PAPER

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To cite this article: Isaac Harris *et al* 2014 *Inverse Problems* **30** 035016

IOP Publishing Inverse Problems

Inverse Problems **30** (2014) 035016 (21pp) doi:10.1088/0266-5611/30/3/035016

Transmission eigenvalues and non-destructive testing of anisotropic magnetic materials with voids

Isaac Harris[1], **Fioralba Cakoni**[1] **and Jiguang Sun**[2]

[1] Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

[2] Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, USA

E-mail: iharris@udel.edu, cakoni@math.udel.edu and jiguangs@mtu.edu

Received 3 November 2013, revised 9 January 2014

Accepted for publication 30 January 2014

Published 21 February 2014

Abstract

In this paper we consider the transmission eigenvalue problem corresponding to the scattering problem for an anisotropic magnetic materials with voids, i.e. subregions with refractive index the same as the background. Here we restrict ourselves to the scalar case of TE-polarization. Under weak assumptions on the material properties, we show that the transmission eigenvalues can be determined from the far field measurements. Then assuming that the contrast on the material properties does not change sign, we prove the existence of at least one transmission eigenvalue for sufficiently small voids. We also show that the first transmission eigenvalue can be used to determining material properties and give qualitative information about the size of the void. Some numerical examples are given to demonstrate the theoretical results.

Keywords: transmission eigenvalues, anisotropic media, inverse scattering problem

(Some figures may appear in colour only in the online journal)

1. Introduction

The non-destructive testing of composite materials using electromagnetic waves is an important problem in engineering. A number of such problems involve complicated materials, in particular anisotropic, hence many methods of reconstructing the matrix refractive index are either unfeasible or computationally expensive. On the other hand for practical purposes it suffices to obtain some partial information on the refractive index in order to evaluate the integrity of the material. The so-called qualitative methods in inverse scattering do just this (see e.g. [4, 19]). In this paper we consider the problem of detecting voids in a known anisotropic dielectric material from electromagnetic measurements in the frequency domain for a range of frequencies. An attempt to reconstruct the voids would involve computing the Green's function for anisotropic media for possibly complicated geometry. On the other hand, our inversion method is based on quantifying the effect that the presence of voids have on the so-called *transmission eigenvalues* (TEVs), which are detectable from the scattering data [7, 20, 23]. TEVs have been used to determine material properties of the scattering media starting with [8] for isotropic inhomogeneities and followed by [3, 5, 9, 17] and [24] for more complicated media.

We begin this paper by showing that the real TEVs can be seen in the far field measured data following the approach in [7]. We note that a rigorous characterization of the real TEVs in terms of the scattering matrix is given in [7] but the analysis there requires restrictive assumptions on the material properties. Next we prove that real TEVs exists for anisotropic magnetic dielectric media with voids. The existing results on this question [6, 15] include only the case of non-magnetic material, i.e. when the magnetic permeability of the media is the same as of the background and the approach used in these papers rely heavily on the fact that the contrast is only on one constitutive parameters of the medium. Our approach to proving the existence of TEVs follows the formulation introduced in [12] with appropriate modifications to allow for the presence of voids. We conclude the paper with numerical examples showing how the volume, shape and the location of voids affect the TEV and what kind of information we can obtain about voids from the first TEV.

2. Formulation of the problem

We begin by considering electromagnetic waves propagating in an inhomogeneous anisotropic dielectric medium in $\mathbb{R}^3$ with electric permittivity $\epsilon = \epsilon(x)$ and magnetic permeability $\mu = \mu(x)$. For time harmonic electromagnetic waves of the form

$$\mathcal{E}(x,t) = \tilde{E}(x)\,\mathrm{e}^{-\mathrm{i}\omega t}, \quad \mathcal{H}(x,t) = \tilde{H}(x)\,\mathrm{e}^{-\mathrm{i}\omega t}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent parts $\tilde{E}$ and $\tilde{H}$ satisfy

$$\nabla \times \tilde{E} - \mathrm{i}\omega\mu(x)\tilde{H} = 0 \quad \text{and} \quad \nabla \times \tilde{H} + \mathrm{i}\omega\epsilon(x)\tilde{E} = 0.$$

Now let us suppose that the inhomogeneity occupies an infinitely long cylinder with cross section D having piece-wise smooth boundary ∂D with ν being the unit outward normal to ∂D. We assume that the axis of the cylinder coincides with the z-axis. We further assume that the conductor is imbedded in a non-conducting homogeneous background, i.e. the electric permittivity $\epsilon_0 > 0$ and the magnetic permeability $\mu_0 > 0$ of the background medium. In addition we assume that inside the inhomogeneous media there is a subregion (possibly multiply-connected) with cross section $\overline{D}_1 \subset D$ that has the same electric permittivity and magnetic permeability as the background, i.e. ϵ_0 and μ_0 respectively. For an orthotropic medium we have that the matrices $\mathcal{A}$ and $\mathcal{N}$ are independent of the z-coordinate and are of the form

$$\mathcal{A} := \frac{\epsilon(x)}{\epsilon_0} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a \end{pmatrix} \quad \mathcal{N} := \frac{\mu(x)}{\mu_0} = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n \end{pmatrix}.$$

Then it is well-known [4] that the only component u of the total magnetic field $\tilde{H} = (0, 0, u)$ polarized perpendicular to the axis of the cylinder satisfies

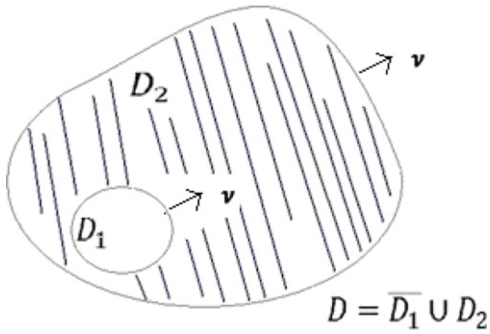

Figure 1. Example of the geometry of a medium with voids.

$$\nabla \cdot A(x)\nabla u + k^2 n(x) u = 0$$

where

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1},$$

and analogously the scattered field u^s satisfies $\Delta u^s + k^2 u^s = 0$ outside the scatter D. We can now rigorously formulate our scattering problem in $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a bounded simply connected open set with piece-wise smooth boundary ∂D. Furthermore assume that we have a symmetric matrix valued function $A(x) \in L^\infty(D, \mathbb{R}^{2\times 2})$ that is uniformly positive definite in D and $n(x) \in L^\infty(D)$ such that $n(x) > n_0 > 0$. We are particularly interested in the case where there exists $\overline{D}_1 \subset D$ (possibly multiple connected) with $A(x) = I$ and $n(x) = 1$ for all $x \in D_1$. Let us denote $D_2 = D \setminus \overline{D}_1$ that is the support of the inhomogeneous media without the voids (see figure 1).

Then the scattering of a plane wave $u^i := \mathrm{e}^{\mathrm{i}kx\cdot d}$ by this anisotropic inhomogeneous media with voids can be formulated as: find $(u^s, u) \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{D}) \times H^1(D)$ such that:

$$\nabla \cdot A(x)\nabla u + k^2 n(x) u = 0 \quad \text{in} \quad D \tag{1}$$

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D} \tag{2}$$

$$u - u^s = \mathrm{e}^{\mathrm{i}kx\cdot d} \quad \text{in} \quad D \tag{3}$$

$$\frac{\partial u}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial}{\partial \nu}\mathrm{e}^{\mathrm{i}kx\cdot d} \quad \text{on} \quad \partial D \tag{4}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - \mathrm{i}k u^s\right) = 0 \tag{5}$$

where $\partial u/\partial \nu_A := \nu \cdot A\nabla u$ and the Sommerfeld radiation condition (5) is assumed uniformly with respect to $\hat{x} = x/r$, $r = |x|$.

It is well-known (see e.g. [4]) that the above system is well-posed. Furthermore, it can be shown that the scattered field u^s assumes the asymptotic behavior

$$u^s(x, d; k) = \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} u_\infty(\theta, \phi; k) + \mathcal{O}(r^{-3/2}) \quad \text{as} \quad r \to \infty$$

where $d := (\cos\phi, \sin\phi)$ is the incident direction and $\hat{x} := (\cos\theta, \sin\theta)$ is the observation direction for $\theta, \phi \in [0, 2\pi]$. Here $u_\infty(\theta, \phi)$ is called the far-field pattern of the scattering problem (1)–(5), which is a function of the observation angle for a given incident angle.

The far-field patterns for all incident directions d defines the far field operator $L^2(0, 2\pi) \to g \in L^2(0, 2\pi)$ by

$$(Fg)(\theta) := \int_0^{2\pi} u_\infty(\theta, \phi) g(\phi)\, \mathrm{d}\phi \quad \text{for} \quad g \in L^2(0, 2\pi).$$

It is also well-known (see e.g. [4], theorem 6.2) that the far-field operator is injective if and only if there does not exist a nontrivial $(w, v) \in H^1(D) \times H^1(D)$ solving:

$$\nabla \cdot A\nabla w + k^2 n w = 0 \quad \text{in} \quad D \tag{6}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad D \tag{7}$$

$$w = v \quad \text{on} \quad \partial D \tag{8}$$

$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D \tag{9}$$

such that v takes the form of a Herglotz function

$$v_g(x) := \int_0^{2\pi} \mathrm{e}^{\mathrm{i}kx\cdot d} g(\phi)\, \mathrm{d}\phi, \quad g \in L^2[0, 2\pi], \quad d := (\cos\phi, \sin\phi).$$

The values of $k \in \mathbb{C}$ for which the *homogeneous interior transmission problem* (6)–(9) has nontrivial solutions are called *TEVs*. The goal of this paper is to obtain information about the voids from a knowledge of the real TEVs. To this end we first show that real TEVs can be determine for the far-field pattern $u_\infty(\theta, \phi)$ for $\theta, \phi \in [0,\ 2\pi]$ (or possibly in a subset of $[0,\ 2\pi]$). Let us introduce the following notation:

$$\inf_{x\in D_2} \inf_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_{\min} \quad \text{and} \quad \inf_{x\in D_2} n(x) = n_{\min}$$

$$\sup_{x\in D_2} \sup_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_{\max} \quad \text{and} \quad \sup_{x\in D_2} n(x) = n_{\max}$$

and

$$\inf_{x\in \mathcal{N}_\delta(\partial D)} \inf_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_\star \quad \text{and} \quad \inf_{x\in \mathcal{N}_\delta(\partial D)} n(x) = n_\star$$

$$\sup_{x\in \mathcal{N}_\delta(\partial D)} \sup_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A^\star \quad \text{and} \quad \sup_{x\in \mathcal{N}_\delta(\partial D)} n(x) = n^\star$$

where $\mathcal{N}_\delta(\partial D)$ is a fixed neighborhood of the boundary. From physical considerations we assume that $A_{\min} > 0$, $n_{\min} > 0$ and $A_{\max} < \infty$, $n_{\max} < \infty$.

3. Determination of transmission eigenvalues from scattering data

We show that the real TEVs corresponding to our problem can be determined from the far-field measurements following the approach in [7] where the same result is proven for the case of isotropic media with $A = I$. To this end we introduce the far field equation (FFE)

$$(Fg_z)(\theta) = \Phi_\infty(\hat{x}, z), \quad z \in D, \quad \hat{x} = (\cos\theta,\ \sin\theta) \tag{10}$$

where $\Phi_\infty(\hat{x}, z) = \gamma \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}$, $\gamma := \mathrm{e}^{\mathrm{i}\pi/4}/\sqrt{8\pi k}$ is the far-field pattern of the radiating fundamental solution to the Helmholtz equation in $\mathbb{R}^2$ given by $\Phi(x, y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x - y|)$. Let F^δ be the far-field operator corresponding to the noisy measurements $u^\delta_\infty(\theta, \phi)$ satisfying $\|u^\delta_\infty - u_\infty\|_{L^2[0,2\pi]} \leqslant \delta \|u_\infty\|_{L^2[0,2\pi]}$. We find the Tikhonov regularized solution $g_{z,\delta} := g^\delta_{z,\epsilon(\delta)}$ of the FFE defined as the unique minimizer of

$$\|F^\delta g - \Phi_\infty(\cdot, z)\|^2_{L^2[0,2\pi]} + \epsilon \|g\|^2_{L^2[0,2\pi]}$$

where the regularization parameter $\epsilon := \epsilon(\delta) \to 0$ as $\delta \to 0$. Provided that F has dense range (which is true in general except for the case when the transmission eigenfunction v takes the form of Herglotz function; see appendix of [7]) the regularized solution $g_{z,\delta}$ is such that

$$\lim_{\delta\to 0} \|F^\delta g_{z,\delta} - \Phi_\infty(\cdot, z)\|_{L^2[0,2\pi]} = 0. \tag{11}$$

Following [1] and using the results developed in section 7.2 in [4] it is possible to show that if k is not a TEV then the Herglotz function $v_{g_{z,\delta}}$ converges in the $H^1(D)$-norm to v where (v, w) solves

$$\nabla \cdot A\nabla w + k^2 n w = 0 \quad \text{in} \quad D \tag{12}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad D \tag{13}$$

$$w - v = \Phi(\cdot, z) \quad \text{on} \quad \partial D \tag{14}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \quad \text{on} \quad \partial D, \tag{15}$$

provided that the solution exists (in other words (12)–(15) is Fredholm with index zero). Let us recall an equivalent variational formulation for the above interior transmission problem analyzed in [2]. To this end, we define a lifting of the essential boundary data into the domain D. Thus we let $\phi_z \in H^1(D)$ be such that $\phi_z = \Phi(\cdot, z)$ on ∂D and attempt to find a solution of the interior transmission problem where $v = v_0 - \phi_z$, and the pair $(w, v_0) \in X(D) := \{w, v_0 \in H^1(D) \,|\, w - v_0 \in H^1_0(D)\}$ satisfies

$$\mathcal{A}_k((w, v_0); (\varphi_1, \varphi_2)) = \ell(\varphi_1, \varphi_2) \quad \text{for all} \quad (\varphi_1, \varphi_2) \in X(D) \tag{16}$$

where the sesqulinear form $\mathcal{A}_k(\cdot\,;\cdot) : X(D) \times X(D) \mapsto \mathbb{C}$ and the conjugate linear functional $\ell(\cdot) : X(D) \mapsto \mathbb{C}$ are given by

$$\mathcal{A}_k((w, v_0); (\varphi_1, \varphi_2)) := \int_D A\nabla w \cdot \nabla\overline{\varphi_1} - \nabla v_0 \cdot \nabla\overline{\varphi_2}\, \mathrm{d}x - k^2 \int_D n w\overline{\varphi_1} - v_0\overline{\varphi_2}\, \mathrm{d}x,$$

$$\ell(\varphi_1, \varphi_2) := \int_{\partial D} \overline{\varphi}_1 \frac{\partial}{\partial \nu}\Phi(x, z)\, \mathrm{d}s_x - \int_D \nabla\phi_z \cdot \nabla\overline{\varphi}_2 - k^2\phi_z\overline{\varphi}_2\, \mathrm{d}x.$$

It has been proven in [2] that the variational problem (16) satisfies the Fredholm property provided $A_\star > 1$, $n_\star > 1$, or $A^\star < 1$, $n^\star < 1$. In particular, if k is a TEV with transmission eigenfunctions (w_k, v_k) then (16) has a solution if and only if the following solvability condition is satisfied

$$\ell(w_k, v_k) = \int_{\partial D} \overline{w}_k \frac{\partial}{\partial \nu}\Phi(x, z)\, \mathrm{d}s_x - \int_D \nabla\phi_z \cdot \nabla\overline{v}_k - k^2\phi_z\overline{v}_k\, \mathrm{d}x = 0. \tag{17}$$

Theorem 3.1. *Let k be a real TEV and assume that $A_\star > 1$, $n_\star > 1$, or $A^\star < 1$, $n^\star < 1$. Then for $z \in D$ (except for possibly a nowhere dense set of points), $||v_{g_{z,\delta}}||_{H^1(D)}$ cannot be bounded as $\delta \to 0$, where $g_{z,\delta}$ satisfies (11).*

Proof. Assume there is a set of positive measure such that $||v_{g_{z,\delta}}||_{H^1(D)}$ is bounded. Hence, a subsequence $v_{g_{z,\delta_n}}$ converges weakly to a $v \in H^1(D)$ satisfying $\Delta v + k^2 v = 0$ in D. Since $Fg_{z,\delta} \to \Phi_\infty(\cdot, z)$, Rellich's lemma implies that $u^s = \Phi(\cdot, z)$ on $\mathbb{R}^2 \setminus \overline{D}$, where u^s is the scattered field with the far-field pattern $Fg_{z,\delta}$. Now, the corresponding total field w in D and v satisfy the interior transmission problem (12)–(15), which gives that there is a solution to the variational problem $\mathcal{A}_k\big((w, v_0); (\varphi_1, \varphi_2)\big) = \ell(\varphi_1, \varphi_2)$. Using integration by parts on the Fredholm solvability condition (17) and using that $\phi_z = \Phi(\cdot, z)$ on ∂D and $\Delta v_k + k^2 v_k = 0$ in D, we have that

$$\int_{\partial D} \overline{w}_k \frac{\partial}{\partial \nu} \Phi(x,z) - \frac{\partial \overline{v}_k}{\partial \nu} \Phi(x,z)\, \mathrm{d}s_x = 0.$$

Notice that since $w_k = v_k$ on ∂D, Green's representation theorem and the unique continuation principle implies that $v_k = 0$ in D. So v_k has zero Cauchy data on ∂D, which implies that $w_k = 0$ and $\frac{\partial w_k}{\partial \nu_A} = 0$ on ∂D, whence $w_k = 0$ in D. Therefore $(w_k, v_k) = (0, 0)$ which contradicts the fact that (w_k, v_k) are eigenfunctions. □

Remark 3.1. Note that the proof of theorem 3.1 carries over to any case of inhomogeneous media for which the corresponding interior transmission problem satisfies the Fredholm property. In particular this holds if $A_{\min} > 1$ or $A_{\max} < 1$, and k^2 is not a Dirichlet eigenvalue for each component of voids D_1.

Remark 3.2. From [2] and [18] we also have the following discreteness result. Assume that either $A_\star > 1$ and $n_\star > 1$, or $A^\star < 1$ and $n^\star < 1$, or $A_{\min} > 1$ or $A_{\max} < 1$ hold. Then the set of TEVs is at most discrete. The real positive TEVs can accumulate only at $+\infty$.

The above analysis indicates that when plotting the $||v_{g_{z,\delta}}||_{H^1(D)}$ against k, where $g_{z,\delta}$ is the Tikhonov regularized solution to the FFE, the TEVs will appear as sharp peaks in the graph. In section 5 we present numerical examples that show the viability of this approach to determine real TEVs from far-field data.

4. Existence of transmission eigenvalues

In this section we consider the existence of TEVs for anisotropic magnetic dielectric media with voids. We apply similar analysis as in [12] which must be modified to account for the presence of voids, that are subregions D_1 where $A = I$ and $n = 1$. For the existence of the TEVs when $A = I$ or $n = 1$ in D see [10, 16, 21].

At this point, we consider the case $(A_{\min} - 1) > 0$ and $(n_{\max} - 1) < 0$, or $(A_{\max} - 1) < 0$ and $(n_{\min} - 1) > 0$. Our goal is to prove the existence of real TEVs, hence we assume that $k^2 \geqslant 0$. To this end, we formulate the TEV problem (6)–(9) as a problem for the difference $u := v - w \in H_0^1(D)$. By subtracting the partial differential equations and boundary conditions for v and w we have that the boundary value problem for v and u is given by

$$\nabla \cdot A \nabla u + k^2 n u = \nabla \cdot (A - I) \nabla v + k^2 (n-1) v \quad \text{in} \quad D \tag{18}$$

$$\frac{\partial u}{\partial \nu_A} = \frac{\partial v}{\partial \nu_A} - \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D. \tag{19}$$

Notice that from $\partial w^+/\partial_A \nu = \partial w^-/\partial \nu$ and the continuity of $\partial v^+/\partial \nu = \partial v^-/\partial \nu$ across ∂D_1 we have that

$$\frac{\partial u^+}{\partial \nu_A} - \frac{\partial u^-}{\partial \nu} = \frac{\partial v^+}{\partial \nu_A} - \frac{\partial v^+}{\partial \nu} \tag{20}$$

where the superscripts $+$ and $-$ indicate approaching the boundary from outside and inside D_1 respectively. Following [12], we will consider (18)–(20) as a Neumann boundary value problem for v which is defined in $D \setminus \overline{D_1}$ where we must incorporate the fact that u is a solution to the Helmholtz equation in D_1. To this end, we need to assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_1 and define the interior Dirichlet to Neumann mapping $T_k : H^{1/2}(\partial D_1) \to H^{-1/2}(\partial D_1)$ by

$$T_k : u\Big|_{\partial D_1} \longmapsto \frac{\partial u}{\partial \nu}\Big|_{\partial D_1} \quad \text{where} \quad \Delta u + k^2 u = 0, \quad \text{in} \quad D_1. \tag{21}$$

With help of T_k we are able to go from boundary terms on ∂D_1 to terms defined in D_1. In particular, an integration by parts gives that

$$\int_{\partial D_1} \overline{\varphi}\, T_k u \,\mathrm{d}s = \int_{D_1} \nabla u \cdot \nabla \overline{\varphi} - k^2 u \overline{\varphi} \,\mathrm{d}x \qquad \forall \varphi \in H^1(D_1). \tag{22}$$

(If D_1 has multiple simply connected components then we define the Dirichlet to Neumann operator component wise). Then for a given $u \in H_0^1(D)$ satisfying the Helmholtz equation inside D_1, we see (18)–(20) as a Neumann boundary value problem for v which can be written in an equivalent variational form as follows

$$\int_{D_2} (A - I)\nabla v \cdot \nabla \overline{\varphi} - k^2(n-1)v\overline{\varphi}\,\mathrm{d}x = \int_{D_2} A\nabla u \cdot \nabla \overline{\varphi} - k^2 n u \overline{\varphi}\,\mathrm{d}x$$
$$+ \int_{\partial D_1} \overline{\varphi}\, T_k u \,\mathrm{d}s \quad \forall \varphi \in H^1(D_2). \tag{23}$$

We use the variational formulation to define a bounded linear operator that maps $u \in H_0^1(D) \mapsto v_u \in H^1(D_2)$. To this end let us define the bounded sesqulinear form and the bounded conjugate linear functional from the variational formulation as:

$$\mathcal{B}_k(v, \varphi) := \int_{D_2} (A - I)\nabla v \cdot \nabla \overline{\varphi} - k^2(n-1)v\overline{\varphi}\,\mathrm{d}x,$$
$$f_u(\varphi) := \int_{D_2} A\nabla u \cdot \nabla \overline{\varphi} - k^2 n u \overline{\varphi}\,\mathrm{d}x + \int_{\partial D_1} \overline{\varphi} T_k u \,\mathrm{d}s.$$

and consider the variational problem of finding $v \in H^1(D_2)$ such that $\mathcal{B}_k(v, \varphi) = f_u(\varphi)$ for all $\varphi \in H^1(D_2)$. We split the solution $v = \widehat{v} + c$ where c is a constant and $\widehat{v} \in \widehat{H}^1(D_2) := \{\widehat{v} \in H^1(D_2) \mid \int_{D_2}(n-1)\widehat{v}\,\mathrm{d}x = 0\}$ equipped with the $H^1(D_2)$ inner-product. It can be shown that functions in $\widehat{H}^1(D_2)$ satisfies the Poincaré inequality, that is $||\widehat{v}||^2_{L^2(D_2)} \leqslant C_p ||\nabla \widehat{v}||^2_{L^2(D_2)}$ for all $\widehat{v} \in \widehat{H}^1(D_2)$. Now letting $\varphi = 1$ for $k^2 \neq 0$ we have that

$$k^2 \int_{D_2} (n-1)v\,\mathrm{d}x = k^2 \int_{D_2} nu\,\mathrm{d}x + \int_{\partial D_1} \frac{\partial u}{\partial \nu}\,\mathrm{d}s = k^2 \int_{D_2} nu\mathrm{d}x - k^2 \int_{D_1} u\,\mathrm{d}x$$

where the latter equality holds due to the fact that u solves the Helmholtz equation in D_1. Using this along with $v = \widehat{v} + c$ we have that $c = \frac{1}{\int_{D_2}(n-1)\,\mathrm{d}x}\left(\int_{D_2} nu\,\mathrm{d}x - \int_{D_1} u\,\mathrm{d}x\right)$. If $k^2 = 0$ we require c to still be defined as in the non-zero case. Now we show the variational problem is well posed in the space $\widehat{H}^1(D_2)$ by proving that $\pm\mathcal{B}_k(\widehat{v}, \widehat{\varphi})$ is $\widehat{H}^1(D_2)$-coercive, when $A_{\min} - 1 > 0$ and $n_{\max} - 1 < 0$, or $A_{\max} - 1 < 0$ and $n_{\min} - 1 > 0$ respectively. If $A_{\min} - 1 > 0$ and $n_{\max} - 1 < 0$

$$\begin{aligned}\mathcal{B}_k(\widehat{v}, \widehat{v}) &= \int_{D_2} (A - I)\nabla \widehat{v} \cdot \nabla \overline{\widehat{v}} - k^2(n-1)|\widehat{v}|^2\,\mathrm{d}x \\ &\geqslant \int_{D_2} (A_{\min} - 1)\nabla \widehat{v} \cdot \nabla \overline{\widehat{v}} + k^2(1 - n_{\max})|\widehat{v}|^2\,\mathrm{d}x \\ &\geqslant (A_{\min} - 1)||\nabla \widehat{v}||^2_{L^2(D_2)} \geqslant C||\widehat{v}||^2_{H^1(D_2)}\end{aligned}$$

where we have used the Poincaré inequality. Similarly we can show that if $A_{\max} - 1 < 0$ and $n_{\min} - 1 > 0$, $-\mathcal{B}_k(\cdot, \cdot)$ is coercive. Having $v_u \in H^1(D_2)$ defined in the annulus D_2 for any $u \in H_0^1(D)$. Since the transmission eigenfunction v solves the Helmholtz equation in the domain D, we insure that v_u can be extended to a solution of the Helmholtz equation in D. Using the Riesz representation theorem we can now define $\mathbb{L}_k u$ by

$$(\mathbb{L}_k u, \varphi)_{H^1(D_2)} = \int_{D_2} \nabla v_u \cdot \nabla \overline{\varphi} - k^2 v_u \overline{\varphi}\,\mathrm{d}x + \int_{\partial D_1} \overline{\varphi}\, T_k v_u \,\mathrm{d}s \quad \forall \varphi \in H_0^1(D_2, \partial D). \tag{24}$$

where

$$H_0^1(D_2, \partial D) = \{u \in H^1(D_2) : \ u = 0 \quad \text{on} \quad \partial D\}.$$

Notice that the mapping $k \mapsto \mathbb{L}_k$ is continuous for $k \in \mathbb{R}$ and k^2 not a Dirichlet eigenvalue. We can now connect the kernel of the operator $\mathbb{L}_k : H_0^1(D_2, \partial D) \to H_0^1(D_2, \partial D)$ to the set of transmission eigenfunctions.

Theorem 4.1. *Assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_1 and assume that either $A_{\min} > 1$ and $n_{\max} < 1$, or $A_{\max} < 1$ and $n_{\min} > 1$. If $w, v \in H^1(D)$ solves (6)–(9) then $u|_{D_2} = w - v \in H_0^1(D_2, \partial D)$ is such that $\mathbb{L}_k u = 0$. Conversely, if $\mathbb{L}_k u = 0$ for $u \in H_0^1(D_2, \partial D)$ then v_u and u can be extended to solution $v, u \in H^1(D)$ of Helmholtz equation in D_1 and the pair $(u + v, v)$ solves (6)–(9).*

Proof. The first part of the Theorem is by construction. Obviously $\mathbb{L}_k u = 0$ since v_u satisfies the Helmholtz equation in D. Conversely, let $\mathbb{L}_k u = 0$ and define $v := v_u \in H^1(D_2)$ as above and in D_1 by

$$\Delta v + k^2 v = 0 \quad \text{in} \quad D_1, \quad v = v_u^+ \quad \text{on} \quad \partial D_1. \tag{25}$$

Since $\mathbb{L}_k u = 0$, (24) implies that $v \in H^1(D)$ and satisfies the Helmholtz equation in D. Furthermore, extending u in D_1 by

$$\Delta u + k^2 u = 0 \quad \text{in} \quad D_1, \quad u = u^+ \quad \text{on} \quad \partial D_1,$$

then (23) implies that $(u + v, v)$ solves (6)–(9). □

The following lemma states some properties of the operator $\mathbb{L}_k$.

Lemma 4.1. *Assume that $k^2 \geqslant 0$ is not a Dirichlet eigenvalue for $-\Delta$ in D_1 and assume that either $A_{\min} > 1$ and $n_{\max} < 1$, or $A_{\max} < 1$ and $n_{\min} > 1$.*

(i) The operator $\mathbb{L}_k : H_0^1(D_2, \partial D) \mapsto H_0^1(D_2, \partial D)$ is self-adjoint.
(ii) The operator $\mathbb{L}_0$ or $-\mathbb{L}_0$ is coercive when $(A_{\min} - 1) > 0$ or $(A_{\max} - 1) < 0$, respectively.
(iii) $\mathbb{L}_k - \mathbb{L}_0$ is a compact.

Proof.

(i) Let u_1 and u_2 be given in $H_0^1(D_2, \partial D)$ and consider v_1 and v_2 in $H^1(D_2)$ satisfying (23) extended inside D as solutions to the Helmholtz equation by (25). Thus, for these functions we have

$$\int_{D_2} (A - I)\nabla v_i \cdot \nabla \overline{\varphi} - k^2(n - 1)v_i \overline{\varphi} \, dx = \int_{D_2} A\nabla u_i \cdot \nabla \overline{\varphi} - k^2 n u_i \overline{\varphi} \, dx$$
$$+ \int_{\partial D_1} \overline{\varphi} \, T_k u_i \, ds \quad \forall \varphi \in H^1(D). \tag{26}$$

By the definition of $\mathbb{L}_k$ we have that

$$\begin{aligned}(\mathbb{L}_k u_1, u_2)_{H^1(D_2)} &= \int_{D_2} \nabla v_1 \cdot \nabla \overline{u_2} - k^2 v_1 \overline{u_2} \, dx + \int_{\partial D_1} \overline{u_2} \, T_k v_1 \, ds \\ &= -\int_{D_2} (A - I)\nabla v_1 \cdot \nabla \overline{u_2} - k^2(n - 1)v_1 \overline{u_2} \, dx \\ &\quad + \int_{D_2} A\nabla v_1 \cdot \nabla \overline{u_2} - k^2 n v_1 \overline{u_2} \, dx + \int_{\partial D_1} \overline{u_2} \, T_k v_1 \, ds.\end{aligned} \tag{27}$$

Taking $i = 2$ and $\varphi = v_1$ and then $i = 1$ and $\varphi = u_2$ in (26) we obtain

$$(\mathbb{L}_k u_1, u_2)_{H^1(D_2)} = -\int_{D_2} A\nabla u_1 \cdot \nabla\overline{u_2} - k^2 n u_1 \overline{u_2}\, dx - \int_{\partial D_1} \overline{u_2}\, T_k u_1\, ds$$
$$+ \int_{D_2} (A-I)\nabla v_1 \cdot \nabla\overline{v_2} - k^2(n-1)v_1\overline{v_2}\, dx - \int_{\partial D_1} v_1 T_k \overline{u_2} - \overline{u_2}\, T_k v_1\, ds$$
$$= -\int_{D_2} A\nabla u_1 \cdot \nabla\overline{u_2} - k^2 n u_1 \overline{u_2}\, dx - \int_{D_1} \nabla u_1\, \nabla\overline{u_2} - k^2 u_1 \overline{u_2}\, dx$$
$$+ \int_{D_2} (A - I)\nabla v_1 \cdot \nabla\overline{v_2} - k^2(n-1)v_1\overline{v_2}\, dx = (u_1, \mathbb{L}_k u_2)_{H^1(D_2)}. \quad (28)$$

Obviously, the right-hand side is a self-adjoint expression of u_1 and u_2 we have that $\mathbb{L}_k$ is a self-adjoint operator and the boundary terms cancel thanks to (22).

(ii) To prove that $\pm\mathbb{L}_0$ is coercive we first assume that $A_{\min} - 1 > 0$ and therefore we consider the operator $\mathbb{L}_0$. Letting $v - u = w$ we have that

$$(\mathbb{L}_0 u, u)_{H^1(D_2)} = \int_{D_2} \nabla v \cdot \nabla\overline{u}\, dx + \int_{\partial D_1} \overline{u}\, T_0 v\, ds$$
$$= \int_{D_2} |\nabla u|^2\, dx + \int_{D_2} \nabla w \cdot \nabla\overline{u}\, dx + \int_{\partial D_1} \overline{u}\, T_0 u\, ds + \int_{\partial D_1} \overline{u}\, T_0 w\, ds.$$

Using (23) for $k^2 = 0$ and $\varphi = w$ and taking the conjugate, we see that

$$\int_{D_2} (A - I)\nabla w \cdot \nabla\overline{v}\, dx = \int_{D_2} A\nabla w \cdot \nabla\overline{u}\, dx + \int_{\partial D_1} w T_0 \overline{u}\, ds.$$

Now once again using that $v = u + w$ we see that

$$\int_{D} \nabla w \cdot \nabla\overline{u}\, dx = \int_{D_2} (A - I)\nabla w \cdot \nabla\overline{w}\, dx - \int_{\partial D_1} w T_0 \overline{u}\, ds.$$

Using the latter equation we see that

$$(\mathbb{L}_0 u, u)_{H^1(D_2)} = \int_{D_2} |\nabla u|^2\, dx + \int_{D_2} (A - I)\nabla w \cdot \nabla\overline{w}\, dx - \int_{\partial D_1} w T_0 \overline{u}\, ds$$
$$+ \int_{\partial D_1} \overline{u} T_0 w\, ds + \int_{\partial D_1} \overline{u} T_0 u\, ds$$

where from

$$\int_{\partial D_1} \overline{u} T_0 w\, ds = \int_{D_1} \nabla w \cdot \nabla\overline{u}\, dx = \int_{\partial D_1} w T_0 \overline{u}\, ds$$

the boundary terms involving w cancel. Now using that $A_{\min} - 1 > 0$ we have that

$$\int_{D_2} (A - I)\nabla w \cdot \nabla\overline{w}\, dx \geqslant (A_{\min} - 1) \int_{D_2} |\nabla w|^2\, dx \geqslant 0.$$

Also notice that integration by parts gives that

$$\int_{\partial D_1} \overline{u} T_0 u\, ds = \int_{D_1} |\nabla u|^2\, dx \geqslant 0.$$

Therefore

$$(\mathbb{L}_0 u, u)_{H^1(D_2)} \geqslant \int_{D_2} |\nabla u|^2\, dx + \int_{\partial D_1} \overline{u} T_0 u\, ds \geqslant \int_{D_2} |\nabla u|^2\, dx$$

proving the coercivity due to the zero boundary condition on ∂D.

Next, assume that $A_{\max} - 1 < 0$, therefore considering the operator $-\mathbb{L}_0$. From (28) we have that

$$(-\mathbb{L}_0 u, u)_{H^1(D_2)} = -\int_{D_2} (A - I)\nabla v \cdot \nabla\overline{v}\, dx + \int_{D_1} |\nabla u|^2\, dx + \int_{D_2} A\nabla u \cdot \nabla\overline{u}\, dx.$$

Now since $A_{\max} - 1 < 0$ we have that:

$$-\int_{D_2} (A - I)\nabla v \cdot \nabla \overline{v} \, \mathrm{d}x \geqslant (1 - A_{\max}) \int_{D_2} |\nabla v|^2 \, \mathrm{d}x \geqslant 0.$$

Therefore $(-\mathbb{L}_0 u, u)_{H^1(D_2)} \geqslant A_{\min} \int_{D_2} |\nabla u|^2 \, \mathrm{d}x$ proving coercivity in this case.

(iii) We now show the compactness of $\mathbb{L}_k - \mathbb{L}_0$. Assume that the sequence $u^j \rightharpoonup 0$ in $H_0^1(D_2, \partial D)$ and therefore we have the existence of $v_k^j \rightharpoonup 0$ and $v_0^j \rightharpoonup 0$ in $H^1(D_2)$, corresponding to solutions of (26). Recall that (24) defines $(\mathbb{L}_k - \mathbb{L}_0)u^j$ in terms of v_k^j and v_0^j. Since zero and k^2 are not Dirichlet eigenvalues, we have that their extension as solution to the Helmholtz equation inside D_1 converge weakly to 0 in D. From the Rellich's embedding theorem, a subsequence of the aforementioned sequences, still denoted by v_k^j and v_0^j converge strongly to zero in $L^2(D)$. We see that the sequences v_k^j and v_0^j satisfy

$$\int_{D_2} (A - I)\nabla v_k^j \cdot \nabla\overline{\varphi} - k^2(n - 1)v_k^j\overline{\varphi} \, \mathrm{d}x = \int_{D_2} A\nabla u^j \cdot \nabla\overline{\varphi} - k^2 n u^j\overline{\varphi} \, \mathrm{d}x + \int_{\partial D_1} \overline{\varphi} T_k u^j \, \mathrm{d}s$$

and

$$\int_{D_2} (A - I)\nabla v_0^j \cdot \nabla\overline{\varphi} \, \mathrm{d}x = \int_{D_2} A\nabla u^j \cdot \nabla\overline{\varphi} \, \mathrm{d}x + \int_{\partial D_1} \overline{\varphi} T_0 u^j \, \mathrm{d}s$$

for all $\varphi \in H^1(D)$. Now using that

$$\int_{\partial D_1} \overline{\varphi} T_k u^j \, \mathrm{d}s = \int_{D_1} \nabla u^j \cdot \nabla\overline{\varphi} - k^2 u^j \overline{\varphi} \, \mathrm{d}x$$

and letting $\tilde{v}^j := v_k^j - v_0^j$ we have that

$$\int_{D_2} (A - I)\nabla\tilde{v}^j \cdot \nabla\overline{\varphi} \, \mathrm{d}x = k^2 \int_{D_2} (n - 1)v_k^j\overline{\varphi} - n u^j\overline{\varphi} \, \mathrm{d}x + k^2 \int_{D_1} u^j\overline{\varphi} \, \mathrm{d}x \forall\varphi \in H^1(D).$$

Letting $\varphi = \tilde{v}^j$ and for either $A - I$ positive or negative definite we obtain that $\tilde{v}^j \to 0$ in $H^1(D_2)$. Now we have that

$$\Delta\tilde{v}^j = -k^2 v_k^j \quad \text{in} \quad D_1 \quad \text{and} \quad \tilde{v}^j = v_k^j - v_0^j \quad \text{on} \quad \partial D_1.$$

Therefore

$$||\tilde{v}^j||_{H^1(D_1)} \leqslant C\big(\big|\big|v_k^j - v_0^j\big|\big|_{H^1(D_2)} + \big|\big|v_k^j\big|\big|_{L^2(D)}\big) \longrightarrow 0$$

where we have used the trace theorem on ∂D_1. Now

$$\begin{aligned}((\mathbb{L}_k - \mathbb{L}_0)u^j, \varphi)_{H^1(D_2)} &= \int_{D_2} \nabla\tilde{v}^j \cdot \nabla\overline{\varphi} - k^2 v_k^j\overline{\varphi} \, \mathrm{d}x + \int_{\partial D_1} \overline{\varphi}\big(T_k v_k^j - T_0 v_0^j\big) \, \mathrm{d}s \\ &= \int_{D} \nabla\tilde{v}^j \cdot \nabla\overline{\varphi} - k^2 v_k^j\overline{\varphi} \, \mathrm{d}x\end{aligned}$$

therefore by the using the Cauchy–Schwartz inequality we have that

$$||(\mathbb{L}_k - \mathbb{L}_0)u^j||_{H^1(D_2)} \leqslant C\big(||\tilde{v}^j||_{H^1(D)} + \big|\big|v_k^j\big|\big|_{L^2(D)}\big).$$

Which proves the claim since the right-hand side tends to zero. □

Notice that the second part of this theorem says that for $k = 0$ the operator $\pm\mathbb{L}_k$ is positive. We now prove that $\pm\mathbb{L}_k$ is positive for a range of values, which gives a lower bound on the TEvs.

Theorem 4.2. *Let $\lambda_1(D)$ be the first Dirichlet eigenvalue of $-\Delta$ in D and let k^2 be a real TEV. Then* *(i) if* $A_{\min} > 1$ *and* $n_{\max} < 1$*, then we have that* $k^2 \geqslant \lambda_1(D)$*,*

(ii) if $A_{\max} < 1$ *and* $n_{\min} > 1$*, then we have that* $k^2 \geqslant \dfrac{A_{\min}}{n_{\max}} \lambda_1(D)$*.*

Proof.

(i) Assume that $A_{\min} - 1 > 0$ and $n_{\max} - 1 < 0$, we have that if u is the difference of eigenfunctions then $(\mathbb{L}_k u, u)_{H^1(D_2)} = 0$. So by the definition of $\mathbb{L}_k$ and by using $v = u + w$ we have that

$$(\mathbb{L}_k u, u)_{H^1(D_2)} = \int_{D_2} \nabla v \cdot \nabla \overline{u} - k^2 v \overline{u} \, \mathrm{d}x + \int_{\partial D_1} \overline{u} T_k v \, \mathrm{d}s = \int_D \nabla v \cdot \nabla \overline{u} - k^2 v \overline{u} \, \mathrm{d}x$$
$$= \int_D |\nabla u|^2 - k^2 |u|^2 \, \mathrm{d}x + \int_D \nabla w \cdot \nabla \overline{u} - k^2 w \overline{u} \, \mathrm{d}x.$$

Now we use the variational form (23) for $\varphi = w$ which gives

$$\int_{D_2} (A - I) \nabla v \cdot \nabla \overline{w} - k^2 (n-1) v \overline{w} \, \mathrm{d}x - \int_{D_2} A \nabla u \cdot \nabla \overline{w} - k^2 n u \overline{w} \, \mathrm{d}x$$
$$= \int_{D_1} \nabla u \cdot \nabla \overline{w} - k^2 u \overline{w} \, \mathrm{d}x.$$

On the left-hand side we once again use that $v = w + u$ and combine the integrals involving both u and w giving that

$$\int_D \nabla u \cdot \nabla \overline{w} - k^2 u \overline{w} \, \mathrm{d}x = \int_{D_2} (A - I) \nabla w \cdot \nabla \overline{w} - k^2 (n-1) |w|^2 \, \mathrm{d}x.$$

Now we look at $(\mathbb{L}_k u, u)_{H^1(D_2)}$ and use the fact that under the assumptions on the coefficients that

$$\int_{D_2} (A - I) \nabla w \cdot \nabla \overline{w} - k^2 (n-1) |w|^2 \, \mathrm{d}x \geqslant 0.$$

Therefore

$$(\mathbb{L}_k u, u)_{H^1(D_2)} = \int_D |\nabla u|^2 - k^2 |u|^2 \, \mathrm{d}x + \int_{D_2} (A - I) \nabla w \cdot \nabla \overline{w} - k^2 (n-1) |w|^2 \, \mathrm{d}x$$
$$\geqslant \int_D |\nabla u|^2 - k^2 |u|^2 \, \mathrm{d}x \geqslant [\lambda_1(D) - k^2] \int_D |u|^2 \, \mathrm{d}x. \quad (29)$$

So if $(\lambda_1(D) - k^2) > 0$, we have that $(\mathbb{L}_k u, u)_{H^1(D_2)} > 0$ which contradicts the fact that $\mathbb{L}_k u = 0$ which implies that all real TEVs must satisfy $k^2 \geqslant \lambda_1(D)$.

(ii) Assuming now that $A_{\max} - 1 < 0$ and $n_{\min} - 1 > 0$, we have that if u is the difference of eigenfunctions then $(-\mathbb{L}_k u, u)_{H^1(D_2)} = 0$. But from (28) we have that

$$(-\mathbb{L}_k u, u)_{H^1(D_2)} = -\int_{D_2} (A - I) \nabla v \cdot \nabla \overline{v} - k^2 (n-1) |v|^2 \, \mathrm{d}x + \int_{D_1} \nabla |u|^2 - k^2 |u|^2 \, \mathrm{d}x$$
$$+ \int_{D_2} A \nabla u \cdot \nabla \overline{u} - k^2 n |u|^2 \, \mathrm{d}x.$$

Notice that under the assumptions on the coefficients we have that

$$-\int_{D_2} (A - I) \nabla v \cdot \nabla \overline{v} - k^2 (n-1) |v|^2 \, \mathrm{d}x \geqslant 0.$$

Therefore

$$
\begin{aligned}
(-\mathbb{L}_k u, u)_{H^1(D_2)} &\geqslant \int_{D_1} |\nabla u|^2 - k^2|u|^2\,\mathrm{d}x + \int_{D_2} A\nabla u\cdot\nabla\overline{u} - k^2 n|u|^2\,\mathrm{d}x \\
&\geqslant \int_{D_1} |\nabla u|^2 - k^2|u|^2\,\mathrm{d}x + A_{\min}\int_{D_2} |\nabla u|\,\mathrm{d}x - k^2 n_{\max}\int_{D_2}|u|^2\,\mathrm{d}x \\
&\geqslant A_{\min}\int_D |\nabla u|\,\mathrm{d}x - k^2 n_{\max}\int_D |u|^2\,\mathrm{d}x \\
&\geqslant [A_{\min}\lambda_1(D) - k^2 n_{\max}]\int_D |u|^2\,\mathrm{d}x.
\end{aligned}
$$

So if $(A_{\min}\lambda_1(D) - k^2 n_{\max}) > 0$, we have that $(-\mathbb{L}_k u, u)_{H^1(D_2)} > 0$ which contradicts the fact that $\mathbb{L}_k u = 0$, which implies all real TEVs satisfy $k^2 \geqslant \frac{A_{\min}}{n_{\max}}\lambda_1(D)$. □

The previous result shows that the operator $\pm\mathbb{L}_k$ is positive for a range of k values. We next show that the operator is non-positive for some k on a subset of $H_0^1(D_2, \partial D)$.

Lemma 4.2. *Provided that the measure of each component of the void D_1 is sufficiently small, there exists a $k > 0$ such that $\mathbb{L}_k$, or $-\mathbb{L}_k$ for $A_{\min} > 1$ and $n_{\max} < 1$, or $A_{\max} < 1$ and $n_{\min} > 1$ respectively, is non-positive on a subspace of $H_0^1(D_2, \partial D)$.*

Proof. Assume that $(A_{\min} - 1) > 0$ and $(n_{\max} - 1) < 0$, and look at the operator $\mathbb{L}_k$. We denote by B_r the ball of radius r. Let R and ϵ be positive constants such that $\overline{B_R} \subset D$, $\overline{D_1} \subset B_\epsilon$ and $R > \epsilon$. By using separation of variables one can see that there exists TEVs for the system (see section 5)

$$
\begin{aligned}
&\Delta\hat{w} + \tau^2\hat{w} = 0 \quad \text{in} \quad B_\epsilon \\
&\nabla\cdot A_{\min}\nabla\hat{w} + \tau^2 n_{\max}\hat{w} = 0 \quad \text{in} \quad B_R\setminus\overline{B_\epsilon} \\
&\Delta\hat{v} + \tau^2\hat{v} = 0 \quad \text{in} \quad B_R \\
&\hat{w}^- = \hat{w}^+ \quad \text{and} \quad \frac{\partial\hat{w}^-}{\partial\nu} = \frac{\partial\hat{w}^+}{\partial\nu_{A_{\min}}} \quad \text{on} \quad \partial B_\epsilon \\
&\hat{w} = \hat{v} \quad \text{and} \quad \frac{\partial\hat{w}}{\partial\nu_{A_{\min}}} = \frac{\partial\hat{v}}{\partial\nu} \quad \text{on} \quad \partial B_R.
\end{aligned}
\tag{30}
$$

Now recall we can only define $\mathbb{L}_k$ when k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_1, so we denote the first eigenvalue as $\lambda_1(D_1)$. Since $\lambda_1(D_1) \to \infty$ as $|D_1| \to 0^+$ we can insure that there is at least one TEV of the form $\tau^2 = k^2(B_R, B_\epsilon, A_{\min}, n_{\max}) < \lambda_1(D_1)$ provided that the measure of each components of D_1 is sufficiently small. So let $\hat{u}$ be the difference of these eigenfunctions with eigenvalue τ^2 giving that from (29)

$$
\int_{B_R} |\nabla\hat{u}|^2 - \tau^2|\hat{u}|^2\,\mathrm{d}x + \int_{B_R\setminus\overline{B_\epsilon}} (A_{\min}-1)\nabla\hat{w}\cdot\nabla\overline{\hat{w}} - \tau^2(n_{\max}-1)|\hat{w}|^2\,\mathrm{d}x = 0.
$$

Therefore $\hat{u} \in H_0^1(B_R)$, so let the extension by zero of $\hat{u}$ to the whole domain D be denoted $\widetilde{u}$. Now since $\overline{D_1} \subset B_\epsilon$ we have that $\Delta\widetilde{u} + \tau^2\widetilde{u} = 0$ in D_1. Since $A_{\min} - 1 > 0$ and $n_{\max} - 1 < 0$, we can construct nontrivial $\widetilde{v} \in H^1(D_2)$ that solve (23) with coefficients A, n in the domain D with void D_1 and let $\widetilde{w} = \widetilde{v} - \widetilde{u}$. Hence from (23) and using that $\widetilde{w} = \widetilde{v} - \widetilde{u}$ we have that:

$$
\begin{aligned}
\int_{D\setminus\overline{D_1}} (A-I)\nabla\widetilde{w}\cdot\nabla\overline{\varphi} - \tau^2(n-1)\widetilde{w}\overline{\varphi}\,\mathrm{d}x &= \int_{B_R} \nabla\hat{u}\cdot\nabla\overline{\varphi} - \tau^2\hat{u}\overline{\varphi}\,\mathrm{d}x \\
&= \int_{B_R\setminus\overline{B_\epsilon}} (A_{\min}-1)\nabla\hat{w}\cdot\nabla\overline{\varphi} - \tau^2(n_{\max}-1)\hat{w}\overline{\varphi}\,\mathrm{d}x.
\end{aligned}
\tag{31}
$$

Therefore for $\varphi = \tilde{w}$ using (31) and the Cauchy–Schwartz inequality we have that

$$\int_{D\setminus\overline{D_1}} (A-I)\nabla\widetilde{w}\cdot\nabla\overline{\widetilde{w}} - \tau^2(n-1)\widetilde{w}\overline{\widetilde{w}}\,\mathrm{d}x = \int_{B_R\setminus\overline{B_\epsilon}} (A_{\min}-1)\nabla\hat{w}\cdot\nabla\overline{\widetilde{w}} - \tau^2(n_{\max}-1)\hat{w}\overline{\widetilde{w}}\,\mathrm{d}x$$
$$\leqslant \left[\int_{B_R\setminus\overline{B_\epsilon}} (A_{\min}-1)|\nabla\hat{w}|^2 - \tau^2(n_{\max}-1)|\hat{w}|^2\,\mathrm{d}x\right]^{\frac{1}{2}}$$
$$\times\left[\int_{B_R\setminus\overline{B_\epsilon}} (A_{\min}-1)|\nabla\widetilde{w}|^2 - \tau^2(n_{\max}-1)|\widetilde{w}|^2\,\mathrm{d}x\right]^{\frac{1}{2}}$$

and using (31) with $\varphi = \tilde{w}$ once more we obtain

$$\int_{D\setminus\overline{D_1}} (A-I)\nabla\widetilde{w}\cdot\nabla\overline{\widetilde{w}} - \tau^2(n-1)|\widetilde{w}|^2\,\mathrm{d}x \leqslant \int_{B_R\setminus\overline{B_\epsilon}} (A_{\min}-1)|\nabla\hat{w}|^2 - \tau^2(n_{\max}-1)|\hat{w}|^2\,\mathrm{d}x.$$

Now we use the definition (29) for the operator $\mathbb{L}_\tau$ with the functions $\widetilde{u}$ and $\widetilde{w}$ to conclude

$$\begin{aligned}(\mathbb{L}_\tau\widetilde{u},\widetilde{u})_{H^1(D_2)} &= \int_D |\nabla\widetilde{u}|^2 - \tau^2|\widetilde{u}|^2\,\mathrm{d}x + \int_{D\setminus\overline{D_1}} (A-I)\nabla\widetilde{w}\cdot\nabla\overline{\widetilde{w}} - \tau^2(n-1)|\widetilde{w}|^2\,\mathrm{d}x\\ &\leqslant \int_{B_R} |\nabla\hat{u}|^2 - \tau^2|\hat{u}|^2\,\mathrm{d}x + \int_{B_R\setminus\overline{B_\epsilon}} (A_{\min}-1)\nabla\hat{w}\cdot\nabla\overline{\hat{w}} - \tau^2(n_{\max}-1)|\hat{w}|^2\,\mathrm{d}x\\ &= 0.\end{aligned}$$

So the operator $\mathbb{L}_\tau$ is non-positive on this one dimensional subspace.

Alternatively, we can construct a finite dimensional subspace of $H_0^1(D_2,\partial D)$ where $\mathbb{L}_\tau$ is non-positive by considering small balls $B_\delta \subset D_2$. In this case let κ, $\hat{w}$ and $\hat{v}$ are the first TEV and corresponding eigenfunctions of the system

$$\begin{aligned}&\nabla\cdot A_{\min}\nabla\hat{w} + \kappa^2 n_{\max}\hat{w} = 0 \quad \text{in} \quad B_\delta\\ &\Delta\hat{v} + \kappa^2\hat{v} = 0 \quad \text{in} \quad B_\delta\\ &\hat{w} = \hat{v} \quad \text{and} \quad \frac{\partial\hat{w}}{\partial\nu_{A_{\min}}} = \frac{\partial\hat{v}}{\partial\nu} \quad \text{on} \quad \partial B_\delta.\end{aligned} \tag{32}$$

Now provided that the measure of each component of D_1 is small enough such that κ^2 is smaller than the first corresponding Dirichlet eigenvalue for $-\Delta$, we can use $\hat{u} = \hat{v} - \hat{w} \in H_0^1(B_\delta)$, and its extension by zero $\tilde{u}$ to the whole domain and the corresponding $\tilde{w}$ and $\tilde{v}$ exactly as above to show that $\mathbb{L}_\tau$ is non-positive in an m-dimensional subspace of $H_0^1(D_2,\partial D)$ where m is the number of balls of radius δ included in D_2.

The same result can be proven for $-\mathbb{L}_k$ exactly in a similar way for the case when $A_{\max}-1 < 0$ and $n_{\min}-1 > 0$ where everywhere $A_{\max}$ is replaced by $A_{\min}$ and $n_{\min}$ is replaced by $n_{\max}$. □

To prove now the existence of TEVs we use the following theorem

Theorem 4.3. *Let $\mathbb{L}_k : H_0^1(D_2,\partial D) \mapsto H_0^1(D_2,\partial D)$ be as defined above. If*

(i) there exists $k_{\min} \geqslant 0$ such that $\theta\mathbb{L}_{k_{\min}}$ is positive on $H_0^1(D_2,\partial D)$

(ii) there exists $k_{\max} < \lambda_1(D_1)$ such that $\theta\mathbb{L}_{k_{\max}}$ is non-positive on a m-dimensional subspace of $H_0^1(D_2,\partial D)$

then there exists m TEVs in $[k_{\min}, k_{\max}]$, where $\theta = 1$ or $\theta = -1$ provided $A_{\min}-1 > 0$ and $n_{\max}-1 < 0$, or $A_{\max}-1 < 0$ and $n_{\min}-1 > 0$, respectively.

For the proof of this theorem see theorem 2.6 [12] (see also theorem 4.7 in [11] or chapter 6 in [4]). In particular the result can be obtained by using min-max condition for the auxiliary eigenvalue problem for the self-adjoint compact operator $\mathbb{I} - \lambda(k)(\theta\mathbb{L}_0)^{-1/2}\theta(\mathbb{L}_k - \mathbb{L}_0)(\theta\mathbb{L}_0)^{-1/2}$.

Now combining lemma 4.2 and theorem 4.3 we can prove the following result.

Theorem 4.4. *Assume that either $A_{\min} > 1$ and $n_{\max} < 1$, or $A_{\max} < 1$ and $n_{\min} > 1$. If the first TEV $\tau > 0$ of (30) is smaller than the first Dirichlet eigenvalue for each of the components of D_1, then there exists one TEV in the interval $(0, \tau)$. If the first TEV $\kappa > 0$ of (32) is smaller than the first Dirichlet eigenvalue for each of the components of D_1 then there exits $m := m(\delta)$ TEV (counting their multiplicity) in the interval $(0, \kappa)$, where m is the number of balls of radius $\delta > 0$ that can fit in D_2.*

Note that the number $m(\delta)$ depends on the size of each components of void D_1 and also on the number of voids and their locations.

Remark 4.1. If $A_{\min} > 1$ and $n_{\max} > 1$, or $A_{\max} < 1$ and $n_{\min} < 1$ it is now obvious how to modify the approach of [12] to prove the existence of TEV. In this case in addition to assuming that the voids are small enough it is necessary to assume that $|n - 1|$ is also small. We omit here the details in order to avoid repetition.

As a by-product of theorems 4.2 and 4.3 we have that the first TEV satisfies the following upper and lower bounds.

Theorem 4.5. *Let $k_1(D, D_1, A, n)$, be the first TEV of the given media with voids and the measure of each component of D_1 is small enough (as discussed above). Then the following inequalities hold:*

(i) If $(A_{\min} - 1) > 0$ and $(n_{\max} - 1) < 0$, then

$$\lambda_1(D) \leqslant k_1^2(D, D_1, A, n) \leqslant \min\left\{k_1^2(B_R, B_\epsilon, A_{\min}, n_{\max}), k_1^2(B_\delta, A_{\min}, n_{\max})\right\}$$

$k_1(B_R, B_\epsilon, A_{\min}, n_{\max})$ and $k_1(B_\delta, A_{\min}, n_{\max})$ are the first TEV corresponding to (30) and the first TEV corresponding to (32), respectively.

(ii) If $(A_{\max} - 1) < 0$ and $(n_{\min} - 1) > 0$, then

$$\frac{A_{\min}}{n_{\max}}\lambda_1(D) \leqslant k_1^2(D, D_1, A, n) \leqslant \min\{(k^2(B_R, B_\epsilon, A_{\max}, n_{\min}), k^2(B_\delta, A_{\max}, n_{\min})\}$$

where $k_1(B_R, B_\epsilon, A_{\max}, n_{\min})$ and $k_1(B_\delta, A_{\min}, n_{\max})$ the first TEV corresponding to (30) and the first TEV corresponding to (32), respectively, with $A_{\max}$ replaced by $A_{\min}$ and $n_{\min}$ replaced by $n_{\max}$.

Here $\lambda_1(D_1)$ is the first Dirichlet eigenvalue of $-\Delta$ in D_1.

We conclude this section by proving a monotonicity result for the first TEV with respect to the size of D_1, which can be useful in identifying voids in known anisotropic material as discussed in section 5. We remark that it is possible to obtain monotonicity results for the first eigenvalue in terms of the material properties, but we do not present them here since the goal of this paper is to detect voids using TEV (see [18] for additional monotonicity results). For given A and n satisfying either $A_{\min} - 1 > 0$ and $n_{\max} - 1 < 0$, or $A_{\max} - 1 < 0$ and $n_{\min} - 1 > 0$ we have the following monotonicity results.

Theorem 4.6. *Let $D_1 \subseteq D_1'$. Then $k_1(D_1) \leqslant k_1(D_1')$ where $k_1(\Omega)$ is the first TEV corresponding to void Ω.*

Proof. Assume that $(A_{\min} - 1) > 0$ and $(n_{\max} - 1) < 0$, and that $\tilde{v}$ and $\tilde{w}$ are the transmission eigenfunctions corresponding to the TEV $k_1(D_1') = \tilde{k}$. Now let $\tilde{u} = \tilde{v} - \tilde{w}$, therefore we have the existence of $v \in H^1(D \setminus \overline{D}_1)$ that solves (23) and define $w = v - \tilde{u}$. Therefore we have from (29) that

$$\int_D |\nabla\tilde{u}|^2 - \tilde{k}^2|\tilde{u}|^2\,\mathrm{d}x + \int_{D\setminus\overline{D}_1'} (A - I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \tilde{k}^2(n-1)|\tilde{w}|^2\,\mathrm{d}x = 0.$$

By the definition of w and (23) we obtain that

$$\int_{D\backslash\overline{D}_1}(A-I)\nabla w\cdot\nabla\overline{\varphi}-\tilde{k}^2(n-1)w\overline{\varphi}\,\mathrm{d}x=\int_D\nabla\tilde{u}\cdot\nabla\overline{\varphi}-\tilde{k}^2\tilde{u}\overline{\varphi}\,\mathrm{d}x$$
$$=\int_{D\backslash\overline{D}'_1}(A-I)\nabla\tilde{w}\cdot\nabla\overline{\varphi}-\tilde{k}^2(n-1)\tilde{w}\overline{\varphi}\,\mathrm{d}x. \quad (33)$$

Therefore letting $\varphi=w$ in (33) and using the Cauchy–Schwartz inequality (in the same way as the equations below (31)) along with $D_1\subseteq D'_1$ we have that

$$\int_{D\backslash\overline{D}_1}(A-I)\nabla w\cdot\nabla\overline{w}-\tilde{k}^2(n-1)|w|^2\,\mathrm{d}x\leqslant\int_{D\backslash\overline{D}'_1}(A-I)|\nabla\tilde{w}|^2-\tilde{k}^2(n-1)|\tilde{w}|^2\,\mathrm{d}x.$$

Now we use the definition (29) for the operator $\mathbb{L}_{\tilde{k}}$ with the functions $\tilde{u}$ to conclude that

$$\begin{aligned}(\mathbb{L}_{\tilde{k}}\tilde{u},\tilde{u})_{H^1(D\backslash\overline{D}_1)}&=\int_D|\nabla\tilde{u}|^2-\tau^2|\tilde{u}|^2\,\mathrm{d}x+\int_{D\backslash\overline{D}_1}(A-I)\nabla w\cdot\nabla\overline{w}-\tilde{k}^2(n-1)|w|^2\,\mathrm{d}x\\&\leqslant\int_D|\nabla\tilde{u}|^2-\tilde{k}^2|\tilde{u}|^2\,\mathrm{d}x+\int_{D\backslash\overline{D}'_1}(A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}}-\tilde{k}^2(n-1)|\tilde{w}|^2\,\mathrm{d}x\\&=0\end{aligned}$$

where $\mathbb{L}_{\tilde{k}}$ is the operator corresponding to the problem with void D_1. Since $\mathbb{L}_{\tilde{k}}$ is non-positive on the subspace spanned by $\tilde{u}$ it means that there is an eigenvalue corresponding to D_1 in $(0,\tilde{k}]$. Therefore the first TEV $k_1(D_1)$ must satisfy $k_1(D_1)\in(0,\tilde{k}]$ which proves the claim. A similar argument holds for when $A_{\max}-1<0$ and $n_{\min}-1>0$, by looking at the operator $-\mathbb{L}_{\tilde{k}}$. □

5. Numerical validation

In this section, we show some numerical examples to show that the first TEV can give information about the voids. We shall address the following issues.

- (i) We check if the TEVs can be determined from scattering data for the case on anisotropic magnetic materials with voids based on the discussion of section 2 (see e.g. [23] for near field data). We confirm that the eigenvalues determined from the far-field data are actually the transmission eigenvalue. In particular, we consider a special case when the scattering object and the void are concentric disks in which case the TEVs can be obtained analytically. For general geometry we compute the TEVs using a continuous finite element method (FEM) with eigenvalues searching technique described in [14] (see also [22] and [24]).
- (ii) We numerically study how the size, location and geometry of voids affects the first TEV.
- (iii) We numerically study the inverse problem of estimating the size of the void(s) using the first TEV. Numerical results indicate that qualitative information can be obtained on the size of the void(s).

Theorem 3.1 suggests that if we solve the FFE and plot the L^2 norm of the solution g against a range of k values, at a TEV the norm of g 'blows up', which should look like a spike in the graph. Below is the numerical procedure with simulated far-field data:

- (a) Solve the direct problem using a cubic FEM with a perfectly matched layer, for a range of k values.
- (b) Evaluate an approximate u_∞ with 1% random noise added (unless otherwise stated).

(c) Using the approximated u_∞ to solve the FFE $Fg_z = \Phi_\infty(\cdot, z)$ for 25 random locations of z in D by a Tikhonov–Morozov regularization strategy.
(d) Plot $||g||_{L^2(0,2\pi)}$ averaged over z versus k. Note that from the estimates derived in the previous section we have a priori knowledge of the interval where the first TEV lies, and we consider this information when choosing the range of k in our computations.

In the following calculations we use N different incident direction ϕ_j and N observation directions θ_i that are uniformly spaced in $[0, 2\pi)$ where $N = 30$ throughout the rest of the paper unless otherwise specified. As explained above the simulated far-field pattern $u_\infty(\theta_i, \phi_j)$ is obtained from solving the direct problem. This would lead to a discretized FFE with $N \times N$ matrix. We then solve the discretized FFE for 25 randomly distributed points in the domain D. Once we have solved this linear systems for $\vec{g}$ which has components $g_i \approx g(\theta_i)$, we plot the average approximation of $||\vec{g}||_{\ell^2}$ over a range of k values.

5.1. Comparison with exact transmission eigenvalues

We will now consider a TEV problem with constant coefficients. For this we assume that $A = \alpha I$ for some constant $\alpha > 0$ and let n be constant such that $n > 0$. Furthermore assume that $D = B_R$ and $D_1 = B_\epsilon$ where $0 < \epsilon < R$. Under these assumptions the TEV problem reads: find nontrivial v, w such that:

$$\Delta w + k^2 w = 0 \quad \text{in} \quad B_\epsilon \tag{34}$$

$$\alpha \Delta w + k^2 n w = 0 \quad \text{in} \quad B_R \setminus \overline{B_\epsilon} \tag{35}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad B_R \tag{36}$$

$$w^- = w^+ \quad \text{and} \quad \frac{\partial w^-}{\partial r} = \alpha \frac{\partial w^+}{\partial r} \quad \text{on} \quad \partial B_\epsilon \tag{37}$$

$$w = v \quad \text{and} \quad \alpha \frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} \quad \text{on} \quad \partial B_R. \tag{38}$$

It can be shown that trying to find transmission eigenfunctions of the form $w(r, \theta) = w_m(r)\mathrm{e}^{\mathrm{i}m\theta}$ and $v(r, \theta) = v_m(r)\,\mathrm{e}^{\mathrm{i}m\theta}$ with $m \in \mathbb{Z}$ gives that the TEVs are given by the roots of $d_m(k)$, where $d_m(k)$ is defined as:

$$d_m(k) := \det \begin{pmatrix} \mathrm{J}_m(k\epsilon) & -\mathrm{J}_m\left(k\sqrt{\frac{n}{\alpha}}\epsilon\right) & -\mathrm{Y}_m\left(k\sqrt{\frac{n}{\alpha}}\epsilon\right) & 0 \\ -\mathrm{J}'_m(k\epsilon) & \sqrt{n\alpha}\mathrm{J}'_m\left(k\sqrt{\frac{n}{\alpha}}\epsilon\right) & \sqrt{n\alpha}\mathrm{Y}'_m\left(k\sqrt{\frac{n}{\alpha}}\epsilon\right) & 0 \\ 0 & \mathrm{J}_m\left(k\sqrt{\frac{n}{\alpha}}R\right) & \mathrm{Y}_m\left(k\sqrt{\frac{n}{\alpha}}R\right) & -\mathrm{J}_m(kR) \\ 0 & -\sqrt{n\alpha}\mathrm{J}'_m\left(k\sqrt{\frac{n}{\alpha}}R\right) & -\sqrt{n\alpha}\mathrm{Y}'_m\left(k\sqrt{\frac{n}{\alpha}}R\right) & \mathrm{J}'_m(kR) \end{pmatrix}$$

where $\mathrm{J}_p(t)$ and $\mathrm{Y}_p(t)$ are the Bessel functions of the first and second kind. To see if the solution of the FFE will capture the TEVs we apply the method discussed above to (34)–(38). We expect to see spikes in the average norm of g at the known TEVs that are the roots of $d_m(k)$. In figure 2 we plot the average $||g||_{L^2(0,2\pi)}$ for the parameters $\alpha = 1/5$, $n = 1$ and $R = 1$ with $\epsilon = 0.1$. Using Newton's method with a centered finite difference approximation for the derivative we can compute the first two roots of $d_0(k)$ given by $k \approx 2.48,\ 5.27$.

We let $A_\alpha = \alpha I$, $n = 1$ and compare the roots of $d_0(k)$ to the spikes in the graph for $||g||_{L^2(0,2\pi)}$ for various values of α and ϵ, where we let the outer radius $R = 1$. These results are shown in table 1. The values agree very well.

Next we consider non-circular domains and compare the TEVs determined from the far field scattering data based on solving the FFE against those computed directly using a FEM. We now compare the reconstructed TEVs using the FFE with the FEM. We fix $A = \mathrm{Diag}(5, 6)$

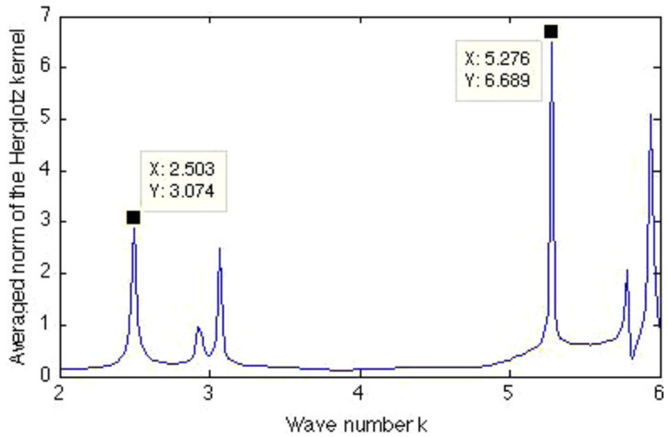

Figure 2. Notice that there are spikes at $k = 2.50,\ 5.27$ on the graph while the first two roots of $d_0(k)$ are 2.48, 5.27. The other spikes in the graph corresponds to roots for $d_m(k)$ where $m \neq 0$. Here $k \in [2, 6]$.

Table 1. Root finding versus far-field equation.

α	ϵ	First root of d_0	Spike in the $\|\|g\|\|_{L^2(0,2\pi)}$
1/2	0.01	7.99	7.80
1/4	0.1	2.91	2.92
1/10	0.05	1.67	1.68

Table 2. Comparison of FFE computation versus FEM calculations.

Method	Domain	First TEV	Second TEV
FFE	Square (2×2)	1.84	6.60
FEM	Square (2×2)	1.84	6.63
FFE	Circle $(R = 1)$	1.98	7.23
FEM	Circle $(R = 1)$	1.98	7.13

Table 3. Limited aperture for disk with $R = 1$, $A =$ Diag$(5, 6)$, $n = 2$. Note that from table 2 we have $k = 1.98$.

Noise	First spike
10^{-3}	1.84
10^{-6}	1.91
10^{-9}	1.98

and $n = 2$ for the rest of the paper. The direct computation by the FEM in table 2 is done by a continuous FEM using the linear Lagrange elements with the mesh size $h \approx 0.01$. The results in table 2 are for domains without the presents of a void.

We now look at the question of partial aperture in using the far field data to compute the TEVs. Partial aperture is where the angles ϕ and θ are not distributed over the entire interval $[0, 2\pi)$, but rather some interval. So in table 3 we use $N = 20$ angles distributed uniformly over $[0, \pi)$. It is known that the smaller the aperture the more unstable the FFE method is for reconstructing that TEVs. To test this we decrease the amount of random noise added to the calculations to see if the first spike computed with partial aperture data coincides with our first FEM computed TEV for sufficiently small noise. The results are shown in table 3.

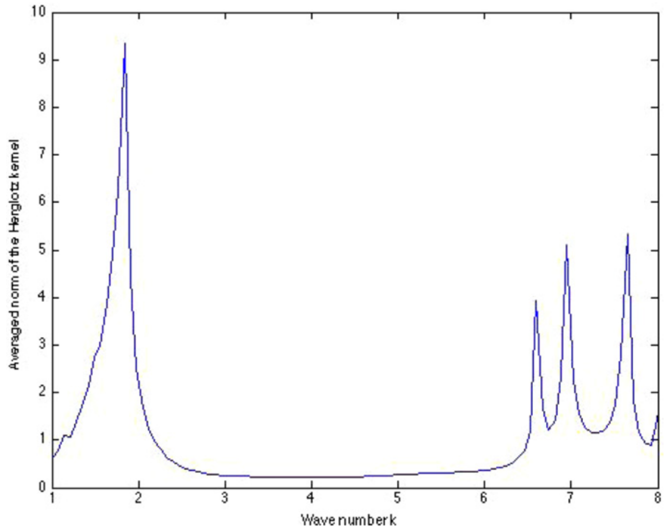

Figure 3. The plot of the average $||g||_{L^2(0,2\pi)}$ for the square (2×2) with no void: $A = \mathrm{Diag}(5, 6)$ and $n = 2$. Here $k \in [1, 8]$.

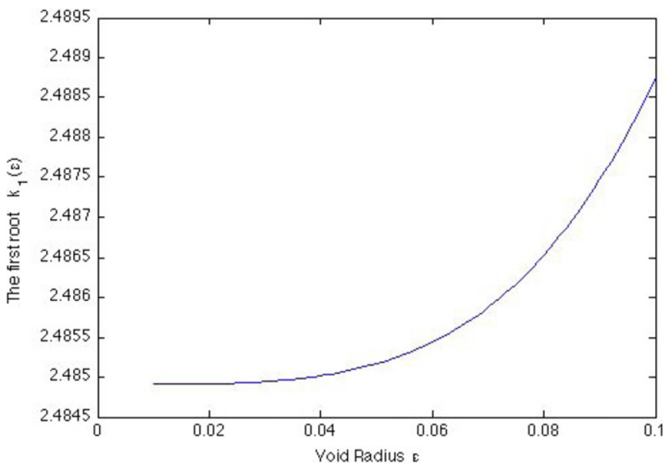

Figure 4. Graph of $k_1(\epsilon)$ versus $[\epsilon_{\min}, \epsilon_{\max}]$ to show the monotonicity of the first TEV with respect to the size of the void.

5.2. Determination of void area

We now consider the inverse problem of determining information about the void D_1 from the first TEV. For fixed A and n theorem 4.6 shows that the first TEV depends monotonically increasing on the size of the void. Indeed, figure 4 is a plot that shows the monotonicity of the first root of $d_0(k)$ with respect to the size of the circular void ϵ where $\alpha = 1/5$, $n = 1$ and $R = 1$.

Figure 5 display the monotonicity of the first TEV in terms of the size of circular void, where $A = \mathrm{diag}(5, 6)$ and $n = 2$ for both D the unit circle and the square $[-1, 1] \times [-1, 1]$. As oppose to the case of figure 4, here the monotonicity is reversed because here $A_{\max} > 1$ and $n > 1$ which is compatible with theoretical investigation.

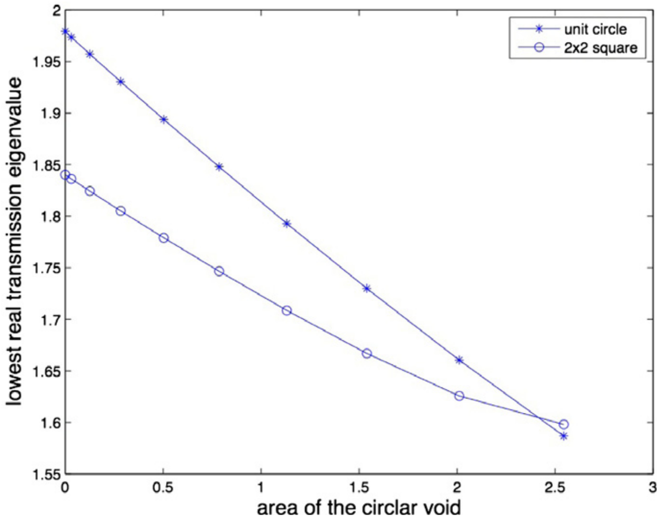

Figure 5. Graph of first TEV k_1 versus the size of a (large) circular void for $A = \text{diag}(5, 6)$ and $n = 2$, and D the unit circle and the square $[-1, 1] \times [-1, 1]$.

Table 4. First TEV for various void sizes computed by the FEM.

ϵ	0.2	0.19	0.18	0.17	0.16	0.15	0.14	0.13	0.12	0.11	0.1
Circle	9.53	9.27	9.02	8.77	8.54	8.31	8.08	7.86	7.64	7.43	7.22
Square	7.76	7.57	7.39	7.21	7.04	6.87	6.70	6.53	6.37	6.21	6.05

Table 5. Dependence of first TEV on void's position.

Location	(0, 0)	(0.6, 0)	(0.3, 0.7)	(−0.2, 0.4)	(0.6, 0.6)
$A_{1/4}; k_1$	2.90	2.92	2.92	2.96	2.92
$A_{1/9}; k_1$	1.77	1.80	1.78	1.80	1.78

This monotonicity dependence is also confirmed by FEM calculations. The results are shown in table 4 for a disk of radius 1 and a 2×2 square $[-1, 1] \times [-1, 1]$ with a void of the form $B_\epsilon(0, 0)$ for various ϵ.

We also numerically investigated the dependence of the first TEV in terms of the location of the void. In particular we fixed the media $A_a = aI$, $n = 1$ with the support D being the unit disc centered at the origin, in which we considered a small circular void of radius $\epsilon = 0.1$ that is centered at (x_1, x_2). In table 5 we see little to no difference if the location of the void is changed, where the first TEV is computed by solving the FFE.

The monotonicity property could be used to obtain information about the volume of the void D_1. Given the first TEV for fixed given material properties, we wish to find information about the size of the void. Hence, we consider the inverse problem of finding the (additive) area of a void(s) from the first TEV and again fix $A = \text{Diag}(5, 6)$, $n = 2$. To do so we find an ϵ^* such that a void of the form $B_{\epsilon^*}(0, 0)$ satisfies $k_1(\text{void(s)}) \approx k_1(B_{\epsilon^*}(0, 0))$. Using this idea we try to reconstruct the area of multiple voids by using the first TEV computed by the FEM. We put two circular voids in the domains considered above, i.e. a disk of radius 1 and a 2×2 square. The voids both have radii 0.1 and be centered at (0, 0) and (0.5, 0.5) respectively.

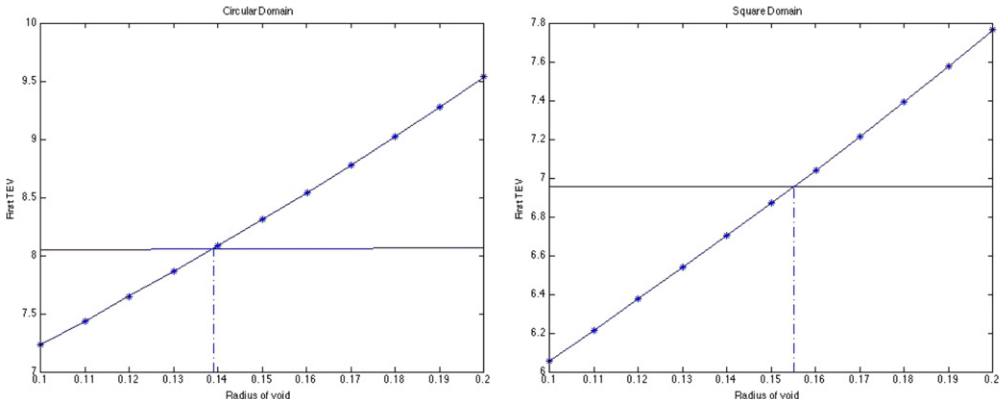

Figure 6. The first TEV of the two domains with a single void versus the radius of the void. The horizontal lines are the k_1's for the two domains with two voids. The vertical dotted lines are the approximated values of ϵ^* such that a void of the form $B_{\epsilon^*}(0, 0)$ gives the same TEV approximately, i.e. $k_1(\text{void(s)}) \approx k_1(B_{\epsilon^*}(0, 0))$.

Table 6. Qualitative reconstruction of area from FF-measurements.

D	D_1	$\lvert B_{\epsilon^*}(0, 0)\rvert$	$\lvert D_1\rvert$
Disk $R = 1$	Disk $r = 0.1$	0.0328	0.0314
	Square	0.0303	0.0300
$[-1, 1] \times [-1, 1]$	Ellipse	0.0613	0.0628
	Square	0.0749	0.1256

We compute the first TEV in each case, then find the area of a single void of the shape of a disk that has the same first TEV. Note that the total area of the two voids is approximately 0.0630. The area of the single void $B_{\epsilon^*}(0, 0)$ is 0.0607 for the unit disk and 0.0775 for the square. These calculations are presented in figure 6. In table 6 we show the results for the area of a void calculated based on the first TEV and on the assumption that (incorrectly) is a disc centered at the origin. These calculations give numerical evidence that the first TEVs can be used to gain qualitative information about the size of the void(s). In those calculations we used the 'exact' TEV computed by the FEM. For this to be useful for industrial applications one of course one need to compute the TEV based on the scattering data.

For voids of large size the shifting of the first eigenvalue is significant as shown in figure 5, which can be a qualitative information about presence of voids. Also if the size of the void become larger the first TEV goes to infinity for $A_{\max} < 1$ and $n_{\min} > 1$ or to zero for $A_{\min} > 1$ and $n_{\max} < 1$. The numerical experiments presented here are preliminary. It is desirable for instance to find a way to use more TEVs in order to obtain addition information about voids (see e.g. [13] for a formula that connects perturbation of eigenvalues to the location and physical information of small non-voids inhomogeneities).

Acknowledgments

The research of IH is supported in part by the NSF grant DMS-1106972 and the University of Delaware Graduate Fellowship. The research of FC is supported in part by the Air Force Office of Scientific Research grant FA9550-13-1-0199 and NSF grant DMS-1106972. The research of JS is supported in part by the NSF under grant DMS-1016092 and the US Army Research Laboratory and the US Army Research Office under cooperative agreement number W911NF-11-2-0046. The authors would also like to thank Peter Monk for providing the code to solve the direct scattering problem and fair field equation.

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