int_{D\backslash\overline{D}_0}\left((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}'^2(n-1)|\tilde{w}|^2 + |\nabla\tilde{u}|^2 - \hat{k}'^2|\tilde{u}|^2\right)dx\\
&= \int_{D\backslash\overline{D}_0}\left((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}'^2(n-1)|\tilde{w}|^2\right)dx\\
&\quad + \int_{B_r}\left(|\nabla\hat{u}|^2 - \hat{k}'^2|\hat{u}|^2\right)dx\\
&\leq \int_{B_r}\left(\left(\frac{\gamma_*}{2}-1\right)|\nabla\hat{w}|^2 + |\nabla\hat{u}|^2 - \hat{k}'^2|\hat{u}|^2\right)dx = 0.
\end{aligned}$$

Thus we can conclude that if $1 < n_* < n < n^* \leq 1 + \frac{\gamma_*\mu}{2\hat{k}'^2}$ there exists a transmission eigenvalue in $(0, \hat{k}']$. □

Remark 3.3. As the area of D_0 goes to 0, in the case when $0 < n_* < n^* < 1$ it is possible to prove the existence of more and more transmission eigenvalues. In this case since the first Dirichlet eigenvalue for $-\Delta$ in D_0 goes to infinity one can take r such that $M(r)$ disjoint balls of radius r are included in $D\backslash\overline{D}_0$ and no Dirichlet eigenvalues are in $[0, \hat{k}]$. This way the assumption (b) of Theorem 3.9 is satisfied in a $M(r)$-dimensional subspace of $H^1_\Gamma(D\backslash\overline{D}_0)$ and thus there exists $M(r)$ transmission eigenvalues in $[0, \hat{k}]$ (counting multiplicity). The smaller the area of D_0 is the smaller r can be chosen and the larger $M(r)$ becomes. The same remark holds true for the case when $1 < n_*$ provided that n^* is small enough, more specifically $n^* < 1 + \frac{\gamma_*\mu}{2\hat{k}'^2}$.

Remark 3.4. The entire argument in the proof of Theorem 3.10 holds true if $\hat{k}$ or $\hat{k}'$ is the first transmission eigenvalue of (22) or (16), respectively, where B_r is replaced with an arbitrary region $B \subset D\backslash\overline{D}_0$ (such transmission eigenvalues are known to exists [12]). Depending on the geometry of $D\backslash\overline{D}_0$ one can choose B such that the corresponding $\hat{k}$ or $\hat{k}'$ are smaller than the ones for the ball B_r (see the estimates on the first transmission eigenvalue in [9], [10] and [12]) which would enable to prove the existence of at least one transmission eigenvalue for larger D_0.

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Received for publication

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