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NEW RESULTS ON TRANSMISSION EIGENVALUES

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Abstract. We consider the interior transmission eigenvalue problem corresponding to the inverse scattering problem for an isotropic inhomogeneous medium. We first prove that transmission eigenvalues exist for media with index of refraction greater or less than one without assuming that the contrast is sufficiently large. Then we show that for an arbitrary Lipshitz domain with constant index of refraction there exists an infinite discrete set of transmission eigenvalues that accumulate at infinity. Finally, for the general case of non constant index of refraction we provide a lower and an upper bound for the first transmission eigenvalue in terms of the first transmission eigenvalue for appropriate balls with constant index of refraction.

1. Introduction

The interior transmission problem is a boundary value problem in a bounded domain which arises in inverse scattering theory for inhomogeneous media. Although simply stated, this problem is not covered by the standard theory of elliptic partial differential equations since as it stands it is neither elliptic nor self-adjoint. Of particular interest is the spectrum associated with this boundary value problem, more specifically the existence of eigenvalues which are called transmission eigenvalues. Besides the theoretical importance of transmission eigenvalues in connection with uniqueness and reconstruction results in inverse scattering theory, recently they have been used to obtain information about the index of refraction from measured data [1], [6]. This is based on the important result that transmission eigenvalues can be determined from the measured far field data which is recently proven in [3]. For information on the interior transmission problem, we refer the reader to [8] and [9].

Up to recently, most of the known results on the interior transmission problem are concerned with when the problem is well-posed. Roughly speaking, two main approaches are available in this direction, namely integral equation methods [7],

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[10], and variational methods typically applied to a fourth order equivalent boundary value problem [2], [5], [15]. However, except for the case of spherically stratified medium [8], [9], until recently little was known about the existence and properties of transmission eigenvalues. Applying the analytic Fredholm theory it was possible to show that transmission eigenvalues form at most a discrete set with infinity as the only possible accumulation point. However, nothing was known in general about the existence of transmission eigenvalues untill the recent important result of Päivärinta and Sylvester [14] who were the first to show that, in the case of (scalar) isotropic media, a finite number of transmission eigenvalues exist provided the index of refraction is bounded away from one. Kirsch [11], and Cakoni and Haddar [5] have extended this existence result to the case of anisotropic media for both the scalar case and Maxwell's equations. Most recently, the analysis of the interior transmission problem and the existence of the corresponding eigenvalues have been established for the case when inside the medium there are subregions with index of refraction equal to one (i.e cavities) which provides the theoretical background for an application of transmission eigenvalues in non-destructive testing [1]. However, the existence of transmission eigenvalues is proven under the restriction that the contrast of the inhomogeneous medium is sufficiently large and the larger the contrast is the more transmission eigenvalues are shown to exist.

The goal of this paper is to first show that for an inhomogeneous medium with bounded support there exists at least one eigenvalue provided that the index of refraction is less than or greater than one inside the medium, thus removing the restriction on the index of refraction being sufficiently large. This is done using the transmission eigenvalues and eigenvectors corresponding to a ball inside the support of the inhomogeneity with constant index of refraction equal to the supremum of the actual index of refraction. In addition, for the case of a homogneous medium, we show that there exists infinitely many transmission eigenvalues with infinity as the only accumulation point. Our analysis makes use of the analytical framework discussed in [5] in particular of an auxilarly eigenvalue problem for a self adjoint coercive operator which depends in a non-linear fashion on a parameter. Specific values of this parameter correspond to transmission eigenvalues. The main tool of our approach is a monotonicity relation that we establish for the eigenvalues of this auxiliary eigenvalue problem with respect to the domain. Finally, as a byproduct of our analysis we obtain a lower and an upper bound for the first transmission eigenvalue for an arbitrary inhomogeneous medium in terms of the first transmission eigenvalue for the smallest ball containing the scatterer and the largest ball contained in the scatterer for the both cases of a positive or negative contrast in the medium. The lower bound is an improvement of the lower bound obtained in [4] and [9] but is implicit in terms of the supremum of the index of refraction.

We conclude by noting that many questions related to the spectrum of interior transmission problems still remain open. In particular, the next step is to show the existence of an infinite discrete set of transmission eigenvalues for the general case of media with non-constant index of refraction and for other scattering problems such as for Maxwell's equations and anisotropic media.

2. Interior transmission eigenvalue problem

The *interior transmission eigenvalue problem* corresponding to the scattering by an isotropic inhomogenous medium in $\mathbb{R}^3$ reads:

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in} \qquad D \tag{1}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \qquad D \tag{2}$$

$$w = v \qquad \text{on} \qquad \partial D \tag{3}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on} \qquad \partial D \tag{4}$$

for $w \in L^2(D)$ and $v \in L^2(D)$ such that $w - v \in H_0^2(D)$ where

$$H_0^2(D) = \left\{ u \in H^2(D) : \; u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial D \right\}.$$

Here we assume that the index of refraction is a real valued function such that $n \in L^\infty(D)$, $1/|n(x) - 1| \in L^\infty(D)$ and $n(x) \geq \delta > 0$ almost everywhere in D. (It is known that for complex index of refraction n this problem has only the zero solution). Furthermore, we assume that $D \subset \mathbb{R}^3$ is a bounded simply connected region with Lipschitz boundary ∂D and denote by ν the outward normal vector to ∂D. We remark that everything in this paper holds true for the same equations in $\mathbb{R}^2$. *Transmission eigenvalues* are the values of $k > 0$ for which the above homogeneous interior transmission problem has non zero solutions. It is possible to write (1)-(4) as an equivalent eigenvalue problem for $u = w - v \in H_0^2(D)$ satisfying the following fourth order equation

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0 \tag{5}$$

In the variational form (5) is formulated as the problem of finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1} (\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v}) \, dx = 0 \qquad \text{for all } v \in H_0^2(D). \tag{6}$$

Following [5] we set $k^2 := \tau$ and define the following bounded sesquilinear forms on $H_0^2(D) \times H_0^2(D)$:

$$\mathcal{A}_\tau(u, v) = \left(\frac{1}{n-1} (\Delta u + \tau u), \, (\Delta v + \tau v) \right)_D + \tau^2 \, (u, \, v)_D \, , \tag{7}$$

$$\begin{aligned} \tilde{\mathcal{A}}_\tau(u, v) \quad &= \quad \left(\frac{1}{1-n} (\Delta u + \tau n u), \, (\Delta v + \tau n v) \right)_D + \tau^2 \, (n u, \, v)_D \\ &= \quad \left(\frac{n}{1-n} (\Delta u + \tau u), \, (\Delta v + \tau v) \right)_D + (\Delta u, \, \Delta v)_D \end{aligned} \tag{8}$$

and

$$\mathcal{B}(u, v) = (\nabla u, \, \nabla v)_D \tag{9}$$

where $(\cdot\,, \cdot)_D$ denotes the $L^2(D)$ inner product. Then (6) can be written as either

$$\mathcal{A}_\tau(u, v) - \tau \mathcal{B}(u, v) = 0 \qquad \text{for all } v \in H_0^2(D), \tag{10}$$

or

$$\tilde{\mathcal{A}}_\tau(u, v) - \tau \mathcal{B}(u, v) = 0 \qquad \text{for all } v \in H_0^2(D). \tag{11}$$

(Note that $\tilde{\mathcal{A}}_\tau = -\mathcal{A}_\tau + 2\tau\mathcal{B}$. We use different notations $\mathcal{A}_\tau$ and $\tilde{\mathcal{A}}_\tau$ to simplify the presentation as it will become clear in the following). In [5] it is proven that if $\frac{1}{n(x)-1} > \gamma > 0$ almost everywhere in D then $\mathcal{A}_\tau$ is a coersive sesquilinear form on $H_0^2(D) \times H_0^2(D)$ whereas if $\dfrac{n(x)}{1-n(x)} > \gamma > 0$ almost everywhere in D then $\tilde{\mathcal{A}}_\tau$ is a coersive sesquilinear form on $H_0^2(D) \times H_0^2(D)$. Using the Riesz representation theorem we now define the bounded linear operators $\mathbb{A}_\tau : H_0^2(D) \to H_0^2(D)$, $\tilde{\mathbb{A}}_\tau : H_0^2(D) \to H_0^2(D)$ and $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ by

$$(\mathbb{A}_\tau u, v)_{H^2(D)} = \mathcal{A}_\tau(u,v), \ \left(\tilde{\mathbb{A}}_\tau u, v\right)_{H^2(D)} = \tilde{\mathcal{A}}_\tau(u,v) \text{ and } (\mathbb{B}u, v)_{H^2(D)} = \mathcal{B}(u,v).$$

As is shown in [5], since n is real the sesquilinear forms $\mathcal{A}_\tau$, $\tilde{\mathcal{A}}_\tau$ and $\mathcal{B}$ are hermitian and therefore the operators $\mathbb{A}_\tau$, $\tilde{\mathbb{A}}_\tau$ and $\mathbb{B}$ are self-adjoint. Furthermore, by definition, $\mathbb{B}$ is a non negative operator and if $\frac{1}{n(x)-1} > \gamma > 0$ then $\mathbb{A}_\tau$ is a positive definite operator, whereas if $\frac{n(x)}{1-n(x)} > \gamma > 0$ then $\tilde{\mathbb{A}}_\tau$ is a positive definite operator. Finally, noting that for $u \in H_0^2(D)$ we have that $\nabla u \in H_0^1(D)^2$, since $H_0^1(D)^2$ is compactly embedded in $L^2(D)^2$ we can conclude that $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ is a compact operator. Also $\mathbb{A}_\tau$ and $\tilde{\mathbb{A}}_\tau$ depend continuously on $\tau \in (0, +\infty)$.

From the above discussion, $k > 0$ is a transmission eigenvalue if for $\tau = k^2$ the kernel of the operator $\mathbb{A}_\tau - \tau\mathbb{B}$ (if $1/(n-1) > \gamma > 0$) or the operator $\tilde{\mathbb{A}}_\tau - \tau\mathbb{B}$ (if $n/(1-n) > \gamma > 0$) is non trivial. In order to analyze the kernel of these operators we consider an auxiliary eigenvalue problem

$$\mathbb{A}_\tau u - \lambda(\tau)\mathbb{B}u = 0 \qquad u \in H_0^2(D) \tag{12}$$

if $1/(n-1) > \gamma > 0$ and

$$\tilde{\mathbb{A}}_\tau u - \lambda(\tau)\mathbb{B}u = 0 \qquad u \in H_0^2(D) \tag{13}$$

if $n/(1-n) > \gamma > 0$.

The eigenvalue problems (12) and (13) fit into the following abstract analytical framework which is discussed in [5]. In particular, let U be a separable Hilbert space with scalar product $(\cdot,\cdot)$, $\mathbb{A}$ be a bounded, positive definite and self-adjoint operator on U and let $\mathbb{B}$ be a non negative, self-adjoint and compact bounded linear operator on U. Then there exists an increasing sequence of positive real numbers $(\lambda_j)_{j\geq 1}$ and a sequence $(u_j)_{j\geq 1}$ of elements of U such that $\mathbb{A}u_j = \lambda_j \mathbb{B}u_j$. The sequence $(u_j)_{j\geq 1}$ form a basis of $(\mathbb{A}\ker(\mathbb{B}))^\perp$ and if $\ker(\mathbb{B})^\perp$ has infinite dimension then $\lambda_j \to +\infty$ as $j \to \infty$ (see Theorem 2.1 in [5]). Furthermore these eigenvalues satisfy a min-max principle (see Corollary 2.1 [5]), namely

$$\lambda_j = \min_{W \subset \mathcal{U}_j} \left(\max_{u \in W\setminus\{0\}} \frac{(\mathbb{A}u, u)}{(\mathbb{B}u, u)} \right) \tag{14}$$

where $\mathcal{U}_j$ denotes the set of all j dimensional subspaces W of U such that $W \cap \ker(\mathbb{B}) = \{0\}$ and $\lambda_1 \leq \lambda_2 \leq \dots$.

The following theorem proved in [5] is needed in our analysis for proving the existence of transmission eigenvalues.

Theorem 2.1. *Let $\tau \longmapsto \mathbb{A}_\tau$ be a continuous mapping from $]0, \infty[$ to the set of self-adjoint and positive definite bounded linear operators on U and let $\mathbb{B}$ be a self-adjoint and non negative compact bounded linear operator on U. We assume that there exists two positive constant $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. $\mathbb{A}_{\tau_0} - \tau_0\mathbb{B}$ *is positive on* U,

2. $\mathbb{A}_{\tau_1} - \tau_1\mathbb{B}$ *is non positive on a m dimensional subspace of* U.

Then each of the equations $\lambda_j(\tau) = \tau$ *for* $j = 1, \dots, m$, *has at least one solution in* $[\tau_0, \tau_1]$ *where* $\lambda_j(\tau)$ *is the* j^{th} *eigenvalue (counting multiplicity) of* $\mathbb{A}_\tau$ *with respect to* $\mathbb{B}$, *i.e.* $\ker(\mathbb{A}_\tau - \lambda_j(\tau)\mathbb{B}) \neq \{0\}$.

Returning to (12) and (13) and setting $U := H_0^2(D)$, we have that (12) and (13) have a countable set of eigenvalues $\{\lambda_k(\tau)\}_{j=1}^{\infty}$ that satisfy the min-max principle (14). Moreover, for the above discussion, each $\lambda_k(\tau)$ is a continuous function of τ and $k > 0$ is a transmission eigenvalue if $\tau := k^2$ is a zero of any of nonlinear equations

$$\lambda_k(\tau) - \tau = 0. \tag{15}$$

Remark 2.1. The multiplicity of transmission eigenvalues is finite since, if k_0 is a transmission eigenvalue, then the kernel of $I - \tau_0\mathbb{A}_{\tau_0}^{-1/2}\mathbb{B}\mathbb{A}_{\tau_0}^{-1/2}$ or $I - \tau_0\tilde{\mathbb{A}}_{\tau_0}^{-1/2}\mathbb{B}\tilde{\mathbb{A}}_{\tau_0}^{-1/2}$ where $\tau_0 := k_0^2$ is finite because the operators $\tau_0\tilde{\mathbb{A}}_{\tau_0}^{-1/2}\mathbb{B}\tilde{\mathbb{A}}_{\tau_0}^{-1/2}$ and $\tau_0\tilde{\mathbb{A}}_{\tau_0}^{-1/2}\mathbb{B}\tilde{\mathbb{A}}_{\tau_0}^{-1/2}$ are compact and self-adjoint (if $1/(n-1) > \gamma > 0$) and (if $n/(1-n) > \gamma > 0$), respectively [12].

Remark 2.2. Based on the analytic Fredholm theory it is shown [8] that transmission eigenvalues can have only $+\infty$ as a possible accumulation point.

We end this section by recalling some well-known results on transmission eigenvalues for a ball. Let B be a ball centered at the origin and let us consider the interior transmission problem for the ball B with constant index of refraction $n > 0$ and $n \neq 1$.

$$\Delta w + k^2 n w = 0 \quad \text{in} \quad B \tag{16}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad B \tag{17}$$

$$w = v \quad \text{on} \quad \partial B \tag{18}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial B \tag{19}$$

By a separation of variables technique, it is possible to show [9] (see also [8]) that (16)-(19) has a countable discrete set of eigenvalues $\{k_j\}_{j=1}^{\infty}$. Let $v_j^{B,n}$ and $w_j^{B,n}$ be a non zero solution of (16)-(19) corresponding to k_j, $j = 1, \dots$. Denoting $u_j^{B,n} := w_j^{B,n} - v_j^{B,n}$ we have that $u_j^{B,n} \in H_0^2(B)$ and

$$\int_B \frac{1}{n-1}(\Delta u_j^{B,n} + k_j^2 u_j^{B,n})(\Delta \overline{u}_j^{B,n} + k_j^2 n \overline{u}_j^{B,n})\, dx = 0. \tag{20}$$

We call $u_j^{B,n}$ a transmission eigenfunction corresponding to the transmission eigenvalue k_j.

3. On the existence of a transmission eigenvalue.

Now we are ready to prove our fist result, namely the proof of the existence of at least one transmission eigenvalue without the restriction that the index of refraction is sufficiently large [5], [14]. To this end, we show that for the equation (12) if $1/(n-1) > \gamma > 0$ and the equation (13) if $n/(1-n) > \gamma > 0$ there exist τ_0 and τ_1 satisfying the assumption 1 and 2, respectively, in Theorem 2.1.

In the following we denote by $n_* = \inf_D(n)$ and $n^* = \sup_D(n)$ and assume that the origin of the coordinative system is inside D.

Theorem 3.1. *Let $n \in L^\infty(D)$ satisfying either one of the following assumptions*
1) $1+\alpha \leq n_* \leq n(x) \leq n^* < \infty$,
2) $0 < n_* \leq n(x) \leq n^* < 1-\beta$.

for some $\alpha > 0$ and $\beta > 0$ positive constants. Then, there exists at least one transmission eigenvalue.

Proof. First assume that assumption 1) holds. This assumption also implies that

$$0 < \frac{1}{n^*-1} \leq \frac{1}{n(x)-1} \leq \frac{1}{n_*-1} < \infty$$

and according to the above, A_τ and B, $\tau > 0$ satisfy the assumptions of Theorem 2.1 with $U = H_0^2(D)$. Using the Poincaré inequality

$$\|\nabla u\|^2_{L^2(D)} \leq \frac{1}{\lambda_0(D)} \|\Delta u\|^2_{L^2(D)} \tag{21}$$

we have that (see [5] for more details)

$$(\mathbb{A}_\tau u - \tau \mathbb{B} u,\, u)_{H_0^2} \geq \left(\gamma - \frac{\gamma^2}{\epsilon} - \frac{\tau}{\lambda_0(D)}\right) \|\Delta u\|^2_{L^2} + \tau(1+\gamma-\epsilon)\|u\|^2_{L^2} \tag{22}$$

with $\gamma = \frac{1}{n^*-1}$ and $\gamma < \epsilon < \gamma+1$. Hence $\mathbb{A}_\tau - \tau\mathbb{B}$ is positive as long as $\tau < \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\lambda_0(D)$. In particular, taking ϵ arbitrary close to $\gamma+1$, the latter becomes $\tau < \frac{\gamma}{1+\gamma}\lambda_0(D) = \frac{\lambda_0(D)}{\sup_D(n)}$. Then any positive number τ_0 smaller then $\frac{\lambda_0(D)}{sup_D(n)}$ satisfies assumption 1 of Theorem 2.1. Next we have that

$$\begin{aligned}(\mathbb{A}_\tau u - \tau \mathbb{B} u,\, u)_{H_0^2(D)} &= \int_D \frac{1}{n-1}|\Delta u + \tau u|^2\,dx + \tau^2 \int_D |u|^2\,dx - \tau \int_D |\nabla u|^2\,dx \\ &\leq \int_D \frac{1}{n_*-1}|\Delta u + \tau u|^2\,dx + \tau^2 \int_D |u|^2\,dx - \tau \int_D |\nabla u|^2\,dx \qquad (23)\end{aligned}$$

Now let $B_1 \subset D$ be a ball contained in D centered at the origin and $u_1 := w_{j_0}^{B_1,n_*} - v_{j_0}^{B_1,n_*} \in H_0^2(B_1)$ be a transmission eigenfunction corresponding to the transmission eigenvalue k_{j_0} for some j_0 for the ball B_1 with index of refraction n_* and let $\tau_1 = k_{j_0}^2$. The extension by zero $\tilde{u}_1$ in the whole D of u_1 is a function in $H_0^2(D)$. Using (20) and (23) we have that

$$(\mathbb{A}_{\tau_1}\tilde{u}_1 - \tau_1\mathbb{B}\tilde{u}_1,\, \tilde{u}_1)_{H_0^2(D)} \leq 0.$$

Hence from Theorem 2.1 there exists a transmission eigenvalue $k > 0$ such that k^2 is between τ_0 and τ_1. Next we assume that assumption 2) holds. The proof for this case uses similar arguments as in the previous case after replacing $\mathbb{A}_\tau$ with $\tilde{\mathbb{A}}_\tau$. In this case we have that

$$0 < \frac{n_*}{1-n_*} \leq \frac{n(x)}{1-n(x)} \leq \frac{n^*}{1-n^*} < \infty,$$

and therefore according to the above, $\tilde{A}_\tau$ and B, $\tau > 0$ satisfy the assumptions of Theorem 2.1 with $U = H_0^2(D)$. Now we have (see [5] for more details)

$$\left(\tilde{\mathbb{A}}_\tau u - \tau\mathbb{B}u,\, u\right)_{H_0^2} \geq \left(1+\gamma-\epsilon-\frac{\tau}{\lambda_0(D)}\right)\|\Delta u\|^2_{L^2} + \tau\left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|u\|^2_{L^2} \tag{24}$$

with $\gamma = \frac{n_*}{1-n_*}$ and $\gamma < \epsilon < \gamma + 1$. Hence $\tilde{A}_\tau - \tau B$ is positive as long as $\tau < (1+\gamma-\epsilon)\,\lambda_0(D)$. In particular letting ϵ be arbitrarily close to γ shows in this case that any $\tau_0 < \lambda_0(D)$ satisfies the assumption 1 in Theorem 2.1. Next we have

$$\left(\tilde{\mathbb{A}}_\tau u - \tau \mathbb{B} u,\, u\right)_{H_0^2(D)} = \int_D \frac{n}{1-n} |\Delta u + \tau u|^2\, dx + \int_D |\Delta u|^2\, dx - \tau \int_D |\nabla u|^2\, dx$$

$$\leq \int_D \frac{n^*}{1-n^*} |\Delta u + \tau u|^2\, dx + \int_D |\Delta u|^2\, dx - \tau \int_D |\nabla u|^2\, dx. \tag{25}$$

Again we consider $B_1 \subset D$ be a ball contained in D centered at the origin and $u_1 := w_{j_0}^{B_1,n^*} - v_{j_0}^{B_1,n^*} \in H_0^2(B_1)$ be a transmission eigenfunction corresponding to the transmission eigenvalue k_{j_0} for some j_0 for the ball B_1 now with index of refraction n^* and let $\tau_1 = k_{j_0}^2$. The extension by zero $\tilde{u}_1$ in the whole D of u_1 is a function in $H_0^2(D)$. Using (20) and (25) we now have that

$$\left(\tilde{\mathbb{A}}_{\tau_1} \tilde{u}_1 - \tau_1 \mathbb{B} \tilde{u}_1,\, \tilde{u}_1\right)_{H_0^2(D)} \leq 0$$

and the existence of a transmission eigenvalue $k > 0$ such that k^2 is between τ_0 and τ_1 is guaranteed by Theorem 2.1. □

3.1. THE EXISTENCE OF AN INFINITE DISCRETE SET OF TRANSMISSION EIGENVALUES FOR CONSTANT INDEX OF REFRACTION. Our next goal is to show that for an arbitrary domain D there exists indeed an infinite countable number of transmission eigenvalues in the case when the index of refraction is a positive constant $n > 0$ such that $n \neq 1$. (Note that for D a ball and spherically stratified medium the result has been proven in [9].) We consider two balls B_1 and B_2 centered at the origin with radius r_1 and r_2, respectively such that $B_1 \subset D \subset B_2$. Next we consider the interior transmission eigenvalue problem for B_1, B_2 and D with index of refraction n, i.e. (16)-(19) with B replaced by B_1 and B_2 , respectively and (1)-(4) with $n(x)$ replaced by the constant n. We denote by $\mathbb{A}_\tau^{B_1}$ on $H_0^2(B_1)$, $\mathbb{A}_\tau^{B_2}$ on $H_0^2(B_2)$ and $\mathbb{A}_\tau^{D}$ on $H_0^2(D)$ the corresponding operators defined in Section 2 and similarly we also have $\tilde{A}_\tau^{B_1}$, $\tilde{A}_\tau^{B_2}$ and $\tilde{A}_\tau^{D}$. The eigenvalues of these operators with respect to the operator $\mathbb{B}$ (defined on the corresponding space), i.e. the eigenvalues of corresponding problems (12) or (13), are denoted by $\lambda_j(\tau, B_1)$, $\lambda_j(\tau, B_2)$ and $\lambda_j(\tau, D)$, $j = 1, \ldots$. The inclusion $B_1 \subset D$ induces a natural embedding $H_0^2(B_1) \subset H_0^2(D)$ just by extending by zero functions in $H_0^2(B_1)$ to the whole D. Similarly, because $D \subset B_2$ we also have $H_0^2(D) \subset H_0^2(B_2)$. In particular, min-max principle (14) implies the following monotonocity

$$\lambda_j(\tau, B_2) \leq \lambda_j(\tau, D) \leq \lambda_j(\tau, B_1). \tag{26}$$

(Note that the kernel of $\mathbb{B}$ contains only constant functions which are not in H_0^2 except for the zero function).

Theorem 3.2. *Assume that the index of refraction $n > 0$ is a positive constant such that $n \neq 1$. Then there exists an infinite discrete set of transmission eigenvalues with $+\infty$ as accumulation point.*

Proof. Let us denote by $a = r_2/r_1 > 1$ and make the change of variable $\tilde{x} = ax$. Obviously, if $x \in B_1$ then $\tilde{x} \in B_2$. Scaling properties of the Helmholz equation give that if k_{j,B_1}, $j = 1, \ldots$ is a transmission eigenvalue for the ball B_1 and $u_j^{B_1}(x) = w_j^{B_1}(x) - v_j^{B_1}(x)$ is a corresponding eigenfunction then $k_{j,B_2} := k_{j,B_1}/a$

is a transmission eigenvalue for the ball B_2 and $\tilde{u}_j^{B_2}(\tilde{x}) := u_j^{B_1}(\tilde{x}/a)$ is a corresponding eigenfunction and conversely. Hence there is a one to one correspondence between transmission eigenvalues for B_1 and B_2 and we count them accordingly. Obviously, from Section 2 we have that for any $j \in \mathbb{N}$ there exists a m_j such that $k^2_{j,B_1} = \lambda_{m_j}(k^2_{j,B_1}, B_1)$. The same scaling property is inherited by the eigenvalue problem (12) or (13). Indeed, if $n > 1$ we have that for $u \in H_0^2(B_1)$ and $\tilde{u} \in H_0^2(B_2)$

$$\begin{aligned}\mathcal{R}(u,\tau,B_1) &= \frac{\frac{1}{n-1}\|\Delta u+\tau u\|^2_{B_1}+\tau^2\|u\|^2_{B_1}}{\|\nabla u\|^2_{B_1}} = \frac{\frac{a^4}{n-1}\|\Delta\tilde{u}+\frac{\tau}{a^2}\tilde{u}\|^2_{B_2}+\tau^2\|\tilde{u}\|^2_{B_2}}{a^2\|\nabla\tilde{u}\|^2_{B_2}}\\ &= a^2\frac{\frac{1}{n-1}\|\Delta\tilde{u}+\frac{\tau}{a^2}\tilde{u}\|^2_{B_2}+\left(\frac{\tau}{a^2}\right)^2\|\tilde{u}\|^2_{B_2}}{\|\nabla\tilde{u}\|^2_{B_2}} = a^2\mathcal{R}\left(\tilde{u},\frac{\tau}{a^2},B_2\right). \qquad (27)\end{aligned}$$

Similarly, for $0 < n < 1$ we have that

$$\begin{aligned}\mathcal{R}(u,\tau,B_1) &= \frac{\frac{n}{1-n}\|\Delta u+\tau u\|^2_{B_1}+\|\Delta u\|^2_{B_1}}{\|\nabla u\|^2_{B_1}} = \frac{\frac{a^4 n}{1-n}\|\Delta\tilde{u}+\frac{\tau}{a^2}\tilde{u}\|^2_{B_2}+a^4\|\Delta\tilde{u}\|^2_{B_2}}{a^2\|\nabla\tilde{u}\|^2_{B_2}}\\ &= a^2\frac{\frac{n}{1-n}\|\Delta\tilde{u}+\frac{\tau}{a^2}\tilde{u}\|^2_{B_2}+\|\Delta\tilde{u}\|^2_{B_2}}{\|\nabla\tilde{u}\|^2_{B_2}} = a^2\mathcal{R}\left(\tilde{u},\frac{\tau}{a^2},B_2\right). \qquad (28)\end{aligned}$$

Hence form (27) and (28) for both cases we have

$$\begin{aligned}\lambda_j(\tau,B_1) &= \min_{W\subset\mathcal{U}_j}\max_{u\in W\setminus\{0\}}\mathcal{R}(u,\tau,B_1) \qquad (29)\\ &= a^2\min_{W\subset\tilde{\mathcal{U}}_j}\max_{\tilde{u}\in W\setminus\{0\}}\mathcal{R}\left(\tilde{u},\frac{\tau}{a^2},B_2\right) = a^2\lambda_j\left(\frac{\tau}{a^2},B_2\right)\end{aligned}$$

where $\mathcal{U}_j$ denotes the set of all j dimensional subspaces W of $H_0^2(B_1)$ and $\tilde{\mathcal{U}}_j$ denotes the set of all j dimensional subspaces W of $H_0^2(B_2)$. Thus, from (29) we can write

$$k^2_{j,B_2} = \frac{1}{a^2}k^2_{j,B_1} = \frac{1}{a^2}\lambda_{m_j}(k^2_{j,B_1},B_1) = \lambda_{m_j}\left(\frac{k^2_{j,B_1}}{a^2},B_2\right) = \lambda_{m_j}(k^2_{j,B_2},B_2).$$

Hence we have proven that for every $j = 1,\dots,$ there exists a m_j such that

$$k^2_{j,B_1} = \lambda_{m_j}(k^2_{j,B_1},B_1) \quad \text{and} \quad k^2_{j,B_2} = \lambda_{m_j}(k^2_{j,B_2},B_2) \qquad (30)$$

where $k_{j,B_1} > 0$ and $k_{j,B_2} > 0$ are the transmission eigenvalue for the ball B_1 and B_2. Recall that $k_{j,B_1} = ak_{j,B_2} > k_{j,B_2}$.

Now as mentioned before transmission eigenvalues for D are the zeros of continuous functions

$$f_j(\tau) := \lambda_j(\tau,D) - \tau \qquad j \in \mathbb{N}.$$

For $\tau = k^2_{j,B_2}$ we have that

$$f_{m_j}(k^2_{j,B_2}) = \lambda_{m_j}(k^2_{j,B_2},D) - k^2_{j,B_2} \geq \lambda_{m_j}(k^2_{j,B_2},B_2) - k^2_{j,B_2} = 0$$

and

$$f_{m_j}(k^2_{j,B_1}) = \lambda_{m_j}j(k^2_{j,B_1},D) - k^2_{j,B_1} \leq \lambda_{m_j}(k^2_{j,B_1},B_1) - k^2_{j,B_2} = 0.$$

Therefore by continuity f_{m_j} has a zero in the interval $[k^2_{j,B_2}, k^2_{j,B_1}]$, whence there exists a transmission eigenvalue in each of the intervals $[k^2_{j,B_2}, k^2_{j,B_1}]$, for $j \in \mathbb{N}$. Since $k_{j,B_2} \to +\infty$ and $k_{j,B_1} \to +\infty$ as $j \to +\infty$, we have shown the existence of an infinite set of transmission eigenvalues. $\square$

Note that form [9] and [13] one has the following asymptotic behavior of the transmission eigenvalues for a ball B of radius r with constant index of refraction n

$$k_{j,B}^2 = \frac{j^2\pi^2}{r^2(\sqrt{n}-1)^2} + O(1)$$

for j large enough. Now for the arbitrary domain D, let $\{k_j\}$ be the sequence of transmission eigenvalues that we recovered in the above analysis. Hence for j large enough we have

$$\frac{j^2\pi^2}{r_2^2(\sqrt{n}-1)^2} + C_1 \leq k_j^2 \leq \frac{j^2\pi^2}{r_1^2(\sqrt{n}-1)^2} + C_2$$

where C_1 and C_2 are two constant (independent of n and j) and r_1 is the radius of the ball $B_1 \subset D$ and r_2 is the radius of the ball $D \subset B_2$.

3.2. Bounds for the first transmission eigenvalue. We end by providing a lower and an upper bound for the first transmission eigenvalue for an arbitrary domain D and index of refraction $n(x)$ in terms of the first eigenvalue for appropriate balls with constant index of refraction.

Corollary 3.1. *Let $n \in L^\infty(D)$ and let B_1 be the largest ball $B_1 \subset D$ and B_2 the smallest ball $D \subset B_2$. Then*

1) *If $1+\alpha \leq n_* \leq n(x) \leq n^* < \infty$ then*
$$0 < k_{1,B_2,n^*} \leq k_{1,D,n(x)} \leq k_{1,B_1,n_*}$$
2) *If $0 < n_* \leq n(x) \leq n^* < 1-\beta$ then*
$$0 < k_{1,B_2,n_*} \leq k_{1,D,n(x)} \leq k_{1,B_1,n^*}$$

where k_{1,B_2,n_*} and k_{1,B_2,n^*} are the first transmission eigenvalue corresponding to the ball B_2 with constant index of refraction n_* and n^* respectively, $k_{1,D,n(x)}$ is the first transmission eigenvalue of D with the given index of refraction $n(x)$ and k_{1,B_1,n^*} and k_{1,B_1,n_*} are the first transmission eigenvalue for the ball B_1 with index of refraction n^* and n_*, respectively.

Proof. Assume first that $1+\alpha \leq n_* \leq n(x) \leq n^* < \infty$. Then obviously for any $u \in H_0^2(D)$ we have

$$\frac{\frac{1}{n^*-1}\|\Delta u + \tau u\|_D^2 + \tau^2\|u\|_D^2}{\|\nabla u\|_D^2} \leq \frac{\int_D \frac{1}{n(x)-1}|\Delta u + \tau u|^2\,dx + \tau^2\|u\|_D^2}{\|\nabla u\|_D^2}$$
$$\leq \frac{\frac{1}{n_*-1}\|\Delta u + \tau u\|_D^2 + \tau^2\|u\|_D^2}{\|\nabla u\|_D^2}. \tag{31}$$

Therefore from (14) we have that for an arbitrary $\tau > 0$

$$\lambda_1(\tau, B_2, n^*) \leq \lambda_1(\tau, D, n^*) \leq \lambda_1(\tau, D, n(x)) \leq \lambda_1(\tau, D, n_*) \leq \lambda_1(\tau, B_1, n_*) \tag{32}$$

where $\lambda_1(\tau, D, n^*)$, $\lambda_1(\tau, D, n(x))$ and $\lambda_1(\tau, D, n_*)$ are the fist eigenvalue of the auxiliary problem for D and n^*, $n(x)$ and n_*, respectively, and $\lambda_1(\tau, B_2, n^*)$ and $\lambda_1(\tau, B_1, n_*)$ are the first eigenvalue of the auxiliary problem for B_2, n^* and B_1, n_*, respectively. Now for $\tau_1 := k_{1,B_1,n_*}$, $B_1 \subset D$ we have that $\lambda_1(\tau, D, n(x)) - \tau \leq 0$ since in the subspace spanned by the extension by zero to the whole D of the the eigenfunction $\tilde{u}_1^{B_1,n_*}$ the Rayleigh quotient minus τ for $\tau = \tau_1$ becomes negative. On the other hand, for $\tau_0 := k_{1,B_2,n^*}$, $B_2 \subset D$ we have $\lambda_1(\tau_0, B_2, n^*) - \tau_0 = 0$ and hence $\lambda_1(\tau_0, D, n(x)) - \tau_0 \geq 0$. Therefore the first eigenvalue $k_{1,D,n(x)}$ corresponding to D and $n(x)$ is between k_{1,B_2,n^*} and k_{1,B_1,n_*}. Note that there is no transmission eigenvalue for D and $n(x)$ that is less then k_{1,B_2,n^*}. Indeed if there is a transmission eigenvalue strictly less then k_{1,B_2,n^*} then by the monotonocity of the eigenvalues of the auxiliary problem with respect to the domain and the fact that for τ small enough there are no transmission eigenvalues we would have found an eigenvalue of the ball B_2 and n^* that is strictly smaller then the first eigenvalue. The case for $0 < n_* \leq n(x) \leq n^* < 1 - \beta$ can be proven in the same way if n_* is replaced by n^*. □

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