THE ELECTROMAGNETIC INVERSE SCATTERING PROBLEM FOR PARTIALLY COATED LIPSCHITZ DOMAINS

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Abstract. We consider the inverse scattering problem of determining the shape of a partially coated obstacle in $\mathbb{R}^3$ from a knowledge of the incident time harmonic electromagnetic plane wave and the electric far field pattern of the scattered wave. A justification is given of the linear sampling method in this case and numerical examples are provided showing the practicality of our method.

Key words. Electromagnetic inverse scattering, Lipschitz domain, mixed boundary conditions, linear sampling method.

1. Introduction. The inverse scattering problem we consider in this paper is to determine the shape of a (possibly disconnected) scattering obstacle from a knowledge of the incident time harmonic electromagnetic plane wave and the electric far field pattern of the scattered wave. Although this problem is a basic one in inverse scattering theory, until now only partial solutions have been obtained [16], [18], [9]. The difficulty lies in what a priori assumptions need to be made on the material properties of the scattering object D, e.g. is D penetrable, perfectly conducting or coated? In particular, for iterative methods such as those of [16] and [18] some a priori knowledge of this type is needed in order to implement the inversion scheme. On the other hand, the linear sampling method used in [9] for solving this problem requires that each component of the scattering object has the same boundary condition although this boundary condition can vary from component to component and does not have to be known a priori. Excluded in this analysis is the case of partially coated obstacles, i.e. the case of possibly mixed boundary conditions on each component and it is this problem that we are concerned with in this paper.

The inverse scattering problem for partially coated obstacles was first considered in [4] for the special case of an infinite cylinder where the Maxwell systems decouples into a two dimensional scalar Helmholtz equation. This problem, and in particular the three dimensional analogue considered in this paper, is particularly important in the use of electromagnetic waves to detect "hostile" objects where the boundary, or more generally a portion of the boundary, is coated with an unknown material in order to avoid detection. Since in general such objects have corners and edges, which are in fact responsible for strong scattering effects, it is important to consider the general case of disconnected obstacles where each component is allowed to have a Lipschitz boundary. These considerations are the motivation for the problem considered in this paper.

The goal of this paper is to establish the validity of the linear sampling method for the solution of the three dimensional electromagnetic inverse scattering problem for partially coated obstacles (c.f. [4] for the two dimensional case). As in [4], in order to accomplish this goal it is first necessary to establish the uniqueness, existence and a priori estimates for the corresponding direct exterior and interior problems. This was done in the scalar case through the use of integral equations of the first kind [4]. However, this approach is not suitable for the three dimensional vector case and hence we establish these results for the vector case in Section 2 of this paper through the use of variational methods. In this section, using the ideas of [14], we also establish the approximation properties of electromagnetic Herglotz pairs which are necessary for the justification of the linear sampling method given in Section 3. Finally, in Section 4, we provide some numerical examples showing the practicality of our method for solving the inverse scattering problem.

2. The direct scattering problem. Let $D \subset \mathbb{R}^3$ be a bounded region with boundary Γ such that $D_e := \mathbb{R}^3 \setminus \overline{D}$ is connected. Each simply connected piece of D is assumed to be a Lipschitz curvilinear polyhedron. Moreover we assume that the boundary $\Gamma = \Gamma_D \cup \Pi \cup \Gamma_I$ is split into two disjoint parts Γ_D and Γ_I having Π as their possible common boundary in Γ and that each part Γ_D and Γ_I can be written as the union of a finite number of open smooth faces $\left(\Gamma_D^j\right)_{j=1,\dots N_D}$ and $\left(\Gamma_I^j\right)_{j=1,\dots,N_I}$, respectively where e_{ij} denotes the common edge of two adjacent faces Γ^i and Γ^j. Let ν denote the unit outward normal defined almost everywhere on Γ.
The direct scattering problem for the scattering of a time harmonic electromagnetic plane wave by a partially

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coated obstacle D is to find an electric field E^t and a magnetic field H^t such that

$$\operatorname{curl} E^t - ikH^t = 0 \tag{2.1}$$

$$\operatorname{curl} H^t + ikE^t = 0 \tag{2.2}$$

in $\mathbb{R}^3 \setminus \overline{D}$ and on the boundary Γ satisfy

$$\nu \times E^t = 0 \quad \text{on} \quad \Gamma_D \tag{2.3}$$

$$\nu \times \operatorname{curl} E^t - i\lambda(\nu \times E^t) \times \nu = 0 \quad \text{on} \quad \Gamma_I \tag{2.4}$$

where $\lambda > 0$ is the surface impedance which is assumed to be a (possibly different) constant on each connected subset of Γ_I. The total fields E^t and H^t are given by

$$E^t = E^i + E \tag{2.5}$$

$$H^t = H^i + H \tag{2.6}$$

where E, H is the scattered field satisfying the Silver-Müller radiation condition

$$\lim_{r\to\infty} (H \times x - rE) = 0 \tag{2.7}$$

uniformly in $\hat{x} = x/|x|$ where $r = |x|$ and the incident field E^i,H^i is given by

$$E^i(x) := \frac{i}{k} \operatorname{curl} \operatorname{curl} p e^{ikx\cdot d} = ik(d \times p) \times d e^{ikx\cdot d} \tag{2.8}$$

$$H^i(x) := \operatorname{curl} p e^{ikx\cdot d} = ikd \times p e^{ikx\cdot d} \tag{2.9}$$

where the wave number k is positive, d is a unit vector giving the direction of propagation and p is the polarization vector.

2.1. Solution of the forward problem. Letting $(H^s(D))^3$, $(H^s_{loc}(D_e))^3$ and $(H^s(\Gamma))^3$, $s \in \mathbb{R}$, denote the product of the standard Sobolev spaces defined on D, D_e, Γ (with the convention $H^0 = L^2$), and

$$\begin{aligned} H(\operatorname{curl}, D) &:= \{u \in (L^2(D))^3 \,:\, \operatorname{curl} u \in (L^2(D))^3\} \\ H^s_t(\Gamma) &:= \{u \in (H^s(\Gamma))^3 \,:\, \nu \cdot u = 0 \quad \text{on} \quad \Gamma\} \\ H^s_t(\Gamma_0) &:= \{u|_{\Gamma_0} \,:\, u \in H^s_t(\Gamma)\} \end{aligned}$$

for an open subset Γ_0 of Γ, we introduce the space

$$X(D, \Gamma_I) := \{u \in H(\operatorname{curl}, D) \,:\, \nu \times u|_{\Gamma_I} \in L^2_t(\Gamma_I)\}, \tag{2.10}$$

equipped with the natural norm

$$\|u\|^2_{X(D,\Gamma_I)} := \|u\|^2_{H(curl,\,D)} + \|\nu \times u\|^2_{L^2(\Gamma_I)}. \tag{2.11}$$

For the exterior domain D_e we define the above spaces in the same way for every $D_e \cap B_R$, with B_R a ball of radius R, and denote these spaces by $H_{loc}(\operatorname{curl}, D_e)$ and $X_{loc}(D_e, \Gamma_I)$, respectively. Furthermore we introduce the trace space of X on the complementary part Γ_D by

$$Y(\Gamma_D) := \left\{ f \in (H^{-1/2}(\Gamma_D))^3 \,:\, \exists\, u \in H_0(\operatorname{curl}, B_R), \quad \begin{array}{l} \nu \times u|_{\Gamma_I} \in L^2_t(\Gamma_I) \\ \text{and} \quad f = \nu \times u|_{\Gamma_D} \end{array} \right\}, \tag{2.12}$$

where the ball B_R contains D and $H_0(\operatorname{curl}, B_R)$ is the space of functions u in $H(\operatorname{curl}, B_R)$ satisfying $\nu \times u|_{\partial B_R} = 0$. It is easy to show that $Y(\Gamma_D)$ is a Banach space with respect to the norm

$$\|f\|^2_{Y(\Gamma_D)} := \inf\{\|u\|^2_{H(curl,B_R)} + \|\nu \times u\|^2_{L^2(\Gamma_I)}\}, \tag{2.13}$$

where the infimum is taken over all functions $u \in H_0(\text{curl}, B_R)$ such that $\nu \times u|_{\Gamma_I} \in L^2_t(\Gamma_I)$ and $f = \nu \times u|_{\Gamma_D}$. Let $X_0(R_R, \Gamma_I) := H_0(\text{curl}, B_R) \cap X(R_R, \Gamma_I)$. By using an extension theorem [7] one can prove that $\|\cdot\|_{Y(\Gamma_D)}$ is equivalent to both of the norms

$$|||f|||_1 := \sup_{\phi \in X(D,\Gamma_I)} \frac{|\langle f, \phi\rangle_1|}{\|\phi\|_{X(D,\Gamma_I)}} \qquad \text{and} \qquad |||f|||_2 := \sup_{\phi \in X_0(B_R \setminus D,\Gamma_I)} \frac{|\langle f, \phi\rangle_2|}{\|\phi\|_{X(B_R \setminus D,\Gamma_I)}},$$

where for $u \in H_0(\text{curl}, B_R)$ such that $\nu \times u|_{\Gamma_I} \in L^2_t(\Gamma_I)$ and $f = \nu \times u|_{\Gamma_D}$

$$\langle f, \phi\rangle_1 := \int_D (\text{curl}\, u \cdot \phi - u \cdot \text{curl}\, \phi)\, dv - \int_{\Gamma_I} \nu \times u \cdot \phi \, ds \quad \phi \in X(D, \Gamma_I), \tag{2.14}$$

$$\langle f, \phi\rangle_2 := \int_{B_R \setminus D} (\text{curl}\, u \cdot \phi - u \cdot \text{curl}\, \phi)\, dv + \int_{\Gamma_I} \nu \times u \cdot \phi \, ds \quad \phi \in X_0(B_R \setminus D, \Gamma_I).$$

In particular, $Y(\Gamma_D)$ is a Hilbert space.

We can now formulate the following *exterior mixed boundary value problem* for the Maxwell equations: given $f \in Y(\Gamma_D)$ and $h \in L^2_t(\Gamma_I)$ find $E \in X_{loc}(D_e, \Gamma_I)$ and $H = \frac{1}{ik}\text{curl}\, E$ satisfying

$$\begin{aligned} &(i) && \text{curl}\,\text{curl}\, E - k^2 E = 0 \quad \text{in} \quad D_e \\ &(ii) && \nu \times E = f \quad \text{on} \quad \Gamma_D \\ &(iii) && \nu \times \text{curl}\, E - i\lambda(\nu \times E) \times \nu = h \quad \text{on} \quad \Gamma_I \\ &(iv) && \lim_{r\to\infty} (H \times x - rE) = 0. \end{aligned} \tag{2.15}$$

Note that the scattered fields E, H in (2.5) and (2.6) satisfy the exterior mixed boundary value problem with $f := -\nu \times E^i$ and $h := -\nu \times \text{curl}\, E^i + i\lambda(\nu \times E^i) \times \nu$.
We will also need to consider the corresponding *interior mixed boundary value problem*: given $f \in Y(\Gamma_D)$ and $h \in L^2_t(\Gamma_I)$ find $E \in X(D, \Gamma_I)$ satisfying

$$\begin{aligned} &(i) && \text{curl}\,\text{curl}\, E - k^2 E = 0 \quad \text{in} \quad D \\ &(ii) && \nu \times E = f \quad \text{on} \quad \Gamma_D \\ &(iii) && \nu \times \text{curl}\, E - i\lambda(\nu \times E) \times \nu = h \quad \text{on} \quad \Gamma_I \end{aligned} \tag{2.16}$$

We begin with establishing uniqueness and existence results for the interior and exterior mixed boundary value problems (2.16) and (2.15).

THEOREM 2.1. *Assume that the impedance part Γ_I is not empty. Then, if $\lambda \neq 0$, the interior mixed boundary value problem (2.16) has at most one solution.*

Proof. Let $E \in X(D, \Gamma_I)$ and $H = \frac{1}{ik}\text{curl}\, E$ be the solution of (2.16) with boundary data $f \equiv 0$ and $h \equiv 0$. Taking the dot product of (2.16(i)), which is understood in the distribution sense, by the complex conjugate of $\overline{E}$, integrating over D and then using integration by parts we obtain

$$\int_D \left(|\text{curl}\, E|^2 - k^2|E|^2\right) dv + \int_{\Gamma_I} \nu \times \text{curl}\, E \cdot \overline{E}_T \, ds = 0, \tag{2.17}$$

where the E_T denotes the tangential component $E_T := (\nu \times E) \times \nu$. Making use of the homogeneous boundary condition $\nu \times \text{curl}\, E = i\lambda E_T$ on Γ_I we have

$$\int_D \left(|\text{curl}\, E|^2 - k^2|E|^2\right) dv + i\lambda \int_{\Gamma_I} |E_T|^2 \, ds = 0. \tag{2.18}$$

Since λ is a real number, by taking the imaginary part of (2.18) we conclude that $E_T \equiv 0$ and $\nu \times \text{curl}\, E \equiv 0$ as functions in $L^2_t(\Gamma_I)$, whence $E \equiv 0$ in D by first using the representation formula to establish the regularity of E across Γ_I and then applying the unique continuation principle ([11], [17]).

Theorem 2.2. *The exterior mixed boundary value problem (2.15) has at most one solution.*
Proof. By doing the same as in the previous theorem but now in the domain $D_e \cap B_R$, where B_R is a ball of radius $R > 0$ containing D we obtain

$$\int\limits_{D_e \cap B_R} \left(|\mathrm{curl}\, E|^2 - k^2 |E|^2\right) dv - ik \int\limits_{S_R} (\nu \times \overline{E}) \cdot H \, ds - i\lambda \int\limits_{\Gamma_I} |E_T|^2 \, ds = 0, \tag{2.19}$$

Taking the imaginary part of (2.19) we now obtain

$$\mathrm{Re} \int\limits_{S_R} (\nu \times \overline{E}) \cdot H \, ds = -\frac{\lambda}{k} \int\limits_{\Gamma_I} |E_T|^2 \, ds \le 0,$$

Hence the uniqueness follows from [11], Theorem 6.10, and the unique continuation principle.□

We now prove the existence of the solution to the exterior and interior mixed boundary value problems. We will write the variational formulation of (2.16) and (2.15), show that this weak formulation is equivalent to our problems and that it has a unique solution. For the sake of conciseness we will consider only the interior problem (2.16) in details and then simply indicate how a similar proof is valid for the exterior problem (2.15).
By using the integration by parts formula and the impedance boundary condition on Γ_I the variational formulation for the electric field of (2.16) becomes:
Find $E \in X(D, \Gamma_I)$ satisfying $\nu \times E = f$ on Γ_D such that

$$\int\limits_D \left(\mathrm{curl}\, E \cdot \mathrm{curl}\, \overline{\phi} - k^2 E \cdot \overline{\phi}\right) dv + i\lambda \int\limits_{\Gamma_I} E_T \cdot \overline{\phi}_T \, ds = -\int\limits_{\Gamma_I} h \cdot \overline{\phi}_T \, ds, \tag{2.20}$$

for every test function

$$\phi \in \tilde{X} := \{u \in H(\mathrm{curl}, D) : \quad \nu \times u|_{\Gamma_D} = 0 \text{ and } \nu \times u|_{\Gamma_I} \in L^2_t(\Gamma_I)\}.$$

From the definition of the space $Y(\Gamma_D)$, there exists a function $U \in X(D, \Gamma_I)$ such that $\nu \times U|_{\Gamma_D} = f$. By subtracting from both sides of (2.20) the expression

$$\int\limits_D \left(\mathrm{curl}\, U \cdot \mathrm{curl}\overline{\phi} - k^2 U \cdot \overline{\phi}\right) dv + i\lambda \int\limits_{\Gamma_I} U_T \cdot \overline{\phi}_T \, ds,$$

which is obviously well defined for $\phi \in \tilde{X}$, we obtain for $W := E - U$ the equation

$$\begin{aligned} &\int\limits_D \left(\mathrm{curl}\, W \cdot \mathrm{curl}\, \overline{\phi} - k^2 W \cdot \overline{\phi}\right) dv + i\lambda \int\limits_{\Gamma_I} W_T \cdot \overline{\phi}_T \, ds \\ &= -\int\limits_{\Gamma_I} h \cdot \overline{\phi}_T \, ds - \int\limits_D \left(\mathrm{curl}\, U \cdot \mathrm{curl}\, \overline{\phi} - k^2 U \cdot \overline{\phi}\right) dv - i\lambda \int\limits_{\Gamma_I} U_T \cdot \overline{\phi}_T \, ds, \end{aligned} \tag{2.21}$$

and $\nu \times W = 0$ on Γ_D. Taking a sufficiently smooth test function ϕ and using a denseness argument one can show that if $W \in \tilde{X}$ solves (2.21) than $E = W + U$ in $X(D, \Gamma_I)$ is a solution of (2.16) and conversely. Hence our problem is to find $W \in \tilde{X}$ such that for every $\phi \in \tilde{X}$

$$a\left(W, \phi\right) = \langle h, \phi \rangle - a\left(U, \phi\right), \tag{2.22}$$

where the sesquilinear form $a : \tilde{X} \times \tilde{X} \to \mathbb{C}$ is defined by

$$a(u, \psi) = (\mathrm{curl}\, u, \mathrm{curl}\, \psi) - k^2 \left(u, \psi\right) + i\lambda \left\langle u_T, \, \psi_T \right\rangle, \quad u, \psi \in \tilde{X},$$

with $(\cdot, \cdot)$ denoting the $L^2(D)$ scalar product, and $\langle \cdot, \cdot \rangle$ the $L^2(\Gamma_I)$ scalar product. The sesquilinear form $a(u, \psi)$ is systematically studied in [20], Chapter 4, in the case when Γ_D and Γ_I are closed manifolds. We will outline the analysis there showing that it remains valid in our case when Γ_D and Γ_I are open subsets of the boundary Γ. Without loss of generality we assume that D is simply connected. First we observe that every function $u \in \tilde{X}$ satisfying $\operatorname{curl} u = 0$ in D and $\nu \times u|_{\Gamma_I} = 0$ satisfies $u = \nabla p$ with $p \in S$ where S is defined by

$$S = \left\{ p \in H^1(\Omega); \quad p = 0 \quad \text{on} \quad \Gamma \right\}.$$

For a similar result for non-simply connected domains see [20], Remark 4.4. Hence the Helmholtz decomposition holds (see e.g. [20] Theorem 4.3)

$$\tilde{X} = X_0 \oplus \nabla S \tag{2.23}$$

where

$$X_0 = \left\{ u \in \tilde{X}; \quad \int_D u \cdot \nabla \overline{\xi} = 0, \quad \text{for all} \quad \xi \in S \right\}.$$

Furthermore, since for $u \in X_0$ both $\nu \times u|_{\Gamma_D} = 0$ and $\nu \times u|_{\Gamma_I} \in L^2_t(\Gamma_I)$ imply $\nu \times u \in L^2_t(\Gamma)$ which from a regularity result of Costabel [15] implies that u is in $H^{\frac{1}{2}}(D)$, we have that X_0 is compactly imbedded in $L^2(D)$. From (2.23) we can now look for our solution in the form $W = W_0 + \nabla p$ with $W_0 \in X_0$ and $p \in S$. Using the fact that $\operatorname{curl}(\nabla p) = 0$ in D and $\nabla p \times \nu = 0$ on Γ and by choosing the test function $\phi = \nabla \xi$ for some $\xi \in S$ we obtain

$$(\nabla p, \nabla \xi) = -\left(U, \nabla \xi\right). \tag{2.24}$$

An application of the Lax-Milgram lemma for the continuous and coercive sesquilinear form $(\nabla p, \nabla \xi)$ implies that there exist a unique $p_0 \in S$ satisfying (2.24) and

$$||\nabla p_0||_{L^2(D)} \leq ||U||_{L^2(D)}. \tag{2.25}$$

Hence determining W is equivalent to determining $W_0 \in X_0$ such that

$$a\left(W_0, \phi\right) = \langle h, \phi \rangle - a\left(U, \phi\right) + k^2 \left(\nabla p_0, \phi\right), \tag{2.26}$$

for all $\phi \in X_0$. We write this sesquilinear form as

$$a\left(u, \phi\right) = b\left(u, \phi\right) - (1 + k^2)\left(u, \phi\right), \tag{2.27}$$

where $b : X_0 \times X_0 \to \mathbb{C}$ is defined by

$$b(u, \phi) := (\operatorname{curl} u, \operatorname{curl} \phi) + (u, \phi) + i\lambda \left\langle u_T \cdot \phi_T \right\rangle, \quad u, \phi \in \tilde{X}.$$

¿From the Cauchy-Schwarz inequality there exists a constant C_1 such that

$$|b(u, \phi)| \leq C_1 ||u||_X ||\phi||_X$$

and by taking the real and the imaginary part there is a constant $C_2 > 0$ such that for $u \in X_0$

$$|b(u, u)| \geq \frac{1}{2} \left(\int_D |\operatorname{curl} u|^2 + |u|^2 \, dv + \lambda \int_{\Gamma_I} |u_T|^2 \, ds \right) \geq C_2 \|u\|_X.$$

Hence by the Lax-Milgram lemma the first term in (2.27) gives rise to a bijective operator and by the compact embedding of X_0 in $L^2(D)$ the second term gives rise to a compact operator. Then a standard argument implies that the Fredholm alternative is applicable which together with the uniqueness Theorem 2.1 shows that there exists a unique solution $E = U + W_0 + \nabla p$ to (2.16), providing $\Gamma_I \neq \emptyset$. Moreover, since $a(U, \phi)$ is bounded and (2.25) holds, we have the norm estimate for this solution

$$\|E\|_X \leq C \left(\|U\|_X + \|h\|_{L^2(\Gamma_I)} \right), \tag{2.28}$$

with some positive constant C independent of h and U. From the definition of the norm of $Y(\Gamma_D)$ (2.13) for every $\epsilon > 0$ we can find a $U_\epsilon \in H(\text{curl}, B_R)$ such that $\nu \times U_\epsilon|_{\Gamma_I} \in L^2_t(\Gamma_I)$ and $f = \nu \times U_\epsilon|_{\Gamma_D}$ satisfies

$$\|U_\epsilon|_D\|^2_{X(D,\Gamma_I)} \leq \|U_\epsilon\|^2_{H(curl,B_R)} + \|U_\epsilon\|^2_{L^2(\Gamma_I)} \leq \|f\|^2_{Y(\Gamma_D)} + \epsilon.$$

Since the unique solution of (2.16) does not depend on the choice of U, the estimate (2.28) implies that, for every $\epsilon > 0$,

$$\|E\|_X \leq C \left(\|f\|_{Y(\Gamma_D)} + \epsilon + \|h\|_{L^2(\Gamma_I)} \right). \tag{2.29}$$

Hence we have proved the following result.

THEOREM 2.3. *Assume that Γ_I is not empty. Then the interior mixed boundary value problem (2.16) has a unique solution which satisfies*

$$\|E\|_{X(D,\Gamma_I)} \leq C \left(\|f\|_{Y(\Gamma_D)} + \|h\|_{L^2(\Gamma_I)} \right), \tag{2.30}$$

for some positive constant C.

Note that for $\Gamma_I = \emptyset$ we have the perfect conductor problem and Theorem 2.3 holds under the assumption that k is not a Maxwell eigenvalue. Furthermore, in this case $Y(\Gamma)$ becomes simply the trace space of $H(\text{curl}, D)$ which for a smooth boundary is known to be $H_{div}^{-\frac{1}{2}}(\Gamma)$, the space of tangential vectors that belong together with their surface divergence to $H^{-\frac{1}{2}}(\Gamma)$. (For the characterization of this space in the case of Lipschitz polyhedra we refer to [1].)

The exterior mixed boundary value problem can be treated in a similar manner but in the domain $D_e \cap B_R$. There are several ways of imposing the boundary condition on the artificial surface ∂B_R basically by incorporating the capacity operator (for details see [21]). Here we only state the desired result.

THEOREM 2.4. *The exterior mixed boundary value problem (2.15) has a unique solution which satisfies*

$$\|E\|_{X(D_e \cap B_R,\Gamma_I)} \leq C \left(\|f\|_{Y(\Gamma_D)} + \|h\|_{L^2(\Gamma_I)} \right), \tag{2.31}$$

for some positive constant C depending on R but not on f and h.

2.2. An approximation property. An *electromagnetic Herglotz pair* is defined to be a pair of vector fields of the form

$$E_g(x) = \int_\Omega e^{ikx\cdot d} g(d) ds(d), \qquad H_g(x) = \frac{1}{ik}\text{curl} E_g(x) \tag{2.32}$$

where the kernel g is a tangential vector field in $L^2_t(\Omega)$. It is easily seen that E_g, H_g is a solution of the Maxwell equations in $\mathbb{R}^3$. Our goal is to prove that the electric field of the solution of the interior mixed boundary value problem (2.16) can be approximated by the electric field of a Herglotz pair in $X(D, \Gamma_I)$.

For the following analysis we need a proper characterization of certain function spaces and the corresponding differential operators defined on the boundary of a Lipschitz polyhedra. To this end, we recall the recent results of [1] and [2] where the situation was first clarified. Let $H^{\frac{1}{2}}_-(\Gamma)$ denote the space of functions $u \in L^2_t(\Gamma)$ such that $u \in H^{\frac{1}{2}}(\Gamma^j_D)$, $j = 1, \dots, N_D$, and $u \in H^{\frac{1}{2}}(\Gamma^j_I)$, $j = 1, \dots, N_I$, and let $H^{-\frac{1}{2}}_-(\Gamma)$ denote the associated dual space with respect to L^2_t scalar product. Then the tangential traces $\nu \times (u \times \nu)$ and $\nu \times u$ of vectors $u \in H^1(D)$ form subspaces of $H^{\frac{1}{2}}_-(\Gamma)$ denoted by $H^{\frac{1}{2}}_{\|}(\Gamma)$ and $H^{\frac{1}{2}}_{\perp}(\Gamma)$, respectively, and these spaces are fully characterized in [1]. Roughly speaking, $H^{\frac{1}{2}}_{\|}(\Gamma)$ contains the tangential surface vectors that are in $H^{\frac{1}{2}}_-(\Gamma)$ and "preserves" a suitable weak tangential continuity across the edges of each smooth face Γ_j, while $H^{\frac{1}{2}}_{\perp}(\Gamma)$

”preserves” a suitable weak normal continuity across the edges of Γ_j. For smooth boundary these spaces coincide with the space of tangential vectors in $H^{\frac{1}{2}}(\Gamma)$. The associated dual spaces with $L^2_t(\Gamma)$ as a pivot space are denoted by $H_{\|}^{-\frac{1}{2}}(\Gamma)$ and $H_{\perp}^{-\frac{1}{2}}(\Gamma)$, respectively. In [1], [2] it is shown that the following sequences are such that the range of one operator is the kernel of the following in the sequence

$$H^{\frac{1}{2}}(\Gamma) \overset{grad_\Gamma}{\longrightarrow} H_{\perp}^{-\frac{1}{2}}(\Gamma) \overset{curl_\Gamma}{\longrightarrow} H^{-\frac{3}{2}}(\Gamma) \longrightarrow \{0\} \tag{2.33}$$

$$H^{\frac{1}{2}}(\Gamma) \overset{curl_\Gamma}{\longrightarrow} H_{\|}^{-\frac{1}{2}}(\Gamma) \overset{div_\Gamma}{\longrightarrow} H^{-\frac{3}{2}}(\Gamma) \longrightarrow \{0\}, \tag{2.34}$$

The mapping $u \to \nu \times (u \times \nu)$ from $H(\mathrm{curl}, D)$ to $H_{\perp curl}^{-\frac{1}{2}}(\Gamma)$ and $u \to \nu \times u$ from $H(\mathrm{curl}, D)$ to $H_{\|div}^{-\frac{1}{2}}(\Gamma)$ are continuous and surjective, where

$$H_{\|div}^{-\frac{1}{2}}(\Gamma) := \left(u \in H_{\|}^{-\frac{1}{2}}(\Gamma) : \quad \mathrm{div}_\Gamma\, u \in H^{-\frac{1}{2}}(\Gamma)\right) \tag{2.35}$$

$$H_{\perp curl}^{-\frac{1}{2}}(\Gamma) := \left(u \in H_{\perp}^{-\frac{1}{2}}(\Gamma) : \quad \mathrm{curl}_\Gamma\, u \in H^{-\frac{1}{2}}(\Gamma)\right). \tag{2.36}$$

The integration by parts formula for functions in $H(\mathrm{curl}, D)$ remains true for Lipshitz polyhedra and naturally defines the duality pairing between $H_{\|div}^{-\frac{1}{2}}(\Gamma)$ and $H_{\perp curl}^{-\frac{1}{2}}(\Gamma)$.
Let $H_{\|div}^{-\frac{1}{2}}(\Gamma_D)$ and $H_{\perp curl}^{-\frac{1}{2}}(\Gamma_D)$ be the spaces of functions in $H_{\|div}^{-\frac{1}{2}}(\Gamma)$ and $H_{\perp curl}^{-\frac{1}{2}}(\Gamma)$, respectively, restricted on Γ_D. The relation (2.14) defines a duality between the space $Y(\Gamma_D)$ defined by (2.12) and a subspace of $H_{\perp,curl}^{-\frac{1}{2}}(\Gamma_D)$. In particular, if $Y(\Gamma_D)'$ denotes the dual space of $Y(\Gamma_D)$ with respect to the duality pairing defined by (2.14), a function $\varphi \in Y(\Gamma_D)'$ can be extended to a function $\tilde{\varphi} \in H_{\perp,curl}^{-\frac{1}{2}}(\Gamma)$ defined on the whole boundary and satisfying $\tilde{\varphi}|_{\Gamma_I} \in L^2_t(\Gamma_I)$.

We now define an operator $\mathcal{H} : L^2_t(\Omega) \to Y(\Gamma_D) \times L^2_t(\Gamma_I)$ by

$$\mathcal{H}g := \begin{cases} \nu \times E_g & \text{on} \quad \Gamma_D \\ \nu \times \mathrm{curl} E_g - i\lambda\nu \times (E_g \times \nu) & \text{on} \quad \Gamma_I \end{cases} \tag{2.37}$$

where E_g is the electric field of an electromagnetic Herglotz pair defined by (2.32). By Theorem 2.1 we see that $\mathcal{H}$ is injective provided $\Gamma_I \neq \emptyset$ and $\lambda \neq 0$.

Theorem 2.5. *Assume that $\Gamma_I \neq \emptyset$ and $\lambda \neq 0$. Then the range of $\mathcal{H}$ is dense in $Y(\Gamma_D) \times L^2_t(\Gamma_I)$.*

Proof. By the change of variables $d \to -d$ it suffices to consider the operator $\mathcal{H}$ with E_g written as

$$E_g(x) = \int_\Omega \mathrm{e}^{-ikx\cdot d} g(d)\, ds(d).$$

Let $H := Y(\Gamma_D) \times L^2_t(\Gamma_I)$ with dual $H^* := Y(\Gamma_D)' \times L^2_t(\Gamma_I)$ in the component-wise duality pairing. Note that $L^2_t(\Gamma_I)$ is considered as the dual space of itself with respect to the L^2 scalar product. The dual operator $\mathcal{H}^\top : H^* \to L^2_t(\Omega)$ of the operator $\mathcal{H}$ is such that for every $(a_1, a_2) \in H^*$ and $g \in L^2_t(\Omega)$ we have

$$\langle \mathcal{H}g, (a_1, a_2)\rangle_{H,H^*} = \left\langle g, \mathcal{H}^\top[a_1, a_2]\right\rangle_{L^2_t(\Omega), L^2_t(\Omega)}.$$

It is enough to show that the dual operator $\mathcal{H}^\top$ is injective. Then the result follows from the fact that the range of $\mathcal{H}$ can be characterized as [19]

$$\overline{(\mathrm{Range}\,\mathcal{H})} = {}^a\mathrm{Kern}\,\mathcal{H}^\top$$

where

$${}^a\mathrm{Kern}\,\mathcal{H}^\top := \left\{(p_1, p_2) \in H : \langle (p_1, p_2), (q_1, q_2)\rangle_{H,H^*} = 0 \quad \forall\, (q_1, q_2) \in \mathrm{Kern}\,\mathcal{H}^\top\right\}.$$

In particular, the injectivity of $\mathcal{H}^\top$ implies that $\overline{(\mathrm{Range}\,\mathcal{H})} = H$. Simple computations shows that the dual operator $\mathcal{H}^\top$ is defined by

$$\mathcal{H}^\top[a_1, a_2] = d \times \Bigg\{ \int\limits_{\Gamma_D} e^{-ikx\cdot d}(a_1 \times \nu)dx$$
$$+ ik\; d \times \int\limits_{\Gamma_I} e^{-ikx\cdot d}(a_2 \times \nu)dx - i\lambda \int\limits_{\Gamma_I} e^{-ikx\cdot d}[\nu \times (a_2 \times \nu)]dx \Bigg\} \times d.$$

We note that $\mathcal{H}^\top[a_1, a_2]$ coincides with the far field pattern of the combined electric and magnetic dipole distributions

$$P(z) = \frac{1}{k^2}\mathrm{curl}\,\mathrm{curl} \int\limits_{\Gamma_D} \Phi(x,z)(a_1 \times \nu)ds_x$$
$$- \mathrm{curl} \int_{\Gamma_I} \Phi(x,z)(a_2 \times \nu)ds_x - i\lambda\frac{1}{k^2}\,\mathrm{curl}\,\mathrm{curl} \int_{\Gamma_I} \Phi(x,z)[\nu \times (a_2 \times \nu)]ds_x,$$

where

$$\Phi(x,z) := \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x-z|}, \quad x \neq z \quad \text{and} \quad x, z \in \mathbb{R}^3. \tag{2.38}$$

The potential P is well defined and satisfies $\mathrm{curlcurl}\,P - k^2P = 0$ in D_e and D.
Now, let us assume that $\mathcal{H}^\top[a_1, a_2] = 0$. This means that the far field pattern of P is zero and from the Rellich lemma $P = 0$ in D_e. Since $a_1 \in Y(\Gamma_D)'$ there is an extension $(\tilde{a}_1 \times \nu) \in H^{-\frac{1}{2}}_{\|div}(\Gamma)$ of $a_1 \times \nu$ such that $(\tilde{a}_1 \times \nu)|_{\Gamma_I} \in L^2_t(\Gamma_I)$. Hence we can write

$$P(z) = \frac{1}{k^2}\mathrm{curl}\,\mathrm{curl} \int\limits_{\Gamma} \Phi(x,z)(\tilde{a}_1 \times \nu)ds_x - \frac{1}{k^2}\mathrm{curl}\,\mathrm{curl} \int\limits_{\Gamma_I} \Phi(x,z)(\tilde{a}_1 \times \nu)ds_x$$
$$- \mathrm{curl} \int_{\Gamma_I} \Phi(x,z)(a_2 \times \nu)ds_x - i\lambda\frac{1}{k^2}\,\mathrm{curl}\,\mathrm{curl} \int_{\Gamma_I} \Phi(x,z)[\nu \times (a_2 \times \nu)]ds_x.$$

If $z \to \Gamma$ the following jump relations hold

$$\nu \times P^+ - \nu \times P^-|_{\Gamma_D} = 0 \tag{2.39}$$
$$\nu \times P^+ - \nu \times P^-|_{\Gamma_I} = -(a_2 \times \nu) \tag{2.40}$$
$$\nu \times \mathrm{curl}\,P^+ - \nu \times \mathrm{curl}\,P^-|_{\Gamma_D} = (a_1 \times \nu) \tag{2.41}$$
$$\nu \times \mathrm{curl}\,P^+ - \nu \times \mathrm{curl}\,P^-|_{\Gamma_I} = -i\lambda[\nu \times (a_2 \times \nu)], \tag{2.42}$$

where by the superscript $+$ and $-$ we distinguish the limit obtained by approaching the boundary Γ from D^e and D, respectively. Note that $\nu \times P^-$ and $\nu \times \mathrm{curl}\,P^- = 0$ exist in the L^2 sense on the whole boundary Γ.
We remark that, since $\tilde{a}_1 \times \nu \in H^{-\frac{1}{2}}_{\|div}(\Gamma)$, then the potential over Γ and the corresponding jump relations are well defined from potential theory for the single layer potentials with $H^{-\frac{1}{2}}$ density [19] (see also [3]), while the jump relations for the potentials over Γ_I with L^2 layer are interpreted in the sense of the L^2 limit [11] p. 172. Therefore combining (2.40) and (2.42) and using the fact that $\nu \times P^+ = \nu \times \mathrm{curl}\,P^+ = 0$ we obtain

$$\nu \times P^-|_{\Gamma_D} = 0 \tag{2.43}$$
$$\left[\nu \times \mathrm{curl}\,P^- + i\lambda\nu \times (P^- \times \nu)\right]\big|_{\Gamma_I} = 0 \tag{2.44}$$

which are understood in the L^2 limit sense. Hence P is such that $\mathrm{curlcurl}\,P - k^2P = 0$ in D and satisfies the boundary conditions (2.43) and (2.44). Using the divergence theorem and a parallel surface argument one can conclude as in Theorem 2.1 that $P = 0$ in D, whence from the jump relations (2.40) and (2.41) and the fact that a_1 and a_2 are tangential fields we obtain that $a_1 = 0$ and $a_2 = 0$. This means that $\mathcal{H}^\top$ is injective, which ends the proof.$\square$

COROLLARY 2.6. *Assume that $\Gamma_I \neq \emptyset$ and $\lambda \neq 0$. Then the electric field E of the solution to the interior mixed boundary value problem (2.16) can be approximated by the electric field of an electromagnetic Herglotz pair with respect to the $X(D, \Gamma_I)$ norm.*

Proof. The result is a consequence of Theorem 2.5 and the a priori estimate (2.30). $\square$

If $\Gamma_I = \emptyset$, then D is a perfect conductor, $X(D, \Gamma_I) = H(\mathrm{curl}, D)$ and the approximation properties of electromagnetic Herglotz pairs follows from the results of [12].

3. Inverse scattering problem. It is known [11] that the radiating solutions E, H to the exterior problem (2.15), have the asymptotic behavior

$$E(x) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad H(x) = \frac{e^{ik|x|}}{|x|}\left\{H_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}$$

as $|x| \to \infty$, where E_∞ and H_∞ are defined on the unit sphere Ω and are known as the *electric far field pattern* and *magnetic far field pattern*, respectively. Moreover they satisfy

$$H_\infty = \hat{x} \times E_\infty, \qquad \text{and} \qquad \hat{x} \cdot E_\infty = \hat{x} \cdot H_\infty = 0.$$

We now consider the scattering of an electromagnetic plane wave by a perfectly conducting obstacle D that is partially coated by a material with surface impedance λ. In this case the scattered fields E, H satisfy (2.15) with $f := -\nu \times E^i$ and $h := -\nu \times \mathrm{curl}\, E^i + i\lambda(\nu \times E^i) \times \nu$, where the incident plane wave is given by (2.8). We indicate the dependence of the electric far field on the incident direction d and polarization p by writing $E_\infty(\hat{x}, d, p)$. The *inverse scattering problem* we will consider in this paper is to determine D from the knowledge of the electric far field $E_\infty(\hat{x}; d, p)$. (Note that we do not assume a priori knowledge of Γ_D, Γ_I or λ!).

3.1. The linear sampling method. The electric far field pattern defines the *electric far field operator* $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))ds(d), \qquad \hat{x} \in \Omega, \tag{3.1}$$

for $g \in L^2_t(\Omega)$. Note that by superposition Fg is the electric far field pattern of the exterior mixed boundary value problem corresponding to the electromagnetic Herglotz pair with kernel ikg as incident field. Now let us consider the electric dipole with polarization q defined by

$$\begin{aligned} E_e(x, z, q) &:= \frac{i}{k}\mathrm{curl}_x\, \mathrm{curl}_x\, q\, \Phi(x, z) \\ H_e(x, z, q) &:= \mathrm{curl}_x\, q\, \Phi(x, z), \end{aligned}$$

where Φ is the fundamental solution of Helmholtz equation defined by (2.38). If $z \in D$ then $E_e(x, z, q)$ and $H_e(x, z, q)$ solve the Maxwell equations in D_e and the corresponding electric far field pattern $E_{e,\infty}(\hat{x}, z, q)$ is given by

$$E_{e,\infty}(\hat{x}, z, q) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x}\, e^{-ik\hat{x}\cdot z}. \tag{3.2}$$

The linear sampling method for solving the inverse problem consists of solving the following linear first kind integral equation which we will call the *far field equation*

$$Fg(\hat{x}) = E_{e,\infty}(\hat{x}, z, q), \tag{3.3}$$

for a set of sampling points $z \in \mathbb{R}^3$ and three linear independent polarizations $q \in \mathbb{R}^3$.

We now define the operator $\mathcal{B} : Y(\Gamma_D) \times L^2_t(\Gamma_I) \to L^2_t(\Omega)$ which maps a pair of functions $(f, h) \in Y(\Gamma_D) \times$

$L^2_t(\Gamma_I)$ onto the electric far field pattern $E_\infty \in L^2_t(\Omega)$ of the solution to (2.15) with boundary data (f,h). The operator $\mathcal{B}$ is a composition of the bounded linear solution operator mapping the boundary data $(f,h) \in Y(\Gamma_D) \times L^2_t(\Gamma_I)$ into the radiating solution to (2.15) (see Theorem 2.4) with the compact operator which takes this solution to the corresponding electric far field (see [11], Theorem 6.8). Hence $\mathcal{B}$ is a bounded injective linear operator and moreover compact.

Theorem 3.1. *The operator $\mathcal{B}$ has dense range, that is the set $\mathcal{B}\left(Y(\Gamma_D) \times L^2_t(\Gamma_I)\right)$ is dense in $L^2_t(\Omega)$.*

Proof. As in the proof of Theorem 2.5 we define $H := Y(\Gamma_D) \times L^2(\Gamma_I)$ and its dual $H^* := Y(\Gamma_D)' \times L^2_t(\Gamma_I)$. Recall that the duality pairing between $Y(\Gamma_D)$ and $Y(\Gamma_D)'$ is defined by (2.14) and the duality pairing between $L^2_t(\Gamma_I)$ and $L^2_t(\Gamma_I)$ is the L^2 scalar product. The dual operator $\mathcal{B}^\top : L^2_t(\Omega) \to H^*$ of the operator $\mathcal{B}$ is such that for every $(f,h) \in H$ and $g \in L^2_t(\Omega)$ we have

$$\langle \mathcal{B}(f,h), g\rangle_{L^2_t(\Omega), L^2_t(\Omega)} = \left\langle (f,h), \mathcal{B}^\top g\right\rangle_{H,H^*}.$$

Next we want to characterize the dual operator $\mathcal{B}^\top$. From [11], Theorem 6.8, we have

$$\mathcal{B}(f,h) := E_\infty = \frac{ik}{4\pi}\hat{x} \times \int_\Gamma \left\{\nu(y) \times E(y) + \frac{1}{ik}[\nu(y) \times \operatorname{curl} E] \times \hat{x}\right\} e^{-ik\hat{x}\cdot y} ds(y),$$

where $E \in X_{loc}(D_e, \Gamma_I)$ is the electric scattered field corresponding to the boundary data (f,h). Hence by changing the order of integration we can write

$$\begin{aligned}\langle \mathcal{B}(f,h), g\rangle = \frac{ik}{4\pi}\int_\Gamma \int_\Omega e^{-ik\hat{x}\cdot y}\Big\{ &[\hat{x} \times (\nu(y) \times E(y))]\cdot g(\hat{x}) \\ &+ \frac{1}{ik}[\hat{x} \times (\nu(y) \times \operatorname{curl} E) \times \hat{x}]\cdot g(\hat{x})\Big\}\, ds(\hat{x})\, ds(y).\end{aligned} \tag{3.4}$$

Let

$$E_g(y) := \int_\Omega g(\hat{x}) e^{-ik\hat{x}\cdot y}\, ds(\hat{x})$$

denote the electric Herglotz wave function with tangential kernel $g \in L^2_t(\Omega)$. Simple calculations show that

$$\begin{aligned}\operatorname{curl}_y E_g(y) &= ik \int_\Omega [g(\hat{x}) \times \hat{x}] e^{-ik\hat{x}\cdot y}\, ds(\hat{x}) \\ \operatorname{curl}_y \operatorname{curl}_y E_g(y) &= k^2 \int_\Omega [\hat{x} \times (g(\hat{x}) \times \hat{x})] e^{-ik\hat{x}\cdot y}\, ds(\hat{x}).\end{aligned}$$

By using the relations

$$\begin{aligned}&[\hat{x} \times (\nu(y) \times E(y))]\cdot g(\hat{x}) = [\nu(y) \times E(y)]\cdot [g(\hat{x}) \times \hat{x}] \\ &[\hat{x} \times (\nu(y) \times \operatorname{curl} E(y)) \times \hat{x}]\cdot g(\hat{x}) = [\nu(y) \times \operatorname{curl} E(y)]\cdot[\hat{x} \times (g(\hat{x}) \times \hat{x})]\end{aligned}$$

and the fact that

$$\operatorname{curl}_y \operatorname{curl}_y E_g(y) = k^2 E_g(y),$$

due to the fact that $g(\hat{x})$ is a tangential vector on the unit sphere, we can rewrite (3.4) as

$$\langle \mathcal{B}(f,h), g\rangle = \frac{1}{4\pi}\int_\Gamma [\nu(y) \times E(y)]\cdot \operatorname{curl} E_g(y) + [\nu(y) \times \operatorname{curl} E(y)]\cdot E_g(y) ds(y). \tag{3.5}$$

Now let $\tilde{E} \in X_{loc}(D_e, \Gamma_I)$ be the solution of the exterior mixed boundary value problem (2.15) with boundary data

$$\begin{aligned} &\nu \times \tilde{E} = \nu \times E_g \quad \text{on} \quad \Gamma_D \\ &\nu \times \operatorname{curl} \tilde{E} - i\lambda(\nu \times \tilde{E}) \times \nu = \nu \times \operatorname{curl} E_g - i\lambda(\nu \times E_g) \times \nu \quad \text{on} \quad \Gamma_I. \end{aligned} \tag{3.6}$$

By splitting the integral in (3.5) into two pieces over Γ_D and Γ_I, and by using the boundary relations (3.6), we obtain

$$\begin{aligned} \langle \mathcal{B}(f,h), g\rangle = &\frac{1}{4\pi} \int\limits_{\Gamma_D} (\nu \times E) \cdot \operatorname{curl} E_g + (\nu \times \operatorname{curl} E) \cdot \tilde{E}\, ds \\ &+ \frac{1}{4\pi} \int\limits_{\Gamma_I} (\nu \times E) \cdot [\operatorname{curl} \tilde{E} + i\lambda(\nu \times \tilde{E})] - i\lambda(\nu \times E) \cdot (\nu \times E_g) + (\nu \times \operatorname{curl} E) \cdot E_g\, ds. \end{aligned} \tag{3.7}$$

Using the relation

$$\int\limits_{\Gamma_D} (\nu \times \operatorname{curl} E) \cdot \tilde{E} ds - \int\limits_{\Gamma_I} (\nu \times \operatorname{curl} \tilde{E}) \cdot E ds = \int\limits_{\Gamma_D} (\nu \times \operatorname{curl} \tilde{E}) \cdot E ds - \int\limits_{\Gamma_I} (\nu \times \operatorname{curl} E) \cdot \tilde{E} ds,$$

which is obtained from Green's formula in D_e for two radiating solutions $E, \tilde{E}$ to the Maxwell equations, and rearranging the terms, we have

$$\begin{aligned} \langle \mathcal{B}(f,h), g\rangle = &\frac{1}{4\pi} \int\limits_{\Gamma_D} (\nu \times E) \cdot (\operatorname{curl} E_g - \operatorname{curl} \tilde{E})\, ds \\ &+ \frac{1}{4\pi} \int\limits_{\Gamma_I} [\nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu] \cdot (E_g - \tilde{E})\, ds, \end{aligned}$$

and finally the boundary condition for E implies

$$\langle \mathcal{B}(f,h), g\rangle = \frac{1}{4\pi} \int\limits_{\Gamma_D} f \cdot (\operatorname{curl} E_g - \operatorname{curl} \tilde{E})\, ds + \frac{1}{4\pi} \int\limits_{\Gamma_I} h \cdot (E_g - \tilde{E})\, ds. \tag{3.8}$$

Hence

$$4\pi \mathcal{B}^\top g = \begin{cases} \nu \times (\operatorname{curl} E_g - \operatorname{curl} \tilde{E}) \times \nu \in Y(\Gamma_D)' & \text{on} \quad \Gamma_D \\ \nu \times (E_g - \tilde{E}) \times \nu \in L^2_t(\Gamma_I) & \text{on} \quad \Gamma_I \end{cases} \tag{3.9}$$

Let $\mathcal{B}^\top g \equiv 0$. Then (3.9) and (3.6) imply that $\nu \times \tilde{E} \equiv \nu \times E_g$ and $\nu \times \operatorname{curl} \tilde{E} \equiv \nu \times \operatorname{curl} E_g$ on the whole boundary Γ. Therefore by using the Stratton-Chu formula (see [11], Theorem 6.6, and justified for Lipschitz boundary in [3], Theorem 3.2 and in [20]) $\tilde{E}$ and $\tilde{H} = \frac{1}{ik} \operatorname{curl} \tilde{E}$ can be extended to a solution of the Maxwell equations in $\mathbb{R}^3$. But since they satisfy the Silver-Müller radiation condition this means that $\tilde{E} \equiv 0$ and hence $E_g \equiv 0$ which can happen only if the kernel $g \equiv 0$.
We can now characterize the range $\mathcal{B}$ as $(\operatorname{range} \mathcal{B})^a = \operatorname{kern} \mathcal{B}^\top$ where $(\,)^a$ denotes the annihilator set [19]. In other words from the injectivity of $\mathcal{B}^\top$ we have

$$\left\{ g \in L^2_t(\Omega) : \langle g, \psi \rangle = 0 \quad \text{for all} \quad \psi \in \operatorname{range} \mathcal{B} \right\} = \{0\},$$

whence the set $\mathcal{B}\left(Y(\Gamma_D) \times L^2_t(\Gamma_I)\right)$ is dense in $L^2_t(\Omega)$. This ends the proof of the theorem. $\square$

In terms of the operator $\mathcal{B}$ the far field equation (3.3) can be written as

$$\mathcal{B}(\Lambda E_g) = -\frac{1}{ik} E_{e,\infty}(\cdot, z, q) \qquad z \in \mathbb{R}^3, \tag{3.10}$$

where Λ denotes the trace operator corresponding to our mixed boundary condition, i.e. $\Lambda u := \nu \times u|_{\Gamma_D}$ on Γ_D and $\Lambda u := \nu \times \operatorname{curl} u - i\lambda(\nu \times u) \times \nu|_{\Gamma_I}$ on Γ_I, and E_g is the electric field of the electromagnetic Herglotz pair given by (2.32). Our goal is to study (3.10) for sampling points $z \in \mathbb{R}^3$.
First let $z \in D$. In this case $E_{e,\infty}(\cdot, z, q)$ is in the range of $\mathcal{B}$ since it is the far field pattern of the electric dipole $E_e(x, z, q)$ which is the solution of the exterior mixed boundary problem (2.15) with boundary data $-ikf_e := \nu \times E_e|_{\Gamma_D}$ and $-ikh_e := \nu \times \operatorname{curl} E_e - i\lambda(\nu \times E_e) \times \nu|_{\Gamma_I}$. Let $E \in X(D, \Gamma_I)$ be the solution of the interior mixed boundary value problem (2.16) satisfying the boundary condition (f_e, h_e). Then if $\Gamma_I \neq \emptyset$, from Theorem 2.5 for every $\epsilon > 0$ there is a $g_\epsilon(\cdot, z) = g_\epsilon(\cdot, z, q) \in L^2_t(\Omega)$ such that the corresponding electric Herglotz function $E_{g_\epsilon(\cdot,z)}$ satisfies

$$\|\Lambda\,(E - E_{g_\epsilon(\cdot,z)})\|_{Y(\Gamma_D)\times L^2_t(\Gamma_I)} < \epsilon. \tag{3.11}$$

The continuity of the operator $\mathcal{B}$ and the fact that $\Lambda\, E \equiv (f_e, h_e)$ imply that

$$\|\mathcal{B}(\Lambda E_{g_\epsilon(\cdot,z)}) + \frac{1}{ik} E_{e,\infty}(\cdot, z, q)\|_{L^2_t(\Omega)} < C\epsilon$$

for some positive constant C. Furthermore, if $z \to \Gamma$ then $\|E_e(\cdot, z, p)\|_{X(D_e \setminus B_R, \Gamma_I)} \to \infty$, whence the well-posedness of the exterior mixed boundary value problem implies

$$\lim_{z\to\Gamma} \|(f_e, h_e)\|_{Y(\Gamma_D)\times L^2_t(\Gamma_I)} = \infty,$$

and so from (3.11)

$$\lim_{z\to\Gamma} \|\Lambda E_{g_\epsilon(\cdot,z)})\|_{Y(\Gamma_D)\times L^2_t(\Gamma_I)} = \infty.$$

Hence the kernel and the corresponding electric Herglotz function satisfy

$$\lim_{z\to\Gamma} \|E_{g_\epsilon(\cdot,z)}\|_{X(D,\Gamma_I)} = \infty, \quad \text{and} \quad \lim_{z\to\Gamma} \|g_\epsilon(\cdot, z)\|_{L^2_t(\Omega)} = \infty.$$

Now let $z \in D_e$. For these points $-\frac{1}{ik} E_{e,\infty}(\cdot, z, q)$ is not in the range of $\mathcal{B}$ because from Rellich's lemma the electric dipole $E_e(x, z, q)$ has to be a solution to Maxwell's equation in D_e which is not possible since it has a singularity at z. However, using Theorem 3.1 and Tikhonov regularization, we can construct a regularized solution to the far field equation (3.10). In particular, there exist functions $(f^\alpha_z, h^\alpha_z) \in Y(\Gamma_D) \times L^2(\Gamma_I)$ corresponding to a parameter $\alpha = \alpha(\delta)$ chosen by a regular regularization strategy (e.g. the Morozov discrepancy principle [11]) such that

$$\|\mathcal{B}(f^\alpha_z, h^\alpha_z) + \frac{1}{ik} E_{e,\infty}(\cdot, z, q)\|_{L^2_t(\Omega)} < \delta, \tag{3.12}$$

for an arbitrary small $\delta > 0$, and

$$\lim_{\alpha\to 0} \left(\|f^\alpha_z\|_{Y(\Gamma_D)} + \|h^\alpha_z\|_{L^2(\Gamma_I)}\right) = \infty. \tag{3.13}$$

Note that in this case we have that $\alpha \to 0$ as $\delta \to 0$. Again assuming that $\Gamma_I \neq \emptyset$, we can use Theorem 2.5 and the continuity of the operator $\mathcal{B}$ to find an electric Herglotz function $E_{g_{\alpha,\epsilon}(\cdot,z)}$ with $g_{\alpha,\epsilon}(\cdot, z) = g_{\alpha,\epsilon}(\cdot, z, q) \in L^2_t(\Omega)$ such that

$$\|\mathcal{B}(\Lambda E_{g_{\alpha,\epsilon}(\cdot,z)}) - \mathcal{B}(f^\alpha_z, h^\alpha_z)\|_{L^2_t(\Omega)} < \epsilon. \tag{3.14}$$

Now combining (3.12) and (3.14) we obtain

$$\|\mathcal{B}(\Lambda E_{g_{\alpha,\epsilon}(\cdot,z)}) + \frac{1}{ik} E_{e,\infty}(\cdot, z, q)\|_{L^2_t(\Omega)} < \epsilon + \delta. \tag{3.15}$$

Furthermore, since $\Lambda E_{g_{\alpha,\epsilon}(\cdot,z)}$ approximates (f_z^α, h_z^α) in $Y(\Gamma_D)\times L_t^2(\Gamma_I)$, (3.13) implies that

$$\lim_{\alpha\to 0}\|\Lambda E_{g_{\alpha,\epsilon}(\cdot,z)}\|_{Y(\Gamma_D)\times L_t^2(\Gamma_I)}=\infty,$$

whence

$$\lim_{\alpha\to 0}\|E_{g_{\alpha,\epsilon}(\cdot,z)}\|_{X(D,\Gamma_I)}=\infty,\quad \text{and}\quad \lim_{\alpha\to 0}\|g_{\alpha,\epsilon}(\cdot,z)\|_{L_t^2(\Omega)}=\infty.$$

We summarize these results in the following main theorem.

THEOREM 3.2. *Assume that $\Gamma_I\neq\emptyset$ and $\lambda>0$. Then if F is the electric far field operator (3.1) corresponding to the mixed boundary value problem problem (2.15), we have that*
1) If $z\in D$ then for every $\epsilon>0$ there exists a solution $g_\epsilon(\cdot,z)=g_\epsilon(\cdot,z,q)\in L_t^2(\Omega)$ satisfying the inequality

$$\|Fg_\epsilon(\cdot,z)-E_{e,\infty}(\cdot,z,q)\|_{L_t^2(\Omega)}<\epsilon.$$

Moreover this solution satisfies

$$\lim_{z\to\Gamma}\|E_{g_\epsilon(\cdot,z)}\|_{X(D,\Gamma_I)}=\infty,\quad \textit{and}\quad \lim_{z\to\Gamma}\|g_\epsilon(\cdot,z)\|_{L_t^2(\Omega)}=\infty.$$

where $E_{g_\epsilon(\cdot,z)}$ is the electric field of the electromagnetic Herglotz pair with kernel g_ϵ, and
2) If $z\in D_e$ then for every $\epsilon>0$ and $\delta>0$ there exists a solution $g_{\delta,\epsilon}(\cdot,z)=g_{\delta,\epsilon}(\cdot,z,q)\in L_t^2(\Omega)$ of the inequality

$$\|Fg_{\delta,\epsilon}(\cdot,z)-E_{e,\infty}(\cdot,z,q)\|_{L_t^2(\Omega)}<\epsilon+\delta,$$

such that

$$\lim_{\delta\to 0}\|E_{g_{\delta,\epsilon}(\cdot,z)}\|_{X(D,\Gamma_I)}=\infty,\quad \textit{and}\quad \lim_{\delta\to 0}\|g_{\delta,\epsilon}(\cdot,z)\|_{L_t^2(\Omega)}=\infty,$$

where $E_{g_{\delta,\epsilon}(\cdot,z)}$ is the electric field of the electromagnetic Herglotz pair with kernel $g_{\delta,\epsilon}$.

We remark that if $\Gamma_I=0$, i.e. D is a perfect conductor, then Theorem 3.2 holds provided k is not a Maxwell eigenvalue.

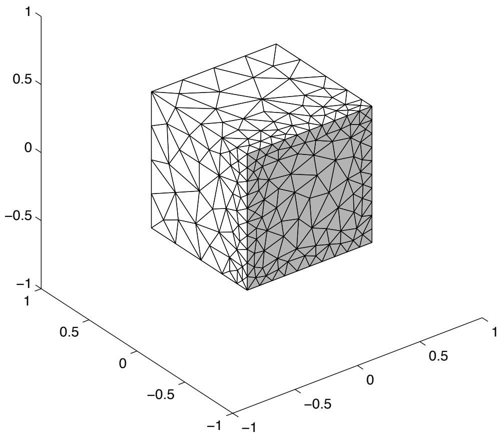
4. Numerical examples. The numerical results in this section are computed in the way detailed in [9]. In summary, for a given test object, the far field pattern is computed using the ultra weak variational formulation of Maxwell's equations given in [6]. The far field data is then perturbed by random noise, and used in a discrete version of the far field equation obtained by applying numerical quadrature to (3.3) using a discrete set of N quadrature points on the unit sphere corresponding to the directions of the incoming waves and the measurement points. Tikhonov regularization and the Morozov discrepancy principle are used in the inversion of the discrete far field equation. We choose z on a uniform grid in the region we are sampling for a scatterer. The region varies depending on the example and can be seen from the figures. In each case we use a 51×51×51 uniform grid. The reconstruction of the two balls shown in Section 4.2 takes 235 seconds on 300MHz Silicon Graphics Origin-2000. The reader is referred to [9] for complete details of the algorithm.

There are three important parameters for the far field data. The first, ϵ controls the amount of random noise added to the data. As in our previous papers we choose $\epsilon=0.01$. The second is the number of incoming waves. This varies between examples. The third parameter is the contour level at which we draw the iso-surface of the reconstruction. Suppose we compute an approximation to $g=g(x,z,q)$ where z and q are the source point location and polarization of the dipole source in (3.2). We define

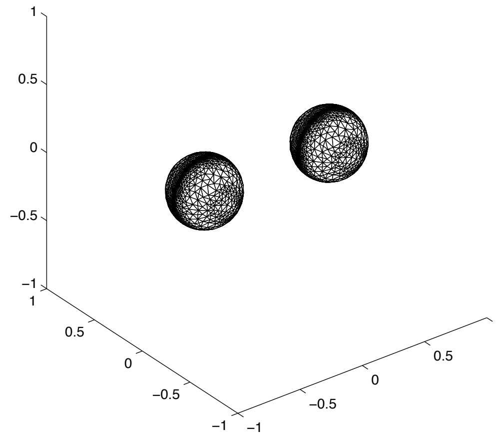
$$\mathcal{G}(z)=\frac{1}{3}\left(\frac{1}{\|g(\cdot,z,e_1)\|_{L^2(\Omega)}}+\frac{1}{\|g(\cdot,z,e_2)\|_{L^2(\Omega)}}+\frac{1}{\|g(\cdot,z,e_3)\|_{L^2(\Omega)}}\right).$$

The iso-surface is then the set of points z such that

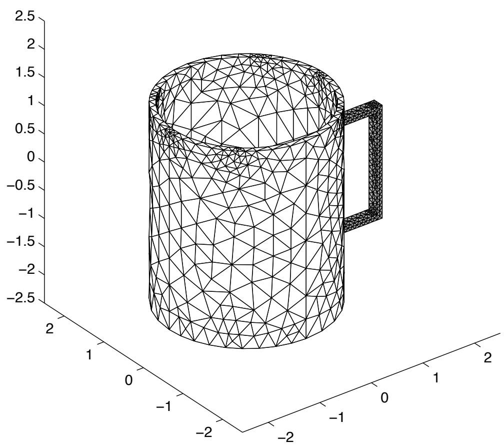
$$\mathcal{G}(z)=0.2\max_z\mathcal{G}(z)$$



(a) Cube



(b) Balls



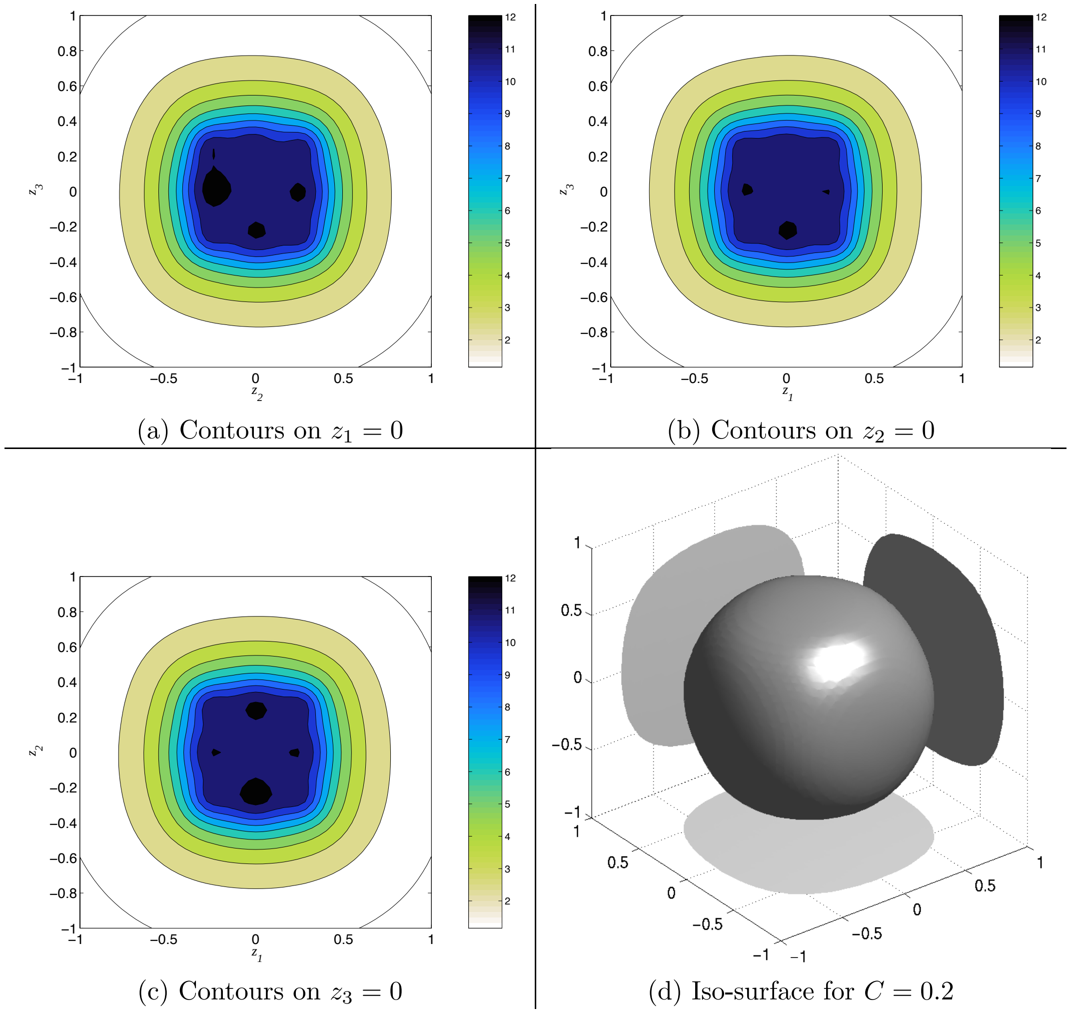
(c) Mug

Fig. 4.1. *Original figures used in this study showing the surface mesh. The shaded region shows where the impedance boundary condition is applied. In each case the mesh is refined towards the curve separating the perfectly conducting and impedance portions of the boundary.*

where the factor 0.2 is chosen "ad hoc".

The three scatterers presented here are shown in Fig. 4.1. The simplest scatterer is just a cube and is a very simple example of a Lipschitz domain. We allow one face to have an impedance boundary condition. The second scatterer, the balls, are disconnected. This example demonstrates that with no modification the LSM can easily reconstruct disconnected objects. The third example, a camping mug, has a metal body and an imperfectly conducting handle. In each case we are interested in investigating the LSM at long wavelengths compared to the object.

4.1. Cube. Our first example is a simple Lipschitz domain, namely the unit cube $[-0.5, 0.5]^3$. We use $N = 42$ incoming waves (as in [9]) and choose $k = 2$. The resulting wavelength of the incident field is π and so the unit cube is less than a third of a wavelength across. Despite this the results shown in Figures 4.2 (where the entire surface is perfectly electrically conducting) and 4.3 (where one face is imperfectly conducting) show that we can obtain a reconstruction of the cube with obvious flattening of the faces. The



(a) Contours on $z_1 = 0$

(b) Contours on $z_2 = 0$

(c) Contours on $z_3 = 0$

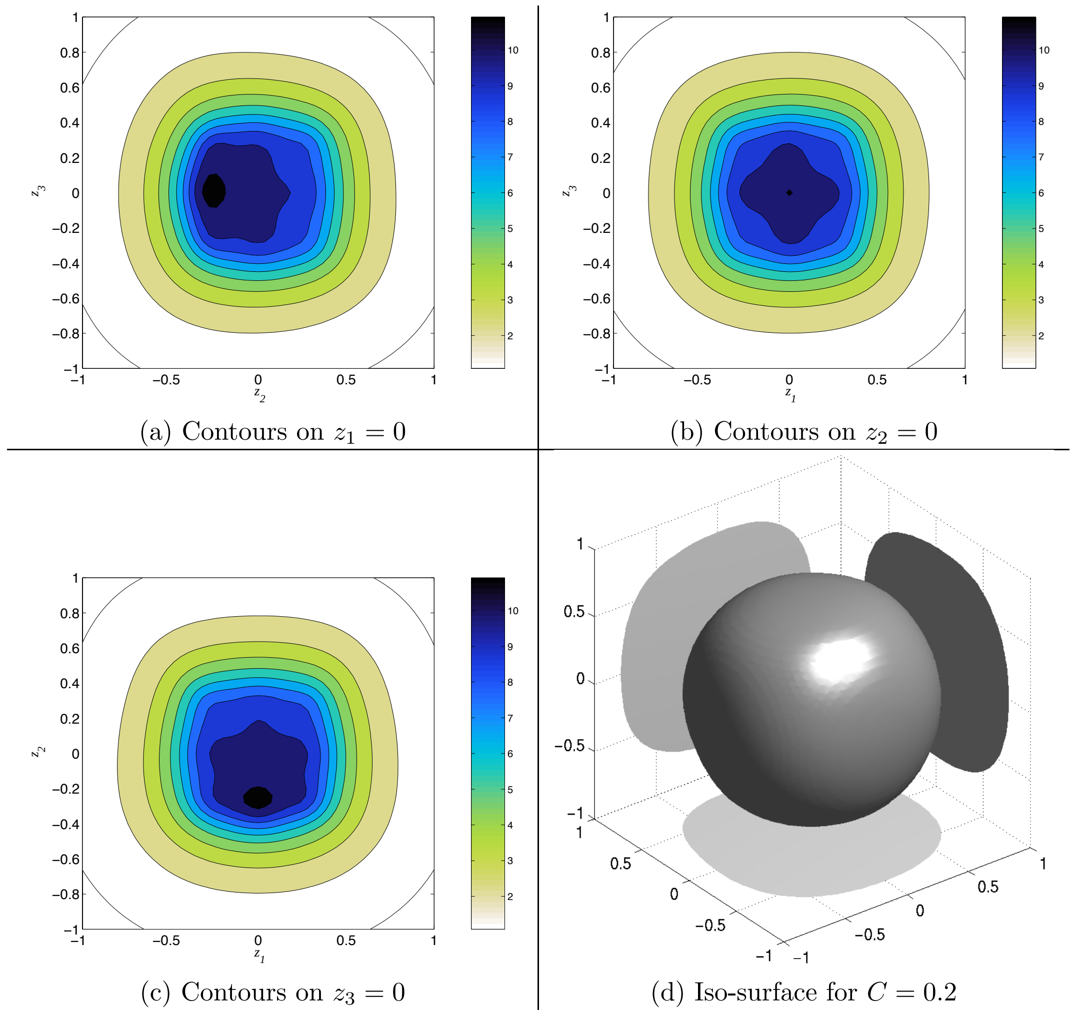
(d) Iso-surface for $C = 0.2$

FIG. 4.2. *Reconstruction of the perfectly conducting cube. The three contour plots show contours of $\mathcal{G}(z)$ as z varies on planes across the reconstruction domain. The three dimensional surface in the lower right hand panel is the surface $\mathcal{G}(z) = 0.2 \max_z \mathcal{G}(z)$. Here we use $k = 2$ and 42 incoming waves. As usual $\epsilon = 0.01$.*

fact that the cube is rounded is not surprising given the long wavelength compared to the size of the cube. Comparing Figures 4.2 and 4.3 we can see that although there are detailed differences in the contour maps for $\mathcal{G}(z)$, the overall three dimensional reconstruction does not differ noticeably in the two cases.

4.2. Two balls. The second example is two balls, one of which is half perfectly conducting and the other of which is a perfect conductor. The original scatterer is shown in panel (b) of Fig. 4.1, and the reconstruction is shown in Fig. 4.4. Since the reconstruction of two perfectly conducting balls is indistinguishable graphically from the reconstruction of the mixed balls, we have not shown it here. In keeping with the previous reconstruction we choose a value of the wavenumber ($k = 4$) such that the diameter of the balls is approximately one third of a wavelength. The reconstruction shows that the LSM can reconstruct disconnected scatterers. The elongation of the reconstructed balls towards one another is seen for long wavelengths as in this case. For shorter wavelengths the elongation decreases.

4.3. Camping mug. The final example is a camping mug shown in panel (c) of Fig. 4.1. The wavenumber is $k = 2$ so the handle of the mug is much less than one wavelength in thickness. Initial attempts to reconstruct the mug with $N = 42$ revealed multiple artifacts in the reconstruction so the number of directions used here is $N = 92$. Assuming an entirely metallic mug and handle, the reconstruction is shown in Fig. 4.5. The handle is suggested by the reconstruction, but the mug appears full! This is likely



(a) Contours on $z_1 = 0$ (b) Contours on $z_2 = 0$ (c) Contours on $z_3 = 0$ (d) Iso-surface for $C = 0.2$

FIG. 4.3. *Reconstruction of the cube with one face having an impedance boundary condition using the same format as Fig. 4.2. Here we use $k = 2$ and 42 incoming waves. As usual $\epsilon = 0.01$. Compare Figure 4.2.*

due to using a long wavelength. With an imperfectly conducting handle, the reconstruction shown in Fig. 4.6 has the same body, but the handle is not visible to any great degree. This suggests that an imperfectly conducting coating does effect the visibility of structures that are already close to the limit of resolution.

5. Conclusion. We have demonstrated by mathematical analysis and numerical results that the linear sampling method can be used to reconstruct scatterers having both perfectly conducting and imperfectly conducting components of the boundary. As is to be expected, the quality of the reconstruction can be influenced by the imperfectly conducting coating, but we have generally not seen much influence of the coating on the quality of reconstruction.

The examples here are all at long wavelength compared to the size of the object. As the wavelength decreases, our experience is that the fidelity of the reconstruction improves. The examples here show that even objects that are less than one third of a wavelength across can be roughly reconstructed.

Acknowledgments
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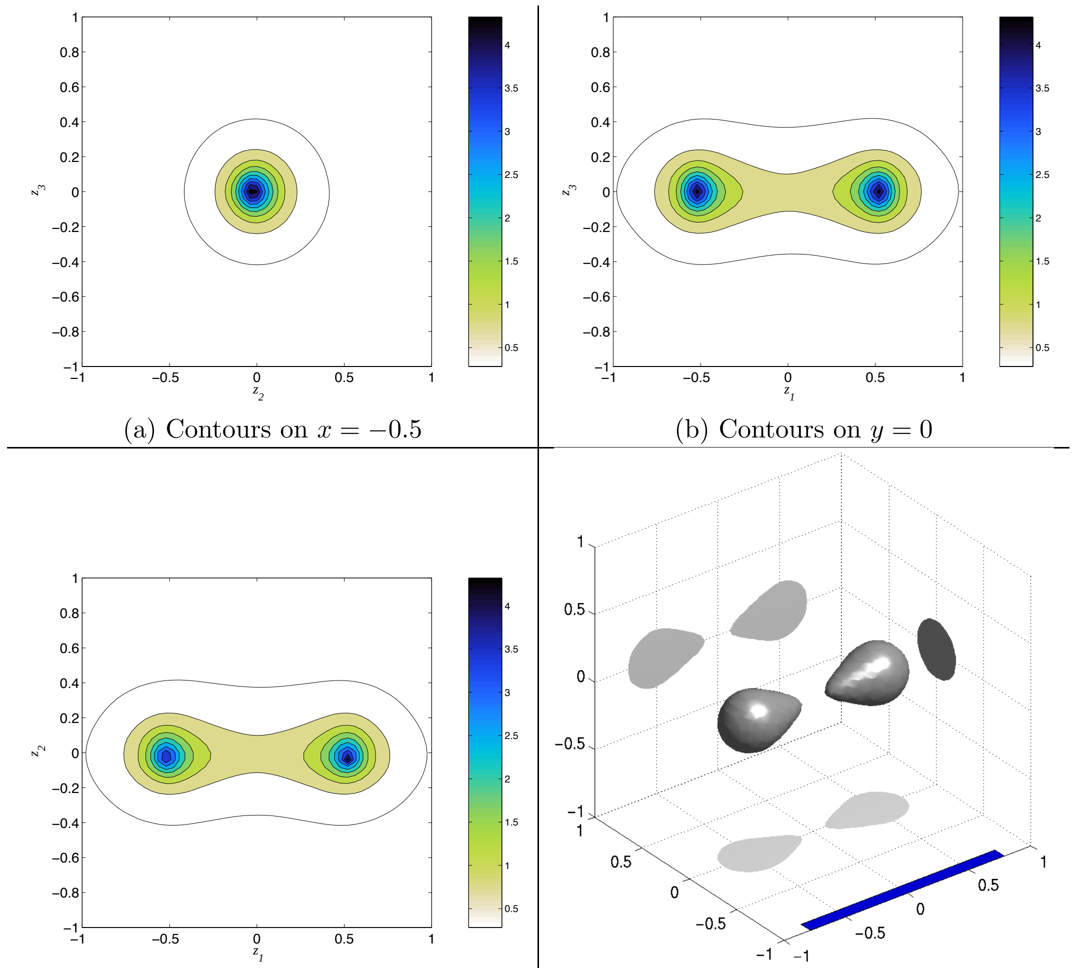

(a) Contours on $x = -0.5$
(b) Contours on $y = 0$
(c) Contours on $z = 0$
(d) Isosurface for $C = 0.2$

FIG. 4.4. *Reconstruction of the balls with one face having an impedance boundary condition using the same format as Fig. 4.2. Here we use* $k = 4$ *and 42 incoming waves. As usual* $\epsilon = 0.01$. *The bar visible in the three dimensional reconstruction shows the wavelength of the incident field.*

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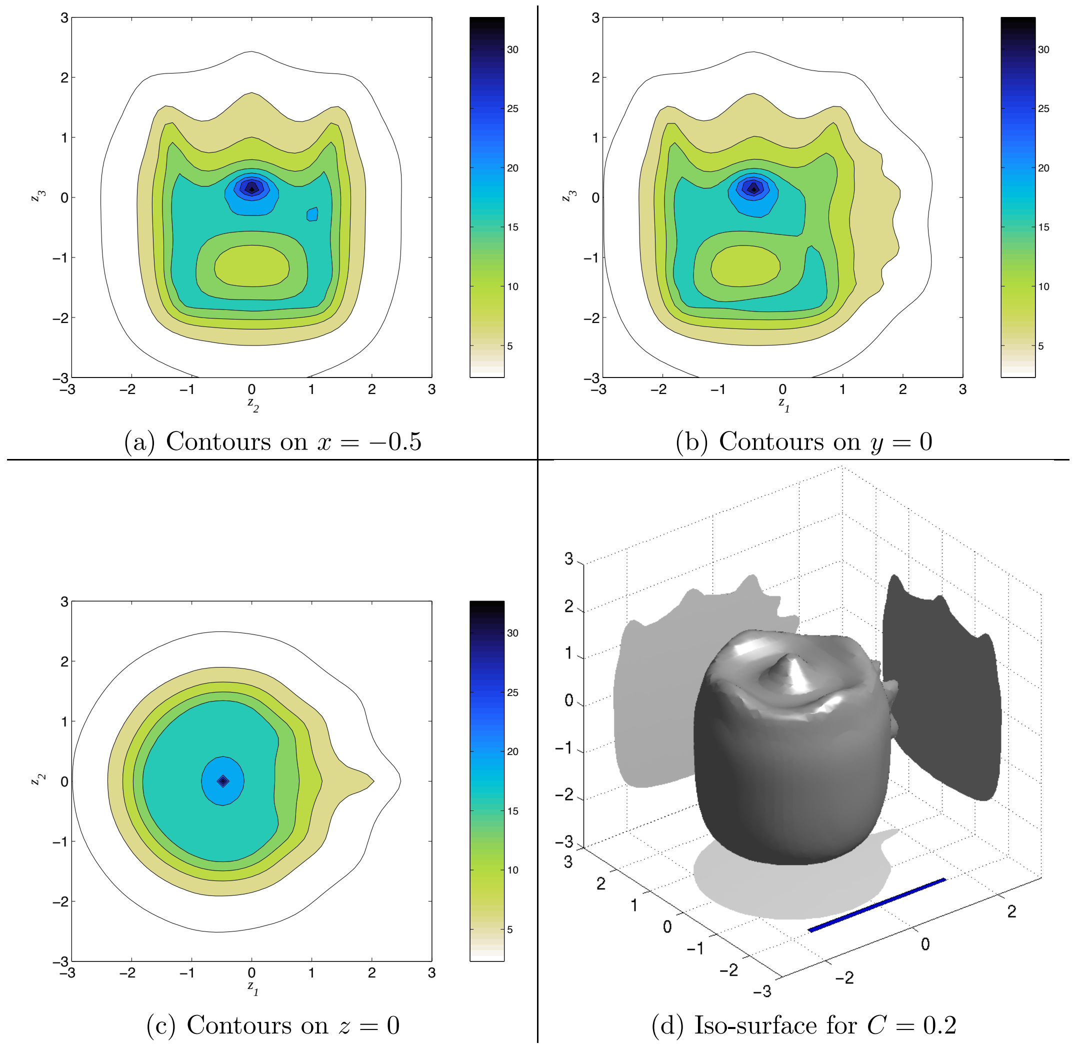

(a) Contours on $x = -0.5$

(b) Contours on $y = 0$

(c) Contours on $z = 0$

(d) Iso-surface for $C = 0.2$

FIG. 4.5. *Reconstruction of the perfectly conducting mug at long wavelength. Here we use* $k = 2$ *and 92 incoming waves. As usual* $\epsilon = 0.01$.

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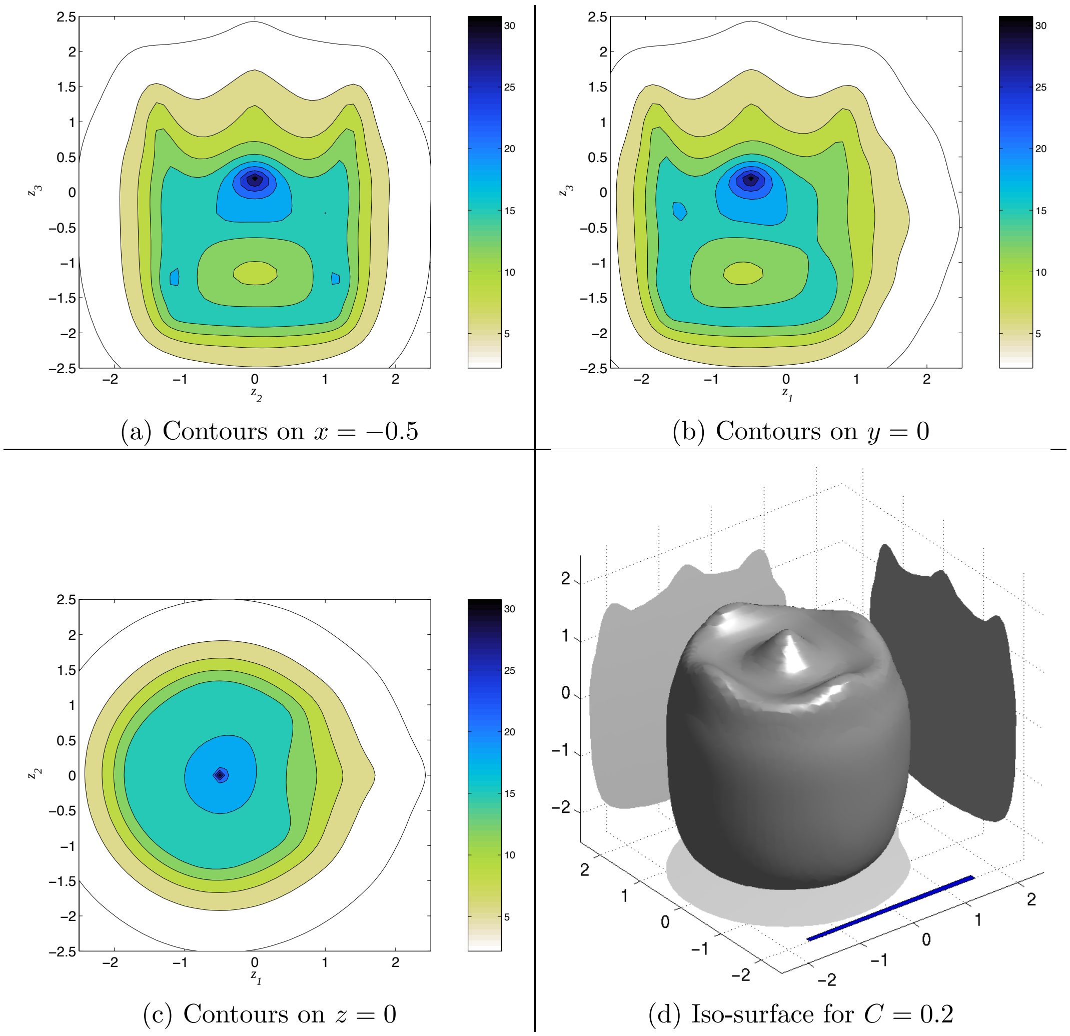

(a) Contours on $x = -0.5$

(b) Contours on $y = 0$

(c) Contours on $z = 0$

(d) Iso-surface for $C = 0.2$

FIG. 4.6. *Reconstruction of the cup with a perfectly conducting body and coated handle at long wavelength. Here we use $k = 2$ and 92 incoming waves. As usual $\epsilon = 0.01$. Note that the coated handle is much less visible than the perfectly conducting handle shown in Figure 4.5.*