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Inequalities in inverse scattering theory

F. Cakoni and D. Colton

Abstract. We consider the scattering of time harmonic electromagnetic plane waves by a bounded, inhomogeneous dielectric medium that is partially coated by a thin metallic layer in $\mathbb{R}^2$. We use far field pattern of the scattered waves at a fixed frequency as data to determine the support D of the inhomogeneous obstacle, the surface conductivity that characterizes the coating and the relative permittivity. No a prior information on the material properties of the scatterer is needed. The support D is determined by the linear sampling method which is based on the approximate solution of the far field equation. This solution is also used to obtain lower bounds for the surface conductivity and relative permittivity. The techniques for solving this inverse scattering problem rely on the analysis of a non standard boundary value problem known as the interior transmission problem.

Key words. Inverse scattering problem, inhomogeneous medium, interior transmission problem, electromagnetic waves, mixed boundary value problems, qualitative approaches in inverse scattering.

AMS classification. 35P25, 35R30, 78A45.

1. Introduction

Until recently, reconstruction algorithms for solving inverse electromagnetic scattering problems have been based on either nonlinear optimization techniques or on linearized models based on weak scattering approximations [6]. In the past ten years a third approach to reconstruction has been developed which comes under the rubric of qualitative methods in inverse scattering theory, the most popular of which is the linear sampling method [1]. In particular, qualitative methods determine the shape of the scattering obstacle without needing any a priori information on the material properties of the scatterer but provide little or no information on the physical properties of the scatterer. However, in the past few years it has been noted that qualitative methods in inverse scattering theory can in certain circumstances provide lower bounds on relevant physical properties of the scatterer [2, 3] and it is to this theme that this paper is directed.

We will illustrate our ideas by considering a simple scattering problem for an infinite dielectric cylinder that is partially coated by a thin metallic coating. For Maxwell's equations in R^3, scattering problems for such coated objects arise when an effort is made to make benign dielectric objects look hostile, e.g. coating wooden decoys to make them appear as tanks to radar interrogation. In general the scatterer is only partially coated and the extent and composition of the coating is unknown. Such situations lead to mixed boundary value problems in scattering theory and particular difficulties

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arise in trying to solve the inverse problem since the boundary conditions on the scattering object are unknown.

The plan of our paper is as follows. In the next section we will formulate the direct and inverse scattering problems for an infinite dielectric cylinder that is partially coated by a highly conductive layer in the case of TM-polarized incident plane waves. We then consider the inverse scattering problem of determining the support D of the cross section of the infinite cylinder, the surface conductivity $\eta(x)$, $x \in \partial D$, that characterizes the coating and the relative permittivity $n(x)$, $x \in D$, of the dielectric cylinder using the far field pattern of the scattered wave as data. Using the linear sampling method, it was shown in [4] that D can be determined in a constructive manner. However, the techniques used in [4] were unable to determine any information on $\eta(x)$ or $n(x)$ due to the fact that the well posedness of the interior transmission problem for coated objects and TM-polarized fields was not established. Hence in Section 3 we will prove the existence of a unique solution to the interior transmission problem for coated objects and TM-polarized fields provided the wave number is not a transmission eigenvalue (For the role played by the interior transmission problem and transmission eigenvalues in inverse scattering theory we refer the reader to [1, 6]). We then use this result to provide lower bounds for the surface conductivity and relative permittivity in terms of data obtained from the far field pattern of the scattered wave.

2. Scattering by a partially coated cylinder

We consider the scattering of a time-harmonic plane wave by a partially coated infinitely long cylinder with axis in the z-direction and assume that the incident field is propagating in a direction perpendicular to the cylinder such that the electric field is parallel to the z-axis. Let the bounded domain $D \subset \mathbb{R}^2$ with Lipschitz boundary Γ be the cross section of the cylinder and assume that the exterior domain $D_e := \mathbb{R}^2 \setminus \bar{D}$ is connected. We denote by ν the unit outward normal to Γ defined almost everywhere on Γ. The boundary Γ has a Lipschitz dissection [8] $\Gamma = \Gamma_1 \cup \Pi \cup \Gamma_2$. Here Γ_1 corresponds to the uncoated part and Γ_2 corresponds to the coated part. Under the above assumptions the electric fields only have a component in the z direction, i.e. the incident field E^i, internal field E^{int} and scattered field E^s are given by $E^i = (0, 0, u^i)$, $E^{int} = (0, 0, v)$ and $E^s = (0, 0, u^s)$. Then the direct scattering problem for the electric field reads: Given $f \in H^{1/2}(\Gamma)$, $h \in H^{-1/2}(\Gamma_1)$ and $h_2 \in H^{-1/2}(\Gamma_2)$ find $v \in H^1(D)$ and $u^s \in H^1_{\text{loc}}(D_e)$ such that

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in } D_e, \tag{2.1}$$

$$\Delta v + k^2 n(x) v = 0 \qquad \text{in } D, \tag{2.2}$$

$$v - u^s = f \qquad \text{on } \Gamma, \tag{2.3}$$

$$\frac{\partial v}{\partial \nu} - \frac{\partial u^s}{\partial \nu} = h_1 \qquad \text{on } \Gamma_1, \tag{2.4}$$

$$\frac{\partial v}{\partial \nu} - \frac{\partial u^s}{\partial \nu} = ik\eta(x) u^s + h_2 \qquad \text{on } \Gamma_2, \tag{2.5}$$

$$\lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial u^s}{\partial v} - iku^s\Big) = 0, \tag{2.6}$$

where $k > 0$ is the wave number, $r = |x|$, $n \in C^1(\bar{D})$, $n > 0$, $\eta \in C(\bar{\Gamma}_2)$, $\eta \geq 0$ and $f := e^{idx\cdot d}|_\Gamma$, $h_1 := (\partial e^{ikx\cdot d}/\partial\nu)|_{\Gamma_1}$, $h_2 := (\partial e^{ikx\cdot d}/\partial v + ik\eta(x)e^{idx\cdot d})|_{\Gamma_2}$ where $d \in \Omega := \{x \in \mathbb{R}^2 : |x| = 1\}$ denotes the direction of the incident plane wave. Here $H^1(D)$ and $H^1_{\text{loc}}(D_e)$ are the usual Sobolev spaces, $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ are the corresponding trace space and its dual respectively and $H^{1/2}(\Gamma_0) := \{u|_{\Gamma_0} : u \in H^{1/2}(\Gamma)\}$ for $\Gamma_0 \subset \Gamma$. The existence of a unique solution to (2.1)–(2.6) was established in [4].

In (2.1)–(2.6), u^s denotes the scattered field and has the asymptotic behavior [1, 6]

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}}\, u_\infty(\hat{x}, d) + O(r^{-3/2})$$

as $r \to \infty$ where $u_\infty(\hat{x}, d)$ is defined for $\hat{x}, d \in \Omega$ and is called the *far field pattern* of u^s. The inverse problem of determining the shape of the scattering object D from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ was considered in [4]. Hence we can assume D is known and address our attention to the problem of determining information on η and n from a knowledge of u_∞. To this end we need to consider the *interior transmission problem* corresponding to (2.1)–(2.6).

3. The interior transmission problem

In this section we will study the *interior transmission problem*

$$\Delta w + k^2 w = 0 \qquad \text{in } D, \tag{3.1}$$

$$\Delta v + k^2 n(x) v = 0 \qquad \text{in } D, \tag{3.2}$$

$$v - w = \varphi \qquad \text{on } \Gamma, \tag{3.3}$$

$$\frac{\partial v}{\partial \nu} - \frac{\partial w}{\partial \nu} = \psi \qquad \text{on } \Gamma_1, \tag{3.4}$$

$$\frac{\partial v}{\partial \nu} - \frac{\partial w}{\partial \nu} = ik\eta(x)w + \psi + \tau \qquad \text{on } \Gamma_2 \tag{3.5}$$

and begin by defining what we mean by a solution for given data φ, ψ and τ. To this end, we first notice that if $\varphi \in H^{3/2}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$ then there exists a lifting function $\Theta \in H^2(D)$ such that $\Theta = \varphi$ and $\partial\Theta/\partial\nu = \psi$ on Γ and there exists a positive constant c such that

$$\|\Theta\|_{H^2(D)} \leq c\big(\|\varphi\|_{H^{3/2}(\Gamma)} + \|\psi\|_{H^{1/2}(\Gamma)}\big).$$

If we define $H_0(D, \Gamma_2)$ by

$$H_0(D, \Gamma_2) := \Big\{u \in H^2(D) : u\big|_{\Gamma_2} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\Gamma_2} = 0\Big\}$$

then the interior transmission problem can be formulated as follows: Given $\varphi \in H^{3/2}(\Gamma)$, $\psi \in H^{1/2}(\Gamma)$ and $\tau \in H^{-1/2}(\Gamma)_2)$, find $w \in L^2(D)$, $v \in L^2(D)$ such that $v - w - \Theta \in H_0(D, \Gamma_2)$ satisfies (3.1) and (3.2) in a distributional sense. If there exists a nontrivial solution of the homogeneous problem associated with (3.1)-(3.5) (i.e. $\varphi = \psi = \tau = 0$) then k is called a *transmission eigenvalue*. It can be shown that the transmission eigenvalues form a discrete set (cf. Corollary 8.21 of [1] for the case of TE-polarized waves).

We first need to show that $w|_{\Gamma_2}$ is well defined. To this end we notice that $w \in L^2(D, \Delta)$ where

$$L^2(D, \Delta) := \{u \in L^2(D) : \Delta u \in L^2(D)\}.$$

Using Green's theorem we have that

$$\int_\Gamma \Big(\frac{\partial w}{\partial \nu}\,\Theta - w\,\frac{\partial \Theta}{\partial \nu}\Big)\,ds = \int_D (\Theta \Delta w - w \Delta \Theta)\,dx \tag{3.6}$$

for $w \in C^\infty(\overline{D})$ and $\Theta \in H^2(D)$. Since $C^\infty(\overline{D})$ is dense in $L^2(D, \Delta)$ and the mapping $\Theta \to (\Theta|_\Gamma \times \partial\Theta/\partial\nu|_\Gamma)$ is injective from $H^2(D)$ into $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$, we can conclude that (3.6) extends the mapping $w \to (w|_\Gamma, \partial w/\partial\nu|_\Gamma)$ to a continuous mapping from $L^2(D, \Delta)$ into $H^{-1/2}(\Gamma) \times H^{-3/2}(\Gamma)$. Hence $w|_{\Gamma_2} \in H^{-1/2}(\Gamma_2)$ is well defined.

We note that, combining (3.3) and (3.5), the transmission boundary condition (3.5) can be replaced by

$$\frac{\partial v}{\partial \nu} - \frac{\partial w}{\partial \nu} = ik\eta(x)v + ik\eta(x)\varphi + \psi + \tau. \tag{3.7}$$

Now let $u = v - w$ and $u_0 = v - w - \Theta$ where Θ is the lifting of φ and ψ. Then

$$\int_D v(\Delta \overline{u'} + k^2 n \overline{u'})\,dx - \int_{\Gamma_2} v\,\frac{\partial \overline{u'}}{\partial \nu}\,ds = 0 \tag{3.8}$$

for any $u' \in H_0(D, \Gamma_2)$ and using the transmission boundary condition (3.7) we have that (assuming $\eta > 0$ for $x \in \overline{\Gamma}_2$)

$$\int_{\Gamma_2} v\,\frac{\partial \overline{u'}}{\partial \nu}\,ds = -\frac{i}{k}\int_{\Gamma_2} \frac{1}{\eta}\,\frac{\partial u}{\partial \nu}\,\frac{\partial \overline{u'}}{\partial \nu}\,ds + \frac{i}{k}\int_{\Gamma_2} \frac{1}{\eta}\,(\psi + \tau)\,\frac{\partial \overline{u'}}{\partial \nu}\,ds + \int_{\Gamma_2} \varphi\,\frac{\partial \overline{u'}}{\partial \nu}\,ds. \tag{3.9}$$

From the fact that

$$v = \frac{1}{k^2(1-n)}\,(\Delta u + k^2 u)$$

and combining (3.9) and (3.8) we have that $u_0 \in H_0(D, \Gamma_2)$ satisfies (assuming $n \neq 1$ for $x \in \overline{D}$)

$$\begin{aligned}
&\int_D \frac{1}{1-n}(\Delta u_0 + k^2 u_0)(\Delta \overline{u'} + k^2 n \overline{u'})\,dx + ik\int_{\Gamma_2} \frac{1}{\eta}\,\frac{\partial u_0}{\partial \nu}\,\frac{\partial \overline{u'}}{\partial \nu}\,ds \\
&= \int_D \frac{1}{1-n}\,(\Delta \Theta + k^2 \Theta)(\Delta \overline{u'} + k^2 n \overline{u'})\,dx + ik\int_{\Gamma_2} \frac{\tau}{\eta}\,\frac{\partial \overline{u'}}{\partial \nu}\,ds + \int_{\Gamma_2} \Theta\,\frac{\partial \overline{u'}}{\partial \nu}\,ds
\end{aligned} \tag{3.10}$$

for every $u' \in H_0(D,\Gamma_2)$. Using the denseness of $C^\infty(\bar{D}$ in $H_0(D,\Gamma_2)$ it is straightforward to show that for $\eta > 0$ and $n \neq 1$ the interior transmission problem is equivalent to (3.10).

We now want to study the solvability of (3.10). To this end we denote the negative of the right hand side of (3.10) by ℓ, which is a bounded conjugate linear functional in $H_0(D,\Gamma_2)$, and define

$$\mathcal{A}_k(u_0,u) := \int_D \Big(\frac{1}{n-1}\,(\Delta u_0 + k^2 u_0)(\Delta \overline{u'} + k^2 \overline{u'}) + k^4 u_0 \overline{u'}\Big)\,dx,$$

$$\tilde{\mathcal{A}}_k(u_0,u) := \int_D \Big(\frac{n}{n-1}\,(\Delta u_0 + k^2 u_0)(\Delta \overline{u'} + k^2 \overline{u'}) + \Delta u \Delta \overline{u'}\Big)\,dx,$$

$$\mathcal{B}(u_0,u) := \int_{\Gamma_2} \frac{1}{\eta}\,\frac{\partial u_0}{\partial \nu}\,\frac{\partial \overline{u'}}{\partial \nu}\,ds, \qquad \mathcal{C}(u_0,u) := \int_D \nabla u_0 \cdot \nabla \overline{u'}\,dx.$$

Note that $\mathcal{A}_k(\,\cdot\,,\,\cdot\,)$, $\tilde{\mathcal{A}}_k(\,\cdot\,,\,\cdot\,)$, $\mathcal{B}(\,\cdot\,,\,\cdot\,)$ and $\mathcal{C}(\,\cdot\,,\,\cdot\,)$ are continuous sesquilinear forms on $H_0(D,\Gamma_2) \times H_0(D,\Gamma_2)$. Let $A_k, \tilde{A}_k, B$ and $C : H_0(D,\Gamma_2) \to H_0(D,\Gamma_2)$ be the bounded linear operators defined by the above sesquilinear forms using the Riesz representation theorem. Due to the compact imbedding of $L^2(D)$ into $H'(D)$, it is easy to see that $C : H_0(D,\Gamma_2) \to H_0(D,\Gamma_2)$ is a compact operator. With the above notation (3.10) can be written as

$$\mathcal{A}_k(u_0,u') - ik\mathcal{B}(u_0,u') - k^2\mathcal{C}(u_0,u') = \ell(u') \tag{3.11}$$

or

$$\tilde{\mathcal{A}}_k(u_0,u') + ik\mathcal{B}(u_0,u') - k^2\mathcal{C}(u_0,u') = -\ell(u'). \tag{3.12}$$

Lemma 3.1. *Let γ be a positive constant and assume that $1/(n-1) \geq \gamma > 0$ (respectively $n/(1-n) \geq \gamma > 0$). Then $\mathcal{A}_k - ik\mathcal{B}$ (respectively $\tilde{\mathcal{A}}_k + ik\mathcal{B}$) is a coercive sesquilinear form on $H_0(D,\Gamma_2) \times H_0(D,\Gamma_2)$.*

Proof. Let $u_0 \in H_0(D,\Gamma_2)$ and assume that $1/(n-1) \geq \gamma > 0$. Then since η is real valued we have that

$$\mathrm{Re}\,(\mathcal{A}_k(u_0,u_0) - ik\mathcal{B}(u_0,u_0)) = \mathcal{A}_k(u_0,u_0).$$

But $\mathcal{A}_k(u_0,u_0) \geq \gamma\|\Delta u_0 + k^2 u_0\|^2_{L^2(D)} + k^4\|u_0\|^2_{L^2(D)}$. Setting $X = \|\Delta u_0\|_{L^2(D)}$, $Y = k^2\|u_0\|_{L^2(D)}$ and integrating by parts we have that

$$\|\Delta u_0 + k^2 u_0\|^2 \geq X^2 + Y^2 - 2XY.$$

Hence

$$\begin{aligned}\mathcal{A}_k(u,u_0) &\geq \gamma X^2 - 2\gamma XY + (\gamma+1)Y^2\\ &= \Big(\gamma + \frac{1}{2}\Big)\Big(Y - \frac{2\gamma}{2\gamma+1}\,X\Big) + \frac{1}{2}\,Y^2 + \frac{\gamma}{1+2\gamma}\,X^2\\ &\geq \frac{\gamma}{1+2\gamma}\,(X^2+Y^2).\end{aligned} \tag{3.13}$$

Again integrating by parts, and using the fact that $u_0 = 0$ on Γ, we have that

$$\|\Delta u_0 + k^2 u_0\|^2_{L^2(D)} = \|\Delta u\|^2_{L^2(D)} - 2k^2\|\nabla u_0\|^2_{L^2(D)} + k^4\|u_0\|^2_{L^2(D)}$$

and hence

$$2k^2\|\nabla u_0\|^2_{L^2(D)} \le \|\Delta u_0\|^2_{L^2(D)} + k^4\|u_0\|^2_{L^2(D)}. \tag{3.14}$$

Combining (3.13) and (3.14) we have that there exists a constant $c_k > 0$ independent of u_0 and γ such that

$$\mathcal{A}_k(u_0, u_0) \ge c_k \frac{\gamma}{1+2\gamma} \|u_0\|^2_{H_0(D,\Gamma_2)}$$

which shows that $\mathcal{A}_k(\,\cdot\,,\,\cdot\,) - ik\mathcal{B}(\,\cdot\,,\,\cdot\,)$ is coercive.

If $n/(1-n) \ge \gamma > 0$ we have that

$$\mathrm{Re}\,(\tilde{\mathcal{A}}_k(u_0, u_0) + ik\mathcal{B}(u_0, u_0)) = \tilde{\mathcal{A}}_k(u_0, u_0).$$

But

$$\tilde{\mathcal{A}}_k(u_0, u_0) \ge \gamma\|\Delta u_0 + k^2 u_0\|^2_{L^2(D)} + k^4\|\Delta u_0\|^2 = (\gamma+1)X^2 - 2\gamma XY + \gamma Y^2.$$

Hence, proceeding in the same way as above, it can be shown that

$$\tilde{\mathcal{A}}_k(u_0, u_0) \ge c_k \frac{\gamma}{1+2\gamma} \|u_0\|^2_{H_0(D,\Gamma_2)}$$

for some positive constant c_k, i.e. $\tilde{\mathcal{A}}_k(\,\cdot\,,\,\cdot\,) + ik\mathcal{B}(\,\cdot\,,\,\cdot\,)$ is coercive. □

Theorem 3.2. *Let α and δ be positive constants and assume that $\alpha \ge n \ge \delta + 1$ or $1 - n \ge \delta$ and $n \ge \alpha$. Then if k is not a transmission eigenvalue there exists a unique solution of the interior transmission problem satisfying*

$$\|w\|_{L^2(D)} + \|v\|_{L^2(D)} \le c(\|\varphi\|_{H^{3/2}(\Gamma)} + \|\psi\|_{H^{1/2}(\Gamma)} + \|\tau\|_{H^{-1/2}(\Gamma_2)})$$

for some positive constant c independent of φ, ψ and τ.

Proof. First consider $\alpha \ge n \ge \delta + 1$. This implies that $1/(n-1) \ge \gamma > 0$. Using the fact that C is a compact operator and from the Lemma $\mathcal{A}_k - ik\mathcal{B}$ is bijective, the result follows from an application of the Fredholm alternative. On the other hand, if $n \ge \alpha$ and $1 - n \ge \delta$ then $n/(1-n) \ge \tilde{\gamma}$ for some positive constant $\tilde{\gamma}$ and the result follows from the fact that $\tilde{\mathcal{A}}_k + ik\mathcal{B}$ is bijective. □

For $g \in L^2(\Omega)$ we define the *Herglotz wave function* with kernel g by

$$v_g(x) := \int_\Omega e^{ikx\cdot d} g(d)\, ds(d)$$

and note that by the Jacobi–Anger expansion $J_n(kr)e^{in\theta}$ are Herglotz wave functions where J_n is a Bessel function of order $n, n = 0, \pm 1, \pm 2, \ldots$.

Theorem 3.3. *The set of Herglotz wave functions v_g, $g \in L^2(\Omega)$, is dense in* $L(D) := \{u \in L^2(D) : \Delta u + k^2 u = 0$ *in the sense of distributions*$\}$.

Proof. Let $u_0 \in L^2(D)$ be orthogonal to v_g, $g \in L^2(\Omega)$, and define

$$u(x) := \int_D \Phi(x,y) u_0(y)\, dy, \qquad x \in \mathbb{R}^2 \tag{3.15}$$

where

$$\Phi(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|) \tag{3.16}$$

and $H_0^{(1)}$ as a Hankel function of the first kind of order zero. Then for $x \in \mathbb{R}^2 \setminus \bar{D}$, the unique continuation property for solutions to the Helmholtz equation and the addition theorem for Bessel functions implies that $u(x) = 0$ and hence, by the continuity properties of volume potentials, $u = \partial u/\partial\nu = 0$ on Γ. Since $\Delta u + k^2 u = -u_0$ in D we have, multiplying by u_0 and integrating by parts, that

$$-\|u_0\|^2_{L^2(D)} = (\Delta u, u_0)_{L^2(D)} + k^2 (u, u_0)_{L^2(D)} = (u, \Delta u_0 + k^2 u_0)_{L^2(D)} = 0$$

and hence $u_0 = 0$. The proof of the theorem now follows immediately. □

Let $\Phi_\infty(\hat{x}, z)$ be the far field pattern of $\Phi(x,z)$, $z \in D$. By Theorems 3.2 and 3.3, for $z \in D$ there exists a Herglotz wave function $v_{g_z^\epsilon}$ with kernel g_z^ϵ an approximate solution of the *far field equation*

$$\int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d) = \Phi_\infty(\hat{x}, z) \tag{3.17}$$

such that $v_{g_z^\epsilon}$ converges to w_z in the $L^2(D)$ norm as $\epsilon \to 0$ where w_z, v_z is the unique solution (assuming that k is not a transmission eigenvalue) to the interior transmission problem

$$\Delta w_z + k^2 w_z = 0 \qquad \text{in } D, \tag{3.18}$$

$$\Delta v_z + k^2 n(x) v_z = 0 \qquad \text{in } D, \tag{3.19}$$

$$v_z - w_z = \Phi(\,\cdot\,, z) \qquad \text{on } \partial D, \tag{3.20}$$

$$\frac{\partial v_z}{\partial\nu} - \frac{\partial w_z}{\partial\nu} = \frac{\partial}{\partial\nu}\Phi(\,\cdot\,, z) \qquad \text{on } \Gamma_1, \tag{3.21}$$

$$\frac{\partial v_z}{\partial\nu} - \frac{\partial w_z}{\partial v} = \frac{\partial}{\partial\nu}\Phi(\,\cdot\,, z) + ik\eta(w_z + \Phi(\,\cdot\,, z)) \qquad \text{on } \Gamma_2. \tag{3.22}$$

Following the approach of [5], we have that the direct scattering problem is well defined for the incident field u^i in $L^2(D)$. Furthermore, for $z \in D$, $u^s := \Phi(\,\cdot\,, z)$ in $\mathbb{R}^2 \setminus \bar{D}$ and $u^s = v_z - w_z$ in D satisfy the direct scattering problem for incident field w_z. Let u^s_ϵ be the scattered field corresponding to $v_{g_z^\epsilon}$. Then $u^z_\epsilon \to u^s = v_z - w_z$ in $H^1(D)$ and $u^s_\epsilon \to \Phi(\,\cdot\,, z)$ in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \bar{D})$. Using this fact and proceeding in the same way as in the last section of [5], it is possible to justify the use of Green's formula applied to w_z and v_z as done in the next section of this paper.

4. Inequalities for η and n

Using the analysis of the previous section, it is now easy to obtain an inequality for η.

Theorem 4.1. *Let $z_0 \in D$ and let v_{z_0}, w_{z_0} be a solution to the interior transmission problem* (3.18)–(3.22)*. Then*

$$\int_{\Gamma_2} \eta(x)|W|^2\, ds = -\frac{2\pi k|\gamma|^2 + \operatorname{Im} w_{z_0}(z_0)}{k}$$

where $\gamma = e^{i\pi/4}/\sqrt{8\pi k}$ and $W = w_{z_0} + \Phi(\,\cdot\,, z_0)$.

Proof. Set $W = w_{z_0} + \Phi(\,\cdot\,, z_0)$. Then, using the fact that n is real,

$$0 = \int_\Gamma \Big(v_{z_0} \frac{\partial \overline{v}_{z_0}}{\partial \nu} - \overline{v}_{z_0} \frac{\partial v_{z_0}}{\partial \nu} \Big)\, ds = \int_\Gamma \Big(W \frac{\partial \overline{W}}{\partial \nu} - \overline{W} \frac{\partial W}{\partial \nu} \Big)\, ds \\ - 2ik \int_{\Gamma_2} \eta(x) W \overline{W}\, ds.$$

But

$$\int_\Gamma \Big(W \frac{\partial \overline{W}}{\partial \nu} - \overline{W} \frac{\partial W}{\partial \nu} \Big)\, ds = \int_\Gamma \Big(\Phi(\,\cdot\,, z_0) \frac{\partial \overline{\Phi(\,\cdot\,, z_0)}}{\partial \nu} - \overline{\Phi(\,\cdot\,, z_0)}\, \frac{\Phi(\,\cdot\,, z_0)}{\partial \nu} \Big)\, ds \\ + \int_\Gamma \Big(w_{z_0} \frac{\partial \overline{\Phi(\,\cdot\,, z_0)}}{\partial \nu} - \overline{\Phi(\,\cdot\,, z_0)}\, \frac{\partial w_{z_0}}{\partial \nu} \Big)\, ds + \int_\Gamma \Big(\Phi(\,\cdot\,, z_0) \frac{\partial \overline{w}_{z_0}}{\partial \nu} - \overline{w}_{z_0} \frac{\partial \Phi(\,\cdot\,, z_0)}{\partial \nu} \Big)\, ds \\ = -2ik \int_\Omega \Phi_\infty(\,\cdot\,, z_0) \overline{\Phi_\infty(\,\cdot\,, z)}\, ds + \overline{w}_{z_0}(z_0) - w_{z_0}(z_0) \\ = -4i\pi k|\gamma|^2 - 2i \operatorname{Im} w_{z_0}(z_0),$$

i.e.

$$\int_{\Gamma_2} \eta(x)|W|^2\, ds = -\frac{2k\pi|\gamma|^2 + \operatorname{Im} w_{z_0}(z_0)}{k}.$$

□

Corollary 4.2. *Let γ and w_{z_0} be as in the above theorem. Then*

$$\sup_{x \in \Gamma_2} \eta(x) \geq -\frac{2k\pi|\gamma|^2 + \operatorname{Im} w_{z_0}(z_0)}{k\|w_{z_0} + \Phi(\,\cdot\,, z_0)\|_{L^2(\Gamma_2)}}$$

with equality holding if $\eta(x)$ is a constant.

Note that from the analyses of the previous section w_{z_0} can be approximated by the Herglotz wave function vg_z^ϵ where g_z^ϵ is an approximate solution to the far field equation (3.17). Since Γ_2 is in general unknown, a more practical lower bound is obtained if in the corollary $L^2(\Gamma_2)$ is replaced by $L^2(\Gamma)$.

Now assume that $\eta = 0$, i.e. the scattering obstacle is uncoated. Then, since D can be determined by the linear sampling method [1, 6], the first eigenvalue $\lambda_0(D)$ of $-\Delta$ in D is known. In this case we have the following theorem [3, 7]:

Theorem 4.3. *Assume that $n - 1 \geq \delta$ for some positive constant δ. Then if k is a transmission eigenvalue we have that*

$$\sup_D n > \frac{\lambda_0(D)}{k^2}.$$

Since the transmission eigenvalues can numerically be determined from the far field pattern, Theorem 4.3 gives a method for determining a lower bound for $n(x)$ from the far field pattern.

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