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THE EXISTENCE OF AN INFINITE DISCRETE SET OF TRANSMISSION EIGENVALUES*

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Abstract. We prove the existence of an infinite discrete set of transmission eigenvalues corresponding to the scattering problem for isotropic and anisotropic inhomogeneous media for both the Helmholtz and Maxwell's equations. Our discussion includes the case of the interior transmission problem for an inhomogeneous medium with cavities, i.e., subregions with contrast zero.

Key words. interior transmission problem, transmission eigenvalues, inhomogeneous medium, inverse scattering

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1. Introduction.

The interior transmission problem arises in inverse scattering theory for inhomogeneous media. It is a boundary value problem for a set of equations defined in a bounded domain coinciding with the support of the scattering object. Of particular interest is the spectrum associated with this boundary value problem, more specifically the existence of eigenvalues which are called transmission eigenvalues. On one hand, in the context of sampling methods for reconstructing the support of the scatterer [2], [17], one needs to avoid those frequencies that correspond to transmission eigenvalues, and hence it is important to know that the transmission eigenvalues form a discrete set. On the other hand, one can use transmission eigenvalues to obtain information about physical properties of the scattering object [1], [4], [6], and therefore it is important to know whether they exist and to understand their connection with the index of refraction. This application is based on the recent results in [3] which justify the numerical observation that transmission eigenvalues can be computed from the far field data. Either way, the investigation of the spectral properties of the interior transmission problem has become an interesting question in inverse scattering theory.

The interior transmission problem was first introduced in [12] in connection with an inverse scattering problem for acoustic waves. Roughly speaking, two main approaches are available in the study of the well posedness of the interior transmission problem, namely, integral equation methods [10], [15] and variational methods [5], [6], [8], [14]. Until recently the only known result on transmission eigenvalues was the fact that they form at most a discrete set with infinity as the only possible accumulation point. The first result about the existence of transmission eigenvalues was announced in [18] for the case of the reduced wave equation in an isotropic inhomogeneous medium where it was shown that there exist a finite number of transmission

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eigenvalues provided that the index of refraction is large enough. This paper was soon followed by [9], [16], where the same result was proven for anisotropic media and Maxwell's equations. Subsequently the difficult case of a medium with cavities, i.e., regions with zero contrast, was investigated in [5]. We refer the reader to [13] for a comprehensive review on the interior transmission problem. Further progress on the question of the existence of transmission eigenvalues was recently made in [7] where the assumption on the size of the index of refraction was removed and for the case of medium with constant index of refraction it was proven that there exist an infinite discrete set of transmission eigenvalues.

In this paper we will extend the ideas of [7] to show that there exists an infinite discrete set of transmission eigenvalues for inhomogeneous isotropic and anisotropic media for both the Helmholtz and Maxwell's equations including the case of media with cavities. The only assumption we impose is that the index of refraction is less than or greater than the index of refraction of the background medium. Our proof employs the analytic framework developed in [6] and [9] and makes use of transmission eigenvalues for balls with constant index of refraction first used in [7]. We will also provide lower and upper bounds for the first transmission eigenvalue.

The plan of our paper is as follows. Having set up the analytic framework, we first show the existence of infinitely many transmission eigenvalues and obtain lower and upper bounds for the first transmission eigenvalue for the case of isotropic inhomogeneous media for the Helmholtz equation. Then in section 2.3 we provide similar results for the case of anisotropic media for both the Helmholtz and Maxwell's equations. Finally we discuss the case of media with voids, i.e., media with subregions having zero contrast, for which we also prove the existence of infinitely many transmission eigenvalues. Of potential use in nondestructive testing [1] is a new upper bound for the first transmission eigenvalue provided by our analysis.

Although the results of this paper provide an important step forward in understanding the spectral properties of the interior transmission problem, many questions still remain. We think that some interesting open problems in this direction are the existence of transmission eigenvalues for the case of media with contrast partly positive and partly negative, the existence of complex transmission eigenvalues, and the completeness of the eigensystem of the interior transmission problem.

2. The existence of an infinite set of transmission eigenvalues.

We consider the interior transmission eigenvalue problem corresponding to the scattering problem for inhomogeneous isotropic and anisotropic media for the Helmholtz equation as well as for Maxwell's equations. Our goal is to prove the existence of infinitely many transmission eigenvalues and provide some estimates for these transmission eigenvalues. Throughout this section, $D \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded simply connected region with piecewise smooth boundary ∂D, and ν denotes the outward unit normal vector to ∂D.

2.1. Abstract analytic framework.

The interior transmission eigenvalue problems we discuss in this paper can be described by the following abstract analytic framework which is introduced in [9]. In particular, let U be a separable Hilbert space with scalar product $(\cdot,\cdot)$, let $\mathbb{A}$ be a bounded, positive definite, and self-adjoint operator on U, and let $\mathbb{B}$ be a nonnegative, self-adjoint, and compact bounded linear operator on U. Then there exists an increasing sequence of positive real numbers $(\lambda_j)_{j\geq 1}$ and a sequence $(u_j)_{j\geq 1}$ of elements of U such that $\mathbb{A}u_j = \lambda_j \mathbb{B}u_j$. The sequence $(u_j)_{j\geq 1}$ forms a basis of $(\mathbb{A}\ker(\mathbb{B}))^\perp$, and if $\ker(\mathbb{B})^\perp$ has infinite dimension, then $\lambda_j \to +\infty$ as $j \to \infty$ (see Theorem 2.1 in [9]). Furthermore, these eigenvalues satisfy a min-max principle (see Corollary 2.1 [9]), namely,

$$\lambda_j = \min_{W\subset\mathcal{U}_j}\left(\max_{u\in W\setminus\{0\}}\frac{(\mathbb{A}u,u)}{(\mathbb{B}u,u)}\right), \tag{2.1}$$

where $\mathcal{U}_j$ denotes the set of all j-dimensional subspaces W of U such that $W\cap\ker(\mathbb{B}) = \{0\}$. These eigenvalues can be ordered in increasing order, i.e., $\lambda_1 \le \lambda_2 \le \dots$.

Let $\tau \longmapsto \mathbb{A}_\tau$ be a continuous mapping from $]0,\infty[$ to the set of self-adjoint and positive definite bounded linear operators on U and consider the generalized eigenvalue problem

$$\mathbb{A}_\tau u - \lambda_j(\tau)\mathbb{B}u = 0, \qquad u\in U. \tag{2.2}$$

Obviously from (2.1) we have that λ_j for every $j\in\mathbb{N}$ is a continuous function of τ in $]0,\infty[$. The following theorem provides the fundamental tool in proving the existence of transmission eigenvalues.

THEOREM 2.1. *Let $\tau \longmapsto \mathbb{A}_\tau$ be a continuous mapping from $]0,\infty[$ to the set of self-adjoint and positive definite bounded linear operators on U and let $\mathbb{B}$ be a self-adjoint and nonnegative compact bounded linear operator on U. We assume that there exist two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. *$\mathbb{A}_{\tau_0} - \tau_0\mathbb{B}$ is positive on U,*
2. *$\mathbb{A}_{\tau_1} - \tau_1\mathbb{B}$ is nonpositive on an m-dimensional subspace of U.*

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1,\dots,m$, has at least one solution in $[\tau_0,\tau_1]$, where $\lambda_j(\tau)$ is the jth eigenvalue (counting multiplicity) of $\mathbb{A}_\tau$ with respect to $\mathbb{B}$; i.e., $\ker(\mathbb{A}_\tau - \lambda_j(\tau)\mathbb{B}) \neq \{0\}$.

Proof. First we can deduce from (2.1) that for all $j\ge 1$, $\lambda_j(\tau)$ is a continuous function of τ. Assumption 1 shows that $\lambda_j(\tau_0) > \tau_0$ for all $j \ge 1$. Assumption 2 implies in particular that $W_k \cap \ker(B) = \{0\}$. Hence, another application of (2.1) implies that $\lambda_j(\tau_1) \le \tau_1$ for $1 \le j \le k$. The desired result is then obtained by applying the intermediate value theorem. ◻

2.2. The scalar isotropic media. The interior transmission eigenvalue problem corresponding to the scattering problem for an isotropic inhomogeneous medium in $\mathbb{R}^d$, $d = 2,3$, reads:

$$\Delta w + k^2 n(x) w = 0 \quad \text{in} \quad D, \tag{2.3}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad D, \tag{2.4}$$

$$w = v \quad \text{on} \quad \partial D, \tag{2.5}$$

$$\frac{\partial w}{\partial\nu} = \frac{\partial v}{\partial\nu} \quad \text{on} \quad \partial D \tag{2.6}$$

for $w\in L^2(D)$ and $v\in L^2(D)$ such that $w - v\in H_0^2(D)$, where

$$H_0^2(D) = \left\{u\in H^2(D):\ u = 0 \quad \text{and} \quad \frac{\partial u}{\partial\nu} = 0 \quad \text{on} \quad \partial D\right\}.$$

Here we assume that for some constant α the positive real valued function n is such that $n(x) > \alpha > 0$ almost everywhere in D, $n\in L^\infty(D)$, and $1/|n-1|\in L^\infty(D)$. Note that these assumptions are relevant from the physical point of view (see the last section for the case of media with voids, i.e., where $n = 1$ in parts of D).

DEFINITION 2.2. *Values of $k > 0$ for which the homogeneous interior transmission problem* (2.3)–(2.6) *has nonzero solutions $w\in L^2(D)$ and $v\in L^2(D)$ such that*

$w - v \in H_0^2(D)$ are called transmission eigenvalues. If $k > 0$ is a transmission eigenvalue we call $u = w - v$ the corresponding eigenfunction where w and v is a nonzero solution of (2.3)–(2.6).

Remark 2.1. Based on the analytic Fredholm theory, it is well known that the set of transmission eigenvalues is at most discrete with $+\infty$ as the only possible accumulation point [11], [15], [20]. The goal of this paper is to prove that indeed there exist infinitely many transmission eigenvalues.

It is possible to write (2.3)–(2.6) as an equivalent eigenvalue problem for $u = w - v \in H_0^2(D)$ for the following fourth order equation:

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0, \tag{2.7}$$

which in variational form is formulated as finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v})\, dx = 0 \qquad \text{for all } v \in H_0^2(D). \tag{2.8}$$

Following [9] we set $\tau := k^2$ and define the following bounded sesquilinear forms on $H_0^2(D) \times H_0^2(D)$:

$$\mathcal{A}_\tau(u, v) = \left(\frac{1}{n-1}(\Delta u + \tau u),\, (\Delta v + \tau v)\right)_D + \tau^2\,(u,\, v)_D\,, \tag{2.9}$$

$$\begin{aligned} \tilde{\mathcal{A}}_\tau(u, v) &= \left(\frac{1}{1-n}(\Delta u + \tau n u),\, (\Delta v + \tau n v)\right)_D + \tau^2\,(nu,\, v)_D \\ &= \left(\frac{n}{1-n}(\Delta u + \tau u),\, (\Delta v + \tau v)\right)_D + (\Delta u,\, \Delta v)_D\,, \end{aligned} \tag{2.10}$$

and

$$\mathcal{B}(u, v) = (\nabla u,\, \nabla v)_D\,, \tag{2.11}$$

where $(\cdot\,,\,\cdot)_D$ denotes the $L^2(D)$ inner product. Using the Riesz representation theorem we now define the bounded linear operators $\mathbb{A}_\tau : H_0^2(D) \to H_0^2(D)$, $\tilde{\mathbb{A}}_\tau : H_0^2(D) \to H_0^2(D)$, and $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ by

$$\begin{aligned} &(\mathbb{A}_\tau u, v)_{H^2(D)} = \mathcal{A}_\tau(u, v), \qquad \left(\tilde{\mathbb{A}}_\tau u, v\right)_{H^2(D)} = \tilde{\mathcal{A}}_\tau(u, v), \\ &\text{and} \qquad (\mathbb{B}u, v)_{H^2(D)} = \mathcal{B}(u, v). \end{aligned} \tag{2.12}$$

In terms of these operators we can rewrite (2.8) as

$$(\mathbb{A}_\tau u - \tau \mathbb{B} u, v)_{H^2(D)} = 0 \quad \text{or} \quad \left(\tilde{\mathbb{A}}_\tau u - \tau \mathbb{B} u, v\right)_{H^2(D)} = 0 \quad \text{for all } v \in H_0^2(D). \tag{2.13}$$

In order to analyze (2.13), we recall the following results from [9] about the properties of the involved operators. To this end, let $\lambda_1(D)$ be the first Dirichlet eigenvalue for $-\Delta$ in D.

Lemma 2.3. *The operators $\mathbb{A}_\tau : H_0^2(D) \to H_0^2(D)$, $\tilde{\mathbb{A}}_\tau : H_0^2(D) \to H_0^2(D)$, $\tau > 0$, and $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ are self-adjoint. If for some constant $\gamma > 0$ and for* *almost all* $x \in D$, $\frac{1}{n(x)-1} > \gamma > 0$, *then* $\mathbb{A}_\tau$ *is a positive definite operator, whereas if* $\frac{n(x)}{1-n(x)} > \gamma > 0$, *then* $\tilde{\mathbb{A}}_\tau$ *is a positive definite operator. In addition,* $\mathbb{B}$ *is a positive compact operator.*

See [9] for the proof.

LEMMA 2.4. *If* $\frac{1}{n(x)-1} > \gamma > 0$ *for some constant* $\gamma > 0$ *and for almost all* $x \in D$, *then*

$$(\mathbb{A}_\tau u - \tau \mathbb{B} u, u)_{H^2} \geq \alpha \|u\|^2_{H^2} > 0 \quad \textit{for all} \quad 0 < \tau < \frac{\lambda_1(D)}{\sup_D(n)} \quad \textit{and} \quad u \in H^2_0(D).$$

If $\frac{n(x)}{1-n(x)} > \gamma > 0$ *for some constant* $\gamma > 0$ *and for almost all* $x \in D$, *then*

$$\left(\tilde{\mathbb{A}}_\tau u - \tau \mathbb{B} u, u\right)_{H^2} \geq \alpha \|u\|^2_{H^2} > 0 \quad \textit{for all} \quad 0 < \tau < \lambda_1(D) \quad \textit{and} \quad u \in H^2_0(D).$$

Proof. For the reader's convenience we show here only the proof for the case of $\frac{1}{n(x)-1} > \gamma > 0$. We have

$$\begin{aligned}(\mathbb{A}_\tau u - \tau \mathbb{B} u,\, u)_{H^2_0} &= \mathcal{A}_\tau(u,u) - \tau \|\nabla u\|^2_{L^2} \\ &\geq \left(\gamma - \frac{\gamma^2}{\epsilon}\right) \|\Delta u\|^2_{L^2} + (1+\gamma-\epsilon)\|u\|^2_{L^2} - \tau \|\nabla u\|^2_{L^2}\end{aligned} \tag{2.14}$$

for $\gamma < \epsilon < \gamma + 1$. Since $\nabla u \in H^1_0(D)$, using the Poincaré inequality we have that

$$\|\nabla u\|^2_{L^2(D)} \leq \frac{1}{\lambda_1(D)} \|\Delta u\|^2_{L^2(D)}, \tag{2.15}$$

and hence we obtain

$$(\mathbb{A}_\tau u - \tau \mathbb{B} u,\, u)_{H^2_0} \geq \left(\gamma - \frac{\gamma^2}{\epsilon} - \frac{\tau}{\lambda_0(D)}\right) \|\Delta u\|^2_{L^2} + \tau(1+\gamma-\epsilon)\|u\|^2_{L^2}.$$

Hence $\mathbb{A}_\tau - \tau \mathbb{B}$ is positive as long as $\tau < (\gamma - \frac{\gamma^2}{\epsilon})\lambda_1(D)$. In particular, choosing $\gamma = \frac{1}{n^*-1}$, where $n^* = \sup_D n(x)$, and taking ϵ arbitrary closed to $\gamma + 1$, the latter becomes $\tau < \frac{\gamma}{1+\gamma}\lambda_1(D) = \frac{\lambda_1(D)}{\sup_D(n)}$. $\square$

Obviously, $\mathbb{A}_\tau$ and $\tilde{\mathbb{A}}_\tau$ depend continuously on $\tau \in (0, +\infty)$. From the above discussion, $k > 0$ is a transmission eigenvalue if for $\tau = k^2$ the kernel of the operator $\mathbb{A}_\tau - \tau \mathbb{B}$ (if $1/(n-1) > \gamma > 0$) or the kernel of the operator $\tilde{\mathbb{A}}_\tau - \tau \mathbb{B}$ (if $n/(1-n) > \gamma > 0$) is nontrivial. In order to analyze the kernel of these operators, we consider the auxiliary eigenvalue problems

$$\mathbb{A}_\tau u - \lambda(\tau) \mathbb{B} u = 0 \qquad u \in H^2_0(D) \qquad \text{if} \quad 1/(n-1) > \gamma > 0 \tag{2.16}$$

and

$$\tilde{\mathbb{A}}_\tau u - \lambda(\tau) \mathbb{B} u = 0 \qquad u \in H^2_0(D) \qquad \text{if} \quad n/(1-n) > \gamma > 0. \tag{2.17}$$

Thus a transmission eigenvalue, $k > 0$ is such that $\tau := k^2$ solves $\lambda(\tau) - \tau = 0$, where $\lambda(\tau)$ is an eigenvalue corresponding to (2.16) or (2.17) in the respective cases. Our goal is now to use Theorem 2.1 to prove the existence of an infinite set of transmission eigenvalues.

Remark 2.2. The multiplicity of transmission eigenvalues is finite since if k_0 is a transmission eigenvalue, then the kernel of $I-\tau_0\mathbb{A}_{\tau_0}^{-1/2}\mathbb{B}\mathbb{A}_{\tau_0}^{-1/2}$ or $I-\tau_0\tilde{\mathbb{A}}_{\tau_0}^{-1/2}\mathbb{B}\tilde{\mathbb{A}}_{\tau_0}^{-1/2}$, where $\tau_0 := k_0^2$ is finite since the operators $\tau_0\mathbb{A}_{\tau_0}^{-1/2}\mathbb{B}\mathbb{A}_{\tau_0}^{-1/2}$ (if $1/(n-1)>\gamma>0$) and $\tau_0\tilde{\mathbb{A}}_{\tau_0}^{-1/2}\mathbb{B}\tilde{\mathbb{A}}_{\tau_0}^{-1/2}$ (if $n/(1-n)>\gamma>0$) are compact and self-adjoint [19]. (Here $\mathbb{A}^{-1/2}$ is defined by $\mathbb{A}^{-1/2}=\int_0^\infty \lambda^{-1/2}dE_\lambda$ where dE_λ is the spectral measure associated with the positive operator $\mathbb{A}$.)

Now let us consider the interior transmission problem corresponding to a ball B_R of radius R centered at zero with constant index of refraction $n_0>0$ such that $n_0\neq 1$, i.e.,

$$\Delta w + k^2 n_0 w = 0 \quad \text{in} \quad B_R, \tag{2.18}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad B_R, \tag{2.19}$$

$$w = v \quad \text{on} \quad \partial B_R, \tag{2.20}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial B_R. \tag{2.21}$$

By a separation of variables technique, it is possible to show [13] (see also [11]) that (2.18)–(2.21) has a countable discrete set of eigenvalues. Let k_{R,n_0} be the first transmission eigenvalue corresponding to B_R and n_0. Typically, this k_{R,n_0} is the first zero of

$$W(k) := \det\begin{pmatrix} J_0(kR) & J_0(k\sqrt{n_0}R) \\ -J_0'(kR) & -\sqrt{n_0}J_0'(k\sqrt{n_0}R)\end{pmatrix} = 0 \qquad \text{in} \quad \mathbb{R}^2, \tag{2.22}$$

where J_0 is the Bessel functions of order zero, and

$$W(k) = \det\begin{pmatrix} j_0(kR) & j_0(k\sqrt{n_0}R) \\ -j_0'(kR) & -\sqrt{n_0}j_0'(k\sqrt{n_0}R)\end{pmatrix} = 0 \qquad \text{in} \quad \mathbb{R}^3, \tag{2.23}$$

where j_0 is the spherical Bessel function of order 0 (if the first zero of the above determinants is not the first transmission eigenvalue, the latter will be a zero of a similar determinant corresponding to higher order Bessel functions or spherical Bessel functions). Let v^{B_R,n_0} and w^{B_R,n_0} be a nonzero solution of (2.18)–(2.21) corresponding to k_{R,n_0}, and denote by $u^{B_R,n_0} := w^{B_R,n} - v^{B_R,n}$ the corresponding eigenfunction. We have that $u^{B_R,n_0}\in H_0^2(B_R)$ and

$$\int_{B_R} \frac{1}{n_0-1}(\Delta u^{B_R,n_0} + k_{R,n_0}^2 u^{B_R,n_0})(\Delta \overline{u}^{B_R,n_0} + k_{R,n_0}^2 n_0 \overline{u}^{B_R,n_0})\,dx = 0. \tag{2.24}$$

In the following we denote by $n_* = \inf_D(n)$ and $n^* = \sup_D(n)$.

THEOREM 2.5. *Let $n\in L^\infty(D)$ satisfy either one of the following assumptions:*

1. $1+\alpha \le n_* \le n(x) \le n^* < \infty$,
2. $0 < n_* \le n(x) \le n^* < 1-\beta$

for some constants $\alpha>0$ and $\beta>0$. Then there exists an infinite set of transmission eigenvalues with $+\infty$ as the only accumulation point.

Proof. First we note that using the analytic Fredholm theory it can be shown [11], [13], [20] that, under assumption 1 or 2, the set of transmission eigenvalues is at most discrete with $+\infty$ as the only possible accumulation point. In the following we show that there exists an infinite countable set of transmission eigenvalues. First let us suppose that assumption 1 holds. This assumption also implies that

$$0 < \frac{1}{n^*-1} \le \frac{1}{n(x)-1} \le \frac{1}{n_*-1} < \infty.$$

Therefore, from Lemma 2.3, $\mathbb{A}_\tau$ and $\mathbb{B}$ defined by (2.12) satisfy the requirement of Theorem 2.1 with $U = H_0^2(D)$, and from Lemma 2.4 they also satisfy the assumption 1 of Theorem 2.1 with $\tau_0 \le \lambda_1(D)/n^*$. Next let k_{1,n_*} be the first transmission eigenvalue for the ball B of radius $R = 1$ and $n_0 := n_*$. This transmission eigenvalue is the first zero of (2.22) in $\mathbb{R}^2$ or (2.23) in $\mathbb{R}^3$ for $R := 1$ and $n_0 := n_*$ (or possibly similar determinants for higher order Bessel functions). By a scaling argument, it is obvious that $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue corresponding to the ball of radius $\epsilon > 0$ with index of refraction n_*. Now take $\epsilon > 0$ small enough such that D contains $m := m(\epsilon) \ge 1$ disjoint balls $B_\epsilon^1, B_\epsilon^2, \dots, B_\epsilon^m$ of radius ϵ; i.e., $\overline{B_\epsilon^j} \subset D$, $j = 1, \dots, m$, and $\overline{B_\epsilon^j} \cap \overline{B_\epsilon^i} = \emptyset$ for $j \ne i$. Then $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue for each of these balls with index of refraction n_* and let $u^{B_\epsilon^j, n_*} \in H_0^2(B_\epsilon^j)$, $j = 1, \dots, m$, be the corresponding eigenfunction. The extension by zero $\tilde{u}^j$ of $u^{B_\epsilon^j, n_*}$ to the whole D is obviously in $H_0^2(D)$ due to the boundary conditions on $\partial B_{\epsilon,n_*}^j$. Furthermore, the vectors $\{\tilde{u}^1, \tilde{u}^2, \dots, \tilde{u}^m\}$ are linearly independent and orthogonal in $H_0^2(D)$ since they have disjoint supports, and from (2.24) we have that

$$0 = \int_D \frac{1}{n_* - 1}(\Delta\tilde{u}^j + k_{\epsilon,n_*}^2 \tilde{u}^j)(\Delta\overline{\tilde{u}}^j + k_{\epsilon,n_*}^2 n_* \overline{\tilde{u}}^j)\, dx \tag{2.25}$$

$$= \int_D \frac{1}{n_* - 1}|\Delta\tilde{u}^j + k_{\epsilon,n_*}^2 \tilde{u}^j|^2\, dx + k_{\epsilon,n_*}^4 \int_D |\tilde{u}^j|^2\, dx - k_{\epsilon,n_*}^2 \int_D |\nabla\tilde{u}^j|^2\, dx \tag{2.26}$$

for $j = 1, \dots, m$. Let us denote by $\mathcal{U}$ the m-dimensional subspace of $H_0^2(D)$ spanned by $\{\tilde{u}^1, \tilde{u}^2, \dots, \tilde{u}^m\}$. Since each $\tilde{u}^j$, $j = 1, \dots, m$ satisfies (2.25) and they have disjoint supports, we have that for $\tau_1 := k_{\epsilon,n_*}^2$ and for every $\tilde{u} \in \mathcal{U}$

$$(\mathbb{A}_{\tau_1}\tilde{u} - \tau_1\mathbb{B}\tilde{u},\, \tilde{u})_{H_0^2(D)} = \int_D \frac{1}{n-1}|\Delta\tilde{u} + \tau_1\tilde{u}|^2\, dx + \tau_1^2 \int_D |\tilde{u}|^2\, dx - \tau_1 \int_D |\nabla\tilde{u}|^2\, dx$$

$$\le \int_D \frac{1}{n_* - 1}|\Delta\tilde{u} + \tau_1\tilde{u}|^2\, dx + \tau_1^2 \int_D |\tilde{u}|^2\, dx - \tau_1 \int_D |\nabla\tilde{u}|^2\, dx = 0. \tag{2.27}$$

This means that assumption 2 of Theorem 2.1 is also satisfied, and therefore we can conclude that there are $m(\epsilon)$ transmission eigenvalues (counting multiplicity) inside $[\tau_0,\, k_{\epsilon,n_*}]$. Note that $m(\epsilon)$ and k_{ϵ,n_*} both go to $+\infty$ as $\epsilon \to 0$. Since the multiplicity of each eigenvalue is finite, we have shown, by letting $\epsilon \to 0$, that there exists an infinite countable set of transmission eigenvalues that accumulate at ∞.

If the index of refraction is such that assumption 2 holds, then we have that

$$0 < \frac{n_*}{1 - n_*} \le \frac{n(x)}{1 - n(x)} \le \frac{n^*}{1 - n^*} < \infty,$$

and therefore according to Lemmas 2.3 and 2.4, $\tilde{\mathbb{A}}_\tau$ and $\mathbb{B}$, $\tau > 0$, satisfy the requirements and assumption 1 of Theorem 2.1 with $U = H_0^2(D)$ for $\tau_0 \le \lambda_1(D)$. In this case we can estimate

$$\left(\tilde{\mathbb{A}}_\tau u - \tau\mathbb{B}u,\, u\right)_{H_0^2(D)} = \int_D \frac{n}{1-n}|\Delta u + \tau u|^2\, dx + \int_D |\Delta u|^2\, dx - \tau \int_D |\nabla u|^2\, dx$$

$$\le \int_D \frac{n^*}{1-n^*}|\Delta u + \tau u|^2\, dx + \int_D |\Delta u|^2\, dx - \tau \int_D |\nabla u|^2\, dx. \tag{2.28}$$

The rest of the proof for checking the validity of assumption 2 of Theorem 2.1 goes exactly in the same way as for the previous case if one replaces n_* by n^*. ∎

Remark 2.3. The argument in the proof of Theorem 2.5 can be carried through if k_{R,n_0} is any transmission eigenvalue corresponding to (2.18)–(2.21).

Remark 2.4. From the proof of Theorem 2.5 it follows that for every $j \in \mathbb{N}$ the equation $\lambda_j(\tau) - \tau = 0$ has at least one solution, where $\lambda_j(\tau)$ is the jth eigenvalue of the auxiliary eigenvalue problem (2.16) or (2.17).

Using Theorem 2.5 we can now obtain the following estimates for transmission eigenvalues. We call B_{r_1} the largest ball of radius r_1 such that $B_{r_1} \subset D$ and B_{r_2} the smallest ball of radius r_2 such that $D \subset B_{r_2}$. For a given $0 < \epsilon \le r_1$ let $m(\epsilon) \in \mathbb{N}$ be the number of balls B_ϵ of radius ϵ that are contained in D. We denote by k_{1,n_*} and k_{1,n^*} the first transmission eigenvalue corresponding to the ball B_1 of radius one with the index of refraction n_* and n^*, respectively.

COROLLARY 2.6. *Assume that $n \in L^\infty(D)$.*

1. *If $1 + \alpha \le n_* \le n(x) \le n^* < \infty$, then*

$$0 < \frac{k_{1,n^*}}{r_2} \le k_{1,D,n(x)} \le \frac{k_{1,n_*}}{r_1}. \tag{2.29}$$

There are at least $m(\epsilon)$ transmission eigenvalues in the interval $[\frac{k_{1,n^*}}{r_2}, \frac{k_{1,n_*}}{\epsilon}]$.

2. *If $0 < n_* \le n(x) \le n^* < 1 - \beta$, then*

$$0 < \frac{k_{1,n_*}}{r_2} \le k_{1,D,n(x)} \le \frac{k_{1,n^*}}{r_1}. \tag{2.30}$$

There are at least $m(\epsilon)$ transmission eigenvalues in the interval $[\frac{k_{1,n_*}}{r_2}, \frac{k_{1,n^*}}{\epsilon}]$. Here $k_{1,D,n(x)}$ is the first transmission eigenvalue corresponding to D and the given index of refraction $n(x)$.

Proof. The upper bounds in (2.29) and (2.30) are a consequence of Theorem 2.5. Next assume first that $1 + \alpha \le n_* \le n(x) \le n^* < \infty$. Then obviously for any $u \in H_0^2(D)$ we have

$$\begin{aligned} \frac{\frac{1}{n^*-1}\|\Delta u + \tau u\|_D^2 + \tau^2\|u\|_D^2}{\|\nabla u\|_D^2} &\le \frac{\int_D \frac{1}{n(x)-1}|\Delta u + \tau u|^2\,dx + \tau^2\|u\|_D^2}{\|\nabla u\|_D^2} \\ &\le \frac{\frac{1}{n_*-1}\|\Delta u + \tau u\|_D^2 + \tau^2\|u\|_D^2}{\|\nabla u\|_D^2}. \end{aligned} \tag{2.31}$$

Therefore from (2.1) we have that for an arbitrary $\tau > 0$

$$\lambda_1(\tau, B_{r_2}, n^*) \le \lambda_1(\tau, D, n^*) \le \lambda_1(\tau, D, n(x)) \le \lambda_1(\tau, D, n_*) \le \lambda_1(\tau, B_{r_1}, n_*), \tag{2.32}$$

where $\lambda_1(\tau, D, n^*)$, $\lambda_1(\tau, D, n(x))$, and $\lambda_1(\tau, D, n_*)$ are the first eigenvalues of the auxiliary problem for D and n^*, $n(x)$, and n_*, respectively, whereas $\lambda_1(\tau, B_{r_2}, n^*)$ and $\lambda_1(\tau, B_{r_1}, n_*)$ are the first eigenvalues of the auxiliary problem for B_{r_2}, n^* and B_{r_1}, n_*, respectively. Now for $\tau_1 := k_{1,n_*}/r_1$, $B_{r_1} \subset D$, from the proof of Theorem 2.5 we have that $\lambda_1(\tau, D, n(x)) - \tau \le 0$. On the other hand, for $\tau_0 := k_{1,n^*}/r_2$, $D \subset B_{r_2}$, we have $\lambda_1(\tau_0, B_{r_2}, n^*) - \tau_0 = 0$ and hence $\lambda_1(\tau_0, D, n(x)) - \tau_0 \ge 0$. Therefore the first eigenvalue $k_{1,D,n(x)}$ corresponding to D and $n(x)$ is between $k_{1,n^*}/r_2$ and $k_{1,n_*}/r_1$. Note that there is no transmission eigenvalue for D and $n(x)$ that is less than $k_{1,n^*}/r_2$. Indeed, if there is a transmission eigenvalue strictly less than $k_{1,n^*}/r_2$, then by the monotonicity of the eigenvalues of the auxiliary problem with respect to the domain and the fact that for τ small enough there are no transmission eigenvalues we would have found an eigenvalue of the ball B_{r_2} and n^* that is strictly smaller than the first eigenvalue. The case for $0 < n_* \leq n(x) \leq n^* < 1-\beta$ can be proven in the same way if n_* is replaced by n^*. ∎

Remark 2.5. From Lemma 2.4 and the above corollary we have that

$$k_{1,D,n(x)} \geq \max\left(\frac{k_{1,n^*}}{r_2}, \sqrt{\frac{\lambda_1(D)}{n^*}}\right)$$

if $1+\alpha \leq n_* \leq n(x) \leq n^* < \infty$ and

$$k_{1,D,n(x)} \geq \max\left(\frac{k_{1,n_*}}{r_2}, \sqrt{\lambda_1(D)}\right)$$

if $0 < n_* \leq n(x) \leq n^* < 1-\beta$, where $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D.

2.3. The anisotropic inhomogeneous media. We now turn our attention to the interior transmission eigenvalue problem corresponding to the scattering problem for anisotropic media. We consider two problems, namely, the interior transmission problem for the anisotropic Maxwell's equations and the interior transmission problem for the anisotropic scalar equation. In the following we show that these eigenvalue problems can be analyzed in the same way as the problem discussed in section 2.2.

DEFINITION 2.7. *A real valued $d \times d$, $d = 2, 3$, matrix function $K \in L^\infty(D, \mathbb{R}^{d\times d})$ is said to be bounded positive definite on D if there exists a constant $\gamma > 0$ such that $\overline{\boldsymbol{\xi}} \cdot K\boldsymbol{\xi} \geq \gamma |\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \mathbb{C}^d$ and a.e. in D.*

Problem 1: *Anisotropic Maxwell's equations.* Let $D \subset \mathbb{R}^3$ satisfy the assumptions stated at the beginning of section 2. In terms of electric fields the interior transmission eigenvalue problem for anisotropic Maxwell's equations (where the magnetic permeability is assumed to be scalar and constant) is formulated as the problem of finding two vector valued functions $\mathbf{E}$ and $\mathbf{E}_0$ satisfying

$$\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 \quad \text{in} \quad D, \tag{2.33}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2 \mathbf{E}_0 = 0 \quad \text{in} \quad D, \tag{2.34}$$

$$\mathbf{E}\times\nu = \mathbf{E}_0 \times \nu \quad \text{on} \quad \partial D, \tag{2.35}$$

$$\operatorname{curl}\mathbf{E}\times\nu = \operatorname{curl}\mathbf{E}_0\times\nu \quad \text{on} \quad \partial D, \tag{2.36}$$

where N is a 3×3 matrix valued function defined on D with $L^\infty(D)$ real valued entries, i.e., $N \in L^\infty(D, \mathbb{R}^{3\times 3})$. To properly formulate this eigenvalue problem, we consider the Hilbert spaces

$$H(\operatorname{curl}, D) := \{\mathbf{u} \in (L^2(D))^3 : \ \operatorname{curl}\mathbf{u} \in (L^2(D))^3\},$$
$$H_0(\operatorname{curl}, D) := \{\mathbf{u} \in H(\operatorname{curl}, D) : \ \mathbf{u}\times\nu = 0 \text{ on } \partial D\}$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\text{curl}} = (\mathbf{u}, \mathbf{v})_D + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v})_D$, where $(\cdot, \cdot)_D$ denotes the $(L^2(D))^3$ scalar product and the corresponding norm $\|\cdot\|_{\text{curl}}$. Next we define

$$\mathcal{U}(D) := \{\mathbf{u} \in H(\operatorname{curl}, D) : \ \operatorname{curl}\mathbf{u} \in H(\operatorname{curl}, D)\},$$
$$\mathcal{U}_0(D) := \{\mathbf{u} \in H_0(\operatorname{curl}, D) : \ \operatorname{curl}\mathbf{u} \in H_0(\operatorname{curl}, D)\}$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathcal{U}} = (\mathbf{u}, \mathbf{v})_{\text{curl}} + (\text{curl}\,\mathbf{u}, \text{curl}\,\mathbf{v})_{\text{curl}}$ and the corresponding norm $\|\cdot\|_{\mathcal{U}}$. We further require that N, N^{-1} and either $(N-I)^{-1}$ or $(I-N)^{-1}$ are bounded positive definite real matrix valued functions on D. Hence a solution of (2.33)–(2.36) is such that $\mathbf{E} \in (L^2(D))^3$, $\mathbf{E}_0 \in (L^2(D))^3$, and $\mathbf{E} - \mathbf{E}_0 \in \mathcal{U}_0(D)$. As it is shown in [9] and [14], (2.33)–(2.36) is equivalent to finding $\mathbf{u} = \mathbf{E} - \mathbf{E}_0 \in \mathcal{U}_0(D)$ such that

$$(\text{curl}\,\text{curl}\, - k^2 N)(N-I)^{-1}(\text{curl}\,\text{curl}\,\mathbf{u} - k^2\mathbf{u}) = 0, \tag{2.37}$$

which in variational form can be written as

$$\int_D (N-I)^{-1}(\text{curl}\,\text{curl}\,\mathbf{u} - k^2\mathbf{u}) \cdot (\text{curl}\,\text{curl}\,\mathbf{v} - k^2 N\mathbf{v})\, dx = 0 \qquad \text{for all } \mathbf{v} \in \mathcal{U}_0(D). \tag{2.38}$$

Problem 2: *Anisotropic scalar equation.* This problem can be stated in $\mathbb{R}^2$ as well as in $\mathbb{R}^3$. Hence the bounded region $D \subset \mathbb{R}^d$, $d = 2, 3$, satisfies the assumptions stated at the beginning of section 2. The interior transmission eigenvalue problem for anisotropic scalar equations reads:

$$\nabla \cdot A\nabla w + k^2\, w = 0 \quad \text{in} \quad D, \tag{2.39}$$

$$\Delta v + k^2\, v = 0 \quad \text{in} \quad D, \tag{2.40}$$

$$w = v \quad \text{on} \quad \partial D, \tag{2.41}$$

$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D, \tag{2.42}$$

where

$$\frac{\partial w}{\partial \nu_A}(x) := \nu(x) \cdot A(x)\nabla v(x), \qquad x \in \partial D. \tag{2.43}$$

Letting $N := A^{-1}$, in terms of new vector valued functions

$$\mathbf{w} = A\nabla w \quad \text{and} \quad \mathbf{v} = \nabla v,$$

the above problem can be written as (see [6] and [9] for details)

$$\nabla(\nabla \cdot \mathbf{w}) + k^2 N\mathbf{w} = 0 \quad \text{in} \quad D, \tag{2.44}$$

$$\nabla(\nabla \cdot \mathbf{v}) + k^2\mathbf{v} = 0 \quad \text{in} \quad D, \tag{2.45}$$

$$\nu \cdot \mathbf{w} = \nu \cdot \mathbf{v} \quad \text{on} \quad \partial D, \tag{2.46}$$

$$\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{v} \quad \text{on} \quad \partial D. \tag{2.47}$$

Here the $d \times d$, $d = 2, 3$, matrix valued function N satisfies the same assumptions as the 3×3 matrix N in Problem 1. The suitable spaces to analyze this problem are

$$H(\text{div}\,, D) := \left\{\mathbf{u} \in (L^2(D))^d : \ \nabla \cdot \mathbf{u} \in L^2(D)\right\}, \qquad d = 2, 3,$$
$$H_0(\text{div}\,, D) := \{\mathbf{u} \in H(\text{div}\,, D) : \ \nu \cdot \mathbf{u} = 0 \text{ on } \partial D\}$$

and

$$\mathcal{H}(D) := \left\{\mathbf{u} \in H(\text{div}\,, D) : \ \nabla \cdot \mathbf{u} \in H^1(D)\right\},$$
$$\mathcal{H}_0(D) := \left\{\mathbf{u} \in H_0(\text{div}\,, D) : \ \nabla \cdot \mathbf{u} \in H^1_0(D)\right\}$$

equipped with the scalar product $(\mathbf{u},\mathbf{v})_{\mathcal{H}(D)} := (\mathbf{u},\mathbf{v})_{L^2(D)} + (\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_{H^1(D)}$ and corresponding norm $\|\cdot\|_{\mathcal{H}}$. Hence a solution $\mathbf{u}$, $\mathbf{v}$ of the interior transmission eigenvalue problem (2.44)–(2.47) is such that $\mathbf{u}\in(L^2(D))^d$, $\mathbf{v}\in(L^2(D))^d$, and $\mathbf{w}-\mathbf{v}\in\mathcal{H}_0(D)$. Similar to the case of Problem 1, (2.44)–(2.47) has an equivalent formulation as a fourth order differential equation for $\mathbf{u}:=\mathbf{w}-\mathbf{v}\in\mathcal{H}_0(D)$,

$$\left(\nabla\nabla\cdot+k^2N\right)(N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}\right)=0 \qquad \text{in} \quad D, \tag{2.48}$$

which can be written in the following variational form:

$$\int_D (N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}\right)\cdot\left(\nabla\nabla\cdot\overline{\mathbf{v}}+k^2N\overline{\mathbf{v}}\right)\,dx=0 \qquad \text{for all } \mathbf{v}\in\mathcal{H}_0(D). \tag{2.49}$$

We note that (2.38) and (2.49) have the same structure, where the operators (curl curl) and $(\nabla\nabla\cdot)$ together with corresponding traces are interchanged.

Definition 2.8. Transmission eigenvalues corresponding to Problem 1 (resp., Problem 2) are the values of $k>0$ for which the homogeneous interior transmission problem (2.33)–(2.36) (resp., (2.44)–(2.47)) has nonzero solutions $\mathbf{w}\in L^2(D)$ and $\mathbf{v}\in L^2(D)$ such that $\mathbf{w}-\mathbf{v}$ is in $\mathcal{U}_0(D)$ (resp., $\mathcal{H}_0(D)$). This solution $\mathbf{u}:=\mathbf{w}-\mathbf{v}$ is the corresponding eigenfunction.

The fact that the set of transmission eigenvalues is at most discrete with $+\infty$ as the only possible accumulation point for both Problems 1 and 2 is proven in [6], [8], [14], and [16]. We now proceed to show that there exist infinitely many transmission eigenvalues. To this end, we notice that both eigenvalue problems (2.38) and (2.49) can be written as an operator equation

$$\mathbb{A}_\tau\mathbf{u}-\tau\mathbb{B}\mathbf{u}=0 \qquad \text{and} \qquad \tilde{\mathbb{A}}_\tau\mathbf{u}-\tau\mathbb{B}\mathbf{u}=0, \qquad \text{for} \quad \mathbf{u}\in\mathcal{S}, \tag{2.50}$$

where $\mathcal{S}$ stands for $\mathcal{U}_0(D)$ if Problem 1 is considered and for $\mathcal{H}_0(D)$ if Problem 2 is considered. Here the bounded linear operators $\mathbb{A}_\tau:\mathcal{S}\to\mathcal{S}$, $\tilde{\mathbb{A}}_\tau:\mathcal{S}\to\mathcal{S}$, and $\mathbb{B}:\mathcal{S}\to\mathcal{S}$ are the operators defined using the Riesz representation theorem (i.e., defined by (2.12) where $H_0^2(D)$ is replaced by $\mathcal{S}$) associated with the sesquilinear forms $\mathcal{A}_\tau$, $\tilde{\mathcal{A}}$, and $\mathcal{B}$ which in the case of Problem 1 are defined by (see [14] for more details)

$$\mathcal{A}_\tau(\mathbf{u},\mathbf{v}):=\left((N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u}-\tau\mathbf{u}),\,(\operatorname{curl}\operatorname{curl}\mathbf{v}-\tau\mathbf{v})\right)_D+\tau^2\,(\mathbf{u},\,\mathbf{v})_D\,, \tag{2.51}$$

$$\begin{aligned}\tilde{\mathcal{A}}_\tau(\mathbf{u},\mathbf{v}):=&\left(N(I-N)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u}-\tau\mathbf{u}),\,(\operatorname{curl}\operatorname{curl}\mathbf{v}-\tau\mathbf{v})\right)_D\\&+(\operatorname{curl}\operatorname{curl}\mathbf{u},\,\operatorname{curl}\operatorname{curl}\mathbf{v})_D\,,\end{aligned} \tag{2.52}$$

and

$$\mathcal{B}(\mathbf{u},\mathbf{v}):=(\operatorname{curl}\mathbf{u},\,\operatorname{curl}\mathbf{v})_D\,, \tag{2.53}$$

respectively, where $(\cdot,\,\cdot)_D$ denotes the $L^2(D)$-inner product, whereas in the case of Problem 2

$$\mathcal{A}_\tau(\mathbf{u},\mathbf{v}):=\left((N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u}+\tau\mathbf{u}\right),\left(\nabla\nabla\cdot\mathbf{v}+\tau\mathbf{v}\right)\right)_D+\tau^2\,(\mathbf{u},\mathbf{v})_D\,, \tag{2.54}$$

$$\begin{aligned}\tilde{\mathcal{A}}_\tau(\mathbf{u},\mathbf{v}):=&\left(N(I-N)^{-1}\left(\nabla\nabla\cdot\mathbf{u}+\tau\mathbf{u}\right),\left(\nabla\nabla\cdot\mathbf{v}+\tau\mathbf{v}\right)\right)_D\\&+(\nabla\nabla\cdot\mathbf{u},\nabla\nabla\cdot\mathbf{v})_D\,,\end{aligned} \tag{2.55}$$

and

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) := (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_D\,, \tag{2.56}$$

respectively.

The properties of these operators are studied in [6], [9], and [14]. Let $\sigma_*(x) > 0$ and $\sigma^*(x) > 0$ be the smallest and the largest eigenvalue, respectively, of the positive definite symmetric $d \times d$, $d = 2, 3$, matrix N. Recall that the largest eigenvalue $\sigma^*(x)$ which coincides with the Euclidean norm $\|N(x)\|_2$ is given by $\sigma^*(x) = \sup_{\|\xi\|=1}(\bar{\xi} \cdot N(x)\,\xi)$ and the smallest eigenvalue $\sigma_*(x)$ is given by $\sigma_*(x) = \inf_{\|\xi\|=1}(\bar{\xi} \cdot N(x)\,\xi)$. In the following we define $n^* := \sup_D \sigma^*(x)$ and $n_* := \inf_D \sigma_*(x)$. Let $\lambda_1(D)$ again be the Dirichlet eigenvalue for $-\Delta$ in D. The following lemma is proven in [6], [9], and [14] (see also the proof of Lemma 2.4).

LEMMA 2.9. *Let $\mathcal{S}$ stand for $\mathcal{U}_0(D)$ if Problem 1 is considered and for $\mathcal{H}_0(D)$ if Problem 2 is considered. The operators $\mathbb{A}_\tau : \mathcal{S} \to \mathcal{S}$, $\tilde{\mathbb{A}}_\tau : \mathcal{S} \to \mathcal{S}$, $\tau > 0$, and $\mathbb{B} : \mathcal{S} \to \mathcal{S}$ are self-adjoint. Furthermore, $\mathbb{B}$ is a positive compact operator. If $(N - I)^{-1}$ is a bounded positive definite matrix function on D (Definition 2.7), then $\mathbb{A}_\tau$ is a positive definite operator and*

$$(\mathbb{A}_\tau u - \tau \mathbb{B} u, u)_{\mathcal{S}} \geq \alpha \|u\|^2_{\mathcal{S}} > 0 \quad \textit{for all} \quad 0 < \tau < \frac{\lambda_1(D)}{n^*} \quad \textit{and} \quad u \in \mathcal{S}.$$

If $N(I - N)^{-1}$ is a bounded positive definite matrix function on D, then $\tilde{\mathbb{A}}_\tau$ is a positive definite operator and

$$\left(\tilde{\mathbb{A}}_\tau u - \tau \mathbb{B} u, u\right)_{\mathcal{S}} \geq \alpha \|u\|^2_{\mathcal{S}} > 0 \quad \textit{for all} \quad 0 < \tau < \lambda_1(D) \quad \textit{and} \quad u \in \mathcal{S}.$$

Note that the kernel of $\mathbb{B} : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is given by

$$\mathrm{Kernel}(\mathbb{B}) = \left\{\mathbf{u} \in \mathcal{U}_0(D) \quad \text{such that } \mathbf{u} := \nabla\varphi,\ \varphi \in H^1(D)\right\},$$

whereas the kernel of $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is given by

$$\mathrm{Kernel}(\mathbb{B}) = \{\mathbf{u} \in \mathcal{H}_0(D) \quad \text{such that } \mathbf{u} := \mathrm{curl}\,\varphi,\ \varphi \in H(\mathrm{curl}\,, D)\}\,.$$

To carry over the approach of section 2.2 to eigenvalue problems for anisotropic medium, namely, Problems 1 and 2, we also need to consider the corresponding interior transmission eigenvalue problems for a ball with a constant index of refraction. To this end, let $B_R \subset \mathbb{R}^3$ be a ball of radius R centered at the origin and $n_0 > 0$ a constant different from one. In [11] it is shown, by using separation of variables, that

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{w} - k^2 n_0 \mathbf{w} = 0 \quad \text{in} \quad B_R, \tag{2.57}$$

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{v} - k^2 \mathbf{v} = 0 \quad \text{in} \quad B_R, \tag{2.58}$$

$$\mathbf{w} \times \nu = \mathbf{v} \times \nu \quad \text{on} \quad \partial B_R, \tag{2.59}$$

$$\mathrm{curl}\,\mathbf{w} \times \nu = \mathrm{curl}\,\mathbf{v} \times \nu \quad \text{on} \quad \partial B_R \tag{2.60}$$

has a countable discrete set of eigenvalues. Let us denote by k_{R,n_0} the first transmission eigenvalue which as in (2.18)–(2.21) is zero of a determinant similar to (2.23) involving spherical Bessel functions of the variable kR and of order greater than or equal to one (see, e.g., [11], p. 263). Let $\mathbf{u}^{B_R,n_0} = \mathbf{w}^{B_R,n_0} - \mathbf{v}^{B_R,n_0}$ be the eigenfunction corresponding to k_{R,n_0}. We have that $\mathbf{u}^{B_R,n_0} \in \mathcal{U}_0(B_R)$ and

(2.61)
$$\int_{B_R} \frac{1}{n_0-1}(\operatorname{curl}\operatorname{curl}\mathbf{u}^{B_R,n_0} - k^2_{R,n_0}\mathbf{u}^{B_R,n_0}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\mathbf{u}}^{B_R,n_0} - k^2_{R,n_0} n_0 \overline{\mathbf{u}}^{B_R,n_0})\, dx = 0.$$

Similarly, for Problem 2, we denote by k_{R,n_0} the first transmission eigenvalue of

$$\Delta w + k^2 n_0 w = 0 \quad \text{in} \quad B_R, \tag{2.62}$$
$$\Delta v + k^2 v = 0 \quad \text{in} \quad B_R, \tag{2.63}$$
$$w = v \quad \text{on} \quad \partial B_R, \tag{2.64}$$
$$\frac{1}{n_0}\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial B_R, \tag{2.65}$$

where here B_R is a two- or three-dimensional ball of radius R [6] (this problem also has infinitely many eigenvalues). The corresponding eigenfunction $\mathbf{u}^{B_R,n_0} = n_0 \nabla w^{B_R,n_0} - \nabla v^{B_R,n_0}$ is in $\mathcal{H}_0(B_R)$, where w^{B_R,n_0}, v^{B_R,n_0} is a nonzero solution to (2.62)–(2.65). Furthermore, we have that

$$\int_{B_R} \frac{1}{n_0-1}(\nabla\nabla\cdot\mathbf{u}^{B_R,n_0} + k^2_{R,n_0}\mathbf{u}^{B_R,n_0}) \cdot (\nabla\nabla\cdot\overline{\mathbf{u}}^{B_R,n_0} + k^2_{R,n_0} n_0 \overline{\mathbf{u}}^{B_R,n_0})\, dx = 0. \tag{2.66}$$

By definition the eigenvectors $\mathbf{u}^{B_R,n_0}$ for (2.57)–(2.60) and (2.57)–(2.60) are not in the kernel of $\mathbb{B} : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ and $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$, respectively. Finally, if $\overline{B}_R \subset D$, then the extension by zero $\tilde{\mathbf{u}}$ of eigenvectors $\mathbf{u}^{B_R,n_0}$ for (2.57)–(2.60) and (2.57)–(2.60) to the whole D is in $\mathcal{U}_0(D)$ and $\mathcal{H}_0(D)$, respectively. Note that, as noticed in Remark 2.3, it is possible to use any transmission eigenvalues corresponding to the ball of radius R.

The above discussion provides all the necessary ingredients to apply Theorem 2.1 to (2.50) in order to prove the existence of an infinite discrete set of transmission eigenvalues for both Problems 1 and 2. The following theorem can now be proven exactly in the same way as Theorem 2.5.

THEOREM 2.10. *Assume that* $N \in L^\infty(D, \mathbb{R}^{d\times d})$, $d = 2, 3$, *satisfies either one of the following assumptions:*

1. $1 + \alpha \le n_* \le (\bar{\xi} \cdot N(x)\,\xi) \le n^* < \infty$,
2. $0 < n_* \le (\bar{\xi} \cdot N(x)\,\xi) \le n^* < 1 - \beta$

for every $\xi \in \mathbb{C}^d$ *such that* $\|\xi\| = 1$ *and some constants* $\alpha > 0$ *and* $\beta > 0$. *Then both Problems* 1 *and* 2 *have an infinite countable set of transmission eigenvalues with* $+\infty$ *as the only accumulation point.*

Theorem 2.10 and the tools developed in this section also enable us to prove similar results as in Corollary 2.6 for the transmission eigenvalues corresponding to anisotropic media.

To this end, let again B_{r_1} be the largest ball of radius r_1 such that $B_{r_1} \subset D$ and B_{r_2} be the smallest ball of radius r_2 such that $D \subset B_{r_2}$. For a given $0 < \epsilon \le r_2$ let $m(\epsilon) \in \mathbb{N}$ be the number of balls B_ϵ of radius ϵ that are contained in D. We denote by k_{1,n_*} and k_{1,n^*} the first transmission eigenvalue of either (2.57)–(2.60) or (2.62)–(2.65) for the ball B_1 of radius one with index of refraction n_* and n^*, respectively. Following exactly the lines of the proof of Corollary 2.6, we can obtain the following estimates for the first transmission eigenvalue corresponding to anisotropic media.

COROLLARY 2.11. *Assume that $N \in L^\infty(D, \mathbb{R}^{d\times d})$, $d = 2, 3$, and let $k_{1,D,N(x)}$ be the first transmission eigenvalue for either Problem* 1 (2.33)–(2.36) *or Problem* 2 (2.44)–(2.47).

1. *If $1 + \alpha \le n_* \le (\bar{\xi} \cdot N(x)\, \xi) \le n^* < \infty$ for every $\xi \in \mathbb{C}^d$ such that $\|\xi\| = 1$, and some constant $\alpha > 0$, then*

$$0 < \frac{k_{1,n^*}}{r_2} \le k_{1,D,N(x)} \le \frac{k_{1,n_*}}{r_1}. \tag{2.67}$$

   Both Problems 1 *and* 2 *have at least $m(\epsilon)$ transmission eigenvalues in the interval $[\frac{k_{1,n^*}}{r_2}, \frac{k_{1,n_*}}{\epsilon}]$.*
2. *If $0 < n_* \le (\bar{\xi} \cdot N(x)\, \xi) \le n^* < 1-\beta$ for every $\xi \in \mathbb{C}^d$ such that $\|\xi\|$, $d = 2, 3$, and some constant $\beta > 0$, then*

$$0 < \frac{k_{1,n_*}}{r_2} \le k_{1,D,N(x)} \le \frac{k_{1,n^*}}{r_1}. \tag{2.68}$$

   Both Problems 1 *and* 2 *have at least $m(\epsilon)$ transmission eigenvalues in the interval $[\frac{k_{1,n_*}}{r_2}, \frac{k_{1,n^*}}{\epsilon}]$.*

Remark 2.6. From Lemma 2.9 and the above corollary we have that

$$k_{1,D,n(x)} \ge \max\left(\frac{k_{1,n^*}}{r_2}, \sqrt{\frac{\lambda_1(D)}{n^*}}\right)$$

if $1 + \alpha \le n_* \le (\bar{\xi} \cdot N(x)\, \xi) \le n^* < \infty$ and

$$k_{1,D,n(x)} \ge \max\left(\frac{k_{1,n_*}}{r_2}, \sqrt{\lambda_1(D)}\right)$$

if $0 < n_* \le (\bar{\xi} \cdot N(x)\, \xi) \le n^* < 1 - \beta$, where $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D.

3. The case of inhomogeneous media with cavities. Motivated by a recent application of transmission eigenvalues to detect cavities inside dielectric materials [1], we now want to show that there are infinitely many transmission eigenvalues for the case of an inhomogeneous dielectric media with cavities, i.e., inhomogeneous media D with regions $D_0 \subset D$, where the index of refraction is the same as the background medium. More precisely, inside D we consider a region $D_0 \subset D$ which can possibly be multiply connected such that $\mathbb{R}^d \setminus \overline{D}_0$, $d = 2, 3$, is connected and assume that its boundary ∂D_0 is piecewise smooth. We denote by ν the unit outward normal to ∂D and ∂D_0. Now we consider the interior transmission eigenvalue problem (2.3)–(2.6) with $n \in L^\infty(D)$ a real valued function such that $n \ge c > 0$, $n = 1$ in D_0, and $|n-1| \ge c > 0$ almost everywhere in $D\setminus\overline{D}_0$. In particular, $1/|n-1| \in L^\infty(D\setminus\overline{D}_0)$. The interior transmission problem for inhomogeneous medium with cavities is investigated in [5]. To prove our result, we need to recall the analytic framework developed in [5]. To this end we introduce the Hilbert space

$$V_0(D, D_0, k) := \{u \in H_0^2(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0\}$$

equipped with the $H^2(D)$ scalar product and look for the solutions v and w both in $L^2(D)$ such that $u = w - v$ in $V_0(D, D_0, k)$. It is shown in [5] that (2.3)–(2.6), with n satisfying the above assumptions, can be written in variational form as

$$\int_{D\setminus\overline{D}_0} \frac{1}{n-1} \left(\Delta + k^2\right) u \,\left(\Delta + k^2\right) \bar{\psi}\, dx + k^2 \int_{D\setminus\overline{D}_0} (\Delta u + k^2 u)\, \bar{\psi}\, dx = 0, \tag{3.1}$$

which is required to be valid for all $\psi \in V_0(D, D_0, k)$.

DEFINITION 3.1. *Values of $k > 0$ for which* (3.1) *has nontrivial solutions $u \in V_0(D, D_0, k)$ are called transmission eigenvalues. These nontrivial solutions are called corresponding eigenfunctions.*

It is important to note that, as is discussed in [5], if k^2 is not both a Dirichlet and a Neumann eigenvalue for $-\Delta$ in D_0 then Definition 3.1 of transmission eigenvalues is equivalent to the existence of a nontrivial weak solution to (2.3)–(2.6). However, if for some k there exists a nontrivial solution to (2.3)–(2.6), then k is a transmission eigenvalue according to Definition 3.1.

Next let us define the following bounded sesquilinear forms on $V_0(D, D_0, k) \times V_0(D, D_0, k)$:

$$\mathcal{A}(u, \psi) = \pm \int_{D \setminus \overline{D}_0} \frac{1}{n-1} \left(\Delta u \, \Delta \bar{\psi} + \nabla u \cdot \nabla \bar{\psi} + u \, \bar{\psi}\right) dx + \int_{D_0} \left(\nabla u \cdot \nabla \bar{\psi} + u \, \bar{\psi}\right) dx \tag{3.2}$$

and

$$\mathcal{B}_k(u, \psi) = \pm k^2 \int_{D \setminus \overline{D}_0} \frac{1}{n-1} \left(u(\Delta \bar{\psi} + k^2 \bar{\psi}) + (\Delta u + k^2 n u)\bar{\psi}\right) dx \mp \int_{D \setminus \overline{D}_0} \frac{1}{n-1} \left(\nabla u \cdot \nabla \bar{\psi} + u \, \bar{\psi}\right) dx - \int_{D_0} \left(\nabla u \cdot \nabla \bar{\psi} + u \, \bar{\psi}\right) dx, \tag{3.3}$$

where the upper sign corresponds to the case when $n - 1 \geq c > 0$ almost everywhere in $D \setminus \overline{D}_0$ and the lower sign corresponds to the case when $1 - n \geq c > 0$ almost everywhere in $D \setminus \overline{D}_0$. Hence k is a transmission eigenvalue if and only if the homogeneous problem

$$\mathcal{A}(u_0, \psi) + \mathcal{B}_k(u_0, \psi) = 0 \text{ for all } \psi \in V_0(D, D_0, k) \tag{3.4}$$

has a nonzero solution. Let $A_k : V_0(D, D_0, k) \to V_0(D, D_0, k)$ and B_k be the self-adjoint operators associated with $\mathcal{A}$ and $\mathcal{B}_k$, respectively, by using the Riesz representation theorem. In [5] it is shown that the operator $A_k : V_0(D, D_0, k) \to V_0(D, D_0, k)$ is positive definite, i.e., $A_k^{-1} : V_0(D, D_0, k) \to V_0(D, D_0, k)$ exists, and the operator $B_k : V_0(D, D_0, k) \to V_0(D, D_0, k)$ is compact. Hence we can define the operator $A_k^{-1/2}$ by $A_k^{-1/2} = \int_0^\infty \lambda^{-1/2} dE_\lambda$, where dE_λ is the spectral measure associated with the positive operator A_k. In particular, $A_k^{-1/2}$ is also bounded, positive definite, and self-adjoint. Thus we have that (2.48) is equivalent to finding $u \in V_0(D, D_0, k)$ such that

$$u + A_k^{-1/2} B_k A_k^{-1/2} u = 0. \tag{3.5}$$

In particular, it is obvious that k is a transmission eigenvalue if and only if the operator

$$I_k + A_k^{-1/2} B_k A_k^{-1/2} : V_0(D, D_0, k) \to V_0(D, D_0, k) \tag{3.6}$$

has a nontrivial kernel where I_k is the identity operator on $V_0(D, D_0, k)$. To avoid dealing with function spaces depending on k, we introduce the orthogonal projection operator P_k from $H_0^2(D)$ onto $V_0(D, D_0, k)$ and the corresponding injection

$R_k : V_0(D, D_0, k) \to H_0^2(D)$. Then one easily sees that $A_k^{-1/2} B_k A_k^{-1/2}$ is injective on $V_0(D, D_0, k)$ if and only if

$$I + R_k A_k^{-1/2} B_k A_k^{-1/2} P_k : H_0^2(D) \to H_0^2(D) \tag{3.7}$$

is injective. Indeed, if $u + R_k A_k^{-1/2} B_k A_k^{-1/2} P_k u = 0$, then by taking the inner product of the latter with the component $w = u - P_k u$ which is orthogonal to $P_k u$, we have that

$$\begin{aligned} 0 &= (u, w)_{H^2} + \left(R_k A_k^{-1/2} B_k A_k^{-1/2} P_k u, w \right)_{H^2} \\ &= (w, w)_{H^2} + \left(A_k^{-1/2} B_k A_k^{-1/2} P_k u, P_k w \right)_{H^2} = \|w\|_{H^2}^2, \end{aligned} \tag{3.8}$$

when $w = 0$. The injectivity of $A_k^{-1/2} B_k A_k^{-1/2}$ now implies the injectivity of (3.7) since the component $P_k u$ is in $V_0(D, D_0, k)$. The converse is obvious. Furthermore, as is discussed in [5], $T_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k : H_0^2(D) \to H_0^2(D)$ is a compact operator, and the mapping $k \to R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ is continuous. Therefore, from the max-min principle for the eigenvalues $\lambda(k)$ of the compact and self-adjoint operator $R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ we can conclude that $\lambda(k)$ is a continuous function of k. Finally, it is clear that the multiplicity of a transmission eigenvalue is finite since it corresponds to the multiplicity of the eigenvalue $\lambda(k) = -1$ [19]. We note that using the analytic Fredholm theory in [5] it is shown that the set of transmission eigenvalues is discrete with $+\infty$ the only accumulation point. The proof of the existence of an infinite set of transmission eigenvalues is based on the following theorem which is proven in [5] (see also [18]). This theorem is a modified version of Theorem 2.1, and the proof is based on the max-min principle for $\lambda(k)$ and the continuity of $\lambda(k)$ on k.

THEOREM 3.2. *Let $T_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$. Assume that*

1. *There is a κ_0 such that $I + T_{\kappa_0}$ is positive on $H_0^2(D)$.*
2. *There is a $\kappa_1 > \kappa_0$ such that $I + T_{\kappa_1}$ is nonpositive on a p-dimensional subspace W_k of $H_0^2(D)$.*

Then there are p transmission eigenvalues in $[\kappa_0, \kappa_1]$ counting their multiplicity.

In the following we set $n_* := \inf_{D \setminus \overline{D}_0}(n)$ and $n^* := \sup_{D \setminus \overline{D}_0}(n)$. Recall that we denote by $\lambda_1(D)$ the first Dirichlet eigenvalue for $-\Delta$ on D.

THEOREM 3.3. *Let $n \in L^\infty(D)$, $n = 1$ in D_0, satisfy either one of the following assumptions:*

1. $1 + \alpha \le n_* \le n(x) \le n^* < \infty$,
2. $0 < n_* \le n(x) \le n^* < 1 - \beta$,

on $D \setminus \overline{D}_0$ for some constants $\alpha > 0$ and $\beta > 0$. Then there exists an infinite set of transmission eigenvalues with $+\infty$ as the only accumulation point.

Proof. First we assume that the assumption 1 holds in which case we have

$$0 < \frac{1}{n^* - 1} \le \frac{1}{n(x) - 1} \le \frac{1}{n_* - 1} < \infty \qquad \text{in} \qquad D \setminus \overline{D}_0.$$

We note that $T_k := I + R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ is positive on $H_0^2(D)$ if and only if $A_k + B_k$ is positive on $V_0(D, D_0, k)$. Hence, combining the terms in (3.1) in a different way, we have that for $u \in V_0(D, D_0, k)$

$$(A_k u + B_k u, u)_{H_0^2(D)} = \int_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta u + k^2 n u|^2\,dx - k^4 \int_{D\setminus\overline{D}_0} n|u|^2\,dx + k^2 \int_{D\setminus\overline{D}_0} |\nabla u|^2\,dx - k^4 \int_{D_0} |u|^2\,dx + k^2 \int_{D_0} |\nabla u|^2\,dx. \tag{3.9}$$

For $n^* = \sup_{D\setminus\overline{D}_0} n > 1$, if the sum of the last four terms in (3.9) is nonnegative, then we have $A_k + B_k$ is positive. Hence we have

$$\begin{aligned} -k^2 \int_{D\setminus\overline{D}_0} n|u_0|^2\,dx + \int_{D\setminus\overline{D}_0} |\nabla u_0|^2\,dx - k^2 \int_{D_0} |u_0|^2\,dx + \int_{D_0} |\nabla u_0|^2\,dx \\ \geq \int_D |\nabla u_0|^2\,dx - k^2 n^* \int_D |u_0|^2\,dx \geq (\lambda_1(D) - k^2 n^*)\|u_0\|^2_{L^2(D)}. \end{aligned} \tag{3.10}$$

Therefore all $\kappa_0 > 0$ such that $\kappa_0^2 \leq \frac{\lambda_1(D)}{n^*}$ satisfy assumption 1 of Theorem 3.2. Next we proceed in the same way as in Theorem 2.5. To this end, take $\epsilon > 0$ small enough such that $D \setminus \overline{D}_0$ contains $m := m(\epsilon) \geq 1$ disjoint balls $B_\epsilon^1, B_\epsilon^2, \ldots, B_\epsilon^m$ of radius ϵ; i.e., $\overline{B_\epsilon^j} \subset D \setminus \overline{D}_0$, $j = 1, \ldots, m$, and $\overline{B_\epsilon^j} \cap \overline{B_\epsilon^i} = \emptyset$ for $j \neq i$. Let $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ be the first transmission eigenvalue for each of these balls with index of refraction n^*, where k_{1,n_*} is the first transmission eigenvalue for the ball B_1 of radius one with index of refraction n_* (i.e., the first eigenvalue corresponding to (2.18)–(2.21) with $R := 1$ and $n_0 := n_*$). Denote by $u^{B_\epsilon^j,n_*} \in H_0^2(B_\epsilon^j)$, $j = 1, \ldots, m$, the eigenfunction corresponding to k_{ϵ,n_*}. The extension by zero $\tilde{u}^j$ of $u^{B_\epsilon^j,n_*}$ to the whole D is obviously in $V_0(D, D_0, k)$ and the vectors $\{\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^m\}$ are linearly independent and orthogonal since they have disjoint supports in $D \setminus \overline{D}_0$. Let us denote by $\mathcal{U}$ the m-dimensional subspace of $V_0(D, D_0, k)$ spanned by $\{\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^m\}$. Since each $\tilde{u}^j$, $j = 1, \ldots, m$, satisfies (2.25) and they have disjoint supports, we have that for $\kappa_1 := k_{\epsilon,n_*}$ and for every $\tilde{u}^j \in \mathcal{U}$ (note that $\tilde{u}^j = 0$ in a neighborhood of D_0)

$$\begin{aligned} &(A_{\kappa_1}\tilde{u} + B_{\kappa_1}\tilde{u}, \tilde{u})_{H_0^2(D)} \\ &= \int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta\tilde{u} + \kappa_1\tilde{u}|^2\,dx + \kappa_1^4 \int\limits_{D\setminus\overline{D}_0} |\tilde{u}|^2\,dx - \kappa_1^2 \int\limits_{D\setminus\overline{D}_0} |\nabla\tilde{u}|^2\,dx \\ &\leq \int\limits_{D\setminus\overline{D}_0} \frac{1}{n_*-1}|\Delta\tilde{u} + \kappa_1^2\tilde{u}|^2\,dx + \kappa_1^4 \int\limits_{D\setminus\overline{D}_0} |\tilde{u}|^2\,dx - \kappa_1^2 \int\limits_{D\setminus\overline{D}_0} |\nabla\tilde{u}|^2\,dx = 0. \end{aligned} \tag{3.11}$$

This means that assumption 2 of Theorem 3.2 is also satisfied, and therefore there are $m(\epsilon)$ transmission eigenvalues (counting multiplicity) inside $[\kappa_0, k_{\epsilon,n_*}]$. Note that $m(\epsilon)$ and k_{ϵ,n_*} both go to $+\infty$ as $\epsilon \to 0$. Since the multiplicity of each eigenvalue is finite we have shown that there exists an infinite countable set of transmission eigenvalues that accumulate at $+\infty$.

Now assume that assumption 2 holds. Similar to the previous case, from the definitions (3.2) and (3.3) of A_k and B_k, we have that

$$\begin{aligned} (A_k u + B_k u, u)_{H_0^2(D)} &= \int_{D\setminus\overline{D}_0} \frac{1}{1-n}|\Delta u + k^2 u|^2\,dx - k^4 \int_{D\setminus\overline{D}_0} |u|^2\,dx \\ &+ k^2 \int_{D\setminus\overline{D}_0} |\nabla u|^2\,dx - k^4 \int_{D_0} |u|^2\,dx + k^2 \int_{D_0} |\nabla u|^2\,dx. \end{aligned} \tag{3.12}$$

Hence we have that $A_k + B_k$ is positive as long as

$$-k^2 \int_{D\setminus\overline{D}_0} n|u|^2\,dx + \int_{D\setminus\overline{D}_0} |\nabla u|^2\,dx - k^2 \int_{D_0} |u|^2\,dx + \int_{D_0} |\nabla u|^2\,dx \\ \geq \int_D |\nabla u|^2\,dx - k^2 \int_D |u|^2\,dx \geq (\lambda_1(D) - k^2)\|u_0\|^2_{L^2(D)} \geq 0. \tag{3.13}$$

Therefore, $\kappa_0 > 0$ such that $\kappa_0^2 \leq \lambda_1(D)$ satisfies assumption 1 of Theorem 3.2. The rest of the proof can be done exactly in the same way as for the first part, where n_* is replaced by n^*. ∎

Finally, we rewrite the main result of Theorem 3.3 in the following corollary. To this end, we call B_{r_1} the largest ball of radius r_1 such that $B_{r_1} \subset D \setminus \overline{D}_0$. For a given $0 < \epsilon \leq r_2$, let $m(\epsilon) \in \mathbb{N}$ be the number of disjoint balls B_ϵ of radius ϵ that are contained in $D \setminus \overline{D}_0$. We denote by k_{1,n_*} and k_{1,n^*} the first transmission eigenvalue for the ball B_1 of radius one with index of refraction n_* and n^*, respectively (i.e., the first eigenvalue corresponding to (2.18)–(2.21) with $R := 1$ and $n_0 := n_*$ and $n_0 := n^*$, respectively). Finally let $\lambda_1(D)$ be the first Dirichlet eigenvalue for $-\Delta$ in D.

Corollary 3.4. *Assume that* $n \in L^\infty(D \setminus \overline{D}_0)$, $n = 1$ *in* D_0, *and let* $k_{1,D,D_0,n(x)}$ *be the first transmission eigenvalue corresponding to* (2.3)–(2.6) *for* D, D_0, *and* $n(x)$.

1. *If* $1 + \alpha \leq n_* \leq n(x) \leq n^* < \infty$ *on* $D \setminus \overline{D}_0$, *then*

$$0 < \sqrt{\frac{\lambda_1(D)}{n^*}} \leq k_{1,D,D_0,n(x)} \leq \frac{k_{1,n_*}}{r_1}. \tag{3.14}$$

There are at least $m(\epsilon)$ *transmission eigenvalues in the interval* $[\sqrt{\frac{\lambda_1(D)}{n^*}}, \frac{k_{1,n_*}}{\epsilon}]$.

2. *If* $0 < n_* \leq n(x) \leq n^* < 1 - \beta$ *on* $D \setminus \overline{D}_0$, *then*

$$0 < \sqrt{\lambda_1(D)} \leq k_{1,D,D_0,n(x)} \leq \frac{k_{1,n^*}}{r_1}. \tag{3.15}$$

There are at least $m(\epsilon)$ *transmission eigenvalues in the interval* $[\sqrt{\lambda_1(D)}, \frac{k_{1,n^*}}{\epsilon}]$.

Alternative lower bounds for the first transmission eigenvalue that involve the geometry of D_0 can be found in [5].

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REFERENCES

[1] F. Cakoni, M. Cayoren, and D. Colton, *Transmission eigenvalues and the nondestructive testing of dielectrics*, Inverse Problems, 26 (2008), pp. 065016.

[2] F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer, Berlin, 2006.

[3] F. Cakoni, D. Colton, and H. Haddar, *On the determination of Dirichlet and transmission eigenvalues from far field data*, C. R. Math., to appear.

[4] F. Cakoni, D. Colton, and P. Monk, *On the use of transmission eigenvalues to estimate the index of refraction from far field data*, Inverse Problems, 23 (2007), pp. 507–522.

[5] F. Cakoni, D. Colton, and H. Haddar, *The interior transmission problem for regions with cavities*, SIAM J. Math Anal., 42 (2010), pp. 145–162.

[6] F. Cakoni, D. Colton, and H. Haddar, *The computation of lower bounds for the norm of the index of refraction in an anisotropic media*, J. Integral Equations Appl., 21 (2009), pp. 203–227.

[7] F. Cakoni and D. Gintides, *New results on transmission eigenvalues*, Inverse Problems Imaging, 4 (2010), pp. 39–48.

[8] F. Cakoni and H. Haddar, *A variational approach for the solution of electromagnetic interior transmission problem for anisotropic media*, Inverse Problems Imaging, 1 (2007), pp. 443–456.

[9] F. Cakoni and H. Haddar, *On the existence of transmission eigenvalues in an inhomogeneous medium*, Appl. Anal., 88 (2009), pp. 475–493.

[10] D. Colton, A. Kirsch, and L. Päivärinta, *Far field patterns for acoustic waves in an inhomogeneous medium*, SIAM J. Math. Anal., 20 (1989), pp. 1472–1483.

[11] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd ed., Applied Mathematical Sciences, Vol. 93, Springer, New York, 1998.

[12] D. Colton and P. Monk, *The inverse scattering problem for acoustic waves in an inhomogeneous medium*, Quart. J. Mech. Appl. Math., 41 (1988), pp. 97–125.

[13] D. Colton, L. Päivärinta, and J. Sylvester, *The interior transmission problem*, Inverse Problems Imaging, 1 (2007), pp. 13–28.

[14] H. Haddar, *The interior transmission problem for anisotropic Maxwell's equations and its applications to the inverse problem*, Math. Methods Appl. Sci., 27, (2004), pp. 2111–2129.

[15] A. Kirsch, *An integral equation approach and the interior transmission problem for Maxwell's equations*, Inverse Problems Imaging, 1 (2007), pp. 159–179.

[16] A. Kirsch, *On the existence of transmission eigenvalues*, Inverse Problems Imaging, 3 (2009), pp. 155–172.

[17] A. Kirsch and N. Grinberg, *The Factorization Method for Inverse Problems*, Oxford University Press, Oxford, 2008.

[18] L. Päivärinta and J. Sylvester, *Transmission eigenvalues*, SIAM J. Math. Anal., 40 (2008), pp. 738–753.

[19] M. Reed and B Simon, *Functional Analysis*, Academic Press, New York, 1980.

[20] B. P. Rynne and B. D. Sleeman, *The interior transmission problem and inverse scattering from inhomogeneous media*, SIAM J. Math. Anal., 22 (1991), pp. 1755–1762.