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ON THE HOMOGENIZATION OF A SCALAR SCATTERING PROBLEM FOR HIGHLY OSCILLATING ANISOTROPIC MEDIA*

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Abstract. We study the homogenization of a transmission problem arising in the scattering theory for bounded inhomogeneities with periodic coefficients modeled by the anisotropic Helmholtz equation. The coefficients are assumed to be periodic functions of the fast variable, specified over the unit cell with characteristic size ϵ. By way of multiple scales expansion, we focus on the $O(\epsilon^k)$, $k = 1, 2$, bulk and boundary corrections of the leading-order ($O(1)$) homogenized transmission problem. The analysis in particular provides the H^1 and L^2 estimates of the error committed by the first-order-corrected solution considering (i) bulk correction only and (ii) boundary and bulk correction. We treat explicitly the $O(\epsilon)$ boundary correction for the transmission problem when the scatterer is a unit square and show it has an L^2-limit as $\epsilon \to 0$, provided that the boundary cutoff of cells is fixed. We also establish the $O(\epsilon^2)$ bulk correction describing the mean wave motion inside the scatterer. The analysis also highlights a previously established, yet scarcely recognized, fact that the $O(\epsilon)$ bulk correction of the mean motion vanishes identically.

Key words. periodic inhomogeneities, scattering, two-scale homogenization, higher-order expansion

AMS subject classifications. 35R30, 35Q60, 35J40, 78A25

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1. Introduction.

Thanks to a broad range of affiliated wave phenomena such as dispersion, anisotropy, and band gaps [7, 16], periodic structures have found use in numerous facets of science and technology, including photonics [28], solid-state physics [22], sound filtering [24], subwavelength imaging [31], design of acoustic lenses [34], cloaking [4, 5], and slow light [6]. In situations when $k\epsilon \ll 1$, where k is the wave number and ϵ is the characteristic size of a periodic cell, the problem is amenable to a long-wavelength approximation [1, 10] that, on including higher-order terms, may capture the incipient wave dispersion and anisotropy due to microstructure. While significant progress has been made on this front considering unbounded periodic media [8, 29, 33], however, little is known about the long-wavelength scattering by periodic structures of finite extent.

To help bridge the gap, we consider the homogenization of a scattering problem for periodic anisotropic anomalies of compact support embedded in a homogeneous background. More precisely, the goal is to obtain a sequence of boundary value problems for the unit cell (resp., scatterer) governing the bulk (resp., boundary) corrections of the multiple-scale asymptotic solution, expanded in powers of ϵ — the characteristic size of the unit cell. While both bulk [2, 8, 17, 20, 21, 25, 29, 33] and

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boundary [1, 10, 19, 26, 27] corrections of the homogenized solution have been considered before, our study focuses on (i) formal boundary correctors of the *transmission* (as opposed to the Neumann or Dirichlet) problem — including rigorous convergence estimates, (ii) *higher-order* boundary corrections, and (iii) higher-order bulk corrections for the transmission problem for the anisotropic Helmholtz equation, none of which have been analyzed rigorously before. See also [32] for related work.

Our work is organized as follows. In section 2 we introduce the transmission problem and neccesary notation. Section 3 recalls the two-scale asymptotic expansion [10] used to tackle the problem and establishes the leading-order bulk and boundary corrections of the homogenized solution, including formal convergence estimates. In section 4 we focus on the higher-order boundary and bulk corrections by providing an explicit treatment of the second-order terms, and by establishing a clear inductive pathway by which third- and higher-order corrections can also be obtained. In doing so, we highlight the previously obtained — yet scarcely recognized — result that the mean of the first-order bulk correction *vanishes identically* [26, 30]. Given the fact that the limit as $\epsilon \to 0$ of the boundary corrector is generally difficult to obtain and in fact may not even exist, in section 5 we provide an explicit example of the boundary corrector limit for the transmission problem assuming the inhomogeneity to be a unit square. For completeness, our analysis concludes by formally deriving a fourth-order, constant-coefficient PDE governing the *mean* of the second-order bulk corrector, which extends the previous analyses [8, 29, 33] to anisotropic periodic media.

2. Preliminaries.

Let $D \subset \mathbb{R}^d$ be a bounded simply connected open set with piecewise-smooth boundary ∂D representing the support of a periodic inhomogeneity. When ∂D is not smooth, we will in addition assume that D is convex [14, 15]. Next, let $\epsilon > 0$ be the characteristic size of a periodic unit cell — assumed to be small both relative to the size of D and the wavelength of the incident field, and let $Y = [0,\, 1]^d$ be the rescaled unit cell. We assume that the physical properties of an obstacle are given by a positive-definite, symmetric, tensor-valued function $a_\epsilon := a(x/\epsilon) \in C^\infty\left(D, \mathbb{R}^{d\times d}\right)$ and a positive scalar function $n_\epsilon := n(x/\epsilon) \in C^\infty(D)$, related (in the context of acoustics) to the mass density and refractive index, respectively. By premise, both coefficients are periodic in $y = x/\epsilon$ with period Y. Note that the featured regularity restrictions on a_ϵ and n_ϵ are imposed primarily for the sake of simplicity, and can be drastically relaxed. In what follows, $x \in D$ is referred to as the slow variable, while $y = x/\epsilon \in Y$ denotes the so-called fast variable [10]. We remark that our convergence analysis applies equally to absorbing media, i.e., to complex valued a_ϵ and n_ϵ; for simplicity, however, we focus our presentation on the case of real-valued coefficients. We further assume that $\inf_{y\in Y} \inf_{|\xi|=1} \xi \cdot a(y)\xi > 0$ and $\inf_{y\in Y} n(y) > 0$. In this setting, the scattering of a time-harmonic incident field u^i by a periodic inhomogeneity D (see Figure 1) can be mathematically formulated for the total field, $u = u^s + u^i$, as

$$
\begin{aligned}
\nabla\cdot\big(a(x/\epsilon)\nabla u\big) + k^2 n(x/\epsilon)u &= 0 \quad \text{in} \quad D,\\
\Delta u^s + k^2 u^s &= 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D},\\
(u^s + u^i) &= u \quad \text{on} \quad \partial D,\\
\nabla(u^s + u^i)\cdot\nu &= a(x/\epsilon)\nabla u\cdot\nu \quad \text{on} \quad \partial D,
\end{aligned}
\tag{1}
$$

where u^s denotes the scattered field; the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \qquad r = |x|, \tag{2}$$

is satisfied uniformly with respect to $\hat{x} := x/r$, and ν is the unit outward normal on ∂D. In what follows we provide an asymptotic expansion for the above problem, including rigorous convergence estimates. The expansion will require boundary corrector functions, which are difficult to analyze. We also characterize the limit of the boundary corrector functions for the case that the scatterer is a square. In this case the limit depends on how the sequence ϵ approaches zero, and the limiting boundary values are described in terms of a boundary cell function on a doubly infinite strip.

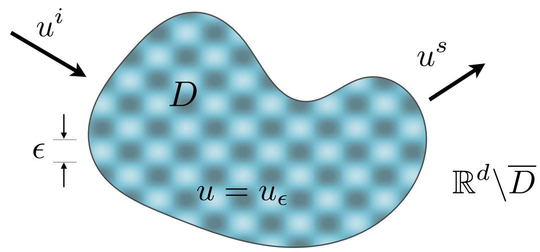

FIG. 1. *Scattering by an obstacle with periodic coefficients.*

3. Leading-order asymptotic expansion of the transmission problem. The above scattering problem for an inhomogeneous obstacle D with periodically varying coefficients can be conveniently formulated as a transmission problem for $u_\epsilon := u$ in D and $u_\epsilon := u^s$ in $\mathbb{R}^d \setminus \overline{D}$, namely,

$$\begin{aligned} \nabla\cdot\big(a(x/\epsilon)\nabla u_\epsilon\big) + k^2 n(x/\epsilon)u_\epsilon &= 0 \quad \text{in } D,\\ \Delta u_\epsilon + k^2 u_\epsilon &= 0 \quad \text{in } \mathbb{R}^d\setminus\overline{D},\\ u_\epsilon^+ - u_\epsilon^- &= f \quad \text{on } \partial D,\\ (\nabla u_\epsilon\cdot\nu)^+ - (a(x/\epsilon)\nabla u_\epsilon\cdot\nu)^- &= g \quad \text{on } \partial D, \end{aligned} \tag{3}$$

where u_ϵ satisfies the Sommerfeld radiation condition (2) at infinity. Here $f := u^i$ and $g := \nu\cdot\nabla u^i$ on ∂D, and the superscripts "+" and "−" denote the respective limits on ∂D from the exterior and interior of D. We are interested in developing the asymptotic theory for this problem as $\epsilon \to 0$, as was done for Dirichlet and Neumann problems on bounded domains [1, 10, 19, 26, 27]. One expects the homogenized or limiting problem to read

$$\begin{aligned} \nabla\cdot(A\nabla u_0) + k^2\overline{n}u_0 &= 0 \quad \text{in } D,\\ \Delta u_0 + k^2 u_0 &= 0 \quad \text{in } \mathbb{R}^d\setminus\overline{D},\\ u_0^+ - u_0^- &= f \quad \text{on } \partial D,\\ (\nabla u_0\cdot\nu)^+ - (A\nabla u_0\cdot\nu)^- &= g \quad \text{on } \partial D, \end{aligned} \tag{4}$$

where u_0 satisfies the Sommerfeld radiation condition (2) at infinity, $\overline{n}$ denotes the unit cell average of n, i.e.,

$$\overline{n} = \int_Y n(y)dy,$$

and A is a constant-valued matrix given by the weighted averages

$$A_{ij} = \int_Y \left(a_{ij}(y) - a_{ik}(y) \frac{\partial \chi^j}{\partial y_k}(y) \right) dy, \tag{5}$$

which make use of Einstein's summation convention. Here the $\chi^j(y)$ are the so-called cell functions which represent the Y-periodic solutions to

$$\frac{\partial}{\partial y_i} \left(a_{ij}(y) - a_{ik}(y) \frac{\partial \chi^j}{\partial y_k}(y) \right) = 0. \tag{6}$$

The additive constant for χ^j is always chosen so that

$$\int_Y \chi^j \, \mathrm{d}y = 0,$$

whereby the solutions to (6) are unique in $H^1_{\#}(Y)$, the space of H^1 functions on the d-dimensional torus, and are themselves C^∞ due to the smoothness of a. To derive this asymptotic limit and prove convergence, we will use the standard technique which regards the solution as that depending on a "slow" variable x, and a "fast" variable $y = x/\epsilon$ [10]. As was done in [26, 27], we write the equation for u_ϵ inside of D as a first-order system

$$\begin{aligned} a(x/\epsilon)\nabla u_\epsilon - v_\epsilon &= 0, \\ \nabla \cdot v_\epsilon + k^2 n(x/\epsilon) u_\epsilon &= 0, \end{aligned} \tag{7}$$

which allows us to obtain (lower regularity) L^2-based estimates of the error. In this setting, an ansatz for the bulk expansions inside of D can be written as

$$\begin{aligned} u_\epsilon &= u_0(x, x/\epsilon) + \epsilon u^{(1)}(x, x/\epsilon) + \epsilon^2 u^{(2)}(x, x/\epsilon) + \cdots, \\ v_\epsilon &= v_0(x, x/\epsilon) + \epsilon v^{(1)}(x, x/\epsilon) + \epsilon^2 v^{(2)}(x, x/\epsilon) + \cdots. \end{aligned} \tag{8}$$

For the bulk expansion in $\mathbb{R}^d \backslash \overline{D}$, on the other hand, it suffices to use

$$u_\epsilon = u_0(x), \qquad v_\epsilon = v_0(x),$$

because there is no microstructure in the exterior. Indeed, one can begin with an expansion as in (8) to show that all terms in the series except the first are zero. We note however that the boundary corrector functions are a completely different story and, as is frequently the case with homogenization problems involving compact support, the boundary corrector functions must be accounted for if one wants to obtain higher-order convergence estimates. These corrector functions solve problems which are substantially more difficult than our original; nonetheless, they are necessary for a full understanding of the behavior of the solution, even in the interior [10, 26]. In fact, the formulation of the boundary corrector for transmission problem (3) is one of the main contributions of this paper, and we pursue this issue in section 5. For now, following the usual procedure, we use the chain rule to write

$$\nabla = \nabla_x + \frac{1}{\epsilon} \nabla_y,$$

substitute (8) into (7), and equate the like powers of ϵ to obtain

$$a(y)\nabla_y u_0 = 0, \tag{9}$$

$$\nabla_y \cdot v_0 = 0, \tag{10}$$

$$a(y)\nabla_y u^{(1)} + a(y)\nabla_x u_0 - v_0 = 0, \tag{11}$$

and

$$\nabla_y \cdot v^{(1)} + \nabla_x \cdot v_0 + k^2 n(y) u_0 = 0. \tag{12}$$

From (9), one concludes that u_0 is independent of y. Equations (10) and (11) then yield the formula (6) for χ^j and the bulk correction

$$u^{(1)} = -\chi^j(y)\frac{\partial u_0}{\partial x_j}, \tag{13}$$

while (11) and taking the Y-average of (12) yield the homogenized PDE in the interior of D (4). We also find the ith component of v_0 to read

$$\begin{aligned}(v_0(x,y))_i &= \big(a(y)\nabla_x u_0(x) + a(y)\nabla_y u^{(1)}(x,y)\big)_i \\ &= \left(a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k}(y)\right)\frac{\partial u_0}{\partial x_j},\end{aligned} \tag{14}$$

whereby the average of v_0 reduces to

$$\overline{v}_0 = \int_Y v_0 \, \mathrm{d}y = A\nabla u_0.$$

From this result and (10), we can derive a candidate for $v^{(1)}$. For the purposes of the first-order analysis, we have some freedom in how we select $v^{(1)}$. We will denote our first choice by $\hat{v}^{(1)}$, to make a distinction from its companion used later for higher-order expansions. To specify $\hat{v}^{(1)}$, let $q(x,y) \in H^1_\#(Y)$ solve

$$\mathrm{rot}_y(q) = v_0 - A\nabla u_0, \tag{15}$$

which exists since the right-hand side has Y-average zero and zero y-divergence. Note that in dimension $d = 2$, q is a scalar and $\mathrm{rot}(\cdot)$ denotes a $\pi/2$ clockwise rotation of the gradient while for $d = 3$, q is a vector and $\mathrm{rot}(\cdot)$ represents the curl operator. Then, we define

$$\hat{v}^{(1)} = \mathrm{rot}_x\,(q) + k^2 a(y)\nabla_y \beta(y) u_0, \tag{16}$$

where β is the unique zero-mean Y-periodic solution to

$$\nabla_y \cdot (a\nabla_y \beta(y)) \;=\; \overline{n} - n(y), \tag{17}$$

which ensures that $\hat{v}^{(1)}$ satisfies (12). To summarize, we have formally derived that

$$\begin{aligned}u_\epsilon &\approx u_0(x) + \epsilon u^{(1)}(x, x/\epsilon), \\ v_\epsilon &\approx v_0(x, x/\epsilon) + \epsilon \hat{v}^{(1)}(x, x/\epsilon),\end{aligned}$$

where u_0, $u^{(1)}$, v_0, and $\hat{v}^{(1)}$ in D are given, respectively, by (4), (13), (14), and (16), while $u^{(1)}$ and $\hat{v}^{(1)}$ are zero in the exterior of D. We also naturally choose $v_0 = \nabla u_0$ in the exterior of D. We note again that our choice of $\hat{v}^{(1)}$ will not be the one to yield the higher-order estimates; however, this particular form will be convenient for proving Theorem 2.1.

When we consider the proposed approximation for u_ϵ, we see the that the correct Dirichlet-type transmission conditions are satisfied by u_0, but that these are now disturbed by the bulk correction $u^{(1)}$. Furthermore, the Neumann-type transmission conditions, which can be viewed as conditions on v_ϵ, are not quite exact due to the variable coefficient $a(x/\epsilon)$, and these are further disturbed by the presence of $\hat{v}^{(1)}$. This motivates the following definition of our boundary corrector function $\hat{\theta}_\epsilon$:

$$\begin{aligned} \nabla \cdot \big(a(x/\epsilon)\nabla\hat{\theta}_\epsilon\big) + k^2 n(x/\epsilon)\hat{\theta}_\epsilon &= 0 \quad \text{in} \;\; D,\\ \Delta\hat{\theta}_\epsilon + k^2\hat{\theta}_\epsilon &= 0 \quad \text{in} \;\; \mathbb{R}^d \setminus \overline{D},\\ \hat{\theta}_\epsilon^+ - \hat{\theta}_\epsilon^- &= u^{(1)} \quad \text{on} \;\; \partial D,\\ (\nabla\hat{\theta}_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla\hat{\theta}_\epsilon \cdot \nu)^- &= \Big(\frac{v_0 - \overline{v}_0}{\epsilon} + \hat{v}^{(1)}\Big)\cdot\nu \quad \text{on} \;\; \partial D, \end{aligned} \tag{18}$$

complemented by the Sommerfeld radiation condition (2) at infinity. By way of (13) and (16), the featured transmission conditions can be rewritten as

$$\begin{aligned} \hat{\theta}_\epsilon^+ - \hat{\theta}_\epsilon^- &= -\chi^j(x/\epsilon)\frac{\partial u_0}{\partial x_j} \quad \text{on} \;\; \partial D,\\ (\nabla\hat{\theta}_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla\hat{\theta}_\epsilon \cdot \nu)^- &= (\mathrm{rot}(q) + k^2 a\nabla_y\beta(y)u_0)\cdot\nu \quad \text{on} \;\; \partial D. \end{aligned} \tag{19}$$

Here, the rot derivative on q denotes the full derivative; for instance in three dimensions ($d = 3$), one has

$$\mathrm{rot} = \nabla\times = \frac{1}{\epsilon}\nabla_y \times + \nabla_x\times,$$

which is $O(\epsilon^{-1})$ since $q = q(x, y)$.

Lemma 1. *Let u_ϵ be the solution to* (3), *u_0 the solution to* (4), *and let the bulk and boundary corrections $u^{(1)}$ and $\hat{\theta}_\epsilon$ be given, respectively, by* (13) *and* (18), *recalling that $u^{(1)} = 0$ outside D. Then for any ball B_{R} of radius $R > 0$ which contains D,*

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon\hat{\theta}_\epsilon)\|_{H^1(B_{\mathrm{R}})} \le C_{\mathrm{R}}\,\epsilon\|u_0\|_{H^2(D)},$$

where C_{R} is a constant independent of ϵ and u_0.

Proof. Introducing the auxiliary error functions in D as

$$z_\epsilon = u_\epsilon - u_0 - \epsilon u^{(1)} \tag{20}$$

and

$$\eta_\epsilon = a(x/\epsilon)\nabla u_\epsilon - v_0 - \epsilon\hat{v}^{(1)}, \tag{21}$$

we find that

$$a(x/\epsilon)\nabla z_\epsilon - \eta_\epsilon = \epsilon(\hat{v}^{(1)} - a(y)\nabla_x u^{(1)}) \tag{22}$$

and

$$\begin{aligned} -\nabla\cdot\eta_\epsilon &= k^2 n(y)(u_\epsilon - u_0) + \epsilon k^2\nabla_x\cdot(a\nabla_y\beta\, u_0) && (23)\\ &= k^2 n(y)z_\epsilon + \epsilon k^2\big(n(y)u^{(1)} + a\nabla_y\beta\cdot\nabla u_0\big). && (24) \end{aligned}$$

This shows that the error pair $(z_\epsilon, \eta_\epsilon)$ satisfies the first-order version of the PDEs with $O(\epsilon)$ residual in the bulk. Outside of D we simply define $z_\epsilon = u_\epsilon - u_0$ and $\eta_\epsilon = \nabla z_\epsilon$, whereby

$$-\nabla \cdot \eta_\epsilon = k^2 z_\epsilon.$$

Now consider, for any $\phi \in C_0^\infty(B_R)$, the integral

$$\int_{B_R} (z_\epsilon - \epsilon\hat{\theta}_\epsilon)\phi\, dx = \int_D \left(u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon\hat{\theta}_\epsilon)\right)\phi\, dx + \int_{B_R\setminus D} \left(u_\epsilon - (u_0 + \epsilon\hat{\theta}_\epsilon)\right)\phi\, dx, \tag{25}$$

and define the auxiliary function $W_\epsilon \in H^1_{loc}(\mathbb{R}^d)$ to solve

$$\begin{aligned} \nabla \cdot a(x/\epsilon)\nabla W_\epsilon + k^2 n(x/\epsilon) W_\epsilon &= \phi \quad \text{in} \quad D,\\ \Delta W_\epsilon + k^2 W_\epsilon &= \phi \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D},\\ W_\epsilon^+ - W_\epsilon^- &= 0 \quad \text{on} \quad \partial D,\\ (\nabla W_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla W_\epsilon \cdot \nu)^- &= 0 \quad \text{on} \quad \partial D, \end{aligned} \tag{26}$$

together with the Sommerfeld radiation condition (2) at infinity. Note that this means that W_ϵ also satisfies the elliptic PDEs *across* ∂D with jumps in the coefficients. Then we have

$$\begin{aligned} \int_{B_R} (z_\epsilon - \epsilon\hat{\theta}_\epsilon)\phi\, dx &= \int_D (z_\epsilon - \epsilon\hat{\theta}_\epsilon)\left(\nabla \cdot a(x/\epsilon)\nabla W_\epsilon + k^2 n(x/\epsilon) W_\epsilon\right)\, dx\\ &\quad + \int_{B_R\setminus D} (z_\epsilon - \epsilon\hat{\theta}_\epsilon)\left(\Delta W_\epsilon + k^2 W_\epsilon\right)\, dx\\ &= -\int_D a(x/\epsilon)\nabla z_\epsilon \nabla W_\epsilon\, dx + \epsilon\int_D a(x/\epsilon)\nabla\hat{\theta}_\epsilon\, \nabla W_\epsilon\, dx\\ &\quad + \int_D (z_\epsilon - \epsilon\hat{\theta}_\epsilon) k^2 n(x/\epsilon) W_\epsilon\, dx + \int_{\partial D} \nabla(z_\epsilon - \epsilon\hat{\theta}_\epsilon)^+ \cdot \nu W_\epsilon\, ds_x\\ &\quad + \int_{\partial B_R} (z_\epsilon - \epsilon\hat{\theta}_\epsilon)\frac{\partial W_\epsilon}{\partial \nu}\, ds_x - \int_{\partial B_R} \frac{\partial(z_\epsilon - \epsilon\hat{\theta}_\epsilon)}{\partial\nu} W_\epsilon\, ds_x, \end{aligned}$$

where we have integrated by parts once on the inside and twice on the exterior, using the fact that $(z_\epsilon - \epsilon\hat{\theta}_\epsilon)$ exhibits no jump across ∂D. We also note that, by a standard argument, one can show that the last two terms on the outer boundary ∂B_R actually sum to zero since both $z_\epsilon - \epsilon\hat{\theta}_\epsilon$ and W_ϵ satisfy Sommerfeld radiation condition (2) at infinity. Indeed, all of the functions in the integrand satisfy the same Helmholtz equation in the exterior, so the last two terms can be integrated over a larger surface at some further radius R_1, on which the Sommerfeld condition can be used to show that the value of these integrals goes to zero as $R_1 \to \infty$. Since no other terms depend on R_1, the original integrals must be zero [11]. Hence we have

$$\begin{aligned} \int_{B_R} (z_\epsilon - \epsilon\hat{\theta}_\epsilon)\phi\, dx &= -\int_D a(x/\epsilon)\nabla z_\epsilon \nabla W_\epsilon dx + \epsilon\int_D a(x/\epsilon)\nabla\hat{\theta}_\epsilon \nabla W_\epsilon\, dx\\ &\quad + \int_D (z_\epsilon - \epsilon\hat{\theta}_\epsilon) k^2 n(x/\epsilon) W_\epsilon\, dx + \int_{\partial D} \nabla(z_\epsilon - \epsilon\hat{\theta}_\epsilon)^+ \cdot \nu\, W_\epsilon\, ds_x. \end{aligned}$$

Now we use the differential equation for $\hat{\theta}_\epsilon$ in the interior to obtain

$$\int_{B_R}(z_\epsilon - \epsilon\hat{\theta}_\epsilon)\phi\,dx = -\int_D a(x/\epsilon)\nabla z_\epsilon \nabla W_\epsilon\,dx + \epsilon\int_{\partial D}(a(x/\epsilon)\nabla\hat{\theta}_\epsilon)^- \cdot \nu\, W_\epsilon\,dx \\ + \int_D z_\epsilon k^2 n(x/\epsilon) W_\epsilon\,dx + \int_{\partial D}\nabla(z_\epsilon - \epsilon\hat{\theta}_\epsilon)^+ \cdot \nu\, W_\epsilon\,ds_x,$$

and apply (22) together with the normal jump for $\hat{\theta}_\epsilon$ to obtain

$$\begin{aligned}
&\int_{B_R}(z_\epsilon - \epsilon\hat{\theta}_\epsilon)\phi\,dx \\
&\quad = -\int_D \eta_\epsilon \nabla W_\epsilon\,dx + \int_D z_\epsilon k^2 n(x/\epsilon) W_\epsilon\,dx + \int_{\partial D}(\nabla z_\epsilon)^+ \cdot \nu\, W_\epsilon\,ds_x \\
&\qquad + \epsilon\int_D \Big(-\hat{v}^{(1)} + a\nabla_x u^{(1)}\Big)\nabla W_\epsilon\,dx + \int_{\partial D}\Big(\overline{v}_0 - v_0 - \epsilon\hat{v}^{(1)}\Big)\cdot \nu\, W_\epsilon\,ds_x \\
&\quad = -\epsilon\int_D k^2\big(nu^{(1)} + a\nabla_y\beta\cdot\nabla u_0\big)W_\epsilon\,dx + \epsilon\int_D\Big(-\hat{v}^{(1)} + a\nabla_x u^{(1)}\Big)\nabla W_\epsilon\,dx \\
&\qquad + \int_{\partial D}\Big((\overline{v}_0 - v_0 - \epsilon\hat{v}^{(1)})^- \cdot \nu - (\eta_\epsilon)^- \cdot \nu + (\nabla z_\epsilon)^+ \cdot \nu\Big) W_\epsilon\,ds_x,
\end{aligned}$$

where in the last step we integrated by parts and used (24). Now, using (21) and the normal jump conditions in (3) and (4), we see that the last boundary term above cancels (note that the boundary correction $\hat{\theta}_\epsilon$ was precisely chosen so that this would happen). Hence

$$\int_{B_R}(z_\epsilon - \epsilon\hat{\theta}_\epsilon)\phi\,dx = -\epsilon\int_D k^2\big(nu^{(1)} + a\nabla_y\beta\cdot\nabla u_0\big)W_\epsilon\,dx \\ +\epsilon\int_D(-\hat{v}^{(1)} + a\nabla u^{(1)})\nabla W_\epsilon\,dx \tag{27}$$

for any $\phi \in C_0^\infty(B_R)$. Since q is defined (15) by solving a PDE in the y-variable only, we can clearly choose q such that

$$\sup_{y\in Y}|\hat{v}^{(1)}| \le C\bigg(\sum_{i,j}2|\frac{\partial u_0}{\partial x_i \partial x_j}| + |u_0|\bigg) \tag{28}$$

for some C independent of ϵ. Furthermore, from definition (13), one obtains the bounds

$$\|nu^{(1)} + a\nabla_y\beta\cdot\nabla u_0\|_{L^2(D)} \le C\|u_0\|_{H^2(D)}, \qquad \|a\nabla_x u^{(1)}\|_{L^2(D)} \le C\|u_0\|_{H^2(D)}.$$

Using the Cauchy–Schwartz inequality in (27) demonstrates that there exists a constant C such that

$$\left|\int_{B_R}(z_\epsilon - \epsilon\hat{\theta}_\epsilon)\phi\,dx\right| \le C\epsilon\|u_0\|_{H^2(D)}\|W_\epsilon\|_{H^1(D)}. \tag{29}$$

We certainly have from standard elliptic estimates that

$$\|W_\epsilon\|_{H^1(D)} \le C_R\|\phi\|_{H^{-1}(B_R)},$$

where C_R depends only on the bounds on the coefficients, from which we obtain the desired result by inserting this into (29) and taking the supremum over all $\phi \in H^{-1}(B_R)$. ∎

The H^1 a priori estimate for the solution of the transmission problem (18) (see, e.g., Theorem 5.24 of [11]), implies that

$$\|\hat{\theta}_\epsilon\|_{H^1(D)} + \|\hat{\theta}_\epsilon\|_{H^1(B_R\setminus\overline{D})} \leq C_R\left(\|u^{(1)}\|_{H^{1/2}(\partial D)} + \left\|\left(\frac{\overline{v}_0 - v_0}{\epsilon} - \hat{v}^{(1)}\right)\cdot\nu\right\|_{H^{-1/2}(\partial D)}\right). \tag{30}$$

Since $u^{(1)}$ contains the oscillating terms $\chi^j(x/\epsilon)$, its $H^{1/2}(\partial D)$ norm is not bounded with respect to ϵ. By a standard argument, we can see that its $H^1(\partial D)$ norm is $O(\epsilon^{-1})$ while its $L^2(\partial D)$ norm is bounded; interpolation between the two yields

$$\|u^{(1)}\|_{H^{1/2}(\partial D)} \leq C\epsilon^{-1/2}\|u_0\|_{H^2(D)}. \tag{31}$$

Also, thanks to our choice of $\hat{v}^{(1)}$, the boundary condition in (18) contains a rot operator (see (19)); so we can eliminate the ϵ^{-1} factor by integrating by parts. In particular for any test function $\phi \in H^1(\partial D)$, one has

$$\int_{\partial D} \mathrm{rot}(q)\cdot\nu\,\phi\,ds_x = -\int_{\partial D} q\ \mathrm{rot}(\phi)\cdot\nu\,ds_x$$

in two dimensions, and

$$\int_{\partial D} \mathrm{rot}(q)\cdot\nu\,\phi\,ds_x = -\int_{\partial D} (q\times\nabla\phi)\cdot\nu\,ds_x$$

in three dimensions. In either case one obtains from (16) and (28) that for fixed y, q is at least in $H^1(D)$ and bounded by the $H^2(D)$-norm of u_0, which implies that its $L^2(\partial D)$-norm is bounded (independently of ϵ) by $\|u_0\|_{H^2(D)}$. On the other hand, it follows from (15), the fact that $q \in H^1_\#(Y)$, and the trace theorem that the $L^2(\partial D)$-norm of q (as a function of y) is likewise bounded by $\|u_0\|_{H^2(D)}$. By invoking the standard duality argument, one obtains

$$\left\|\left(\frac{\overline{v}_0 - v_0}{\epsilon} - \hat{v}^{(1)}\right)\cdot\nu\right\|_{H^{-1}(\partial D)} \leq C\|u_0\|_{H^2(D)}. \tag{32}$$

We also have

$$\left\|\left(\frac{\overline{v}_0 - v_0}{\epsilon} - \hat{v}^{(1)}\right)\cdot\nu\right\|_{L^2(\partial D)} \leq C\epsilon^{-1}\|u_0\|_{H^2(D)}. \tag{33}$$

Note that the bound for the leading term in (33), namely, $\epsilon^{-1}\mathrm{rot}_y q$, follows from the definition (15), which in particular implies that the $L^2(\partial D)$-norm of $\mathrm{rot}_y q$ is bounded by $\|u_0\|_{H^2(D)}$ independently of ϵ. Considering the second term $\mathrm{rot}_x q$ (whose x-divergence is zero for fixed y), its normal trace on the boundary is defined and bounded in $L^2(\partial D)$ by the $L^2(D)$-norm of $\mathrm{rot}_x q$, which is in turn bounded by $\|u_0\|_{H^2(D)}$ thanks to (28). By interpolating between the two estimates (32) and (33) and inserting into (30), we find

$$\|\hat{\theta}_\epsilon\|_{H^1(D)} + \|\hat{\theta}_\epsilon\|_{H^1(B_R\setminus D)} \leq C_R\,\epsilon^{-1/2}\|u_0\|_{H^2(D)} \tag{34}$$

for some constant C_R independent of ϵ. In the appendix we show that the above transmission boundary layer is locally bounded in L^2 independently of ϵ; more precisely, we show that

$$\|\hat{\theta}_\epsilon\|_{L^2(B_R)} \leq C_R\|u_0\|_{H^2(D)}. \tag{35}$$

From the above bounds we can obtain our first result which truly validates the asymptotic expansion.

THEOREM 2. *Let u_ϵ be the solution to* (3), u_0 *the solution to* (4), *and let the bulk correction $u^{(1)}$ be given by* (13) *in the interior of D and zero on the exterior of D. Then for any ball $B_{\rm R}$ of radius $R > 0$ which contains D,*

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)})\|_{H^1(D)} + \|u_\epsilon - u_0\|_{H^1(B_R\setminus D)} \leq C_{\rm R}\,\epsilon^{1/2}$$

and

$$\|u_\epsilon - u_0\|_{L^2(B_{\rm R})} \leq C_{\rm R}\,\epsilon,$$

where $C_{\rm R}$ is a constant independent of ϵ.

Remark 3. While the bulk correction is necessary to obtain the H^1 convergence, it does not in general improve upon the L^2 estimate. That is, unless the boundary correction approaches zero (which is generally not the case), or is somehow otherwise accounted for,

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)})\|_{L^2(B_{\rm R})} \leq C_{\rm R}\,\epsilon$$

is the best that one can obtain.

4. Higher-order terms. In this section we pursue the asymptotic expansion further. We find the next terms in the bulk expansion, and in the process we present a proof that the first-order mean field correction discussed in [3, 10] vanishes in general. This was previously shown for problems with no lower-order terms [30, 26]; however, we reproduce the calculation both for emphasis and consider the more general case of nonzero refraction index n . We also present the equation for the important first-order transmission boundary correction. This boundary layer, or some approximation to it, is necessary to obtain an approximation that is correct up to $O(\epsilon^2)$.

To clearly demonstrate that we account for all necessary terms in the expansion, we will from the beginning assume a more general form of $u^{(1)}$, namely,

$$u^{(1)} = -\chi^j(y)\frac{\partial u_0}{\partial x_j} + \hat{u}(x), \tag{36}$$

in place of (13). Note that the introduction of $\hat{u}$ does not affect (9)–(12), nor does it affect the estimates in the previous section since it is a term of order ϵ in $H^1(D)$. However, we will indeed show that $\hat{u} = 0$ during the derivation of higher-order estimates. Continuing with our ansatz and equating the like powers of ϵ in (7), we further obtain

$$a\nabla_x u^{(1)} + a\nabla_y u^{(2)} = v^{(1)}, \tag{37}$$

$$-\nabla_x \cdot v^{(1)} - \nabla_y \cdot v^{(2)} = k^2 n(y) u^{(1)}. \tag{38}$$

If we apply a divergence operator $\nabla_y\cdot$ to (37), we obtain

$$\nabla_y \cdot \left(a\nabla_x u^{(1)} + a\nabla_y u^{(2)}\right) = -\nabla_x \cdot v_0 - k^2 n(y) u_0, \tag{39}$$

thanks to (12). Using the homogenized equation for u_0, (39) yields

$$\nabla_y \cdot \left(a\nabla_x u^{(1)} + a\nabla_y u^{(2)}\right) = -\nabla_x \cdot v_0 + \nabla \cdot A\nabla u_0 + k^2(\overline{n} - n(y))u_0, \tag{40}$$

which, along with (36) and (14), provides the equation for $u^{(2)}$, i.e.,

$$\begin{aligned}\nabla_y \cdot \left(a\nabla_y u^{(2)}\right) =& \Big(- a_{ij} + a_{ik}\frac{\partial \chi^j}{y_k} + \frac{\partial}{\partial y_k}(a_{ki}\chi^j) + A_{ij}\Big)\frac{\partial u_0}{\partial x_i \partial x_j} \\ &- \frac{\partial a_{ij}}{\partial y_i}\frac{\partial \hat{u}}{\partial x_j} + k^2(\overline{n} - n(y))u_0,\end{aligned} \tag{41}$$

since all other terms are known. To construct a solution to (41), we conveniently set

$$b_{ij}(y) = -a_{ij} + a_{ik}\frac{\partial \chi^j}{\partial y_k} + \frac{\partial}{\partial y_k}(a_{ki}\chi^j), \tag{42}$$

and note that $\bar{b}_{ij} = -A_{ij}$. We then introduce higher-order cell functions χ^{ij} to be the Y-periodic solutions to

$$\nabla_y \cdot \left(a\nabla_y \chi^{ij}\right) \;=\; b_{ij}(y) - \bar{b}_{ij}, \tag{43}$$

and recall the cell function (17) corresponding to the lower-order term $\overline{n} - n(y)$. With such definitions, one verifies that

$$u^{(2)}(x,y) = \chi^{ij}(y)\frac{\partial^2 u_0}{\partial x_i \partial x_j} - \chi^j(y)\frac{\partial \hat{u}}{\partial x_j} + k^2\beta(y)u_0 + \bar{u}^{(2)}(x) \tag{44}$$

indeed satisfies (39). Note that we have added the (not yet determined) mean field $\bar{u}^{(2)}(x)$ here for completeness; its value does not affect the estimates in this section. Now, instead of $\hat{v}^{(1)}$, we define the first-order correction $v^{(1)}$ of v_ϵ by way of (37), i.e.,

$$v^{(1)} = a\nabla_x u^{(1)} + a\nabla_y u^{(2)}. \tag{45}$$

Note that (12) is still satisfied, and that

$$\begin{aligned}\nabla_x \cdot v^{(1)} = &\left(-a_{ki}\chi^j + a_{kl}\frac{\partial \chi^{ij}}{\partial y_l}\right)\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}\\ &+ k^2 a_{ki}\frac{\partial \beta}{\partial y_i}\frac{\partial u_0}{\partial x_k} + (a_{ij} - a_{ik}\frac{\partial \chi^j}{\partial y_k})\frac{\partial^2 \hat{u}}{\partial x_i \partial x_j}.\end{aligned} \tag{46}$$

On averaging (38) with respect to y, the term with the y derivative of $v^{(2)}$ becomes zero and we obtain the equation for $\hat{u}$, namely,

$$\nabla\cdot A\nabla\hat{u} + k^2\overline{n}\hat{u} + \left(-\overline{a_{ki}\chi^j} + \overline{a_{kl}\frac{\partial \chi^{ij}}{\partial y_l}}\right)\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} + k^2(\overline{a_{ki}\frac{\partial \beta}{\partial y_i}} - \overline{n\chi^k})\frac{\partial u_0}{\partial x_k} = 0. \tag{47}$$

For the case of $k = 0$, the above equation for the mean field was derived, for example, in [10] and discussed again in [3]. It was shown in [26, 30], however, that the source term

$$\left(-\overline{a_{ki}\chi^j} + \overline{a_{kl}\frac{\partial \chi^{ij}}{\partial y_l}}\right)\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} = 0. \tag{48}$$

To demonstrate this, we integrate by parts

$$\begin{aligned}\int_Y a_{kl}\frac{\partial \chi^{ij}}{\partial y_l}dy = -\int_Y \frac{\partial a_{lk}}{\partial y_l}\chi^{ij}dy &= -\int_y (\nabla_y \cdot a\nabla_y \chi^k)\chi^{ij}dy\\ &= -\int_Y \chi^k \nabla_y \cdot a\nabla_y \chi^{ij}dy.\end{aligned} \tag{49}$$

Now we can use the equation for χ^{ij} to obtain

$$\begin{aligned}\int_Y a_{kl}\frac{\partial \chi^{ij}}{\partial y_l} &= \int_y a_{ij}\chi^k dy - \int_Y a_{il}\chi^k\frac{\partial \chi^j}{\partial y_l} - \int_Y \chi^k\frac{\partial}{\partial y_l}(a_{li}\chi^j)dy - \int_Y \chi^k A_{ij}dy\\ &= \int_y a_{ij}\chi^k dy - \int_Y a_{il}\chi^k\frac{\partial \chi^j}{\partial y_l} + \int_Y a_{li}\chi^j\frac{\partial \chi^k}{\partial y_l}dy,\end{aligned} \tag{50}$$

whereby

$$\left(- \overline{a_{ki}\chi^j} + \overline{a_{kl}\frac{\partial \chi^{ij}}{\partial y_l}} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} = \left(-\overline{a_{ik}\chi^j} + \overline{a_{ij}\chi^k} - \overline{a_{il}\chi^k \frac{\partial \chi^j}{\partial y_l}} + \overline{a_{il}\chi^j \frac{\partial \chi^k}{\partial y_l}} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}. \tag{51}$$

For each fixed i, the terms are skew symmetric in j and k, hence the sum is zero due to the symmetry in the third-order derivatives — which proves (48). Furthermore, one easily obtains via the equation for β and integration by parts that

$$\overline{a_{ik}\frac{\partial \beta}{\partial y_i}} = -\overline{\frac{\partial a_{ik}}{\partial y_i}\beta} = -\overline{\nabla_y \cdot (a\nabla_y \chi^k)\beta} = -\overline{\chi^k \nabla_y \cdot (a\nabla_y \beta)} = -\overline{\chi^k(\overline{n} - n)} = \overline{\chi^k n} \tag{52}$$

so that the entire source term for $\hat{u}$ in (47) cancels. Accordingly, it suffices to take $\hat{u} = 0$ since we account for the perturbation in the transmission condition with our boundary layer θ_ϵ. Note, however, that this is generally not the case with the higher-order mean fields — those carry the well-known dispersive properties of periodic media [29, 33] — a subject that we discuss further in section 6.

We now find $v^{(2)}$ which satisfies (38). To this end, consider the auxiliary functions χ^{ijk} and β^k as the Y-periodic and zero-mean solutions to

$$\nabla_y \cdot \left(a\nabla_y \chi^{ijk} \right) \ = \ c_{ijk}(y) - \bar{c}_{ijk} \tag{53}$$

and

$$\nabla_y \cdot \left(a\nabla_y \beta^k \right) = n\chi^k - a_{ki}\frac{\partial \beta}{\partial y_i}, \tag{54}$$

where

$$c_{ijk}(y) = a_{ki}\chi^j - a_{kl}\frac{\partial \chi^{ij}}{\partial y_l}. \tag{55}$$

Then, if we take

$$v^{(2)} = a\nabla_y \chi^{ijk} \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} + k^2 a\nabla_y \beta^k \frac{\partial u_0}{\partial x_k} \tag{56}$$

and recall (48), we indeed have that the pair $v^{(2)}$, $v^{(1)}$ satisfies (38). We are now ready to define the first- and second-order boundary layers, which we denote by θ_ϵ and ϕ_ϵ, respectively. Note that our first-order boundary corrector function is close to $\hat{\theta}_\epsilon$ from (18); in fact, it differs only in that it now uses a more appropriate choice of $v^{(1)}$. With reference to (18), the first-order boundary corrector function now satisfies

$$\begin{aligned} \nabla \cdot \left(a(x/\epsilon)\nabla\theta_\epsilon \right) + k^2 n(x/\epsilon)\theta_\epsilon &= 0 \quad \text{in} \ \ D, \\ \Delta\theta_\epsilon + k^2\theta_\epsilon &= 0 \quad \text{in} \ \ \mathbb{R}^d \setminus \overline{D}, \\ \theta_\epsilon^+ - \theta_\epsilon^- &= u^{(1)} \quad \text{on} \ \ \partial D, \\ (\nabla\theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla\theta_\epsilon \cdot \nu)^- &= \left(\frac{v_0 - \overline{v}_0}{\epsilon} + v^{(1)} \right) \cdot \nu \quad \text{on} \ \ \partial D, \end{aligned} \tag{57}$$

while its second-order counterpart solves

$$
\begin{aligned}
\nabla \cdot \big(a(x/\epsilon)\nabla\phi_\epsilon\big) + k^2 n(x/\epsilon)\phi_\epsilon &= 0 \quad \text{in} \quad D, \\
\Delta\phi_\epsilon + k^2\phi_\epsilon &= 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D}, \\
\phi_\epsilon^+ - \phi_\epsilon^- &= u^{(2)} \quad \text{on} \quad \partial D, \\
(\nabla\phi_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla\phi_\epsilon \cdot \nu)^- &= v^{(2)} \cdot \nu \quad \text{on} \quad \partial D,
\end{aligned} \tag{58}
$$

noting for clarity that

$$
a\nabla_x u^{(2)} + a\nabla_y u^{(3)} = v^{(2)} \tag{59}
$$

due to (7). For convenience, let us also summarize the other terms we use in our expansion:

$$
u^{(1)} = -\chi^j(y)\frac{\partial u_0}{\partial x_j}, \tag{60}
$$

$$
u^{(2)} = \chi^{ij}(y)\frac{\partial^2 u_0}{\partial x_i \partial x_j} + k^2\beta(y)u_0 + \bar{u}^{(2)}(x), \tag{61}
$$

$$
v_0 = a(y)\nabla_x u_0(x) - a(y)\nabla_y \chi^j \frac{\partial u_0}{\partial x_j}, \tag{62}
$$

$$
v^{(1)} = -a\chi^j\nabla_x\frac{\partial u_0}{\partial x_j} + a\nabla_y\chi^{ij}\frac{\partial u_0}{\partial x_i \partial x_j} + k^2 a\nabla_y\beta\, u_0. \tag{63}
$$

The following theorem gives us a true second-order estimate of a solution to the transmission problem (3), assuming we have H^4 (as opposed to merely H^2) regularity on the homogenized solution.

LEMMA 4. *Let u_ϵ be the solution to* (3)*, u_0 the solution to* (4)*, and let the bulk and boundary corrections $u^{(1)}$, $u^{(2)}$, θ_ϵ, and ϕ_ϵ be given, respectively, by* (60), (61), (57)*, and* (58)*. Then for any ball B_{R} of radius $R > 0$ which contains D,*

$$
\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon\theta_\epsilon + \epsilon^2 u^{(2)} + \epsilon^2\phi_\epsilon)\|_{H^1(B_{\mathrm{R}})} \le C_{\mathrm{R}}\,\epsilon^2\|u_0\|_{H^4(D)},
$$

where C_{R} is a constant independent of ϵ and u_0.

Proof. The proof is very similar to the proof of Lemma 1. We again define the error functions in D, but this time include the second-order bulk corrections

$$
z_\epsilon = u_\epsilon - u_0 - \epsilon u^{(1)} - \epsilon^2 u^{(2)}, \tag{64}
$$

$$
\eta_\epsilon = a(x/\epsilon)\nabla u_\epsilon - v_0 - \epsilon v^{(1)} - \epsilon^2 v^{(2)}. \tag{65}
$$

In this case, one finds that

$$
a(x/\epsilon)\nabla z_\epsilon - \eta_\epsilon = \epsilon^2(v^{(2)} - a(y)\nabla_x u^{(2)}) \tag{66}
$$

and

$$
\begin{aligned}
-\nabla \cdot \eta_\epsilon &= k^2 n(y)(u_\epsilon - u_0 - \epsilon u^{(1)}) + \epsilon^2\nabla_x \cdot v^{(2)} &(67)\\
&= k^2 n(y) z_\epsilon + \epsilon^2\big(k^2 n(y)u^{(2)} + \nabla_x \cdot v^{(2)}\big). &(68)
\end{aligned}
$$

This shows that the error pair $(z_\epsilon, \eta_\epsilon)$ now satisfies the first order version of the PDE with an $O(\epsilon^2)$ residual in the bulk. Outside of D we simply define $z_\epsilon = u_\epsilon - u_0$ and $\eta_\epsilon = \nabla z_\epsilon$, whereby

$$-\nabla \cdot \eta_\epsilon = k^2 z_\epsilon.$$

Again consider, for any $\phi \in C_0^\infty(B_R)$, the integral

$$\begin{aligned}\int_{B_R}(z_\epsilon - \epsilon\theta_\epsilon - \epsilon^2\phi_\epsilon)\phi\,dx &= \int_D \big(u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \epsilon\theta_\epsilon + \epsilon^2\phi_\epsilon)\big)\phi\,dx \\ &\quad + \int_{B_R\setminus D}\big(u_\epsilon - (u_0 + \epsilon\theta_\epsilon + \epsilon^2\phi_\epsilon)\big)\phi\,dx,\end{aligned} \tag{69}$$

and define the auxiliary function $W_\epsilon \in H^1_{loc}(\mathbb{R}^d)$ to solve (26) as before. Then, using the Sommerfeld radiation conditions to eliminate the outer boundary and the fact that $z_\epsilon - \epsilon\theta_\epsilon - \epsilon^2\phi_\epsilon$ has no jump on ∂D, we find that

$$\begin{aligned}\int_{B_R}(z_\epsilon - \epsilon\theta_\epsilon - \epsilon^2\phi_\epsilon)\phi\,dx &= -\int_D a(x/\epsilon)\nabla z_\epsilon \nabla W_\epsilon\,dx + \epsilon\int_D a(x/\epsilon)\nabla\theta_\epsilon\nabla W_\epsilon\,dx \\ &\quad + \epsilon^2\int_D a(x/\epsilon)\nabla\phi_\epsilon\nabla W_\epsilon\,dx + \int_D (z_\epsilon - \epsilon\theta_\epsilon - \epsilon^2\phi_\epsilon)k^2 n(x/\epsilon)W_\epsilon\,dx \\ &\quad + \int_{\partial D}\nabla(z_\epsilon - \epsilon\theta_\epsilon - \epsilon^2\phi_\epsilon)^+ \cdot \nu\, W_\epsilon\,ds_x.\end{aligned}$$

We now apply the differential equations for θ_ϵ and ϕ_ϵ, their jump conditions, (66) and (67), which yield

$$\begin{aligned}\int_{B_R}(z_\epsilon - \epsilon\theta_\epsilon - \epsilon^2\phi_\epsilon)\phi\,dx &= -\int_D \eta_\epsilon\nabla W_\epsilon\,dx + \int_D z_\epsilon k^2 n(x/\epsilon)W_\epsilon\,dx \\ &\quad + \int_{\partial D}(\nabla z_\epsilon)^+\cdot\nu W_\epsilon\,ds_x + \epsilon^2\int_D(-v^{(2)} + a\nabla_x u^{(2)})\nabla W_\epsilon\,dx \\ &\quad + \int_{\partial D}(\overline{v}_0 - v_0 - \epsilon v^{(1)} - \epsilon^2 v^{(2)})\cdot\nu W_\epsilon\,ds_x \\ &= -\epsilon^2\int_D(k^2 n(x/\epsilon)u^{(2)} + \nabla_x\cdot v^{(2)})W_\epsilon\,dx + \epsilon^2\int_D(-v^{(2)} + a\nabla_x u^{(2)})\nabla W_\epsilon\,dx \\ &\quad + \int_{\partial D}\Big((\overline{v}_0 - v_0 - \epsilon v^{(1)} - \epsilon^2 v^{(2)})^- \cdot\nu - (\eta_\epsilon)^-\cdot\nu + (\nabla z_\epsilon)^+\cdot\nu\Big)W_\epsilon\,ds_x \\ &= -\epsilon^2\int_D(k^2 n(x/\epsilon)u^{(2)} + \nabla_x\cdot v^{(2)})W_\epsilon\,dx + \epsilon^2\int_D(-v^{(2)} + a\nabla_x u^{(2)})\nabla W_\epsilon\,dx,\end{aligned}$$

where in the last step the boundary integral vanishes thanks to (3), (4), (64), and (65) canceling as intended. Note that here we are using more stringent regularity requirements on u_0 in order to secure boundedness of the above integrals. Indeed, the L^2 bounds on the featured integrands (in particular, that on $\nabla_x \cdot v^{(2)}$) will require fourth-order L^2 derivatives, and we obtain

$$\left|\int_{B_R}(z_\epsilon - \epsilon\theta_\epsilon - \epsilon^2\phi_\epsilon)\phi\,dx\right| \le C\epsilon^2\|u_0\|_{H^4(D)}\|W_\epsilon\|_{H^1(D)}. \tag{70}$$

Finally, the claim of the lemma follows from the bound

$$\|W_\epsilon\|_{H^1(D)} \le C_R\|\phi\|_{H^{-1}(D)}$$

by taking the supremum over all $\phi \in H^{-1}(D)$. ∎

Following the same reasoning as that used to obtain the bound (34) on $\hat{\theta}_\epsilon$, we have

$$\|\phi_\epsilon\|_{H^1(D)} + \|\phi_\epsilon\|_{H^1(B_R\setminus D)} \leq C_R\,\epsilon^{-1/2}\|u_0\|_{H^4(D)} \tag{71}$$

and

$$\|\phi_\epsilon\|_{L^2(B_R)} \leq C_R\,\|u_0\|_{H^4(D)}. \tag{72}$$

Hence the following theorem is a straightforward corollary of the above lemma.

THEOREM 5. *Let u_ϵ be the solution to* (3)*, u_0 the solution to* (4)*, and let the bulk and boundary corrections $u^{(1)}$ and θ_ϵ be given by* (13) *and* (18)*, respectively. We also note that in the definition of θ_ϵ, $v^{(1)}$ must be given by* (63)*. Then for any ball B_R of radius $R > 0$ which contains D, we have*

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon\theta_\epsilon)\|_{H^1(D)} + \|u_\epsilon - (u_0 + \epsilon\theta_\epsilon)\|_{H^1(B_\mathrm{R}\setminus D)} \leq C_\mathrm{R}\,\epsilon^{3/2}\|u_0\|_{H^4(D)}$$

and

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon\theta_\epsilon)\|_{L^2(B_\mathrm{R})} \leq C_\mathrm{R}\,\epsilon^{2}\|u_0\|_{H^4(D)},$$

where C_R is a constant independent of ϵ and u_0.

5. An example of a boundary layer limit. The first-order boundary corrector function θ_ϵ given by (57) is of great interest for the analysis of scattering by periodic media, in particular, because its presence is *necessary* to obtain estimates of order ϵ^2. Unfortunately, (57) is harder to solve than our original equation for u_ϵ, rendering θ_ϵ useless as a straightforward numerical correction. However, we would overcome this problem if we could find its limit. Presently, there are a number of works on the limit (when it exists) of the analogue of θ_ϵ for Dirichlet problems on rational polygons [26, 30], Neumann problems [27], and more recently works which extend the existence of the limit to more general domains [12, 13]. In this section we will borrow certain techniques from [26, 30] to find the limit of our transmission boundary corrector θ_ϵ for a specific geometry. This requires a new analysis since the boundary layer and its limit take a different form.

For simplicity, consider the case where the domain D is a unit square $(0,1)\times(0,1)$. Note that the present discussion can also be extended to convex polygonal domains with sides of rational slope [26]. To find the limit of θ_ϵ, one can consider the transmission data on one side of the square at a time. We first note that the leading-order part of the conormal jump data (which has the factor of $\frac{1}{\epsilon}$) in (57), namely,

$$(v_0 - \overline{v}_0)\cdot\nu = \mathrm{rot}_y q\cdot\nu,$$

is a tangential y derivative on the boundary cells, and so will have integral zero on a boundary period cell. We will see below that when we have the $\frac{1}{\epsilon}$ factor, we need the oscillatory part to have zero boundary average for the limit to exist. That is, its weak limit needs to be zero. So, on the right side of the square $\partial D\cap\{x_1 = 1\}$,

$$\begin{aligned}(v_0 - \overline{v}_0)\cdot\nu &= \left(a_{1j}(x/\epsilon) - a_{1k}(x/\epsilon)\frac{\partial\chi^j}{\partial y_k}(x/\epsilon) - A_{1j}\right)\frac{\partial u_0}{\partial x_j}\\ &=: g_1(y)\frac{\partial u_0}{\partial x_1} + g_2(y)\frac{\partial u_0}{\partial x_2}\end{aligned}$$

which is a y_2 derivative. Unless slow parts $\frac{\partial u_0}{\partial x_1}, \frac{\partial u_0}{\partial x_2}$ are linearly dependent, this implies that g_1 and g_2 both separately have a boundary cell average of zero. Were it to be the case that the slow parts were linearly dependent, we could group them together as a single boundary term. Without loss of generality we will assume this is not the case.

Consider also the term with $v^{(1)}$ in the boundary data (57), and define its boundary cell average on $\partial D \cap \{x_1 = 1\}$,

$$\overline{v^{(1)}}^{\partial} := \int_0^1 v^{(1)}(y, x) dy_2,$$

which on that boundary will depend only on the slow variable. We write

$$v^{(1)} := \overline{v^{(1)}}^{\partial} + (v^{(1)} - \overline{v^{(1)}}^{\partial})$$

and note that the second term, which has a boundary cell average of zero but no factor of $\frac{1}{\epsilon}$, will not contribute in the limit to this first-order boundary correction since its H^1-weak limit is zero and hence its strong L^2 limit will be zero. It will contribute to the higher-order correction limit. Hence instead of (57) we consider the following system

$$\begin{aligned} \nabla \cdot \big(a(x/\epsilon)\nabla\theta_\epsilon\big) + k^2 n(x/\epsilon)\theta_\epsilon &= 0 \quad \text{in } \ D, \\ \Delta\theta_\epsilon + k^2\theta_\epsilon &= 0 \quad \text{in } \ \mathbb{R}^2 \setminus \overline{D}, \\ \theta_\epsilon^+ - \theta_\epsilon^- &= 1_{\{x_1=1\}}\, \chi^j(x/\epsilon)\frac{\partial u_0}{\partial x_j} \quad \text{on } \ \partial D, \end{aligned}$$

$$(\nabla\theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla\theta_\epsilon \cdot \nu)^- = 1_{\{x_1=1\}} \left(\epsilon^{-1} g_j(x/\epsilon)\frac{\partial u_0}{\partial x_j} + \overline{v^{(1)}}^{\partial}\right) \quad \text{on } \ \partial D, \tag{73}$$

together with the Sommerfeld radiation condition (2) at infinity. Here $1_M(x)$ is the characteristic function equaling 1 for $x \in M$ and zero otherwise, and

$$g_j(x/\epsilon) = a_{1j}(x/\epsilon) - a_{1k}(x/\epsilon)\frac{\partial \chi^j}{\partial y_k}(x/\epsilon) - A_{1j}. \tag{74}$$

Note that in the above problem, the transmission data on the right side of the square depend strongly on the choice of ϵ. If, for example $\epsilon_m = 1/m$ for integer m, this boundary layer problem would see only a boundary slice of the periodic functions $\chi^j(y)$, $g_j(y)$, and $v^{(1)}$. It is for this reason that one can expect different limits of the boundary layer function for different sequences of ϵ going to zero. This phenomenon was first noticed in [30] and further understood in [26]. Let us therefore assume that ϵ_m is a sequence going to zero for which the boundary cutoff is fixed. That is, we assume that the fractional part of $1/\epsilon_m$ is constant, i.e.,

$$\frac{1}{\epsilon_m} - \left\lfloor \frac{1}{\epsilon_m} \right\rfloor = \delta$$

for all m, and we abuse notation a bit to set our oscillatory boundary functions to their restrictions:

$$\chi^j(y_2) = \chi^j(\delta, y_2), \quad g_j(y_2) = g_j(\delta, y_2), \quad \overline{v^{(1)}}^{\partial} = \overline{v^{(1)}}^{\partial}(\delta, x). \tag{75}$$

In order to describe the limit of the above boundary layer problem, we need to introduce auxiliary problems on a strip $G = \{-\infty < y_1 < \infty; y_2 \in [0,1]\}$ with its two halves

$$G^+ = \{y_1 > 0; y_2 \in [0,1]\} \qquad \text{and} \qquad G^- = \{y_1 < 0; y_2 \in [0,1]\}.$$

Let $\hat{w}_j(y_1, y_2)$ solve

$$\begin{aligned} \nabla_y \cdot \big(a(y_1+\delta, y_2)\nabla \hat{w}_j\big) &= 0 \quad \text{in } G^-, \\ \Delta_y \hat{w}_j &= 0 \quad \text{in } G^+, \\ \hat{w}_j(0,y_2)^+ - \hat{w}_j(0,y_2)^- &= \chi^j(y_2), \\ \partial_{y_1}\hat{w}_j(0,y_2)^+ - a_{1i}(\delta,y_2)\partial_{y_i}\hat{w}_j(0,y_2)^- &= g_j(y_2), \\ \hat{w}_j \ \ [0,1]\text{-periodic in} &\quad y_2, \\ \text{there exists } \gamma > 0 \text{ such that} \quad e^{\pm\gamma y_1}\nabla \hat{w}_j &\in L^2(G^\pm). \end{aligned} \tag{76}$$

By Theorem 9, such a solution $\hat{w}_j$ exists and is unique up to an additive constant across the entire strip G. Note the exponential decay of all derivatives in both directions at infinity; this ensures that $\hat{w}_j$ approaches a constant as $y_1 \to \pm\infty$. Next, set

$$d_j^+ = \lim_{y_1\to\infty} \hat{w}_j \qquad \text{and} \qquad d_j^- = \lim_{y_1\to-\infty} \hat{w}_j.$$

We can now define our limiting boundary value jump as

$$\chi_j^* = d_j^+ - d_j^-, \tag{77}$$

where one notes that its value is independent of the choice of the additive constant for $\hat{w}_j$. Then, we can similarly define $w_j(y_1, y_2)$ via

$$\begin{aligned} \nabla_y \cdot \big(a(y_1+\delta, y_2)\nabla w_j\big) &= 0 \quad \text{in } G^-, \\ \Delta_y w_j &= 0 \quad \text{in } G^+, \\ w_j(0,y_2)^+ - w_j(0,y_2)^- &= \chi^j(y_2) - \chi_j^*, \\ \partial_{y_1} w_j(0,y_2)^+ - a_{1i}(\delta,y_2)\partial_{y_i} w_j(0,y_2)^- &= g_j(y_2), \\ w_j \ \ [0,1]\text{-periodic in} &\quad y_2, \\ \text{there exists } \gamma > 0 \text{ such that} \quad e^{\pm\gamma y_1}\nabla w_j &\in L^2(G^\pm). \end{aligned} \tag{78}$$

Note that w_j, the solution to (78), is itself unique only up to an additive constant. However, from the definition of χ_j^*, w_j will always approach the same constant limit in both directions. (To see this, just subtract off from a given choice of $\hat{w}_j$ the piecewise constant function d_j^+ in G^+, d_j^- in G^-.) Therefore an additive constant can be chosen so that w_j itself also decays to zero as $|y_1| \to \infty$. With the above results in place, we have the following convergence theorem.

THEOREM 6. *Let $D = (0,1) \times (0,1)$ be the unit square and let ϵ_m be a sequence approaching zero such that $\frac{1}{\epsilon_m} - \lfloor \frac{1}{\epsilon_m} \rfloor = \delta$ for all m. Then if θ_{ϵ_m} solves* (73) *for $\epsilon = \epsilon_m$, we have that $\theta_{\epsilon_m} \to \theta^*$ strongly in $L^2_{loc}(\mathbb{R}^2)$, where θ^* solves*

$$\begin{aligned} \nabla \cdot A\nabla\theta^* + k^2 \overline{n}\theta^* &= 0 \quad \text{in } D, \\ \Delta\theta^* + k^2\theta^* &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ (\theta^*)^+ - (\theta^*)^- &= 1_{\{x_1=1\}}\, \chi_j^* \frac{\partial u_0}{\partial x_j} \quad \text{on } \partial D, \\ (\nabla\theta^* \cdot \nu)^+ - (A\nabla\theta^* \cdot \nu)^- &= 1_{\{x_1=1\}} \left(\overline{a_{12}w_j}^{\partial}\, \frac{\partial^2 u_0}{\partial x_j \partial x_2} + \overline{v^{(1)}}^{\partial} \right) \quad \text{on } \partial D, \end{aligned} \tag{79}$$

where A is the homogenized matrix given by (5),

$$\overline{a_{12}w_j}^{\partial} := \overline{a_{12}(\delta, y_2)w_j(0, y_2)^-}$$

denotes the average in the y_2 direction of $a_{12}w_j$ at $y_1 = 0$ coming from the left side of the strip, and the χ_j^* are given by (77).

Proof. Without loss of generality, we can assume that the Dirichlet part of the jump data is zero in a neighborhood of the corners. The reason that we can assume this is because in the proof of Theorem 10, if we were to only have small-support Dirichlet jumps, and no Neumann jumps, the L^2 norm would go to zero; see (126). Let us decompose θ_ϵ solving (73) into $\theta_\epsilon = \psi_\epsilon^{(1)} + \psi_\epsilon^{(2)}$, where $\psi_\epsilon^{(1)}$ satisfies

$$\begin{aligned}
\nabla \cdot \left(a(x/\epsilon)\nabla\psi_\epsilon^{(1)}\right) + k^2 n(x/\epsilon)\psi_\epsilon^{(1)} &= 0 \quad \text{in} \;\; D, \\
\Delta\psi_\epsilon^{(1)} + k^2\psi_\epsilon^{(1)} &= 0 \quad \text{in} \;\; \mathbb{R}^2 \setminus \overline{D}, \\
\left(\psi_\epsilon^{(1)}\right)^+ - \left(\psi_\epsilon^{(1)}\right)^- &= 1_{\{x_1=1\}}\, \chi_j^* \frac{\partial u_0}{\partial x_j} \quad \text{on} \;\; \partial D, \\
\left(\nabla\psi_\epsilon^{(1)} \cdot \nu\right)^+ - \left(a(x/\epsilon)\nabla\psi_\epsilon^{(1)} \cdot \nu\right)^- &= 1_{\{x_1=1\}} \left(\overline{a_{12}w_j}^{\partial} \frac{\partial^2 u_0}{\partial x_j \partial x_2} + \overline{v^{(1)}}^{\partial}\right) \quad \text{on} \;\; \partial D,
\end{aligned}$$

and $\psi_\epsilon^{(2)}$ solves

$$\begin{aligned}
\nabla \cdot \left(a(x/\epsilon)\nabla\psi_\epsilon^{(2)}\right) + k^2 n(x/\epsilon)\psi_\epsilon^{(2)} &= 0 \quad \text{in} \;\; D, \\
\Delta\psi_\epsilon^{(2)} + k^2\psi_\epsilon^{(2)} &= 0 \quad \text{in} \;\; \mathbb{R}^2 \setminus \overline{D}, \\
\left(\psi_\epsilon^{(2)}\right)^+ - \left(\psi_\epsilon^{(2)}\right)^- &= 1_{\{x_1=1\}} \left(\chi_j(x/\epsilon) - \chi_j^*\right) \frac{\partial u_0}{\partial x_j} \quad \text{on} \;\; \partial D, \\
\left(\nabla\psi_\epsilon^{(2)} \cdot \nu\right)^+ - \left(a(x/\epsilon)\nabla\psi_\epsilon^{(2)} \cdot \nu\right)^- &= 1_{\{x_1=1\}} \left(\epsilon^{-1} g_j(x/\epsilon)\frac{\partial u_0}{\partial x_j} - \overline{a_{12}w_j}^{\partial} \frac{\partial^2 u_0}{\partial x_j \partial x_2}\right) \\
&\qquad \text{on} \;\; \partial D.
\end{aligned}$$

Note from the analysis in the previous sections that $\psi_\epsilon^{(1)} \to \theta^*$ strongly in $L^2_{loc}(\mathbb{R}^2)$, since this is an example of a standard homogenization problem. It therefore suffices to show that $\psi_\epsilon^{(2)} \to 0$ strongly in $L^2_{loc}(\mathbb{R}^2)$. We will consider the boundary data when $j = 1$ only; the case $j = 2$ will follow in exactly the same way. Define $V(x_2)$ to be the restriction of $\partial u_0/\partial x_1$ to $\partial D \cap \{x_1 = 1\}$, extended as a constant in the x_1-direction, and extended by zero for x_2 outside of $(0, 1)$. Recall that we assumed that $V(x_2)$ is zero in a neighborhood of $x_2 = 1$ and $x_2 = 0$, so that this extension is smooth. Next, define $\phi(x_1)$ to be a smooth cutoff function (constant in x_2) such that $\phi \equiv 1$ for $x_1 \geq 1$ and $\phi \equiv 0$ for $x_1 \leq 0$. Let

$$\psi_\epsilon^{(3)} = w_1\left(\frac{x_1 - 1}{\epsilon}, \frac{x_2}{\epsilon}\right),$$

where w_1 solves (78) with $j = 1$, and with the constant chosen so that w_1 goes to zero as $y_1 \to \pm\infty$. Note that χ_1^* was chosen precisely so that such a w_1 exists which decays to zero in both directions. Then we set

$$\psi_\epsilon^{(4)} = \psi_\epsilon^{(3)} V(x_2)\phi(x_1),$$

and note that $\psi_\epsilon^{(4)} \to 0$ in $L^2_{loc}(\mathbb{R}^2)$ thanks to the exponential decay of w_1.

Now we will show that $\psi_\epsilon^{(2)} - \psi_\epsilon^{(4)}$ is small by looking at its residual when the differential operators are applied. Inside of D, we have

$$
\begin{aligned}
(80)\qquad \big(\nabla\cdot\big(a(x/\epsilon)\nabla\big) + k^2 n(x/\epsilon)\big)(\psi_\epsilon^{(2)} - \psi_\epsilon^{(4)}) &= -(\nabla\cdot a\nabla\psi_\epsilon^{(3)})V\phi - k^2 n\psi_\epsilon^{(3)}V\phi \\
&\quad - a\nabla\psi_\epsilon^{(3)}\cdot\nabla(V\phi) - (\nabla\cdot(a\psi_\epsilon^{(3)}))\cdot\nabla(V\phi) - \psi_\epsilon^{(3)} a : \nabla\nabla(V\phi).
\end{aligned}
$$

Above, the tensor notation " :" represents the multiple contraction of two tensors producing a scalar; in this case $A : B = A_{ij}B_{ij}$, using Einstein summation. The first term on the right-hand side of (80) is zero while the second, fourth, and last term go to zero strongly in $H^{-1}(D)$ due to the exponential decay of w_1. The third term also goes to zero strongly in $H^{-1}(D)$ as $\epsilon \to 0$, but it takes a little more effort to see this; we sketch the same argument as [26, p. 1287]. Note that from the exponential decay of w_1 we have

$$
|\nabla\psi_\epsilon^{(3)}| \le \frac{C}{\epsilon}\exp\Big(-\gamma\frac{|x_1 - 1|}{\epsilon}\Big).
$$

Consider any test function $\Phi \in C_0^\infty(D)$; then

$$
(81)\qquad \Big|\int_D a\nabla\psi_\epsilon^{(3)}\cdot\nabla(V\phi)\Phi dx\Big| \le C\int_D \frac{1}{\epsilon}\exp\Big(\gamma\frac{x_1 - 1}{\epsilon}\Big)|\Phi| dx.
$$

Considering for fixed $0 < x_2 < 1$ the integral on the right-hand side, i.e.,

$$
(82)\qquad \int_0^1 \frac{1}{\epsilon}\exp\Big(\gamma\frac{x_1 - 1}{\epsilon}\Big)|\Phi| dx_1,
$$

integrating by parts ($|\Phi|$ will have a weak derivative), and using the Cauchy–Schwartz inequality one obtains (see p. 1287 in [26] for details)

$$
(83)\qquad \int_0^1 \frac{1}{\epsilon}\exp\Big(\gamma\frac{x_1 - 1}{\epsilon}\Big)|\Phi| dx_1 \le C\epsilon^{1/2}\left(\int_0^1 \Big|\frac{\partial}{\partial x_1}\Phi(x_1, x_2)\Big|^2 dx_1\right)^{1/2}.
$$

Integrating this result over x_2 and again using the Cauchy–Schwartz inequality, we have

$$
(84)\qquad \int_D \frac{1}{\epsilon}\exp\Big(\gamma\frac{x_1 - 1}{\epsilon}\Big)|\Phi| dx \le C\epsilon^{1/2}\|\Phi\|_{H_0^1(D)}
$$

which shows that the third term and thus the entire right-hand side of (80) is small in $H^{-1}(D)$. At the same time, outside of D, we have

$$
\begin{aligned}
\big(\Delta + k^2\big)(\psi_\epsilon^{(2)} - \psi_\epsilon^{(4)}) &= -\Delta\psi_\epsilon^{(3)}V\phi - k^2\psi_\epsilon^{(3)}V\phi \\
(85)\qquad &\quad -2\nabla\psi_\epsilon^{(3)}\cdot\nabla(V\phi) - \psi_\epsilon^{(3)}\Delta(V\phi).
\end{aligned}
$$

Note that for $x_1 \le 0$, $x_2 \ge 1$, or $x_2 \le 0$, all terms are zero thanks to the cutoff functions. For $x_1 \ge 1$, the first term is zero, and all other terms go to zero in H^{-1} of any bounded domain, due again to the exponential decay of $\psi_\epsilon^{(3)}$. Hence both inside and outside of D the operator residuals go to zero in H^{-1}. Let us now examine the transmission data for $(\psi_\epsilon^{(2)} - \psi_\epsilon^{(4)})$ across ∂D. Due to the presence of cutoff functions, both Dirichlet and Neumann jumps are all zero across the top, bottom, and left side of the square. The Dirichlet jump is also clearly zero across the right side of the square due to the matching of our jump data. For the jump in the conormal derivative across the right side of the square, recalling that $\nu = (1,0)$ and $x_1 = 1$, one has

$$\begin{aligned}(\nabla\psi_\epsilon^{(4)}\cdot\nu)^+ - (a\nabla\psi_\epsilon^{(4)}\cdot\nu)^- &= \left(\frac{\partial\psi_\epsilon^{(3)}}{\partial x_1}\right)^+(V\phi) - a_{1k}\left(\frac{\partial\psi_\epsilon^{(3)}}{\partial x_k}\right)^-(V\phi) - a_{12}(\psi_\epsilon^{(3)})^-\frac{\partial V}{\partial x_2}\phi\\ &= \frac{1}{\epsilon}\left(\frac{\partial w_1}{\partial y_1}\right)^+ V - \frac{1}{\epsilon}a_{1k}\left(\frac{\partial w_1}{\partial y_k}\right)^- V - a_{12}w_1^- V'\\ &= \frac{1}{\epsilon}g_1 V - a_{12}(\delta, y_2)(w_1(0,y_2))^- V',\end{aligned} \tag{86}$$

so that the conormal jump also matches with that of $\psi_\epsilon^{(2)}$ for $a(x/\epsilon)$ isotropic. Hence in the isotropic case, all transmission data vanish. In the anisotropic case, we have an oscillating bounded conormal jump,

$$(\nabla(\psi_\epsilon^{(4)} - \psi_\epsilon^{(2)})\cdot\nu)^+ - (a\nabla(\psi_\epsilon^{(4)} - \psi_\epsilon^{(2)})\cdot\nu)^- = (\overline{a_{12}w_1} - a_{12}w_1^-)\frac{\partial^2 u_0}{\partial x_1\partial x_2},$$

which is small in $H^{-1/2}(\partial D)$ since it has zero average. In particular, since its average along a boundary cell is zero, it can be written as a y_2 derivative, for which we can integrate by parts. Furthermore, due to the exponential decay in all directions, $(\psi_\epsilon^{(2)} - \psi_\epsilon^{(4)})$ will also satisfy the Sommerfeld radiation condition at infinity. Standard regularity results then yield convergence to zero in H^1 on bounded domains, and hence in L^2_{loc}. ∎

Remark 7. By doing the above on all four sides of the square D, we have found the limit of $O(\epsilon)$ boundary corrector θ_ϵ. The above result, however, will also allow us to find the limit of the $O(\epsilon^2)$ boundary correction, and indeed that of higher orders. The first thing one needs to note is that the oscillatory part of $v^{(1)}$ (which we have thrown away at first order) will need to come in at second order. This is actually convenient, as it has the same slow part as the second-order Dirichlet jump. In light of this, a more appropriate definition for the second-order boundary correction may be $\tilde\phi_\epsilon$ where $\tilde\phi_\epsilon$ satisfies

$$\begin{aligned}\nabla\cdot\big(a(x/\epsilon)\nabla\tilde\phi_\epsilon\big) + k^2 n(x/\epsilon)\tilde\phi_\epsilon &= 0 \quad \text{in } D,\\ \Delta\tilde\phi_\epsilon + k^2\tilde\phi_\epsilon &= 0 \quad \text{in } \mathbb{R}^d\setminus\overline{D},\\ \tilde\phi_\epsilon^+ - \tilde\phi_\epsilon^- &= u^{(2)} \quad \text{on } \partial D,\\ (\nabla\tilde\phi_\epsilon\cdot\nu)^+ - (a(x/\epsilon)\nabla\tilde\phi_\epsilon\cdot\nu)^- &= \frac{1}{\epsilon}(v^{(1)} - \overline{v^{(1)}}^{\partial})\cdot\nu + \overline{v^{(2)}}^{\partial}\cdot\nu \quad \text{on } \partial D\end{aligned} \tag{87}$$

with θ_ϵ also appropriately modified. The oscillatory parts of the data will have the same slow factors, allowing us to use the same proof for the limit on a square. Note that the mean field of $u^{(2)}$, $\overline{u}^{(2)}(x)$, will affect the far field precisely through this boundary layer.

Remark 8. In the special (but relevant) case where a is constant and the periodic structure is only in the lower-order term n, the cell functions χ^j and the first-order bulk correction are all zero. The first-order boundary correction θ_ϵ may be almost zero; indeed its only nonzero data are the conormal jump given by $v_1\cdot\nu = k^2 a\nabla_y\beta(y)\cdot\nu u_0$. Obviously from our analysis in the previous sections, in this case the boundary correction is bounded in the H^1-norm. However, as shown by this example, in the case when D is a square, the limit of the boundary corrector is not zero and in fact it is not even unique. Indeed, for an appropriately chosen subsequence ϵ_k, the subsequent limit of the boundary corrector will take a simple form with the only contribution coming from the boundary average $\overline{v^{(1)}}^{\partial} \cdot \nu$ within the cell. In the case of general domains with smooth boundary without flat parts, the first-order boundary corrector for a-constant should have zero limit (since the term $k^2 a \nabla_y \beta(y) \cdot \nu u_0$ appearing in the Neumann data has Y cell average zero, its boundary weak limit is zero, and hence the boundary corrector limit should be zero).

6. Higher-order PDE governing the mean behavior of u_ϵ. Let us in this section go back to the transmission problem (3), and consider the bulk expansion of u_ϵ inside D as in (8). Here we take a different approach, where we consider the mean behavior of u_ϵ by taking cell averages of the terms in its expansion. That is, we take the Y-average of u_ϵ, to write

$$\bar{u}_\epsilon \;=\; u_0(x) + \epsilon \bar{u}^{(1)}(x) + \epsilon^2 \bar{u}^{(2)}(x) + \cdots , \tag{88}$$

where

$$\bar{u}^{(n)} = \int_Y u^{(n)}(x,y)\, dy. \tag{89}$$

We now know that u_0 solves (4), and

$$u^{(1)}(x,y) = -\chi^j(y) \frac{\partial u_0}{\partial x_j} \tag{90}$$

as shown in section 4. Here the goal is to expose the PDEs governing the mean response, $\bar{u}_\epsilon$, inside D when considering the higher-order approximations.

Thanks to the identity $\bar{\chi}^j = 0$, one immediately finds from (90) that $\bar{u}^{(1)} = 0$. To uncover the behavior of $\bar{u}^{(2)}$, on the other hand, it can be shown by substituting (8) into (7) and collecting the terms of order ϵ^0 that

$$k^2 n u_0 + \nabla_x \cdot \big(a(\nabla_x u_0 + \nabla_y u^{(1)})\big) + \nabla_y \cdot \big(a(\nabla_x u^{(1)} + \nabla_y u^{(2)})\big) \;=\; 0. \tag{91}$$

By virtue of this result, (6) and (90), one finds as in [10, 33] that $u^{(2)}$ admits the representation

$$u^{(2)}(x,y) \;=\; \bar{u}^{(2)}(x) + \psi^{ij}(y) \frac{\partial^2 u_0}{\partial x_i \partial x_j}, \tag{92}$$

where ψ^{ij} are the zero-mean, Y-periodic solutions to

$$\nabla_y \cdot \big(a \nabla_y (\psi^{ij} + A_{ij}\beta(y)/\bar{n})\big) \;=\; b_{ij}(y) - \bar{b}_{ij}, \tag{93}$$

where β and b_{ij} are given by (17) and (42), respectively. A comparison between (43) and (93) reveals that in fact $\psi^{ij} = \chi^{ij} - A_{ij}\beta/\bar{n}$.

At order $O(\epsilon^m)$, $m \geqslant 1$, one similarly finds from (7) and (8) that

$$k^2 n u^{(m)} + \nabla_x \cdot \big(a(\nabla_x u^{(m)} + \nabla_y u^{(m+1)})\big) + \nabla_y \cdot \big(a(\nabla_x u^{(m+1)} + \nabla_y u^{(m+2)})\big) \;=\; 0. \tag{94}$$

Taking $m = 1$ and recalling (48), it can be shown from (94) that $u^{(3)}$ can be written as

$$u^{(3)}(x,y) \;=\; \bar{u}^{(3)}(x) - \chi^j(y) \frac{\partial \bar{u}^{(2)}}{\partial x_j} + \psi^{ijk}(y) \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}, \tag{95}$$

where ψ^{ijk} are Y-periodic, have zero mean, and solve

$$\nabla_y \cdot \big(a\nabla_y(\psi^{ijk} - A_{ij}\beta^k(y)/\bar{n})\big) \;=\; c_{ijk}(y) - \bar{c}_{ijk} - \frac{\partial}{\partial y_l}\big(a_{lk}\psi^{ij}\big), \tag{96}$$

featuring c_{ijk} that are given by (55); cf. (53) that is solved by χ^{ijk}.

On computing the Y-average of (91), using (90) and (92), and exploiting the periodicity of χ^j and χ^{ij}, one can show that

$$A : \nabla\nabla u_0 + k^2\bar{n}\, u_0 \;=\; 0 \tag{97}$$

as in (4), where the multiple tensor contraction produces a scalar (in this case $A : \nabla\nabla u_0 = \nabla\cdot A\nabla u_0$). On the other hand the Y-averaging of (94) for $m = 2$, combined with the fact that $\bar{u}^{(1)} = 0$ and the use of (90), (92), and (95) yields

$$A : \nabla\nabla\bar{u}^{(2)} + k^2\bar{n}\,\bar{u}^{(2)} \;=\; -\Big(\mathcal{A} : \nabla\nabla\nabla\nabla u_0 + k^2\mathcal{N} : \nabla\nabla u_0\Big), \tag{98}$$

where $\mathcal{A}$ and $\mathcal{N}$ are, respectively, the fourth- and second-order constant tensors given by

$$\mathcal{A}_{ijkl} \;=\; \overline{a_{kl}\psi^{ij} + a_{kr}\frac{\partial\psi^{ijk}}{\partial y_r}}, \qquad \mathcal{N}_{ij} \;=\; \overline{n\psi^{ij}},$$

and

$$\mathcal{A} : \nabla\nabla\nabla\nabla u_0 \;=\; \mathcal{A}_{ijkl}\,\frac{\partial^4 u_0}{\partial x_i\partial x_j\partial x_k\partial x_l};$$

see also [33]. Referring back to (88), (97) and (98) — combined with the result $\bar{u}^{(1)} = 0$ obtained earlier — demonstrate that the mean of the field inside D in (4) *formally* satisfies

$$A : \nabla\nabla\bar{u}_\epsilon + k^2\bar{n}\,\bar{u}_\epsilon \;=\; -\,\epsilon^2\Big(\mathcal{A} : \nabla\nabla\nabla\nabla\bar{u}_\epsilon + k^2\mathcal{N} : \nabla\nabla\bar{u}_\epsilon\Big) + o(\epsilon^2). \tag{99}$$

As examined in [33], the $O(\epsilon^2)$-terms in (99) bring additional length scales into the problem that are responsible for describing the *incipient wave dispersion* due to small-scale periodic fluctuations in $a(x/\epsilon)$ and $n(x/\epsilon)$.

Appendix A. Some auxiliary results.

A.1. On a transmission problem in the strip. In this section, we study the solvability of a transmission problem in a strip which is used to construct the limit of the boundary corrector. To study this problem we adapt here the approach of section 10.4 in [23] used for a similar problem in a half-strip. For the sake of reader's convenience we sketch here the main lines of the analysis. To formulate the problem under consideration, we recall the notation

$$G^+ = \{y_1 > 0; y_2 \in [0,1]\} \quad \text{and} \quad G^- = \{y_1 < 0; y_2 \in [0,1]\}$$

and consider h a smooth function defined on [0, 1] and $a := (a_{ij})_{2\times 2}$ with entries $a_{ij} \in C^1(\overline{G^+})$ such that $\xi\cdot a\xi \geq \alpha|\xi|^2$, $\xi \in \mathbb{R}^2$ and $\alpha > 0$. Assuming that $h(y_2)$ as well as the coefficient $a_{ij}(y_1, y_2)$ are extended periodically in y_2 with period [0, 1], the transmission problem we would like to solve is

$$\nabla_y \cdot \big(a(y_1,y_2)\nabla w^-\big) = 0, \qquad y_1 < 0,\ \ -\infty < y_2 < +\infty, \tag{100}$$

$$\Delta_y w^+ = 0, \qquad y_1 > 0,\ \ -\infty < y_2 < +\infty, \tag{101}$$

$$w^+(0,y_2) - w^-(0,y_2) = 0, \qquad -\infty < y_2 < +\infty, \tag{102}$$

$$\partial_{y_1} w^+(0,y_2) - a_{1i}(0,y_2)\partial_{y_i} w^-(0,y_2) = h, \qquad -\infty < y_2 < +\infty, \tag{103}$$

such that w^+, w^- are periodic in y_2 and $e^{-\gamma y_1}\nabla w^- \in L^2(G^-)$ and $e^{\gamma y_1}\nabla w^+ \in L^2(G^+)$ for some $\gamma > 0$. It is natural to look for a solution to this problem in the spaces

$$H^1_{per}[0,\,1] := \left\{\varphi \in H^1[0,\,1], \quad \text{periodic with period } [0,\,1]\right\},$$

$$V^+ := \left\{u(y_1, y_2) \in L^2((0,R), H^1_{per}[0,\,1]) \ \ \forall R > 0 \text{ such that } e^{\gamma y_1}\nabla u \in L^2(G^+)\right\},$$

$$V^- := \left\{u(y_1, y_2) \in L^2((-R,0), H^1_{per}[0,\,1]) \ \forall R > 0 \text{ such that } e^{-\gamma y_1}\nabla u \in L^2(G^-)\right\}.$$

In general this strip problem does not have a solution, and when it has a solution (w^+, w^-), any $(w^+ + c, w^- + c)$ is also a solution. Hence a necessary solvability condition on the data h is

$$\int_0^1 h(y_2)dy_2 = 0 \tag{104}$$

(this is easily seen by integrating by parts both equations against the test function 1, which is in the solution space, and using the transmission condition along with the periodicity). This solvability condition is also interpreted as an orthogonality condition with the kernel of the adjoint problem.

To find a solution to the strip problem we narrow the solution space by adding a side condition which in principle eliminates constants. To this end we let

$$\overline{u}(y_1) := \int_0^1 u(y_1, y_2)\, dy_2, \tag{105}$$

and define

$$V_0^{\pm} := \left\{u \in V^{\pm} \text{ such that } \overline{u}(0) = 0\right\},$$

where $u(0, y_2)$ is understood in the sense of trace. Thanks to the zero-mean condition at $y_1 = 0$, $V_0^{\pm}$ are Hilbert spaces equipped with the norms

$$\|u\|_{V_0^+} := \|e^{\gamma y_1}\nabla u\|_{L^2(G^+)} \qquad \text{and} \qquad \|u\|_{V_0^-} := \left\|e^{-\gamma y_1}\nabla u\right\|_{L^2(G^-)}.$$

In this framework, we are looking for a solution $(w^-, w^+) \in X(G)$ to (100)–(103), where

$$X(G) := \left\{(u^-, u^+) \in V_0^- \times V_0^+, \text{ such that}\, u^-(0, y_2) = u^+(0, y_2), \ \ y_2 \in [0,\,1]\right\}.$$

To control the behavior of the function itself as $y_1 \to \pm\infty$ we need the following test space

$$\tilde{V}_0^{\pm} := \left\{u \in V_0^{\pm}, \text{ such that } \ e^{\pm\gamma y_1} u \in L^2(G^{\pm})\right\}$$

equipped with the norm

$$\|u\|^2_{\tilde{V}_0^{\pm}} = \|u\|^2_{V_0^{\pm}} + \|e^{\pm\gamma y_1} u\|^2_{L^2(G^{\pm})}.$$

Now multiplying (100) and (101) by $e^{-2\gamma y_1} v^-$ and $e^{2\gamma y_1} v^+$, respectively, for $(v^-, v^+) \in \tilde{X}(G)$, where

$$\tilde{X}(G) := \left\{(u^-, u^+) \in \tilde{V}_0^- \times \tilde{V}_0^+, \text{ such that } u^-(0, y_2) = u^+(0, y_2), \ \ y_2 \in [0,\,1]\right\},$$

integrating by parts, and using (103), we may put (100)–(103) in the following variational form: find $w := (w^-, w^+) \in X(G)$, both periodic in y_2, satisfying

$$\int_{G^-} a(y_1, y_2)\nabla w^- \cdot \nabla(e^{-2\gamma y_1} v^-)\, dy_1 dy_2 + \int_{G^+} \nabla w^+ \cdot \nabla(e^{2\gamma y_1} v^+)\, dy_1 dy_2$$
$$= \int_0^1 h v^+\, dy_2 \tag{106}$$

for every $v := (v^-, v^+) \in \tilde{X}(G)$. Note that $\tilde{X}(G) \subset X(G)$ and since the solution space and the test space are different, from Tartar's lemma, section 10.3 in [23], to prove the solvability of (106) it suffices to prove that the continuous bilinear form $A : X(G) \times \tilde{X}(G) \to \mathbb{R}$ defined by

$$A(w, v) := \int_{G^-} a(y_1, y_2)\nabla w^- \cdot \nabla(e^{-2\gamma y_1} v^-)\, dy_1 dy_2$$
$$+ \int_{G^+} \nabla w^+ \cdot \nabla(e^{2\gamma y_1} v^+)\, dy_1 dy_2 \tag{107}$$

is M-coercive, that is there exists a bounded linear and onto mapping $M : X(G) \to \tilde{X}(G)$ such that

$$A(w, Mw) \geq c\|w\|_{X(G)} \quad \text{for all } w \in X(G) \text{ and for some } c > 0.$$

Our construction of the linear mapping M makes use of the construction by Lions in [23, section 10.4] of a similar mapping in the half-strip. In particular, we define $M = (M^+, M^-)$, where

$$M^+ u^+ = u^+ - 2\gamma e^{-2\gamma y_1} * \overline{u}^+, \quad y_1 > 0,$$

and

$$M^- u^- = u^- + 2\gamma e^{2\gamma y_1} * \overline{u}^-, \quad y_1 < 0,$$

where $\overline{u}^\pm$ is defined by (105) and $*$ denotes the convolution in y_1 where the functions are extended by zero on the other half-line and this does not cause any problem since $\overline{u}^\pm(0) = 0$. In [23, section 10.4] it is shown that M^+ is bounded linear and onto mapping from V_0^+ to $\tilde{V}_0^+$ (the only difference between our spaces and the spaces in [23, section 10.4] is the condition at $y_1 = 0$ and this does not change anything in the proof). By making the change of variable $y_1 \to -y_1$ the same argument implies that M^- is bounded linear and onto mapping from V_0^- to $\tilde{V}_0^-$. Furthermore since

$$M^+ u^+(0, y_2) = M^- u^-(0, y_2), \qquad y_2 \in [0, 1],$$

we can conclude that $M : X(G) \to \tilde{X}(G)$ in bounded linear and onto. Next one can calculate

$$A(w, Mw) = \int_{G^-} a(y_1, y_2) e^{-2\gamma y_1} \nabla w^- \cdot \nabla w^-\, dy_1 dy_2 \tag{108}$$
$$-2\gamma \int_{G^-} a_{1i} e^{-2\gamma y_1} \frac{\partial w^-}{\partial y_i}(w^- - \overline{w}^-) dy_1 dy_2$$
$$+ \int_{G^+} e^{2\gamma y_1} \nabla w^+ \cdot \nabla w^+\, dy_1 dy_2$$
$$+2\gamma \int_{G^+} a_{1i} e^{2\gamma y_1} \frac{\partial w^+}{\partial y_i}(w^+ - \overline{w}^+) dy_1 dy_2.$$

By Poincaré's inequality we have that

$$\|w^{\pm}(y_1,y_2)-\overline{w}^{\pm}(y_1)\|_{L^2[0,\,1]} \le c_p \left\|\frac{\partial w^{\pm}}{\partial y_2}\right\|_{L^2[0,\,1]} \tag{109}$$

and since the derivatives of $w^{\pm}$ "decay exponentially" as $y_1 \to \pm\infty$ it follows that $e^{\pm\gamma y_1}(w^{\pm}-\overline{w}^{\pm}) \in L^2(G^{\pm})$. From the definition of the norm $X(G)$ and the positive definiteness of a, the first and third terms on the right-hand side of (108) are positive and bounded below:

$$\int_{G^-} a(y_1,y_2)e^{-2\gamma y_1}\nabla w^- \cdot \nabla w^- \, dy_1 dy_2 + \int_{G^+} e^{2\gamma y_1}\nabla w^+ \cdot \nabla w^+ \, dy_1 dy_2 \ge \alpha \|w\|^2_{X(G)}.$$

Further, using (109), the second and the fourth terms of (108) have absolute values bounded above by $c\gamma\|w\|^2_{X(G)}$. Hence

$$A(w, Mw) \ge (\alpha - c\gamma)\|w\|^2_{X(G)}$$

which implies M-coercivity for small enough γ (since α and c are independent of γ). From Tartar's lemma, this completes the proof of the existence of a unique solution to (100)–(103) in the space $X(G)$. Any other solution to (100)–(103) in $V^+ \times V^-$ differs from this found solution by a constant. Therefore, if h satisfies (104), a solution to (100)–(103) exists, and it is unique up to an additive constant.

THEOREM 9. *Assume that* $\chi \in C^1[0,1]$, $g \in C^1[0,1]$, *and* $\int_0^1 g(y_2)dy_2 = 0$. *Then there exists a* $w \in H^1([-R,R]\times[0,\,1])$ *for all* $R > 0$, *and periodic in* y_2 *with period* $[0,\,1]$, *unique up to a additive constant, satisfying*

$$\nabla_y \cdot \big(a(y_1,y_2)\nabla w\big) = 0, \qquad y_1 < 0,\ -\infty < y_2 < +\infty, \tag{110}$$

$$\Delta_y w = 0, \qquad y_1 > 0,\ -\infty < y_2 < +\infty, \tag{111}$$

$$w^+(0,y_2) - w^-(0,y_2) = \chi(y_2), \qquad -\infty < y_2 < +\infty, \tag{112}$$

$$\partial_{y_1} w^+(0,y_2) - a_{1i}(\delta,y_2)\partial_{y_i} w^-(0,y_2) = g(y_2), \quad -\infty < y_2 < +\infty, \tag{113}$$

along with $e^{\gamma y_1}\nabla w \in L^2(G^+)$ *for* $y_1 > 0$ *and* $e^{-\gamma y_1}\nabla w \in L^2(G^-)$ *for* $y_1 < 0$ *for some* $\gamma > 0$, *where* $\chi(y_2)$ *and* $g(y_2)$ *are extended periodically in* y_2 *with period* $[0,\,1]$.

Proof. Let $\tilde{w}$ be the solution of the following problem,

$$\Delta \tilde{w} = 0 \qquad y_1 > 0,\ -\infty < y_2 < +\infty, \tag{114}$$

$$\tilde{w}(0,y_2) = \chi(y_2) \qquad -\infty < y_2 < +\infty, \tag{115}$$

$$e^{\gamma y_1}\nabla \tilde{w} \in L^2(G^+). \tag{116}$$

This problem can be explicitly solved and the solution is

$$\tilde{w} = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos 2n\pi y_2 + b_n \sin 2n\pi y_2\right) e^{-2n\pi y_1},$$

where a_0, a_n, b_n are the Fourier coefficients of χ. In particular, we have that

$$\int_0^1 \frac{\partial \tilde{w}}{\partial y_1}(0,y_2)\, dy_2 = 0.$$

Then $w^- := w|_{G^-} \in V_0^-$ and $w^+ := w|_{G^+} - \tilde{w} \in V_0^+$ are the unique solution of (100)–(103) with Neuman data h replaced by

$$h(y_2) := g(y_2) - \frac{\partial \tilde{w}}{\partial y_1}(0, y_2). \tag{117}$$

From the above and the assumption on g we have that h given by (117) is such that $\overline{h(0)} = 0$, i.e, it has mean zero at $[0, y_2]$. This means that the solvability condition for the strip problem is satisfied and the theorem follows from the discussion in section A.1 prior to this theorem. ∎

A.2. L^2-boundedness of the boundary corrector. Although L^2 estimates (see [9] and [18]) are, in general, expected for the transmission problems satisfied by the boundary correctors discussed above, here we only show that the boundary corrector θ_ϵ given by (57) is bounded in $L^2(B_R)$-norm for any ball B_R of radius R containing D. In particular we use here an approach by duality following [27].

THEOREM 10. *Let $u_0 \in H^2(B_\mathrm{R})$ be the solution of* (4)*, and let θ_ϵ be given by* (57)*. Then*

$$\|\theta_\epsilon\|_{L^2(B_\mathrm{R})} \le C_\mathrm{R} \|u_0\|_{H^2(D)},$$

where B_R is an arbitrary ball of radius R containing D and C which depends only on R.

Proof. Without loss of generality we prove our theorem for the boundary corrector $\hat{\theta}_\epsilon$ given by (18). Let $\phi \in L^2(B_R)$ and let $W_\epsilon \in H^1_{loc}(\mathbb{R}^d)$ be the solution of (26) where ϕ is extended by zero outside B_R. Let W_0, $W^{(1)}$, and Θ_ϵ be analogous to u_0, u_1, and $\hat{\theta}_\epsilon$ corresponding to W_ϵ in the same type of homogenization analysis as for u_ϵ in section 3. Integrating by parts, using the equations for W_ϵ, and arguing in the same way as in the proof of Lemma 1 to cancel the boundary terms on ∂B_R we have

$$\begin{aligned}\int_{B_R} \hat{\theta}_\epsilon\, \phi\, dx = &- \int_D a(x/\epsilon) \nabla \hat{\theta}_\epsilon \cdot \nabla W_\epsilon\, dx + k^2 \int_D n(x/\epsilon) \hat{\theta}_\epsilon\, W_\epsilon\, dx \\ &+ \int_{\partial D} \nu \cdot a(x/\epsilon) \nabla W_\epsilon^-\, \hat{\theta}_\epsilon^-\, ds_x \\ &+ \int_{\partial D} \nu \cdot \nabla \hat{\theta}_\epsilon^+ \cdot W_\epsilon^+\, ds_x - \int_{\partial D} \nu \cdot \nabla W_\epsilon^+ \cdot \hat{\theta}_\epsilon^+\, ds_x.\end{aligned} \tag{118}$$

Integrating again by parts in D, using the equation for θ_ϵ, the transmission conditions for θ_ϵ across ∂D, and the continuity of W_ϵ and its conormal derivative across ∂D from (26) we have that

$$\begin{aligned}\int_{B_R} \hat{\theta}_\epsilon\, \phi\, dx &= \int_{\partial D} \nu \cdot \left(\frac{v_0 - \overline{v}_0}{\epsilon} + \hat{v}^{(1)} \right) W_\epsilon^+\, ds_x - \int_{\partial D} u^{(1)} \left(\nu \cdot \nabla W_\epsilon^+ \right) ds_x \\ &= \int_{\partial D} \nu \cdot \left(\mathrm{rot}(q) + k^2 a \nabla_y \beta(y) u_0 \right)\, W_\epsilon^+\, ds_x \\ &\quad + \int_{\partial D} \chi^j(x/\epsilon) \frac{\partial u_0}{\partial x_j} \left(\nu \cdot \nabla W_\epsilon^+ \right) ds_x.\end{aligned} \tag{119}$$

The same analysis as in section 3 applies to W_ϵ. In particular by the analogue of Lemma 1

$$\|W_\epsilon - (W_0 + \epsilon W^{(1)} + \epsilon \Theta_\epsilon)\|_{H^1(B_R)} \le C_R\, \epsilon \|W_0\|_{H^2(D)}, \tag{120}$$

where C_R is a constant independent of ϵ and W_0. By continuity of the trace mapping $\gamma^+ : H^1(B_R \setminus \overline{D}) \to H^{1/2}(\partial D)$, (120) also holds in the $H^{1/2}(\partial D)$-norm. In particular, since the bulk correction $W^{(1)}$ is zero outside D, (26) takes the form

$$\|W_\epsilon - (W_0 + \epsilon\Theta_\epsilon)\|_{H^1(B_R\setminus\overline{D})} \le C_R\,\epsilon\|W_0\|_{H^2(D)}.$$

Now from the equations satisfied by W_ϵ, W_0, and Θ_ϵ, we have that their Laplacian is in $L^2(B_R \setminus \overline{D})$ hence they are in $H^1(B_R \setminus \overline{D}, \Delta)$. For such functions u, it is well known (see, e.g., Theorem 5.7 in [11]) that $\gamma_1^+ : u \mapsto \nu \cdot \nabla u$ can be extended as a bounded linear mapping from $H^1(B_R \setminus \overline{D}, \Delta) \to H^{-1/2}(\partial D)$, which implies that

$$\|\nu \cdot \nabla(W_\epsilon^+ - W_0^+ - \epsilon\Theta_\epsilon^+)\|_{H^{-1/2}(\partial D)} \le C_R\,\epsilon\|W_0\|_{H^2(D)}. \tag{121}$$

Hence we can estimate

$$\begin{aligned}\left|\int_{\partial D} \nu\cdot\big(\mathrm{rot}(q) + k^2 a\nabla_y\beta(y)u_0\big)\; W_0^+\,ds_x\right| &\le \left|\int_{\partial D} q\,\mathrm{rot}(W_0^+)\cdot\nu\,ds_x\right| \\ &+k^2\left|\int_{\partial D}\nu\cdot a\nabla_y\beta(y)u_0\,W_0^+\,ds_x\right| \le C_1\|u_0\|_{H^2(D)}\|W_0\|_{H^2(D)}\end{aligned} \tag{122}$$

since by (15) q in $H^{-1}(\partial D)$ is bounded by the $H^2(D)$-norm of u_0 independent of ϵ and by (17) $a\nabla_y\beta(y)$ is bounded in $H^{1/2}(\partial D)$ independent of ϵ. Also we know that

$$\|u^{(1)}\|_{L^2(\partial D)} \le C\|u_0\|_{H^2(D)},$$

and hence we have

$$\begin{aligned}\left|\int_{\partial D} u^{(1)}\,(\nu\cdot\nabla W_0^+)\,ds_x\right| &\le C\|u^{(1)}\|_{L^2(\partial D)}\|\nabla W_0\|_{L^2(\partial D)} \\ &\le C_2\|u_0\|_{H^2(D)}\|W_0\|_{H^2(D)}.\end{aligned} \tag{123}$$

Next

$$\begin{aligned}\left|\int_{\partial D}\left(\frac{v_0-\overline{v}_0}{\epsilon} + \hat{v}^{(1)}\right)\epsilon\Theta_\epsilon\,ds_x\right| &\le \epsilon C_2\left\|\frac{v_0-\overline{v}_0}{\epsilon} + \hat{v}^{(1)}\right\|_{H^{-1/2}(\partial D)}\|\Theta_\epsilon\|_{H^{1/2}(\partial D)} \\ &\le C_3\|u_0\|_{H^2(D)}\|W_0\|_{H^2(D)}\end{aligned} \tag{124}$$

since by the analogue of (34)

$$\|\Theta_\epsilon\|_{H^{1/2}(\partial D)} \le C\epsilon^{-1/2}\|W_0\|_{H^2(D)}$$

and by (34)

$$\left\|\frac{v_0-\overline{v}_0}{\epsilon} + \hat{v}^{(1)}\right\|_{H^{-1/2}(\partial D)} \le C\epsilon^{-1/2}\|u_0\|_{H^2(D)}.$$

Similarly, since in addition we have

$$\|u^{(1)}\|_{H^{1/2}(\partial D)} \le C\epsilon^{-1/2}\|u_0\|_{H^2(D)},$$

$$\begin{aligned}\left|\int_{\partial D} u^{(1)}\,\epsilon(\nu\cdot\nabla\Theta_\epsilon^+)\,ds_x\right| &\le C\epsilon\|u^{(1)}\|_{H^{1/2}(\partial D)}\|\Theta_\epsilon\|_{H^{1/2}(\partial D)} \\ &\le C_4\|u_0\|_{H^2(D)}\|W_0\|_{H^2(D)}.\end{aligned} \tag{125}$$

Finally, again using (34) and its analogue for Θ_ϵ we have

$$\begin{aligned}\Big| -\epsilon \int_D a(x/\epsilon)\nabla\hat{\theta}_\epsilon \cdot \nabla\Theta_\epsilon \, dx + k^2\epsilon \int_D n(x/\epsilon)\hat{\theta}_\epsilon \, \Theta_\epsilon \, dx + \epsilon \int_{\partial D} \nu \cdot a(x/\epsilon)\nabla\Theta_\epsilon^- \, \hat{\theta}_\epsilon^- \, ds_x \\ +\epsilon \int_{\partial D} \nu \cdot \nabla\hat{\theta}_\epsilon^+ \cdot \Theta_\epsilon^+ \, ds_x - \epsilon \int_{\partial D} \nu \cdot \nabla\Theta_\epsilon^+ \cdot \hat{\theta}_\epsilon^+ \, ds_x \Big| \\ \le \epsilon C \|\theta_\epsilon\|_{H^1(B_R)} \|\Theta_\epsilon\|_{H^1(B_R)} \le C_5 \|u_0\|_{H^2(D)} \|W_0\|_{H^2(D)}. \end{aligned} \tag{126}$$

Now combining (122), (123), (124), (125), and (126) together with the fact that the remainder of (120) is of order ϵ, and since $\|W_0\|_{H^2(D)} \le \|\phi\|_{L^2(B_R)}$ from the homogenized equations for W_0, we obtain that

$$\|\hat{\theta}_\epsilon\|_{L^2(B_R)} \le C\|u_0\|_{H^2(D)}.$$

Now, we notice that the difference between $\hat{\theta}_\epsilon$ and θ_ϵ is in the jump of derivatives across the boundary ∂D, which in a similar way to the second term of (122) can be bounded in the $L^2(B_R)$-norm by $\|u_0\|_{H^2(D)}$ independently of ϵ. This proves the theorem. ∎

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CORRECTION

The author of reference [32] is incorrect. The corect reference is as follows.

V. Vinoles, *Problèmes d'interface en presence de métamatériaux: Modélisation, analyse et simulations*, Thèse de doctoratès mathématiques, Université Paris-Saclay, 2016.