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IDENTIFICATION OF PARTIALLY COATED ANISOTROPIC BURIED OBJECTS USING ELECTROMAGNETIC CAUCHY DATA

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ABSTRACT. We consider the three dimensional electromagnetic inverse scattering problem of determining information about a target buried in a known inhomogeneous medium from a knowledge of the electric and magnetic fields corresponding to time harmonic electric dipoles as incident fields. The scattering object is assumed to be an anisotropic dielectric that is (possibly) partially coated by a thin layer of highly conducting material. The data is measured at a given surface containing the object in its interior. Our concern is to determine the shape of this scattering object and some information on the surface conductivity of the coating without any knowledge of the index of refraction of the inhomogeneity. No a priori assumption is made on the extent of the coating, i.e., the object can be fully coated, partially coated or not coated at all. Our method, introduced in [**14, 17**], is based on the linear sampling method and reciprocity gap functional for reconstructing the shape of the scattering object. The algorithm consists in solving a set of linear integral equations of the first kind for several sampling points and three linearly independent polarizations. The solution of these integral equations is also used to determine the surface conductivity.

1. Introduction. The inverse scattering problem we consider in this paper is to determine the shape and surface conductivity of an anisotropic dielectric that is partially coated by a thin conducting material from a knowledge of the scattered electromagnetic wave due to time-harmonic point sources. The scattering object is embedded in a known inhomogeneous background. Such problems arise in the detection of chemical waste deposits as well as certain problems arising in the nondestructive evaluation of urban infrastructure, testing the integrity of coatings, etc. The literature on this subject is particularly rich, see, e.g., [**4, 14, 26** and the references therein], and for a scholarly review of some aspects of its history we refer the reader to [**4**].

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Typically in such applications, the material properties of the scattering object are not known *a priori* and the newly developed class of electromagnetic imagining techniques, described as “qualitative methods in inverse scattering theory” [**11**], are designed to overcome this difficulty [**18, 21, 24, 26**]. These methods avoid any weak scattering assumption but, as opposed to nonlinear optimization techniques, only seek limited information about the scattering object and do not rely on any a priori knowledge of the geometry and physical properties of the scatterer. We focus our attention on linear sampling methods. A further problem arising in the electromagnetic imagining of buried objects is related to the ability to compute accurately the Green’s function of the background medium or, in other words, being able to distinguish between the scattered field due to the target and scattered field due to the background medium including interfaces, the antenna, etc. This task can be rather expensive and sometimes practically impossible. To address this issue, a new version of the *linear sampling method* based on the *reciprocity gap functional* (RG-LSM) was introduced by Colton and Haddar [**17**] for the scalar case and by Cakoni, Fares and Haddar [**14**] for the vector case.

The RG-LSM, like the classical linear sampling method, is based on the study of an *ill-posed integral equation of the first kind* but as opposed to the linear sampling method this equation does not involve the Green’s function for the background medium. This benefit is paid for by the need to measure the tangential component of *both* electric and magnetic total fields on the boundary of a bounded region containing the scatterer. Due to the knowledge of the electromagnetic Cauchy data of the total field on the boundary, it suffices to consider the scattering problem only in this region. In particular, if the medium inside this region is homogeneous, which is the case in this paper, all that is needed is the fundamental solution of an equation with constant coefficients. In addition, RG-LSM has the advantage of offering a more flexible mathematical framework than the classical linear sampling method. We note that the reciprocity gap functional is used in different ways to solve other inverse problems [**1, 27**].

This paper analyzes the RG-LSM for the case of a buried partially coated dielectric. The problem is to determine *both* the support of the inhomogeneity and the surface conductivity using the solution of a vector integral equation of the first kind for a set of sampling points.

This can be done without any a priori assumption on the coating and the index of refraction of the object. However, no information about the index of refraction and the support of the coating can be obtained by this method. The case of a buried partially coated perfect conductor was considered in [**8**] where the analysis was based on a variety of recent results on the mixed boundary value problems for electromagnetic inverse scattering [**9, 10, 13, 21**]. The analytical justification of the RG-LSM for the inverse problem considered in this paper is much more difficult and is only possible due to the recent results obtained by Cakoni and Haddar [**12**] on the well posedness of the *interior transmission problem* with mixed transmission conditions which generalize the results of Haddar [**23**]. As in [**14**], we will first consider the case when the electric and magnetic fields are both known on the entire boundary of an absorbing homogeneous region of the background media that is known a priori to contain the target. The case of an object buried in the earth is then handled by assuming that the part of the boundary below the surface of the earth is far away from the incident sources and hence we can assume that the total electric and magnetic fields are very small on this portion of the boundary.

2. Formulation of the direct and inverse scattering problem. We consider the scattering of a time-harmonic electromagnetic field of frequency ω by a scattering object embedded in a piecewise homogeneous background medium in $\mathbf{R}^3$. We assume that the magnetic permeability $\mu_0 > 0$ of the background medium is a positive constant whereas the electric permittivity $\epsilon(x)$ and conductivity $\sigma(x)$ are piecewise constant. Moreover, we assume that for $|x| = r > R$, for R sufficiently large, $\sigma = 0$ and $\epsilon(x) = \epsilon_0$. Then the electric field $\widetilde{\mathcal{E}}$ and magnetic field $\widetilde{\mathcal{H}}$ in the background medium satisfy the time-harmonic Maxwell's equations

$$\nabla \times \widetilde{\mathcal{E}} - i\omega\mu_0\widetilde{\mathcal{H}} = 0, \qquad \nabla \times \widetilde{\mathcal{H}} + (i\omega\epsilon(x) - \sigma(x))\widetilde{\mathcal{E}} = 0.$$

After an appropriate scaling [**20**] and elimination of the magnetic field we now have that in the background medium $\mathcal{E}$ satisfies

$$\operatorname{curl}\operatorname{curl}\mathcal{E} - k^2 n(x)\mathcal{E} = 0,$$

where $\widetilde{\mathcal{E}} = 1/\sqrt{\epsilon_0}\mathcal{E}$, $k^2 = \epsilon_0\mu_0\omega^2$ and $n(x) = 1/\sqrt{\epsilon_0}\,(\epsilon(x) + i\sigma(x)/\omega)$. Note that the piecewise constant function $n(x)$ satisfies $n(x) = 1$ for

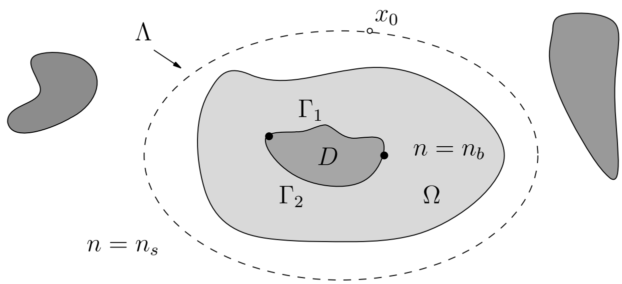

FIGURE 1. Example of the geometry of the scattering problem.

$r > R$, $\mathcal{R}e\,(n) > 0$ and $\mathcal{I}m\,(n) \geq 0$. The surfaces across which $n(x)$ is discontinuous are assumed to be piecewise smooth and closed.

The incident field is considered to be an electric dipole located at $x_0 \in \Lambda$ with polarization $p \in \mathbf{R}^3$, where Λ is an open surface (to be made precise later on) situated in a layer with constant index of refraction n_s, given by

$$E_e(x, x_0, p, k_s) := \frac{i}{k_s} \text{curl}_x \text{curl}_x\, p\, \frac{e^{ik_s|x-x_0|}}{4\pi|x-x_0|} \tag{1}$$

where $k_s^2 = k^2 n_s$. We denote by $\mathbf{G}(x, x_0)$ the free space Green's tensor of the background medium and define $E^i(x) := E^i(x, x_0, p) = \mathbf{G}(x, x_0)p$ which satisfies

$$\text{curl curl}\, E^i(x) - k^2 n(x) E^i(x) = p\, \delta(x - x_0) \;\text{ in } \mathbf{R}^3, \tag{2}$$

where δ denotes the Dirac distribution. Note that E^i can be written as

$$E^i(x) = E_e(x, x_0, p, k_s) + E_b^s(x) \tag{3}$$

where $E_b^s = E_b^s(\cdot, x_0, p)$ is the electric scattered field due to the background medium.

Now let D denote a scattering object embedded in the above piecewise homogeneous background such that $\mathbf{R}^3 \setminus \overline{D}$ is connected with piecewise smooth boundary ∂D. We denote by ν the outward unit normal defined almost everywhere on ∂D. Furthermore, we assume that the boundary $\partial D = \Gamma_1 \cup \Gamma_2$ is split into two open disjoint parts Γ_1 and Γ_2. The domain D is the support of an anisotropic object that is partially coated on the portion Γ_2 of the boundary by a very thin homogeneous layer of a highly conductive material. Again, after an appropriate scaling [**20**], the index of refraction of the scattering object is represented by a symmetric matrix-valued function denoted by $N(x)$, $x \in \overline{D}$, whose entries are bounded complex-valued functions such that

$$(4) \qquad \left.\begin{array}{r} \bar{\xi} \cdot \mathcal{I}m\,(N)\,\xi \;\geq 0 \;\text{ and} \\ \bar{\xi} \cdot \mathcal{R}e\,(N)\,\xi \geq \gamma|\xi|^2 \end{array}\right\} \quad \text{for all} \quad \left\{\begin{array}{l} \xi \in \mathbf{C}^3 \;\text{ and} \\ x \in \overline{D} \end{array}\right.$$

where γ is a positive constant. The surface conductivity of the coating is described by the positive constant $\eta > 0$, [**2**]. Note that we assume that the magnetic permitivity of the scattering object is the same as that of the background medium.

We now consider a bounded domain Ω such that $\overline{D}$ is contained in Ω and the open surface Λ is contained in $\mathbf{R}^3 \setminus \Omega$. Let $\partial\Omega$ denote the piecewise smooth boundary of Ω. Note that Λ may be a subset of $\partial\Omega$. We make the assumption that the medium inside the domain Ω containing the scattering object D is homogeneous with constant index of refraction n_b and define $k_b^2 = k^2 n_b$, see Figure 1 for the geometry of the problem.

Under the above assumptions we have that the interior electric field E^{int} and the exterior total electric field

$$E = E^s + E^i$$

satisfy the following transmission problem

$$(5) \qquad \operatorname{curl}\operatorname{curl} E^{\text{int}} - k^2 N(x) E^{\text{int}} = 0 \;\text{ in } D,$$

$$(6) \qquad \operatorname{curl}\operatorname{curl} E - k^2 n(x) E = p\,\delta(x - x_0) \;\text{ in } \mathbf{R}^3 \setminus \overline{D}$$

$$(7) \qquad \nu \times E - \nu \times E^{int} = 0 \;\text{ on } \partial D$$

$$(8) \qquad \nu \times \operatorname{curl} E - \nu \times \operatorname{curl} E^{int} = 0 \;\text{ on } \Gamma_1$$

$$(9) \qquad \nu \times \operatorname{curl} E - \nu \times \operatorname{curl} E^{int} = ik\eta(\nu \times E) \times \nu \;\text{ on } \Gamma_2.$$

In addition, the scattered field E^s satisfies the Silver Müller radiation condition

$$(10) \qquad \lim_{|x| \to \infty} (\operatorname{curl} E^s \times x - ik|x|E^s) = 0$$

uniformly in $\hat{x} = x/|x|$.

In order to set up the function spaces for the above scattering problem, we introduce the following definitions.

$$
\begin{aligned}
H(\mathrm{curl}, D) &:= \{u \in (L^2(D))^3 : \nabla \times u \in (L^2(D))^3\} \\
H_t^s(\partial D) &:= \{u \in (H^s(\partial D))^3 : \nu \cdot u = 0 \text{ on } \partial D\}, \quad s \geq 0 \\
L_t^2(\Gamma_2) &:= \{u|_{\Gamma_2} : u \in L_t^2(\partial D)\}
\end{aligned}
$$

where $L_t^2(\partial D)$ is the space of square integrable tangential vector-valued functions defined on ∂D. The space of solutions is constructed with aid of

$$
X(D, \Gamma_2) := \{u \in H(\mathrm{curl}, D) : \nu \times u|_{\Gamma_2} \in L_t^2(\Gamma_2)\} \tag{11}
$$

equipped with the norm

$$
\|u\|^2_{X(D,\Gamma_2)} = \|u\|^2_{H(\mathrm{curl},\, D)} + \|\nu \times u\|^2_{L^2(\Gamma_2)}. \tag{12}
$$

For the exterior domain $D_e := \mathbf{R}^3 \setminus \overline{D}$ we define the Fréchet spaces $H_{\mathrm{loc}}(\mathrm{curl}, D_e)$, respectively $X_{\mathrm{loc}}(D_e, \Gamma_2)$, consisting of functions belonging to $H(\mathrm{curl}, D_e \cap B_R)$, respectively $X(D_e \cap B_R, \Gamma_2)$, for every ball B_R containing D. It is known that the traces $\nu \times u|_{\partial D}$ and $\nu \times (u \times \nu)|_{\partial D}$ of $u \in H(\mathrm{curl}, D)$ (or $u \in H_{\mathrm{loc}}(\mathrm{curl}, D_e)$) are in the Hilbert spaces

$$
\begin{aligned}
H_{\mathrm{div}}^{-\frac{1}{2}}(\partial D) &:= \left\{u \in (H^{-\frac{1}{2}}(\partial D))^3,\ \nu \cdot u = 0,\ \mathrm{div}_{\partial D}\, u \in H^{-\frac{1}{2}}(\partial D)\right\} \\
H_{\mathrm{curl}}^{-\frac{1}{2}}(\partial D) &:= \left\{u \in (H^{-\frac{1}{2}}(\partial D))^3,\ \nu \cdot u = 0,\ \mathrm{curl}_{\partial D}\, u \in H^{-\frac{1}{2}}(\partial D)\right\}
\end{aligned}
$$

with $\mathrm{div}_{\partial D}$ and $\mathrm{curl}_{\partial D}$ denoting the surface curl and the surface divergence, respectively. Note that by an integration by parts we can define a duality relation between $H_{\mathrm{div}}^{-\frac{1}{2}}(\partial D)$ and $H_{\mathrm{curl}}^{-\frac{1}{2}}(\partial D)$ (see [**29**] in the case when the boundary is smooth, and [**5, 6**] in the case when the boundary is piecewise smooth).

The *direct scattering problem* can be formulated as follows: given E^i defined by (3) find $E^{\mathrm{int}} \in X(D, \Gamma_2)$ and $E^s = E - E^i \in X_{\mathrm{loc}}(\mathbf{R}^3 \setminus \overline{D}, \Gamma_2)$ such that (E^{int}, E) satisfies (5)–(6) in the distributional sense, (7)–(9) in the sense of traces and such that E^s satisfies the radiation condition (10). We remark that (5) and (6) are satisfied in the sense of distributions which implies the continuity of tangential components of E, $E^{\rm int}$ and $\operatorname{curl} E$, $\operatorname{curl} E^{\rm int}$ across the interface where the index of refraction has jumps. Using the variational approach developed in [**10**], it can be shown that the direct scattering problem has a unique solution.

Remark 2.1. It is also possible to consider the problem of objects buried in an unbounded multi-layered medium. In this case, the radiation condition and mathematical analysis of the forward problem becomes more complicated (see [**22**] for the case of two layered medium). However the following analysis of the inverse scattering problems remains the same.

In order to formulate the *inverse problem* we assume that *both* the tangential components of the total electric field $E = E(\cdot, x_0, p)$ and magnetic field $H = (1/(ik))\operatorname{curl} E$, respectively, are known on $\partial\Omega$. Furthermore, without loss of generality, we assume that Λ is a closed surface surrounding Ω situated in a layer with index of refraction n_s. By an analyticity argument the following analysis also holds true if the point sources are located on an open analytic surface provided it can be extended to a closed (analytic) surface as above.

The *inverse scattering problem* we are interested in is to determine the support D of the anisotropic inhomogeneity and η from a knowledge of the tangential components $\nu \times E$ and $\nu \times \operatorname{curl} E$ measured on $\partial\Omega$ for all points $x_0 \in \Lambda$ and two linearly independent polarizations p_1, $p_2 \in T_{x_0}$ where T_{x_0} is the tangent plane to Λ at x_0. Here ν denotes the outward unit normal to $\partial\Omega$. We remark that in what follows ν is always the outward unit normal to the surface under consideration unless otherwise stated. For later use, we shall denote

$$\mathcal{U} := \{E(\cdot, x_0, p),\ x_0 \in \Lambda,\ p \in T_{x_0}\}, \tag{13}$$

which represents the set of electric fields corresponding to the measurements.

In this paper we shall not study the uniqueness question of the inverse problem stated above. However, for the reader's convenience we note that, in the case of a homogeneous background with far field data, the uniqueness for D is proved in [**10**] whereas the uniqueness for η is shown in [**13**]. There is no uniqueness proof for D and η in the current case but we believe that the ideas of [**10, 13**] can be adapted to the inverse problem under consideration.

The reconstruction method we shall present is based on the knowledge of a solution to the so-called *interior transmission problem* associated with D and N. This is why we shall present this problem first and state some recent results on its solution. Another important ingredient for this method is a density result that is also proved in following section.

3. The interior transmission problem. In this section we recall some results on the *interior transmission problem* and develop some related properties which will be very useful in the analysis of the inverse problem. The interior transmission problem reads as follows: Given ϕ, ψ and τ, find E^0 and E^{int} such that

$$\operatorname{curl}\operatorname{curl}E^0 - k^2 n_b E^0 = 0 \ \text{ in } D \tag{14}$$

$$\operatorname{curl}\operatorname{curl}E^{\text{int}} - k^2 N(x) E^{\text{int}} = 0 \text{ in } D \tag{15}$$

$$\nu \times E^0 - \nu \times E^{int} = \phi \ \text{ on } \partial D \tag{16}$$

$$\nu \times \operatorname{curl}E^0 - \nu \times \operatorname{curl}E^{\text{int}} = \psi \ \text{ on } \Gamma_1 \tag{17}$$

$$\nu \times \operatorname{curl}E^0 - \nu \times \operatorname{curl}E^{\text{int}} = ik\eta\nu \times (E^0 \times \nu) + \psi + \tau \text{ on } \Gamma_2. \tag{18}$$

The study of this nonstandard boundary value problem is not straightforward and requires special attention in the case of Maxwell's equations due to the vectorial aspect of the unknowns, the anisotropy, and the curl curl operator. A variational technique based on a reformulation of this problem as a fourth order boundary value problem has been employed in [**23**] to study the existence and uniqueness in the case $\eta = 0$ and $n_b = 1$. This technique has been generalized in [**12**] to treat the present case and we shall now present the main results from this paper.

We first introduce the Hilbert space

$$H_{\text{inc}}(D) := \{u \in L^2(D)^3, \text{ s.t. curl curl}\, u - k^2 n_b u = 0 \text{ in } D\}$$

equipped with the $L^2(D)^3$ norm. We remark that if $u \in H_{\text{inc}}(D)$ then $\operatorname{curl}\operatorname{curl} u \in L^2(D)^3$. Therefore, if $u \in H_{\text{inc}}(D)$, one can define the traces $u \times \nu_{|\partial D}$ and $\operatorname{curl} u \times \nu_{|\partial D}$ as functions of $H_t^{-\frac{1}{2}}(\partial D)$ and

$H_t^{-\frac{3}{2}}(\partial D)$, respectively, see [**12**]. The proof of this is based on Stokes' formulas and the lifting result of Lemma 3.1 in [**23**]. We next define

$$H_{\text{inc}}(D,\Gamma_2) := \{u \in H_{\text{inc}}(D) \text{ such that } u \times \nu_{|\Gamma_2} \in L_t^2(\Gamma_2)\}$$

which is a Hilbert space equipped with the norm

$$\|u\|^2_{H_{\text{inc}}(D,\Gamma_2)} = \|u\|^2_{L^2(D)} + \|\nu \times u\|^2_{L_t^2(\Gamma_2)}.$$

The solution to the interior transmission problem is defined as functions $E^{\text{int}} \in L^2(D)$, $E^0 \in H_{\text{inc}}(D,\Gamma_2)$ such that $E^{\text{int}} - E^0 \in H(\text{curl}, D)$, curl $(E^{int} - E^0) \in H(\text{curl}, D)$ and (E^{int}, E^0) satisfies [**14–18**].

With this functional setting, and for data

$$\phi \in H_t^{\frac{3}{2}}(\partial D),\ \psi \in H_t^{\frac{1}{2}}(\partial D) \text{ and } \tau \in L_t^2(\Gamma_2),$$

by slightly modifying the analysis in [**12**] to account for the n_b complex, one can show that the Fredholm alternative applies to [**14–18**] under the following three conditions.

Condition 1:

$$M := (n_b I - N)^{-1} \text{ is a bounded matrix,} \tag{19}$$

Condition 2:

$$\begin{cases} \mathcal{I}m\,(M) & \text{is nonnegative on } D, \text{ and} \\ \mathcal{I}m\,(\widetilde{M}) - \{\mathcal{I}m\,(n_b M)\}^2\{\mathcal{I}m\,(M)\}^{-1} & \text{is nonnegative on } D, \end{cases} \tag{20}$$

and

Condition 3: Letting $\widetilde{M} := n_b N M$ *either*

$$\mathcal{R}e\,(M) \text{ and } \mathcal{R}e\,(\widetilde{M}) \text{ are nonnegative on } D \tag{21}$$

and the two matrices

$$\begin{cases} \mathcal{R}e\,(\widetilde{M}) - \{\mathcal{R}e\,(NM)\}^2\{\mathcal{R}e\,(M)\}^{-1}, \text{ and} \\ \mathcal{R}e\,(M) - \{\mathcal{R}e\,(NM)\}^2\{\mathcal{R}e\,(\widetilde{M})\}^{-1}, \end{cases} \tag{22}$$

are uniformly positive definite on D, *or*

$$-\mathcal{R}e\,(M) \text{ and } -\mathcal{R}e\,(\widetilde{M}) \text{ are nonnegative on } D \tag{23}$$

and the two matrices

$$\begin{cases} \{\mathcal{R}e\,(n_b M)\}^2\{\mathcal{R}e\,(M)\}^{-1} - \mathcal{R}e\,(\widetilde{M}), \text{ and} \\ \{\mathcal{R}e\,(n_b M)\}^2\{\mathcal{R}e\,(\widetilde{M})\}^{-1} - \mathcal{R}e\,(M), \end{cases} \tag{24}$$

are uniformly positive definite on D.

We remark that these conditions also apply to the cases where n_b is a matrix that commutes with N. When n_b and N are real scalars, one can easily verify that the first condition is equivalent to $0 < N < n_b$ and the third set of conditions is equivalent to $0 < n_b < N$. If only n_b is a real scalar the first set conditions is equivalent to $\mathcal{I}m\,(N) > 0$ on D.

Furthermore, one can prove that if the second matrix in (20) is uniformly positive definite on D, then the uniqueness of solutions holds true. In general, we shall exclude in our subsequent analysis the set of frequencies for which this uniqueness result is not valid. This leads us to the following definition.

Definition 3.1. The values of k for which the homogeneous interior transmission problem, i.e., [**14–18**] with $\phi = \psi = \tau = 0$, has a nontrivial solution are called *transmission eigenvalues.*

We remark that the questions of whether or not transmission eigenvalues exist and if so whether they form a discrete set are in general open.

In order to connect the interior transmission problem with a scattering problem we need to define appropriately the scattered field corresponding to the incident field in $H_{inc}(D, \Gamma_2)$. In particular, for $E^0 \in H_{\text{inc}}(D, \Gamma_2)$, let E^s satisfy

$$\text{curl curl}\, E^s - k^2 n_b E^s = 0 \text{ in } \mathbf{R}^3 \setminus \overline{D} \tag{25}$$

$$\text{curl curl}\, E^s - k^2 N E^s = k^2(N - n_b I)E^0 \text{ in } D \tag{26}$$

$$\nu \times E^s_+ - \nu \times E^s_- = 0 \text{ on } \partial D \tag{27}$$

$$\nu \times \operatorname{curl} E^s_+ - \nu \times \operatorname{curl} E^s_- = 0 \text{ on } \Gamma_1 \tag{28}$$

$$\nu \times \operatorname{curl} E^s_+ - \nu \times \operatorname{curl} E^s_- = ik\eta(E^s_+ + E^0)_\top \text{ on } \Gamma_2 \tag{29}$$

$$\lim_{r\to\infty} (\operatorname{curl}\ E^s \times x - ik_b r E^s) = 0 \tag{30}$$

where we used the notation $u_\top := \nu \times (u \times \nu)$, and E^s_+ and E^s_- denote the limit of E^s approaching ∂D from $\mathbf{R}^3 \setminus \overline{D}$ and D, respectively. Setting $\widetilde{N} = N$ in D and $\widetilde{N} = n_b$ outside D, it can be shown, see [**10, 28**], that the above transmission problem is equivalent to the following variational problem: Find $E^s \in X_{\text{loc}}(\mathbf{R}^3 \setminus \overline{\Gamma}_2, \Gamma_2)$ satisfying

$$\int_{\mathbf{R}^3} (\operatorname{curl} E^s \cdot \operatorname{curl} U - k^2 \widetilde{N} E^s \cdot U)\, ds - ik\eta \int_{\Gamma_2} E^s_\top \cdot U_\top\, ds \\ = k^2 \int_D (N - n_b I) E^0 \cdot U\, dv + ik\eta \int_{\Gamma_2} E^0_\top \cdot U_\top\, ds, \tag{31}$$

together with the radiation condition (30), for all $U \in X_{\text{loc}}(\mathbf{R}^3 \setminus \overline{\Gamma}_2, \Gamma_2)$ with compact support. The bilinear form corresponding to the above variational problem is studied in detail in [**10**], see also [**28**]. Here one needs to use the Dirichlet to Neumann mapping in order to reduce the problem in a bounded domain. The analysis of the variational problem shows that the Fredholm alternative can be applied to the equation (31) and therefore to (25)–(30). In particular, the solution E^s satisfies the following a priori estimate:

$$\|E^s\|_{X(B_R \setminus \overline{\Gamma}_2, \Gamma_2)} \le C \left(\|E^0\|_{L^2(D)} + \|\nu \times E^0\|_{L^2_t(\Gamma_2)} \right) \tag{32}$$

where $C > 0$ is a positive constant independent of E^0 and where B_R is a ball of radius R.

We now show that $H_{\text{inc}}(D, \Gamma_2)$ is the closure of the space of entire solutions to Maxwell's equations. Let us introduce

$$M^m_n(x) := \operatorname{curl}(x u^m_n(x)) \quad \text{and} \\ u^m_n(x) := j_n(k_b|x|) Y^m_n(x/|x|) \\ \text{where } \{Y^m_n,\ m = -n, \dots, n,\ n = 0, 1, \dots\}$$

is the set of orthonormal spherical harmonics and j_n denotes the spherical Bessel function of order n. The following lemma is fundamental for the theoretical foundation of our inverse scheme.

Lemma 3.2. *The space*

$$H := \text{span}\,\{M_n^m,\, \text{curl}\, M_n^m :\; n = 1, 2, \ldots, m = -n, \ldots, n\}$$

is dense in $H_{\text{inc}}(D, \Gamma_2)$.

Proof. The proof follows the ideas of the proof of Lemma 4.3 in [**23**]. Let $\overline{H}$ be the closure of H in $H_{\text{inc}}(D, \Gamma_2)$, and let $E^0 \in H_{\text{inc}}(D, \Gamma_2)$ be in the orthogonal complement of $\overline{H}$. We define

$$E(x) = \int_D \mathcal{G}(x,y) E^0(y)\, dy + \int_{\Gamma_2} \mathcal{G}(x,y) \left[(\nu \times E^0) \times \nu\right] ds(y),$$
$$x \in \mathbf{R}^3 \setminus \overline{\Gamma}_2$$

where

$$\mathcal{G}(x,y) = \Phi(x,y,k_b) I + \frac{1}{k_b^2}\, \text{grad}_x\, \text{div}_x\, \Phi(x,y,k_b) I \tag{33}$$

with

$$\Phi(x,y,k_b) := \frac{1}{4\pi} \frac{e^{ik_b|x-y|}}{|x-y|}, \quad x \neq y.$$

By definition we have that

$$\text{curl curl}\, E - k_b^2 E = E_0 \text{ in } D \tag{34}$$
$$\text{curl curl}\, E - k_b^2 E = 0 \text{ in } \mathbf{R}^3 \setminus \overline{D}. \tag{35}$$

Furthermore, using the jump relations of single layer potential with L_t^2 densities [**20, 25**] we also have

$$\nu \times E_+ - \nu \times E_- = 0 \text{ on } \partial D \tag{36}$$
$$\nu \times \text{curl}\, E_+ - \nu \times \text{curl}\, E_- = 0 \text{ on } \Gamma_1 \tag{37}$$
$$\nu \times \text{curl}\, E_+ - \nu \times \text{curl}\, E_- = (\nu \times E^0) \times \nu \text{ on } \Gamma_2 \tag{38}$$

where E_+ and E_- denote the limit of E approaching ∂D from $\mathbf{R}^3 \setminus \overline{D}$ and D, respectively. Now, since in $\mathbf{R}^3 \setminus \overline{D}$

$$E(x) = \frac{1}{k_b^2} \text{curl curl} \left(\int_D \Phi(x,y,k_b) E^0(y)\, dy \right.$$
$$\left. + \int_{\Gamma_2} \Phi(x,y,k_b) \left[(\nu \times E^0) \times \nu\right] ds(y) \right),$$

from the expansion of Theorem 6.27 in [**20**] for $\Phi(x,y,k_b)$ and the fact that E^0 is orthogonal to H with respect to the inner product $(\cdot,\cdot)_{L^2(D)} + (\cdot,\cdot)_{L^2_t(\Gamma_2)}$ we conclude that $E = 0$ in $\mathbf{R}^3 \setminus \overline{D}$. Hence, taking the $L^2(D)$ inner product of (34) with E^0 and the $L^2(\Gamma_2)$ inner product of (38) with $(\nu \times E^0) \times \nu$, we obtain that

$$\|E^0\|^2_{L^2(D)} + \|\nu \times E^0\|^2_{L^2_t(\Gamma_2)} = \left(\operatorname{curl}\operatorname{curl} E - k_b^2\, E,\, E_0\right)_{L^2(D)} + \left(\nu \times \operatorname{curl} E,\, E^0\right)_{L^2_t(\Gamma_2)}. \tag{39}$$

Finally, in view of the zero boundary conditions (36)–(37) and the fact that the test functions are dense in $X(D,\Gamma_2)$, we obtain after integrating by parts that the right hand side of (39) is zero since $\operatorname{curl}\operatorname{curl} E^0 - k_b^2 E^0 = 0$ in the distribution sense. Hence $E^0 = 0$ which ends the proof. □

4. The reciprocity gap operator. Our inverse scheme is based on the construction of approximating solutions to the interior transmission problem from the boundary data of the inverse problem. These solutions are computed by solving an integral equation of the first kind constructed from the so-called *reciprocity gap operator*. This section is devoted to the definition and description of some properties of this operator. Let us define

$$\mathbf{H}(\Omega) := \left\{W \in H(\operatorname{curl},\Omega),\ \text{such that}\ \operatorname{curl}\operatorname{curl} W - k_b^2 W = 0\ \text{in}\ \Omega\right\}.$$

The expression of the reciprocity gap operator is obtained from the so-called *gap reciprocity functional* $\mathcal{R}$ defined on $\mathcal{U} \times \mathbf{H}(\Omega)$ by

$$\mathcal{R}(E,W) := \int_{\partial\Omega} \{(\nu \times E)\cdot \operatorname{curl} W - (\nu \times W)\cdot \operatorname{curl} E\}\, ds, \tag{40}$$

where the integrals are interpreted in the sense of the duality between $H^{-\frac{1}{2}}_{\text{div}}(\partial\Omega)$ and $H^{-\frac{1}{2}}_{\text{curl}}(\partial\Omega)$. Notice that in the absence of a scattering object D, the right hand side of (40) is zero for all point sources, whereas if D is present, this right hand side defines a nonzero function of the source location x_0 and the source polarization p. This observation motivates the idea of using (40) to set up an integral equation whose solution is an indicator function for D. To this end, we define the *reciprocity gap operator* $R : \mathbf{H}(\Omega) \to L^2_t(\Lambda)$ by

$$R(W)(x_0)\cdot p = \mathcal{R}(E(\cdot,x_0,p),W) \tag{41}$$

for all $x_0 \in \Lambda$ and $p \in T_{x_0}$. Notice that this definition makes sense since E depends linearly on the polarization p and so does $\mathcal{R}$. It is easy to prove, see e.g., [**11**, Theorem 4.8] for a similar result in the scalar case, that the operator $R : \mathbf{H}(\Omega) \to L^2_t(\Lambda)$ is compact. We shall prove in the following two lemmas that this operator is also injective with dense range if there are no eigenvalues of the interior transmission problem.

Lemma 4.1. *Assume that k is not a transmission eigenvalue for D. Then the operator $R : \mathbf{H}(\Omega) \to L^2_t(\Lambda)$ defined by (41) is injective.*

Proof. From (41), $RW = 0$ means $\mathcal{R}(E(\cdot, x_0, p), W) = 0$ for all $(x_0, p) \in \Lambda \times T_{x_0}$. Using the second vector Green's formula and the transmission conditions (7)–(10), we have that

$$\begin{aligned} 0 &= \int_\Gamma (\nu \times E)\cdot \operatorname{curl} W - (\nu \times W)\cdot \operatorname{curl} E \, ds \\ &= \int_{\partial D} (\nu \times E)\cdot \operatorname{curl} W - (\nu \times W)\cdot \operatorname{curl} E \, ds \\ &= \int_{\partial D} (\nu \times E^{\text{int}})\cdot \operatorname{curl} W - (\nu \times W)\cdot \operatorname{curl} E^{\text{int}} \, ds \\ &\quad + ik\eta \int_{\Gamma_2} (\nu \times E)\cdot(\nu \times W)\, ds. \end{aligned} \tag{42}$$

Now (see [**10**]) let $\widetilde{E}^{\text{int}} \in X(D, \Gamma_2)$ and $\widetilde{E} \in X_{\text{loc}}(\mathbf{R}^3 \setminus \overline{D}, \Gamma_2)$ be the unique solution to

$$\operatorname{curl}\operatorname{curl} \widetilde{E}^{\text{int}} - k^2 N(x)\widetilde{E}^{\text{int}} = 0 \text{ in } D \tag{43}$$

$$\operatorname{curl}\operatorname{curl} \widetilde{E} - k^2 n(x)\widetilde{E} = 0 \text{ in } \mathbf{R}^3 \setminus \overline{D} \tag{44}$$

$$\nu \times (\widetilde{E} + W) - \nu \times \widetilde{E}^{\text{int}} = 0 \text{ on } \partial D \tag{45}$$

$$\nu \times \operatorname{curl}(\widetilde{E} + W) - \nu \times \operatorname{curl} \widetilde{E}^{\text{int}} = 0 \text{ on } \Gamma_1 \tag{46}$$

$$\nu \times \operatorname{curl}(\widetilde{E} + W) - \nu \times \operatorname{curl} \widetilde{E}^{\text{int}} = ik\eta \left[\nu \times (\widetilde{E} + W)\right] \times \nu \text{ on } \Gamma_2 \tag{47}$$

$$\lim_{r\to\infty} \left(\operatorname{curl} \widetilde{E} \times x - ikr\widetilde{E}\right) = 0.$$

Expressing W in the last equation of (42) in terms of $\widetilde{E}$ and $\widetilde{E}^{\text{int}}$ using (45), (46) and (47), and using the fact that E^{int} and $\widetilde{E}^{\text{int}}$ satisfy the same equation in D, we obtain that

$$
\begin{aligned}
0 = \int_{\partial D} & (\nu \times \widetilde{E}) \cdot \operatorname{curl} E^{\mathrm{int}} - (\nu \times E^{\mathrm{int}}) \cdot \operatorname{curl} \widetilde{E} \, ds \\
& - ik\eta \int_{\Gamma_2} (\nu \times \widetilde{E}) \cdot (\nu \times E^{\mathrm{int}}) \, ds.
\end{aligned} \tag{48}
$$

Next, expressing E^{int} in terms of the total exterior field

$$E = E^s + \mathbf{G}(\cdot, x_0)p$$

using the transmission conditions (7)–(10), using the fact that E^s and $\widetilde{E}$ are radiating solutions to the same equation outside D and, finally, using the Stratton-Chu representation formula outside D [**28**] we can rewrite (48) as

$$
\begin{aligned}
0 &= \int_{\partial D} (\nu \times \widetilde{E}) \cdot \operatorname{curl} (E^s + \mathbf{G}(\cdot, x_0)p) \\
&\quad - [\nu \times (E^s + \mathbf{G}(\cdot, x_0)p)] \cdot \operatorname{curl} \widetilde{E} \, ds \\
&= \int_{\partial D} (\nu \times \widetilde{E}) \cdot \operatorname{curl} \mathbf{G}(\cdot, x_0)p) - (\nu \times \mathbf{G}(\cdot, x_0)p) \cdot \operatorname{curl} \widetilde{E} \, ds \\
&= -p \cdot \widetilde{E}(x_0).
\end{aligned}
$$

Since p is an arbitrary polarization in the tangent plane to Λ at x_0, we obtain that $\nu \times \tilde{E}(x_0) = 0$ for all $x_0 \in \Lambda$. Furthermore, since $\tilde{E}$ is a radiating solution to $\operatorname{curl} \operatorname{curl} \widetilde{E} - k^2 n(x) \widetilde{E} = 0$ outside the domain bounded by Λ and satisfies $\nu \times \widetilde{E} = 0$ on Λ, we can conclude by the uniqueness theorem for scattering by a perfect conductor that $\widetilde{E} = 0$ outside the domain bounded by Λ. Finally, from the unique continuation principle, we have that $\widetilde{E} = 0$ outside D as well. Therefore, $E_0 := W$ and $E^{\mathrm{int}} := \widetilde{E}^{\mathrm{int}}$ satisfy the homogeneous interior transmission problem, i.e., (14)–(18) with $\phi = \psi = \tau = 0$, whence from the assumption that k is not a transmission eigenvalue, we finally obtain that $W = 0$ in D. This proves the lemma. □

Lemma 4.2. *Assume that k is not a transmission eigenvalue for D. Then the operator $R : \mathbf{H}(\Omega) \to L^2_t(\Lambda)$ defined by* (41) *has dense range.*

Proof. Consider $\beta \in L^2_t(\Lambda)$, and assume that

$$(RW, \beta)_{L^2_t(\Lambda)} = 0 \text{ for all } W \in \mathbf{H}(\Omega).$$

From (41) and the bi-linearity of $\mathcal{R}$, one has

$$(RW, \beta)_{L^2_t(\Lambda)} = \int_\Lambda \mathcal{R}(E(\cdot, x_0, \alpha(x_0)), W)\, ds(x_0) = \mathcal{R}(\mathcal{E}, W),$$

where

$$\mathcal{E}(x) = \int_\Lambda E(x, x_0, \alpha(x_0))\, ds(x_0) \tag{49}$$

and $\alpha = (\beta \cdot p)\, p$. Letting

$$\mathcal{E}^{\text{int}}(x) = \int_\Lambda E(x, x_0, \alpha(x_0))\, ds(x_0), \tag{50}$$

by linearity we have that $\mathcal{E}$ and $\mathcal{E}^{\text{int}}$ satisfy the scattering problem (5)–(10). Using the second vector Green's formula and the transmission conditions for $\mathcal{E}$ and $\mathcal{E}^{\text{int}}$, one concludes that

$$\begin{aligned} 0 = \mathcal{R}(\mathcal{E}, W) = k^2 \int_D (N - n_b I)\, \mathcal{E}^{\text{int}} \cdot W\, dx \\ + ik\eta \int_{\Gamma_2} (\nu \times \mathcal{E})(\nu \times W)\, ds \end{aligned} \tag{51}$$

for all $W \in \mathbf{H}(\Omega)$. Since $\mathbf{H}(\Omega)$ contains the space H of Lemma 3.2, we conclude from this lemma and (5.1) that $\mathcal{E}^{\text{int}} = 0$ in D and $\nu \times \mathcal{E}|_{\Gamma_2} = 0$. Then the transmission conditions imply that both $\nu \times \mathcal{E} = 0$ and $\nu \times \text{curl } \mathcal{E} = 0$ on ∂D. This means that the extension of $\mathcal{E}$ by 0 inside D satisfies Maxwell's equations inside the domain bounded by Λ with the index n set equal to n_b inside D. From the unique continuation principle one has that $\mathcal{E}$ is 0 inside the domain bounded by Λ and outside D. Noting that

$$\mathcal{E}(x) = \int_\Lambda (E^s(x, x_0, \alpha(x_0)) + \mathbf{G}(x, x_0)\alpha(x_0))\, ds(x_0),$$

one concludes that $\mathcal{E} \times \nu$ is continuous across Λ. The uniqueness theorem for the exterior problem for Maxwell's equations with boundary data $\nu \times \mathcal{E} = 0$ on Λ implies that $\mathcal{E} = 0$ outside the domain bounded by Λ as well. Finally, from the jump relations of the vector potential across Λ [**20**], we have that

$$0 = \operatorname{curl} \mathcal{E}|_{\Lambda^+} - \operatorname{curl}\ \mathcal{E}|_{\Lambda^-} = -\alpha \text{ on } \Lambda.$$

Hence $(\beta \cdot p)\, p = 0$ for all p tangential to Λ which implies that $\beta = 0$. $\square$

5. The inverse scheme.

5.1. *The sampling integral operator.* The inverse scheme is based on the construction of an integral equation using the reciprocity operator of the previous section and a parametric family of solutions in $\mathbf{H}(\Omega)$ which satisfy certain properties to be made precise later. To fix our ideas we shall consider here the case of single layer potentials $A\varphi$ defined by

$$(A\varphi)(x) := \operatorname{curl}\operatorname{curl} \int_{\tilde{\Lambda}} \varphi(y)\Phi(x, y, k_b)\, ds, \quad \varphi \in L^2_t(\tilde{\Lambda}) \tag{52}$$

where $\tilde{\Lambda}$ is a part of the analytic boundary of some simply connected domain containing Ω in its interior. The *sampling integral operator*, $S : L^2_t(\tilde{\Lambda}) \to L^2_t(\Lambda)$ is defined by

$$S\varphi := R\, A\varphi \text{ for } \varphi \in L^2_t(\tilde{\Lambda}). \tag{53}$$

Using the definition of R and interchanging the order of integration, it is readily seen that S is an integral operator whose (matrix) kernel $s(x_0, y)$ is defined by

$$(s(x_0, y) \cdot q) \cdot p = \mathcal{R}(E(\cdot, x_0, p), \operatorname{curl}\operatorname{curl}\ (q\, \Phi(\cdot, y, k_b)))$$

for $(x_0, y) \in \Lambda \times \tilde{\Lambda}$ and $(p, q) \in T_{x_0} \times \tilde{T}_y$ where $\tilde{T}_y$ denotes the tangent plane to $\tilde{\Lambda}$ at y.

The key property needed for the parametric family of solutions is that it is a dense subset of $H_{\text{inc}}(D, \Gamma_2)$. This is the case for single layer potentials as shown by the following lemma, which uses the density result of Lemma 3.2.

Lemma 5.1. *The set* $\{A\varphi,\, \varphi \in L^2_t(\tilde{\Lambda})\}$ *is dense in* $H_{\rm inc}(D, \Gamma_2)$.

Proof. It suffices only to prove that the set of $\{A\varphi,\, \varphi \in L^2_t(\tilde{\Lambda})\}$ is complete in $L^2_t(\partial B_R)$ where B_R is a large ball containing D and contained in the domain bounded by the analytic extension of $\tilde{\Lambda}$ such that k_b is not a Maxwell eigenvalue for B_R (which is not a restriction since we can always find such a ball!). The result of the lemma is then obtained by combining Lemma 3.2 and Theorem 7.9 in [**20**]. To this end, noting that

$$\mathrm{curl}_x\, \mathrm{curl}_x \int_{\tilde{\Lambda}} \varphi(y)\Phi(x, y, k_b)\, ds(y) = -ik_b \int_{\tilde{\Lambda}} \mathcal{G}(x, y)^\top \varphi(y)\, ds(y)$$

where $\mathcal{G}$ is given by (33) and $\top$ denotes the transposed matrix, we take $a \in L^2_t(\partial B_R)$ such that

$$\int_{\partial B_R} \overline{a}(x) \cdot \int_{\tilde{\Lambda}} \mathcal{G}(x, y)^\top \varphi(y)\, ds(y)\, ds(x) = 0 \tag{54}$$

for every $\varphi \in L^2_t(\tilde{\Lambda})$. We want to show that $a = 0$. By interchanging the order of integration we arrive at

$$\int_{\tilde{\Lambda}} \varphi(y) \cdot \int_{\partial B_R} \mathcal{G}(x, y)\overline{a}(x)\, ds(x)\, ds(y) = 0$$

for every $\varphi \in L^2_t(\tilde{\Lambda})$. This implies that

$$\nu \times \int_{\partial B_R} \mathcal{G}(x, y)\overline{a}(x)\, ds(x) = 0 \text{ on } \tilde{\Lambda},$$

which by analyticity holds true on the closed analytic extension of $\tilde{\Lambda}$. Hence, using the uniqueness of the exterior Maxwell problem and analytic continuation we have that the surface potential

$$(Va)(y) := \int_{\partial B_R} \mathcal{G}(x, y)\overline{a}(x)\, ds(x), \quad y \in \mathbf{R}^3 \setminus \partial B_R, \quad a \in L^2_t(\partial B_R)$$

is zero outside ∂B_R. By continuity of the tangential component of Va across ∂B_R and the fact that k_b is not a Maxwell eigenvalue for B_R, we conclude that $Va = 0$ in B_R as well. Finally by applying the jump relation for $\nu \times \nabla \times (Va)$ across ∂B_R [**20**], we obtain that $a \equiv 0$. This ends the proof. □

A consequence of this Lemma is that the sampling operator S has dense range provided that k is not a transmission eigenvalue. The proof of this result follows that of Lemma 4.2 where W is replaced by $A\varphi$ and the density result of Lemma 3.2 is replaced with that of Lemma 5.1. On the other hand, as an easy exercise on the use of the unique continuation principle and the uniqueness of a solution to the exterior Dirichlet problem for Maxwell's equations, one can verify that $A : L^2_t(\tilde{\Lambda}) \to \mathbf{H}(\Omega)$ is injective. Lemma 4.1 therefore implies that S is injective provided that k is not a transmission eigenvalue. We summarize these results in the following lemma.

Lemma 5.2. *The sampling integral operator $S : L^2_t(\tilde{\Lambda}) \to L^2_t(\Lambda)$ is compact. It is also injective with dense range provided that k is not a transmission eigenvalue for D.*

Remark 5.1. Alternatively one can use, instead of the single layer potential, the electric Herglotz function $\mathcal{H}g$ defined by

$$\mathcal{H}g(x) := \int_{S^2} g(d) e^{ik_b d\cdot x}\, ds(d), \quad g \in L^2_t(S^2) \tag{55}$$

where S^2 is the unit sphere, and define the sampling operator as

$$\widetilde{S} : L^2_t(S^2) \longrightarrow L^2_t(\Lambda), \quad \text{such that} \quad \widetilde{S}g = R\mathcal{H}g.$$

Subsection 3.2 in [**7**] together with Lemma 3.2 imply that the set $\{\mathcal{H}g,\, g \in L^2_t(S^2)\}$ is dense in $H_{\mathrm{inc}}(D, \Gamma_2)$ as well. Therefore, the analysis which follows also holds with S replaced by $\widetilde{S}$.

5.2. *The determination of D.* We are now in possession of all ingredients to describe a sampling algorithm to determine D without knowing N and η. The only requirement is that the interior transmission problem is well posed within the functional framework defined in Section 3.

Let $z \in \mathbf{R}^3$ be a sampling point and let $q \in \mathbf{R}^3 \setminus \{0\}$ be an arbitrary vector (that will be kept fixed). Let

$$E_e(x, z, q, k_b) := \frac{i}{k_b} \operatorname{curl}_x \operatorname{curl}_x q\, \Phi(x, z, k_b), \tag{56}$$

be the electric field of the electric dipole corresponding to k_b. We associate with this dipole the function $\ell_z \in L^2_t(\Lambda)$ defined by

$$\ell_z(x_0) \cdot p = \mathcal{R}(E(\cdot, x_0, p),\ E_e(\cdot,\ z, q, k_b)) \tag{57}$$

for $x_0 \in \Lambda$ and $p \in T_{x_0}$. Our proposed sampling algorithm consists in seeking for each sampling point z an approximate solution of the ill-posed integral equation

$$S\varphi_z = \ell_z \text{ in } L^2_t(\Lambda). \tag{58}$$

In view of Lemma 5.2, an approximate solution can be constructed using any regular regularization, see [**20**]. The following theorem suggests that the norm of this approximate solution should be much larger for z outside D than for z in D, allowing us to use this norm as an indicator of the location and the shape of D. The idea behind the different behavior of this approximate solution for z inside and outside D is that when $z \in D$ an approximate solution to (58) can be constructed from a sequence $A\varphi_z^\varepsilon$ approaching E_z^0 where $(E_z^0, E_z^{\text{int}})$ is a solution to the interior transmission problem

$$\operatorname{curl} \operatorname{curl} E_z^0 - k^2 n_b E_z^0 = 0 \text{ in } D \tag{59}$$

$$\operatorname{curl} \operatorname{curl} E_z^{\text{int}} - k^2 N(x) E_z^{\text{int}} = 0 \text{ in } D \tag{60}$$

$$\nu \times E_z^0 - \nu \times E_z^{\text{int}} = \nu \times E_e(\cdot, z, q, k_b) \text{ on } \partial D \tag{61}$$

$$\nu \times \operatorname{curl} E_z^0 - \nu \times \operatorname{curl} E_z^{\text{int}} = \nu \times \operatorname{curl} E_e(\cdot, z, q, k_b) \text{ on } \Gamma_1 \tag{62}$$

$$\nu \times \operatorname{curl} E_z^0 - \nu \times \operatorname{curl} E_z^{\text{int}} = ik\eta\nu \times \left[\left(E_z^0 - E_e(\cdot, z, q, k_b)\right) \times \nu\right] + \nu \times \operatorname{curl} E_e(\cdot, z, q, k_b) \text{ on } \Gamma_2, \tag{63}$$

whereas for $z \notin D$ this construction cannot hold. Let us remark that (58) can be equivalently written, using the definition of S and ℓ_z, in the form

$$\mathcal{R}(E, A\varphi_z) = \mathcal{R}(E, E_e(\cdot,\ z, q, k_b)) \text{ for all } E \in \mathcal{U}. \tag{64}$$

We also emphasize that, as opposed to the classical linear sampling method, the background Green's function $\mathbf{G}(\cdot, x_0)p$ does not appear in the integral equation (64).

Theorem 5.3. *Assume that k, N and n_b are such that the interior transmission problem* (59)–(63) *is well posed* (*for instance, under conditions* (19)–(20)). *Then*

1. *For $z \in D$ and a given $\epsilon > 0$, there exists a $\varphi_z^\epsilon \in L_t^2(\tilde{\Lambda})$ such that*

$$\|S\varphi_z^\epsilon - \ell_z\|_{L_t^2(\Lambda)} < \epsilon$$

and the corresponding single layer potential $A\varphi_z^\epsilon$ converges to E_z^0 in $H_{\mathrm{inc}}(D,\Gamma_2)$ as $\epsilon \to 0$ where $(E_z^0, E_z^{\mathrm{int}})$ is the solution of (59)–(63).

Moreover, for a fixed $\epsilon > 0$, we have that

$$\lim_{z\to\partial D} \|A\varphi_z^\epsilon\|_{H_{\mathrm{inc}}(D,\Gamma_2)} = \infty \quad \text{and} \quad \lim_{z\to\partial D} \|\varphi_z^\epsilon\|_{L_t^2(\tilde{\Lambda})} = \infty.$$

2. *For $z \in \mathbf{R}^3 \setminus \overline{\Delta}$ and a given $\epsilon > 0$, every $\varphi_z^\epsilon \in L_t^2(\tilde{\Lambda})$ that satisfies*

$$\|S\varphi_z^\epsilon - \ell_z\|_{L_t^2(\Lambda)} < \epsilon$$

is such that

$$\lim_{\epsilon\to 0} \|A\varphi_z^\epsilon\|_{H_{\mathrm{inc}}(D,\Gamma_2)} = \infty \quad \text{and} \quad \|\varphi_z^\epsilon\|_{L_t^2(\tilde{\Lambda})} = \infty.$$

Proof. Consider $z \in D$, and let E_z^0 and E_z^{int} be the solution to the interior transmission problem (59)–(63). Since both $W \in \mathbf{H}(\Omega)$ and $E_e(\cdot, z, q, k_b)$ satisfy $\operatorname{curl}\operatorname{curl} U - k_b U = 0$ in $\Omega \setminus \overline{D}$, integrating by parts and using the equations satisfied by the total electric field we have that

$$\mathcal{R}(E,W) = k^2 \int_D (N - n_b I) E^{\mathrm{int}} \cdot W \, dx + ik \int_{\Gamma_2} \eta \, (\nu \times E)(\nu \times W) \, ds. \tag{65}$$

From Lemma 4.1 we see that $\mathcal{R}(E,W) = \mathcal{R}(E, E_e(\cdot, z, q, k_b))$ has a unique solution W if and only W coincides with E_z^0 in D. But this is in general not possible. However, from Lemma 5.1, for every $\epsilon > 0$ we can find a $\varphi_z^\epsilon \in L_t^2(\tilde{\Lambda})$ such that

$$\|E_z^0 - A\varphi_z^\epsilon\|_{H_{\text{inc}}(D,\Gamma_2)} < \epsilon$$

which implies

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, W)\|_{L_t^2(\Lambda)} < C\epsilon$$

for some positive constant $C > 0$, whence

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot,\, z, q, k_b))\|_{L_t^2(\Lambda)} < \epsilon.$$

Furthermore, by construction, $A\varphi_z^\epsilon$ converges to E_z^0 in the $H_{\text{inc}}(D, \Gamma_2)$ norm as $\epsilon \to 0$. Next we observe that $E^s := -E_e(\cdot,\, z, q, k_b)$ in $\mathbf{R}^3 \setminus \overline{D}$ and $E^s := E_z^{\text{int}} - E_z^0$ in D satisfy the scattering problem (25)–(30) with $E^0 := E_z^0$. From the discussion in Section 2 we have that

$$\|E_e(\cdot,\, z, q, k_b)\|_{X(B_R\setminus\overline{D})} \le C\|E^0\|_{H_{\text{inc}}(D,\Gamma_2)}.$$

Hence, due to the singularity of the electric dipole, we can conclude that $\|E^0\|_{H_{\text{inc}}(D,\Gamma_2)} \to \infty$ as $z \to \partial D$ and hence so does $\|A\varphi_z^\epsilon\|_{H_{\text{inc}}(D,\Gamma_2)}$ and $\|\varphi_z^\epsilon\|_{L_t^2(\tilde{\Lambda})}$.

Now we consider $z \in \Omega \setminus \overline{D}$ and let φ_z^ϵ and its corresponding single layer potential $A\varphi_z^\epsilon$ be such that

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot,\, z, q, k_b))\|_{L_t^2(\Lambda)} < \epsilon. \tag{66}$$

Note that from Lemma 4.2 we can always find such a $A\varphi_z^\epsilon$. Assume to the contrary that $\|A\varphi_z^\epsilon\|_{H_{\text{inc}}(D,\Gamma_2)} < C$ where the positive constant C is independent of ϵ. Noting that the total field can be written as $E(\cdot,\, x_0, p)E^s(\cdot,\, x_0, p) + \mathbf{G}(\cdot, x_0)p$ and integrating by parts, we obtain that

$$\begin{aligned}
\mathcal{R}(E, E_e(x, z, q, k_b)) = &\int_{\partial\Omega} (\nu \times E^s(x, x_0, p)) \cdot \operatorname{curl} E_e(x, z, q, k_b)\, ds_x \\
&- \int_{\partial\Omega} (\nu \times E_e(x, z, q, k_b)) \cdot \operatorname{curl} E^s(x, x_0, p)\, ds_x \\
&+ \int_{\partial\Omega} (\nu \times \mathbf{G}(x, x_0)p) \cdot \operatorname{curl} E_e(x, z, q, k_b)\, ds_x \\
&- \int_{\partial\Omega} (\nu \times E_e(x, z, q, k_b)) \cdot \operatorname{curl} \mathbf{G}(x, x_0)p\, ds_x.
\end{aligned}$$

Due to the symmetry of the background Green's function, $E^s(x, x_0, p)$ as a function of x_0 satisfies

$$\operatorname{curl}_{x_0} \operatorname{curl}_{x_0} E^s(x, x_0, p) - k^2 n(x_0) E^s(x, x_0, p) = 0$$

in the domain bounded by Λ and ∂D. Hence, the first two integrals in the above equation give a solution $W(x_0)$ to the same equation as the one satisfied by $E^s(\cdot, x_0, p)$, whereas the last two integrals add up to $-\mathbf{G}(z, x_0)p$ by the Stratton-Chu formula and the fact that $E_e(x, z, q, k_b)$ is the fundamental solution of

$$\operatorname{curl} \operatorname{curl} E - k_b^2 E = 0.$$

On the other hand, we have that

$$\mathcal{R}(E, A\varphi_z^\epsilon) = k^2 \int_D (N - n_b I)\, E^{\text{int}} \cdot A\varphi_z^\epsilon \, dx + ik\eta \int_{\Gamma_2} (\nu \times E)(\nu \times A\varphi_z^\epsilon)\, ds.$$

Combining the above equalities we obtain that

$$\begin{aligned}(67)\quad \mathcal{R}(E, A\varphi_z^\epsilon) &- \mathcal{R}(E, E_e(\cdot,\, z, q, k_b)) \\ &= -W(x_0) + \mathbf{G}(z, x_0)p + k^2 \int_D (N - n_b I)\, E^{int} \cdot A\varphi_z^\epsilon \, dx \\ &\quad + ik\eta \int_{\Gamma_2} (\nu \times E^{int})(\nu \times A\varphi_z^\epsilon)\, ds.\end{aligned}$$

Now since $\|A\varphi_z^\epsilon\|_{H_{\text{inc}}(D,\Gamma_2)} < C$ there exists a subsequence, still denoted by $A\varphi_z^\epsilon$, that converges weakly to a $V \in H_{\text{inc}}(D, \Gamma_2)$ as $\epsilon \to 0$. Now set

$$\begin{aligned}\widetilde{W}(x_0) = \lim_{\epsilon \to 0} \mathcal{R}(E, A\varphi_z^\epsilon) = k^2 \int_D (N - n_b I)\, E^{\text{int}} \cdot V\, dx \\ + ik\eta \int_{\Gamma_2} (\nu \times E)(\nu \times V)\, ds, \quad x_0 \in \Lambda.\end{aligned}$$

From (66) we now have that

$$(68)\qquad \widetilde{W}(x_0) = W(x_0) + \mathbf{G}(z, x_0)p, \quad x_0 \in \Lambda.$$

Since $\widetilde{W}(x_0)$ and $W(x_0)$ can be continued as radiating solutions to

$$\mathrm{curl}_{x_0}\,\mathrm{curl}_{x_0}\,E^s(x,x_0,p) - k^2 n(x_0)E^s(x,x_0,p) = 0$$

outside the domain bounded by Λ, we deduce by uniqueness and the unique continuation principle that (68) holds true in $\mathbf{R}^3 \setminus (\overline{D} \cup \{z_0\})$. We now arrive at a contradiction by letting $x_0 \to z$. Hence $A\varphi_z^\epsilon$ is unbounded in the $H_{\mathrm{inc}}(D,\Gamma_2)$ norm as $\epsilon \to 0$, which proves the theorem. □

Theorem 5.3 provides a characterization of the boundary ∂D of the scattering object D in terms of the behavior of $\|A\varphi_z^\epsilon\|_{H_{\mathrm{inc}(D,\Gamma_2)}}$, which is a norm depending on the unknown region D, and therefore not useful numerically. Instead, one can use the behavior of $\|\varphi_z^\epsilon\|_{L_t^2(\tilde{\Lambda})}$ which follows that of $\|A\varphi_z^\epsilon\|_{H_{\mathrm{inc}(D,\Gamma_2)}}$. In particular, given a discrepancy $\epsilon > 0$ and φ_z^ϵ the ϵ-approximate solution of (64), the boundary of the scatterer is reconstructed as the set of points z where the $L_t^2(\tilde{\Lambda})$ norm of φ_z^ϵ becomes large. In practice, since (64) is severely ill-posed due to the compactness of the operator S, one uses regularization methods to obtain a solution to (64). Obviously, an important question is whether this regularized solution will exhibit the properties of the ϵ-approximate solution provided by Theorem 5.3. In general, this question is still open (however, see [**3**] for an answer to this question in the case of the scalar problem for a perfect conductor in homogeneous background using far field data). Numerical examples for similar reconstruction methods have shown in these cases that the computed regularized solution behaves in the way that the theory predicts [**14, 15, 18, 24**]. Notice that the method determines D without any a priori knowledge of N, Γ_1, Γ_2 or η.

5.3. *The determination of η.* Assuming now that the support of the inhomogeneity D is known (an approximation of D is obtained as above), we want to use the approximate solution of (64) to estimate the surface conductivity η without determining N. In particular, we will obtain a lower bound for η and if the object is fully coated we will reconstruct η, see formula (77) in Theorem 5.4 below. All this is done provided that $\mathcal{I}m\,(N) \neq 0$ (but N is unknown) since the estimate for η is obtained from the absorption property of the inclusion.

Our formula for η involves the solution of (59)–(63). Let $z \in D$, and let $(E_z^0, E_z^{\text{int}})$ be the unique solution of (59)–(63), assuming that k is not a transmission eigenvalue for D. As mentioned above,

$$E^s := -E_e(\cdot,\, z, q, k_b) \text{ in } \mathbf{R}^3 \setminus \overline{D} \text{ and}$$
$$E^s := E_z^{\text{int}} - E_0^z \text{ in D}$$

satisfy the scattering problem (25)–(30) with $E_0 = E_0^z$. The idea behind the formulas for η is to eliminate the unknown E_z^{int} from the imaginary part of the energy identity associated with (59)–(63). The difficulty here is to justify the use of this energy identity for solutions with weak regularity as defined in Section 3. We shall accomplish this by using single layer potentials as approximating solutions.

Let $A\varphi_z^\epsilon$ approximate E_0^z in the $H_{\text{inc}}(D, \Gamma_2)$ norm with discrepancy $\epsilon > 0$, and let $E_z^{s,\epsilon}$ be the solution of (25)–(30) with $E_0 = A\varphi_z^\epsilon$. From (32) we have that

$$E_z^{s,\epsilon} \to -E_e(\cdot,\, z, q, k_b) \text{ in } X(B_R \setminus \overline{D}, \Gamma_2) \text{ and}$$
$$E_z^{s,\epsilon} \to (E_z^{\text{int}} - E_0^z) \text{ in } X(D, \Gamma_2) \text{ as } \epsilon \to 0.$$

We define $E_z^{\text{int},\epsilon} := E_z^{s,\epsilon} + A\varphi_z^\epsilon$ in D. Applying the second vector Green's formula (note that $E^{\text{int},\epsilon}$ has the needed regularity) we obtain that

$$\int_{\partial D} \left(\nu \times E_z^{\text{int},\epsilon} \cdot \operatorname{curl} \overline{E_z^{\text{int},\epsilon}} - \nu \times \overline{E_z^{\text{int},\epsilon}} \cdot \operatorname{curl} E_z^{\text{int},\epsilon}\right) ds = 2ik^2 \int_D E_z^{\text{int},\epsilon} \cdot \mathcal{I}m\,(N) \overline{E_z^{\text{int},\epsilon}}\, dx. \tag{69}$$

On the other hand, from the boundary conditions for $E_z^{s,\epsilon}$, we have that

$$\begin{aligned}\int_{\partial D} &\left(\nu \times E_z^{\text{int},\epsilon} \cdot \operatorname{curl} \overline{E_z^{\text{int},\epsilon}} - \nu \times \overline{E_z^{\text{int},\epsilon}} \cdot \operatorname{curl} E_z^{\text{int},\epsilon}\right) ds \\ &= \int_{\partial D} \nu \times (E_z^{s,\epsilon} + A\varphi_z^\epsilon) \cdot \operatorname{curl}(\overline{E_z^{s,\epsilon}} + \overline{A\varphi_z^\epsilon}) - \nu \times (\overline{E_z^{s,\epsilon}} + \overline{A\varphi_z^\epsilon}) \\ &\quad \cdot \operatorname{curl}(E_z^{s,\epsilon} + A\varphi_z^\epsilon)\, ds - 2ik\eta \int_{\Gamma_2} |\nu \times (E_z^{s,\epsilon} + A\varphi_z^\epsilon)|^2\, ds.\end{aligned} \tag{70}$$

But

$$\begin{aligned}
\int_{\partial D} \nu \times (E_z^{s,\epsilon} + A\varphi_z^\epsilon) \cdot \operatorname{curl}(\overline{E_z^{s,\epsilon}} + \overline{A\varphi_z^\epsilon}) \\
\qquad - \nu \times (\overline{E_z^{s,\epsilon}} + \overline{A\varphi_z^\epsilon}) \cdot \operatorname{curl}(E_z^{s,\epsilon} + A\varphi_z^\epsilon)\, ds \\
= \quad & \int_{\partial D} \left(\nu \times E_z^{s,\epsilon} \cdot \operatorname{curl} \overline{E_z^{s,\epsilon}} - \nu \times \overline{E_z^{s,\epsilon}} \cdot \operatorname{curl} E_z^{s,\epsilon}\right) ds \\
& + \int_{\partial D} \left(\nu \times E_z^{s,\epsilon} \cdot \operatorname{curl} \overline{A\varphi_z^\epsilon} - \nu \times \overline{A\varphi_z^\epsilon} \cdot \operatorname{curl} E_z^{s,\epsilon}\right) ds \\
& + \int_{\partial D} \left(\nu \times A\varphi_z^\epsilon \cdot \operatorname{curl} \overline{E_z^{s,\epsilon}} - \nu \times \overline{E_z^{s,\epsilon}} \cdot \operatorname{curl} A\varphi_z^\epsilon\right) ds \\
& + \int_{\partial D} \left(\nu \times A\varphi_z^\epsilon \cdot \operatorname{curl} \overline{A\varphi_z^\epsilon} - \nu \times \overline{A\varphi_z^\epsilon} \cdot \operatorname{curl} A\varphi_z^\epsilon\right) ds \\
= \quad & \mathbf{I}_1^\epsilon + \mathbf{I}_2^\epsilon + \mathbf{I}_3^\epsilon + \mathbf{I}_4^\epsilon.
\end{aligned}$$

Hence combining (69)–(70) we have that

$$\begin{aligned}
(71) \quad 2ik^2 \int_D E_z^{\mathrm{int},\epsilon} \cdot \mathcal{I}m\,(N) \overline{E_z^{\mathrm{int},\epsilon}}\, dx \\
+ 2ik\eta \int_{\Gamma_2} |\nu \times (E_z^{s,\epsilon} - A\varphi_z^\epsilon)|^2\, ds = \mathbf{I}_1^\epsilon + \mathbf{I}_2^\epsilon + \mathbf{I}_3^\epsilon + \mathbf{I}_4^\epsilon
\end{aligned}$$

where the above integrals are interpreted in the sense of duality between $H_{\mathrm{curl}}^{-\frac{1}{2}}(\partial D)$ and $H_{\mathrm{div}}^{-\frac{1}{2}}(\partial D)$. Since $E_z^{s,\epsilon}$ tends to $-E_e(\cdot, z, q, k_b)$ in $X(B_R \setminus \overline{D}, \Gamma_2)$, we have that

$$\begin{aligned}
(72) \qquad \lim_{\epsilon \to 0} \mathbf{I}_1^\epsilon = & \int_{\partial D} \Big(\nu \times E_e(\cdot, z, q, k_b) \cdot \operatorname{curl} \overline{E_e(\cdot, z, q, k_b)} - \nu \\
& \times \overline{E_e(\cdot, z, q, k_b)} \cdot \operatorname{curl} E_e(\cdot, z, q, k_b)\Big)\, ds \\
= & \int_{\partial \Omega} \Big(\nu \times E_e(\cdot, z, q, k_b) \cdot \operatorname{curl} \overline{E_e(\cdot, z, q, k_b)} - \nu \\
& \times \overline{E_e(\cdot, z, q, k_b)} \cdot \operatorname{curl} E_e(\cdot, z, q, k_b)\Big)\, ds \\
& + 2i\, \mathcal{I}m\, \left(k_b^2\right) \int_{\Omega \setminus \overline{D}} |E_e(y, z, q, k_b)|^2\, dy.
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
(73)\quad & \lim_{\epsilon\to 0} \mathbf{I}_2^\epsilon \\
&= -\lim_{\epsilon\to 0}\int_{\partial D}\Big(\nu\times E_e(\cdot,z,q,k_b)\cdot \mathrm{curl}\overline{A\varphi_z^\epsilon} - \nu\ \times\overline{A\varphi_z^\epsilon}\cdot \mathrm{curl}E_e(\cdot,z,q,k_b)\Big)\,ds \\
&\quad + \lim_{\epsilon\to 0}\int_{\partial D}\nu\times(E_z^{s,\epsilon}+E_e(\cdot,z,q,k_b))\cdot\mathrm{curl}\,\overline{A\varphi_z^\epsilon}-\nu \\
&\quad \times\overline{A\varphi_z^\epsilon}\cdot\mathrm{curl}\,(E_z^{s,\epsilon}+E_e(\cdot,z,q,k_b))\,ds \\
&= -\lim_{\epsilon\to 0} ik_bq\cdot\overline{A\varphi_z^\epsilon}(z)+0 = -ik_bq\cdot\overline{E_z^0}(z)
\end{aligned}
$$

where the last limit can be deduced using the mean value theorem for the Helmholtz equation [**16**], see also [**19**, page 602]. Note that using the transmission conditions across ∂D, we have that

$$
\begin{aligned}
\int_{\partial D} & \nu\times(E_z^{s,\epsilon}+E_e(\cdot,z,q,k_b))\cdot\mathrm{curl}\,\overline{A\varphi_z^\epsilon}-\nu \\
&\qquad \times\overline{A\varphi_z^\epsilon}\cdot\mathrm{curl}\,(E_z^{s,\epsilon}+E_e(\cdot,z,q,k_b))\,ds \\
&= k^2\int_D\Big[N(E_z^{s,\epsilon}-E_z^{\mathrm{int}}+E_z^0) \\
&\qquad \cdot\overline{A\varphi_z^\epsilon}+(N-n_bI)(E_z^0-A\varphi_z^\epsilon)\Big]\cdot\overline{A\varphi_z^\epsilon}\,dv \\
&\qquad - k^2\int_D\overline{n_b}(\overline{E_z^{s,\epsilon}}-\overline{E_z^{\mathrm{int}}}+\overline{E_z^0})\cdot A\varphi_z^\epsilon\,dv \\
&\qquad + ik\eta\int_{\Gamma_2}\nu\times\big(E_z^{s,\epsilon}+E_e(\cdot,z,q,k_b)+A\varphi_z^\epsilon-E_z^0\big) \\
&\qquad \cdot\overline{A\varphi_z^\epsilon}\,ds\longrightarrow 0\quad\text{as}\quad\epsilon\to 0
\end{aligned}
$$

since $(E_z^{s,\epsilon}-E_z^{\mathrm{int}}+E_z^0)\to 0$ in $L^2(D)$, $A\varphi_z^\epsilon\to E_z^0$ in $H_{\mathrm{inc}}(D,\Gamma_2)$ and $\nu\times(E_z^{s,\epsilon}+E_e(\cdot,z,q,k_b))\to 0$ in $L_t^2(\Gamma_2)$. Obviously, since $\mathbf{I}_3=-\overline{\mathbf{I}}_2$, we obtain that

$$
(74)\qquad \lim_{\epsilon\to 0}\mathbf{I}_3^\epsilon = -i\overline{k_b}q\cdot E_z^0(z).
$$

Finally,

$$
\begin{aligned}
(75)\qquad \lim_{\epsilon\to 0}\mathbf{I}_4^\epsilon &= \lim_{\epsilon\to 0}2i\,\mathcal{I}m\,\left(k_b^2\right)\int_D\left|A\varphi_z^\epsilon\right|^2\,dy \\
&= 2i\,\mathcal{I}m\,\left(k_b^2\right)\int_D\left|E_z^0(y)\right|^2\,dy.
\end{aligned}
$$

Hence, taking the limit as $\epsilon \to 0$ in (71) and using (72), (73), (74) and (75), we obtain

$$
\begin{aligned}
(76)\quad & 2ik^2 \int_D E_z^{\rm int} \cdot \mathcal{I}m\,(N) \overline{E_z^{\rm int}}\, dx \\
&\quad + 2ik\eta \int_{\Gamma_2} |\nu \times (E_z^0 - E_e(\cdot, z, q, k_b))|^2 ds \\
&= \int_{\partial\Omega} \Big(\nu \times E_e(\cdot, z, q, k_b) \cdot \operatorname{curl} \overline{E_e(\cdot, z, q, k_b)} - \nu \\
&\quad \times \overline{E_e(\cdot, z, q, k_b)} \cdot \operatorname{curl} E_e(\cdot, z, q, k_b)\Big)\, ds \\
&\quad + 2i\mathcal{I}m\, \left(k_b^2\right) \int_{\Omega \setminus \overline{D}} |E_e(y, z, q, k_b)|^2\, dy - 2i\mathcal{R}e\,(k_b q \cdot E_z^0) \\
&\quad + 2i\mathcal{I}m\, \left(k_b^2\right) \int_D \left|E_z^0(y)\right|^2 dy
\end{aligned}
$$

Thus, we have the following theorem.

Theorem 5.4. *Assume that k, N and n_b are such that the interior transmission problem* (59)–(63) *is well posed (for instance, under conditions* (19)–(24)*). Let $z \in D$ be fixed and E_z^0 be such that $E_z^0, E_z^{\rm int}$ is the unique solution of* (59)–(63)*. Then*

$$
\begin{aligned}
(77)\quad & \eta \int_{\Gamma_2} \left|\nu \times [E_z^0 - E_e(\cdot,\, z, q, k_b)]\right|^2 ds \\
&\leq A(z, \Omega, k_b, q) - \mathcal{R}e\,(\sqrt{n_b} q \cdot E_z^0) \\
&\quad + k\mathcal{I}m\, (n_b) \left\{ \int_{\Omega \setminus \overline{D}} |E_e(\cdot, z, q, k_b)|^2\, dy + \int_D \left|E_z^0\right|^2 dy \right\}
\end{aligned}
$$

where the constant $A(z, \Omega, k_b, q)$ is given by

$$
\begin{aligned}
(78)\quad & A(z, \Omega, k_b, q) \\
&:= \frac{1}{k} \int_{\partial\Omega} \mathcal{I}m\, \left(\nu \times E_e(\cdot, z, q, k_b) \cdot \operatorname{curl} \overline{E_e(\cdot, z, q, k_b)}\right) ds.
\end{aligned}
$$

If $\mathcal{I}m\,(N) = 0$, then inequality (77) *becomes an equality.*

Let us indicate once again that, numerically, one has access to an approximation to E_z^0 by $A\varphi_z$ where φ_z is the nearby (regularized) solution of the integral equation (64). We conclude this section with few remarks concerning further practical aspects of this theorem.

1. A drawback of (77) is that in general the extent of the coating Γ_2 is not known. Thus, in practice, this expression provides only a lower bound for η. However, if the object is fully coated, that is $\Gamma_2 = \Gamma$, and if the inhomogeneities is a dielectric, then we can indeed compute an approximation of η. For an idea of how accurate this lower bound is for η, we refer the reader to [**21**] where numerical examples for η are given in the case of homogeneous background using the linear sampling method with far field data.

2. The above analysis for the determination of D can be carried out exactly in the same way if η is an $L_\infty(\Gamma_2)$ function such that $\eta(x) \geq \eta_0 > 0$. For instance, when $\mathcal{I}m\,(N) = 0$, formula (77) becomes an integral equation of the first kind for η with kernel

$$\left|\nu(y) \times [E_z^0(y) - E_e(y,\, z, q, k_b)]\right|^2, \quad (y, z) \in \Gamma_2 \times D.$$

One can use this equation to determine $\|\eta\|_{L_\infty(\Gamma_2)}$. In this regard, see [**13**] for the scalar case and [**9**] for the case of a partially coated perfect conductor for Maxwell's equation.

3. The above analysis for solving (64) requires the measured tangential component of the total electric and magnetic field on the whole boundary $\partial\Omega$ of Ω. The case of an object buried in a layered medium, for instance in the earth, is handled by assuming that the part of $\partial\Omega$ below the surface of the earth is far away from the incident sources and hence we can assume that the total electric and magnetic fields are very small on this portion of the boundary.

Numerical examples for determining D and η for partially coated buried penetrable objects using the above results will follow in a forthcoming paper. However, for the implementation procedure and examples in similar situations, see [**14, 15, 21**].

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