Nature of the transmission eigenvalue spectrum for elastic bodies

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Abstract

This study develops a spectral theory of the interior transmission problem (ITP) for heterogeneous and anisotropic elastic solids. The importance of this subject stems from its central role in a certain class of inverse scattering theories (the so-called qualitative methods) involving penetrable scatterers. Although simply stated as a coupled pair of elastodynamic wave equations, the ITP for elastic bodies is neither self-adjoint nor elliptic. To help deal with such impediments, earlier studies have established the well-posedness of an elastodynamic ITP under notably restrictive assumptions on the contrast in elastic parameters between the scatterer and the background solid. Due to the lack of problem self-adjointness, however, these studies were successful in substantiating only the discreteness of the relevant eigenvalue spectrum, but not its existence. The aim of this work is to provide a systematic treatment of the ITP for heterogeneous and anisotropic elastic bodies that transcends the limitations of earlier treatments. Considering a broad range of material-contrast configurations (both in terms of elastic tensors and mass densities), this paper investigates the questions of the solvability of the ITP, the discreteness of its eigenvalues and, for the first time, of the actual existence of such eigenvalue spectrum. Necessitated by the breadth of material configurations studied, the relevant claims are established through the development of a suite of variational formulations, each customized to meet the needs of a particular subclass of eigenvalue problems. As a secondary result, the lower and upper bounds on the first transmission eigenvalue are obtained in terms of the elasticity and mass density contrasts between the obstacle and the background. Given the fact that the transmission eigenvalues are computable from experimental observations of the scattered field, such estimates may have significant potential toward exposing the nature (e.g. compliance) of penetrable scatterers in elasticity.

1 Introduction

The interior transmission problem (ITP), which appears in inverse scattering theory for inhomogeneous media, is a boundary-value problem formulated as a pair of governing field equations over bounded domain $D \subset \mathbb{R}^3$ representing the support of a scatterer, that are coupled through the Cauchy data on ∂D. A salient feature of this problem is that it entails the physical properties of both the obstacle and the background medium. Assuming steady-state wave propagation and scattering, the existence of a non-trivial solution to the *homogeneous* ITP amounts to that of an incident wave field, illuminating inhomogeneity D, that generates no scattered field. The excitation frequencies at which this phenomenon occurs form the set of the so-called *transmission eigenvalues*. In the context of qualitative approaches to inverse scattering [7] such as the linear sampling method [22] and the factorization method [34], an attempt to reconstruct an obstacle at these frequencies turns out to be futile owing to their intimate relationship to the solution of the corresponding ITP. Over the last two decades, a mounting body of studies on the interior transmission problem has shown that, due to its lack of self-adjointness and ellipticity at any frequency, the treatment of the ITP is not tractable by any classical theory of partial differential equations. A survey of particular issues and difficulties associated with the treatment of this non-traditional boundary-value problem can be found in [25]. Here one should mention that the early works on the ITP have primarily focused on the question of its well-posedness via either integral equation methods [20, 23, 32, 39] or customized variational formulations [5, 8, 18, 28] applied to Helmholtz, Maxwell, and Navier equations. These studies have in particular established a number of conditions, stated in terms of (contrast between) the physical properties of the scatterer and the background, under which the existence and uniqueness of a solution to the ITP can be ensured. Moreover, for certain configurations the set of transmission eigenvalues characterizing the ITP has been shown, by making recourse to the analytic Fredholm theory, to be at most countable with infinity as the only possible accumulation point.

Recently, a leap toward understanding the nature of the transmission eigenvalues has been made in [38], where the issue of the *existence* of such frequency values has been addressed for the first time. This development has been followed by a number of works on the spectral theory of the ITP [12–14, 16, 17, 19, 33]. In [6, 9–11] it was further shown that the transmission eigenvalues, originally seen as frequencies at which the linear sampling and factorization methods break down, can actually be used to obtain (in a non-iterative fashion) a qualitative information about the physical characteristics of a hidden scatterer. This result constitutes a significant advancement on the qualitative approaches to inverse scattering that have hitherto been designed and used exclusively for geometrical obstacle reconstruction.

The impetus to study the ITP for elastic bodies stems from the development of the linear sampling method [2, 4, 20, 26, 37] and the factorization method [21] for inverse scattering problems in *elastodynamics*. In this case, the aforementioned impediments plaguing the mathematical treatment of the ITP [25] are compounded by the *tensorial* structure of the elastic wave equation, which in particular features a fourth-order elastic tensor which may have up to six distinct eigenvalues in a general anisotropic case. The existing literature on the elastic ITP is relatively scarce [5, 18–20] and provides, for a limited number of physical configurations (specified in terms of the elasticity and mass density contrasts between the obstacle and the scatterer), *sufficient conditions* under which the problem is well-posed and has, at most, a countable set of eigenvalues.

To help complete the theoretical foundation of qualitative methods for inverse elastodynamic scattering, this work aims to establish a comprehensive treatment of the interior transmission eigenvalue problem for elastic bodies. By generalizing upon the methodologies developed for the Helmholtz and Maxwell equations, the goal is build a spectral theory of the elastic ITP for all possible material configurations, specified in terms of obstacle-background “contrasts” between the respective elastic tensors and mass densities. In this setting, the emphasis is made on i) solvability of the ITP; ii) discreteness of its transmission eigenvalue set, and iii) existence of such eigenvalues. With the aid of such fundamental results, the relationship between the first transmission eigenvalue (observable from the scattered field data) and the bounds on elastic and mass density characteristics of a hidden scatterer is exposed for the first time.

The article is organized as follows. The interior transmission problem for elastic bodies is introduced in Section 2, followed by the reference analytical (spherically symmetric) example for which the existence of transmission eigenvalues can be proved explicitly. Making use of this critical result, the featured eigenvalue problem is investigated for a comprehensive set of material-contrast configurations. In particular, Section 3 addresses the configurations with material similitude, i.e. situations where either the mass densities or the elastic tensors of the obstacle and the background solid coincide. The analysis is completed in Section 4 which deals with generic configurations without material similitude, i.e. those where the non-vanishing (mass density and elasticity) obstacle-background contrasts are considered to be either of the same or opposite sign. Owing to the complexity of the problem which turns out to be resilient to a unified treatment, each class of material configurations is dealt with via a custom-designed variational formulation.

Preliminaries

2 Preliminaries

Consider the time-harmonic vibrations of a bounded domain $D \subset \mathbb{R}^3$, with smooth boundary ∂D, at frequency ω. For clarity, all quantities in this study are interpreted as *dimensionless* by making reference to the characteristic length d_0, reference elastic modulus μ_0, and reference mass density ρ_0. Next, let $(\boldsymbol{\mathcal{C}}, \rho) \in L^\infty(D)$ and $(\boldsymbol{\mathcal{C}}_*, \rho_*) \in L^\infty(D)$ denote two sets of bounded material-parameter distributions over D, where $\boldsymbol{\mathcal{C}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{C}}_*(\boldsymbol{x})$ are real-valued, symmetric, fourth-order elastic tensor fields, while $\rho(\boldsymbol{x})$ and $\rho_*(\boldsymbol{x})$ are mass density distributions such that

$$\begin{array}{c} \mathsf{c}|\boldsymbol{\xi}|^2 \leqslant \boldsymbol{\xi} : \boldsymbol{\mathcal{C}}(\boldsymbol{x}) : \bar{\boldsymbol{\xi}} \leqslant \mathsf{C}|\boldsymbol{\xi}|^2, \\ \mathsf{c}_*|\boldsymbol{\xi}|^2 \leqslant \boldsymbol{\xi} : \boldsymbol{\mathcal{C}}_*(\boldsymbol{x}) : \bar{\boldsymbol{\xi}} \leqslant \mathsf{C}_*|\boldsymbol{\xi}|^2, \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{p} \leqslant \rho(\boldsymbol{x}) \leqslant \mathsf{P}, \\ \mathsf{p}_* \leqslant \rho_*(\boldsymbol{x}) \leqslant \mathsf{P}_*, \end{array} \qquad \boldsymbol{x} \in D. \tag{1}$$

minmax

Here $\boldsymbol{\xi}$ is a complex-valued, second-order tensor, while $\mathsf{c}, \mathsf{c}_*, \mathsf{p}, \mathsf{p}_*$ and $\mathsf{C}, \mathsf{C}_*, \mathsf{P}, \mathsf{P}_*$ are strictly positive constants, signifying respectively the *infima* and *suprema* of the associated scalar quantities. With reference to (1), it is further noted that $\mathsf{c}, \mathsf{C}, \mathsf{c}_*$ and C_* represent the bounds on the extreme eigenvalues of $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$, computed with respect to double contraction with a second-order tensor. In the most general anisotropic case $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$, which are endowed with major symmetry [5], may each have up to six distinct eigenvalues.

Hereon, it is assumed that the two distributions of material properties are "*non-intersecting*" in the sense that either

$$\mathsf{c}_* \geqslant 1 \geqslant \mathsf{C} \qquad \text{or} \qquad \mathsf{c} \geqslant 1 \geqslant \mathsf{C}_* \qquad \text{or} \qquad \boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_* \ \text{ in } D, \tag{2}$$

nonc

and either

$$\mathsf{p}_* \geqslant 1 \geqslant \mathsf{P} \qquad \text{or} \qquad \mathsf{p} \geqslant 1 \geqslant \mathsf{P}_* \qquad \text{or} \qquad \rho = \rho_* \ \text{ in } D, \tag{3}$$

nonr

with the unity as a point of demarcation achieved via suitable choice of the normalization constants μ_0 and ρ_0. Note that the strict equalities in (2) and (3) are, when applicable, assumed to hold almost everywhere in D, with the additional constraint

$$\big(\mathsf{c}_* = \mathsf{C} \,\vee\, \mathsf{c} = \mathsf{C}_* \,\vee\, \boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_* \text{ in } D\big) \,\wedge\, \big(\mathsf{p}_* = \mathsf{P} \,\vee\, \mathsf{p} = \mathsf{P}_* \,\vee\, \rho = \rho_* \text{ in } D\big) = \perp, \tag{4}$$

nonq

imposed on (2) and (3) to facilitate the variational analysis of the ensuing eigenvalue problem.

2.1 Interior transmission eigenvalue problem

With the above definitions the interior transmission eigenvalue problem (ITEP), that arises in a variety of inverse scattering problems [24], can be stated as a task of finding the non-trivial pair $(\boldsymbol{u}, \boldsymbol{u}_*) \in H^1(D) \times$

$H^1(D)$ that solves the homogeneous interior transmission problem

$$\begin{aligned} &\nabla\cdot[\mathcal{C}:\nabla \boldsymbol{u}] + \rho\omega^2 \boldsymbol{u} = \boldsymbol{0} \quad \text{in } D, && \boldsymbol{u}-\boldsymbol{u}_* = \boldsymbol{0} \quad \text{on } \partial D,\\ &\nabla\cdot[\mathcal{C}_*:\nabla \boldsymbol{u}_*] + \rho_*\omega^2 \boldsymbol{u}_* = \boldsymbol{0} \quad \text{in } D, && \boldsymbol{n}\cdot(\mathcal{C}:\nabla\boldsymbol{u} - \mathcal{C}_*:\nabla\boldsymbol{u}_*) = \boldsymbol{0} \quad \text{on } \partial D, \end{aligned} \tag{5}$$ ITPref

where $H^1 := W^{1,2}$ denotes the usual Sobolev space, and $\boldsymbol{n}$ is the unit normal on ∂D oriented toward the exterior of D.

TE **Definition 1.** *Values of ω^2 for which homogeneous problem (5) permits non-trivial solution $(\boldsymbol{u},\boldsymbol{u}_*)\in H^1(D)\times H^1(D)$ are called the transmission eigenvalues corresponding to transmission eigenfunctions $(\boldsymbol{u},\boldsymbol{u}_*)$.*

The ITEP plays a central role in the development of qualitative techniques for obstacle reconstruction such as the linear sampling method [7, 22] and the factorization method [34], that commonly revolve around the behavior of the so-called *measurements operator* which maps a set of incident wave patterns onto the set of scattered wavefields. To provide specificity for the discussion, let $(\mathcal{C}_*,\rho_*)$ and $(\mathcal{C},\rho)$ hereon denote respectively the material properties of a hidden obstacle $D\subset\Omega$ and the background domain Ω (e.g. $\mathbb{R}^3$ or a half-space). With such premise, it can be shown that the scattering operator characterizing D is injective with dense range providing that there does not exist a non-trivial solution $(\boldsymbol{u},\boldsymbol{u}_*)$ to homogeneous boundary value problem (5), where $\boldsymbol{u}$ is in the form of a single-layer potential over Ω whose source density is distributed over the source surface. Thus, if ω^2 is a transmission eigenvalue of (5), the scattering operator fails to be one-to-one and the linear sampling and factorization methods can no longer be applied.

The difficulties plaguing the study of the above-described ITEP stem from the structure of the boundary conditions prescribed over ∂D whereby (5) is neither self-adjoint, nor elliptic at any frequency (see [25] in the context of the scalar Helmholtz equation). These impediments are reflected in the fact that the existing studies of the ITEP for elastic bodies [5, 18, 19] are each formulated under fairly restrictive conditions in terms of the "contrast" between $(\mathcal{C}_*,\rho_*)$ and $(\mathcal{C},\rho)$. To shed further light on the problem, this investigation aims to generalize upon the recent developments for the Helmholtz equation and Maxwell equations [13, 16, 17, 30, 33, 38] toward: a) studying the solvability of (5) in situations when the contrast between $(\mathcal{C}_*,\rho_*)$ and $(\mathcal{C},\rho)$ transcends the restrictions imposed by earlier studies, and b) establishing, for the first time, the *existence* of transmission eigenvalues in elasticity. To this end, the task of investigating the ITEP for elastic bodies is recast as that of characterizing the kernel of a differential-trace operator $\mathbb{J}-f(\omega)\mathbb{K}$ that synthesizes the left-hand side of (5), constructed such that i) $\mathbb{J}$ and $\mathbb{K}$ are both self-adjoint, and ii) $\mathbb{K}$ is compact. Such decomposition in turn permits the analysis to proceed by focusing on the so-called "material ellipticity conditions" under which operator $\mathbb{J}$ is invertible.

RmkMatSym **Remark 1.** *The reference problem (5) is symmetric in material pairs $(\mathcal{C},\rho)$ and $(\mathcal{C}_*,\rho_*)$. Thus, in each case studied in this article only one material configuration among (2) and (3) will be stated, and lemmas and theorems will be generalized owing to a symmetry argument.*

RefCase 2.2 Analytical example

To help lay the foundation for the ensuing analysis, consider first the canonical case where D is a *ball* of radius R, while pairs $(\mathcal{C},\rho)$ and $(\mathcal{C}_*,\rho_*)$ each correspond to a *homogeneous isotropic* solid. By virtue of its simplicity, this example allows one to explicitly demonstrate the existence of a countable set of transmission eigenvalues associated with radially-symmetric eigenfunctions.

In the isotropic case, the fourth-order elastic tensors $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ can be synthesized in terms of the respective Lamé parameters (λ, μ) and (λ_*, μ_*). Under such restriction $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ have only two distinct eigenvalues [35], given respectively by $\{2\mu, 3\lambda+2\mu\}$ and $\{2\mu_*, 3\lambda_*+2\mu_*\}$, and their strong ellipticity is ensured by the well-known inequalities

$$\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad \mu_* > 0, \qquad 3\lambda_* + 2\mu_* > 0. \tag{6}$$

For completeness, it is noted that λ and λ_* are sign-indefinite by virtue of the fact that $\text{sign}(\lambda)\!=\!\text{sign}(\nu)$ and $\text{sign}(\lambda_*)\!=\!\text{sign}(\nu_*)$, where $\nu\in(-1,\frac{1}{2})$ and $\nu_*\in(-1,\frac{1}{2})$ are the Poisson's ratios affiliated respectively with $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$. In what follows, it is for simplicity assumed that $\nu \geqslant 0$ and $\nu_* \geqslant 0$. With such hypothesis, one has

$$\lambda = \frac{\mathsf{C}-\mathsf{c}}{3} \geqslant 0, \qquad \mu = \frac{\mathsf{c}}{2} > 0, \qquad \lambda_* = \frac{\mathsf{C}_* - \mathsf{c}_*}{3} \geqslant 0, \qquad \mu_* = \frac{\mathsf{c}_*}{2} > 0. \tag{7}$$

Lame

When the solution to the interior transmission problem is sought in the form of radially-symmetric vector fields $\boldsymbol{u}(\boldsymbol{x})\!=\!u(r)\boldsymbol{e}_r$ and $\boldsymbol{u}_*(\boldsymbol{x})\!=\!u_*(r)\boldsymbol{e}_r$ such that $r\!=\!|\boldsymbol{x}|$ and $\boldsymbol{e}_r\!=\!\boldsymbol{x}/r$, the field equations (5a) and (5b) can next be reduced for $r\in[0,R)$ as

$$u''(r) + \frac{2}{r}u'(r) + \left(\frac{\omega^2}{c^2} - \frac{2}{r^2}\right)u(r) = 0, \qquad u_*''(r) + \frac{2}{r}u_*'(r) + \left(\frac{\omega^2}{c_*^2} - \frac{2}{r^2}\right)u_*(r) = 0, \tag{8}$$

ITPsphere

where $c\!=\!\sqrt{(\lambda\!+\!2\mu)/\rho}$; $c_*\!=\!\sqrt{(\lambda_*\!+\!2\mu_*)/\rho_*}$, while f' and f'' denote respectively first and second derivative of $f(r)$ with respect to its argument. As a result, the solution to (5) can be written in terms of the spherical Bessel functions of the first order $j_1\big(\frac{\omega}{c}r\big)$ and $j_1\big(\frac{\omega}{c_*}r\big)$, which exposes the existence of a non-trivial solution when ω^2 is a transmission eigenvalue satisfying the characteristic equation

$$F(\omega) := \begin{vmatrix} j_1\big(\frac{\omega}{c}R\big) & j_1\big(\frac{\omega}{c_*}R\big) \\ \omega\sqrt{\rho(\lambda+2\mu)}\,j_1'\big(\frac{\omega}{c}R\big) + \frac{2\lambda}{R}j_1\big(\frac{\omega}{c}R\big) & \omega\sqrt{\rho_*(\lambda_*+2\mu_*)}\,j_1'\big(\frac{\omega}{c_*}R\big) + \frac{2\lambda_*}{R}j_1\big(\frac{\omega}{c_*}R\big) \end{vmatrix} = 0. \tag{9}$$

AnalyticTE

Previous studies of the ITEP for elastic solids [5, 18, 19] have consistently shown that the transmission eigenvalues, when they exist, can only accumulate at infinity. Accordingly, it is natural to investigate the asymptotic behavior of $F(\omega)$ as $\omega\to\infty$. To this end, one may employ the relationships

$$j_1(t) \underset{t\to\infty}{=} -\frac{\cos(t)}{t} + O\left(\frac{1}{t^2}\right) \qquad \text{and} \qquad j_1'(t) \underset{t\to\infty}{=} \frac{\sin(t)}{t} + O\left(\frac{1}{t^2}\right), \tag{10}$$

to find

$$F(\omega) \underset{\omega\to\infty}{=} \frac{c\,c_*}{\omega R^2}\left[\sqrt{\rho(\lambda\!+\!2\mu)}\sin\!\left(\frac{\omega}{c}R\right)\cos\!\left(\frac{\omega}{c_*}R\right) - \sqrt{\rho_*(\lambda_*\!+\!2\mu_*)}\cos\!\left(\frac{\omega}{c}R\right)\sin\!\left(\frac{\omega}{c_*}R\right)\right] + O\left(\frac{1}{\omega^2}\right). \tag{11}$$

AsympF

Assuming non-zero material contrast between $(\boldsymbol{\mathcal{C}},\rho)$ and $(\boldsymbol{\mathcal{C}}_*,\rho_*)$, one finds that the leading terms in (11) are nearly-periodic functions of frequency as $\omega\to\infty$, and so is F [31]. Thus, expansion (11) demonstrates that F has infinitely many zeros, i.e. that the set of transmission eigenvalues stemming from (9) is indeed countable. In concluding the example, it is noted that (8)–(11) represent an elastic-solid analogue of the well known spherically-symmetric study of the scalar Helmholtz equation, see e.g. [17, 25].

Vanish 3 Configurations with material similitude

In what follows, let $\mathcal{D}_\rho := (\rho_* - \rho)^{-1}$ and $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} := (\boldsymbol{\mathcal{C}}_*^{-1} - \boldsymbol{\mathcal{C}}^{-1})^{-1}$ quantify respectively the contrasts in mass density and elasticity between the two materials. With such notation, this section is devoted to investigating the ITEP for elastic solids in situations where either $\mathcal{D}_\rho$ or $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}}$ vanishes identically in D. Following the approach suggested in [9, 15, 27, 39], the problem at hand can be conveniently formulated as a system of fourth-order differential equations that is amenable to eigen-analysis in terms of variational methods.

For clarity of the ensuing developments, it is important to recall the underpinning analytical framework introduced in [16]. To this end, let $\mathbb{J}$ a bounded, positive definite, self-adjoint linear operator on separable Hilbert space W, and let $\mathbb{K}$ be a non-negative, self-adjoint, compact bounded linear operator on W. With such hypotheses, it can be shown that there exists an increasing sequence of positive real numbers λ_n and associated sequence of elements $\boldsymbol{w}_n \in W$ such that $\mathbb{J}\boldsymbol{w}_n = \lambda_n \mathbb{K}\boldsymbol{w}_n$. Next, letting $\tau \mapsto \mathbb{J}_\tau$ be a continuous mapping from $(0, +\infty)$ to the set of self-adjoint, positive definite, bounded linear operators on W, consider the eigenvalue problem of finding $\boldsymbol{w} \in W$ such that $(\mathbb{J}_\tau - \lambda_n(\tau)\mathbb{K})\, \boldsymbol{w} = \boldsymbol{0}$.
The following theorem, established in [16], is a fundamental tool toward demonstrating the existence of transmission eigenvalues.

ThRef1 **Theorem 1.** *Let $\tau \mapsto \mathbb{J}_\tau$ be a continuous mapping from $(0, +\infty)$ to the set of self-adjoint, positive definite, bounded linear operators on W, and let $\mathbb{K}$ be a non-negative, self-adjoint, compact bounded linear operator on W. Assume the existence of two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. *$\mathbb{J}_{\tau_0} - \tau_0 \mathbb{K}$ is positive on W, and*
2. *$\mathbb{J}_{\tau_1} - \tau_1 \mathbb{K}$ is non-positive on an m-dimensional subspace of W.*

Then each of the equations $\lambda_n(\tau) = \tau$, $n = 1, 2 \ldots, m$ has at least one solution for $\tau \in [\tau_0, \tau_1]$ where $\lambda_n(\tau)$ is the nth eigenvalue (counting multiplicity) of $\mathbb{J}_\tau$ with respect to $\mathbb{K}$, i.e. $\ker(\mathbb{J}_\tau - \lambda_n(\tau)\mathbb{K}) \neq \{\boldsymbol{0}\}$.

VanishC 3.1 Equal elastic tensors

In this section, it postulated that $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}}$ vanishes (i.e. $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_*$) while $\mathcal{D}_\rho \neq 0$ almost everywhere in D according to (4). On introducing the Sobolev space of vector fields with zero Cauchy data on ∂D, namely

$$H_0^2(D) = \big\{ \boldsymbol{\varphi} \in H^2(D) \colon \; \boldsymbol{\varphi} = \boldsymbol{0} \; \text{ and } \; \boldsymbol{n} \cdot \boldsymbol{\mathcal{C}} \colon \boldsymbol{\nabla}\boldsymbol{\varphi} = \boldsymbol{0} \; \text{ on } \partial D \big\}, \qquad (12)$$ Sobolev1

and assuming that pair $(\boldsymbol{u}, \boldsymbol{u}_*) \in L^2(D) \times L^2(D)$ solves the interior transmission problem (5) where $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_*$ one finds that, when $\rho_* \neq \rho$ and $\omega > 0$, the solution difference $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{u}_* \in H_0^2$ solves the fourth-order equation

$$\big(\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}} \colon \boldsymbol{\nabla}] + \rho\,\omega^2\big)\, \mathcal{D}_\rho \,\big(\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}} \colon \boldsymbol{\nabla}] + \rho_*\omega^2\big)\, \boldsymbol{v} = \boldsymbol{0} \qquad \text{in } D. \qquad (13)$$ Fourth1

The variational formulation of (13) consists in finding $\boldsymbol{v} \in H_0^2(D)$ such that

$$\int_D \mathcal{D}_\rho \big(\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}} \colon \boldsymbol{\nabla}\boldsymbol{v}] + \rho_*\omega^2 \boldsymbol{v}\big) \cdot \big(\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}} \colon \boldsymbol{\nabla}\bar{\boldsymbol{\varphi}}] + \rho\,\omega^2 \bar{\boldsymbol{\varphi}}\big)\, \mathrm{d}\boldsymbol{x} = 0 \qquad \forall \boldsymbol{\varphi} \in H_0^2(D). \qquad (14)$$ Var1

To facilitate the treatment of the variational problem at hand, let $\tau := \omega^2$, and define the auxiliary bounded sesquilinear forms on $H^2_0(D)\times H^2_0(D)$ as

$$\begin{aligned} \mathcal{A}_\tau(\boldsymbol{\varphi},\boldsymbol{\psi}) \;&:=\; \big\langle \mathcal{D}_\rho \left(\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\varphi}] + \rho\,\tau\boldsymbol{\varphi}\right), \left(\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\psi}] + \rho\,\tau\boldsymbol{\psi}\right) \big\rangle_{L^2(D)} \;+\; \tau^2 \left\langle \rho\,\boldsymbol{\varphi}, \boldsymbol{\psi}\right\rangle_{L^2(D)}, \\ \mathcal{B}(\boldsymbol{\varphi},\boldsymbol{\psi}) \;&:=\; \left\langle \boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\psi}\right\rangle_{L^2(D)}, \end{aligned} \tag{15}$$

SesF1

for all $(\boldsymbol{\varphi},\boldsymbol{\psi}) \in H^2_0(D)\times H^2_0(D)$, where the inner product between two nth-order tensors is understood in the sense of n-tuple contraction. On exercising (15) and the divergence theorem, (14) can be equivalently formulated as a task of finding $\boldsymbol{v}\in H^2_0(D)$ that satisfies

$$\mathcal{A}_\tau(\boldsymbol{v},\boldsymbol{\varphi}) \,-\, \tau\,\mathcal{B}(\boldsymbol{v},\boldsymbol{\varphi}) = 0 \qquad \forall \boldsymbol{\varphi}\in H^2_0(D). \tag{16}$$

vint1

Note that the boundedness of the featured operators is a consequence of tensor $\boldsymbol{\mathcal{C}}$ being positive definite and bounded. To expose the sufficient conditions for the ellipticity of $\mathcal{A}_\tau$, the latter can be conveniently recast as

$$\begin{aligned} \mathcal{A}_\tau(\boldsymbol{\varphi},\boldsymbol{\psi}) = \;& \big\langle \rho\mathcal{D}_\rho \left(\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\varphi}] + \tau\boldsymbol{\varphi}\right), \left(\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\psi}] + \tau\boldsymbol{\psi}\right) \big\rangle_{L^2(D)} \\ &+ \big\langle (1-\rho)\mathcal{D}_\rho \boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\varphi}], \boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\psi}]\big\rangle_{L^2(D)} \;+\; \tau^2 \big\langle \rho\mathcal{D}_\rho(\rho_*-1)\boldsymbol{\varphi},\boldsymbol{\psi}\big\rangle_{L^2(D)}. \end{aligned} \tag{17}$$

SesF1bis

LemCoer1 **Lemma 1.** *Assuming $\boldsymbol{\mathcal{C}}=\boldsymbol{\mathcal{C}}_*$ and restrictions on the contrast in mass densities $\mathsf{p}_*\geqslant 1\geqslant \mathsf{P}$ and $\mathcal{D}_\rho\neq 0$, then $\mathcal{A}_\tau$ is a coercive sesquilinear form on $H^2_0(D)\times H^2_0(D)$.*

Proof. The stated hypotheses of the Lemma imply the existence of real-valued constants α, α_* and γ such that in D one has

$$1-\rho \geqslant \alpha \geqslant 0, \qquad \rho_* - 1 \geqslant \alpha_* \geqslant 0, \qquad \mathcal{D}_\rho \geqslant \gamma > 0, \tag{18}$$

Cst1

where α and α_* cannot vanish simultaneously.

When $\boldsymbol{\varphi}\in H^2_0(D)$, one finds by virtue of (17a), (18), the Cauchy-Schwarz inequality, and triangle inequality that

$$\mathcal{A}_\tau(\boldsymbol{\varphi},\boldsymbol{\varphi}) \;\geqslant\; \mathsf{p}\,\gamma \left\{ \left(1+\frac{\alpha}{\mathsf{p}}\right) x^2 + (1+\alpha_*)\, y^2 - 2xy \right\}, \tag{19}$$

where $x := \|\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\varphi}]\|_{L^2(D)}$ and $y := \tau\,\|\boldsymbol{\varphi}\|_{L^2(D)}$. In this setting, several combinations in terms of α and α_* must be considered separately to provide a valid lower bound for $\mathcal{A}_\tau$. In particular, it can be shown that

$$\mathcal{A}_\tau(\boldsymbol{\varphi},\boldsymbol{\varphi}) \;\geqslant\; \mathsf{p}\gamma \left\{ \frac{\alpha}{\mathsf{p}} x^2 + \alpha_* y^2 + (x-y)^2 \right\} \qquad \text{when} \qquad \begin{array}{c} \alpha > 0, \\ \alpha_* > 0, \end{array} \tag{20}$$

coer11

$$\mathcal{A}_\tau(\boldsymbol{\varphi},\boldsymbol{\varphi}) \;\geqslant\; \mathsf{p}\gamma \left\{ \left(1-\frac{1}{\delta_*}\right) x^2 + (1+\alpha_*-\delta_*)\, y^2 + \delta_* \left(y - \frac{x}{\delta_*}\right)^2 \right\} \qquad \text{when} \qquad \begin{array}{c} \alpha = 0, \\ \alpha_* > 0, \end{array} \tag{21}$$

coer12

assuming $\delta_*\in(1, 1{+}\alpha_*)$, and

$$\mathcal{A}_\tau(\boldsymbol{\varphi},\boldsymbol{\varphi}) \;\geqslant\; \mathsf{p}\gamma \left\{ \left(1+\frac{\alpha}{\mathsf{p}}-\delta\right) x^2 + \left(1-\frac{1}{\delta}\right) y^2 + \delta \left(x - \frac{y}{\delta}\right)^2 \right\} \qquad \text{when} \qquad \begin{array}{c} \alpha > 0, \\ \alpha_* = 0, \end{array} \tag{22}$$

coer13

where $\delta\in(1, 1{+}\alpha/\mathsf{p})$.

From the lower bound in (1) on elastic tensor $\boldsymbol{\mathcal{C}}$, on the other hand, there exists a constant $\beta > 0$ such that

$$\|\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\varphi}]\|^2_{L^2(D)} \;+\; \|\boldsymbol{\varphi}\|^2_{L^2(D)} \;\geqslant\; \beta\,\|\boldsymbol{\varphi}\|^2_{H^2(D)}, \tag{23}$$

betab

see, e.g., [36]. On dropping the squared-difference terms on the right-hand sides of (20)–(22), one finally concludes from (23) that there exists a constant $C_\tau > 0$ (dependent on τ) such that $\mathcal{A}_\tau(\boldsymbol{\varphi},\boldsymbol{\varphi}) \geqslant C_\tau \|\boldsymbol{\varphi}\|^2_{H^2(D)}$ which concludes the proof. $\square$

On employing the Riesz representation theorem and identifying $H^2_0(D)$ with its dual, one can introduce bounded linear operators $\mathbb{A}_\tau, \mathbb{B} : H^2_0(D) \to H^2_0(D)$ such that for all $(\boldsymbol{\varphi},\boldsymbol{\psi}) \in H^2_0(D) \times H^2_0(D)$

$$\big\langle \mathbb{A}_\tau \boldsymbol{\varphi}, \boldsymbol{\psi} \big\rangle_{H^2_0(D)} = \mathcal{A}_\tau(\boldsymbol{\varphi},\boldsymbol{\psi}), \qquad \big\langle \mathbb{B} \boldsymbol{\varphi}, \boldsymbol{\psi} \big\rangle_{H^2_0(D)} = \mathcal{B}(\boldsymbol{\varphi},\boldsymbol{\psi}). \tag{24}$$

LinF1

As a result, (16) can be rewritten as $\big\langle (\mathbb{A}_\tau - \tau\mathbb{B})\boldsymbol{v}, \boldsymbol{\varphi} \big\rangle_{H^2_0(D)} = 0$ for all $\boldsymbol{\varphi} \in H^2_0(D)$. Thus if ω^2 is a transmission eigenvalue associated with (5) when $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_*$ then, recalling that $\tau = \omega^2$, one has that $\ker(\mathbb{A}_\tau - \tau\mathbb{B}) \neq \{\boldsymbol{0}\}$.

PropGen1 **Lemma 2.** *Assuming $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_*$, linear operator $\mathbb{A}_\tau : H^2_0(D) \to H^2_0(D)$ is positive definite, self-adjoint, and depends continuously on $\tau > 0$ when $\mathsf{p}_* \geqslant 1 \geqslant \mathsf{P}$ and $\mathcal{D}_\rho \neq 0$ hold almost everywhere in D. Further, $\mathbb{B} : H^2_0(D) \to H^2_0(D)$ is a self-adjoint and positive compact linear operator.*

Proof. Since ρ, ρ_* and $\boldsymbol{\mathcal{C}}$ are by premise real-valued and $\boldsymbol{\mathcal{C}}$ possesses the major symmetry, the sesquilinear forms $\mathcal{A}_\tau$ and $\mathcal{B}$ are Hermitian which requires that operators $\mathbb{A}_\tau$ and $\mathbb{B}$ be self-adjoint. The positive definite character of $\mathbb{A}_\tau$ is a direct consequence of (24) and Lemma 1, while its continuous dependence on $\tau > 0$ arises from the premise that $\mathcal{A}_\tau$ depends continuously on $\tau > 0$.

To establish the claim that $\mathbb{B}$ is compact, consider a bounded sequence $\boldsymbol{\varphi}_n \in H^2_0(D)$, such that there exists a subsequence $\tilde{\boldsymbol{\varphi}}_n$ that weakly converges with respect to the $H^2_0(D)$-norm to $\boldsymbol{\varphi}_\mathrm{o} \in H^2_0(D)$. Since $\tilde{\boldsymbol{\varphi}}_n \in H^2_0(D)$, it follows that $\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}_n \in H^1(D)$. By virtue of the compact embedding of $H^1(D)$ in $L^2(D)$, one accordingly finds that $\boldsymbol{\nabla}\tilde{\boldsymbol{\varphi}}_n$ converges strongly to $\boldsymbol{\nabla}\boldsymbol{\varphi}_\mathrm{o}$ with respect to the $L^2(D)$-norm. Finally using the definition of $\mathbb{B}$ in (24), the Cauchy-Schwarz inequality, and the boundedness of elastic tensor $\boldsymbol{\mathcal{C}}$, one finds that

$$\|\mathbb{B}(\tilde{\boldsymbol{\varphi}}_n - \boldsymbol{\varphi}_\mathrm{o})\|_{H^2_0(D)} \;\leqslant\; \mathsf{C}\,\|\boldsymbol{\nabla}(\tilde{\boldsymbol{\varphi}}_n - \boldsymbol{\varphi}_\mathrm{o})\|_{L^2(D)}, \tag{25}$$

which implies that $\mathbb{B}$ is compact since $\mathbb{B}\tilde{\boldsymbol{\varphi}}_n$ strongly converges to $\mathbb{B}\tilde{\boldsymbol{\varphi}}_\mathrm{o}$ with respect to the $H^2_0(D)$-norm. With this result in place, the proof of the lemma can now be completed by noting that $\mathbb{B}$ is positive owing to the positive definiteness of $\boldsymbol{\mathcal{C}}$ stipulated in (1). $\square$

The ensuing theorem establishes a lower bound for the transmission eigenvalues. To this end consider the negative Laplace operator $-\Delta$ for which, as shown by classical eigenvalue theory [29], there exist an increasing sequence of real-valued, positive Dirichlet eigenvalues $\lambda_n(D)$ and a sequence of corresponding first-order eigentensors $\boldsymbol{\varphi}_n$ satisfying

$$-\Delta\boldsymbol{\varphi}_n = \lambda_n(D)\boldsymbol{\varphi}_n \quad \text{in } D, \qquad \boldsymbol{\varphi}_n = \boldsymbol{0} \quad \text{on } \partial D. \tag{26}$$

In this setting $\lambda_1(D) > 0$ denotes the first, i.e. the smallest Dirichlet eigenvalue of the negative Laplace operator.

Theorem 2. *If either $\mathsf{p}_* \geqslant 1 \geqslant \mathsf{P}$ or $\mathsf{p} \geqslant 1 \geqslant \mathsf{P}_*$ while $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} = \boldsymbol{0}$ and $\mathcal{D}_\rho \neq 0$ hold almost everywhere in D, the set of transmission eigenvalues affiliated with (5) is discrete, with infinity being the only possible* *accumulation point. Moreover, every feasible transmission eigenvalue ω^2 is such that*

$$\omega^2 \;>\; \lambda_1(D)\,\frac{\mathsf{c}}{\max(\mathsf{P},\mathsf{P}_*)}.$$

Proof. When $\mathsf{p}_* \geqslant 1 \geqslant \mathsf{P}$ and $\mathcal{D}_\rho \neq 0$ holds almost everywhere in D, linear operator $\mathbb{A}_\tau$ is invertible due to Lemma 2 and, since $\mathbb{B}$ is a compact operator, so is $\mathbb{A}_\tau^{-1}\mathbb{B}$. On denoting by $\mathbb{I}$ the identity operator on $H^2_0(D)$, the Fredholm alternative applies [40] whereby $\mathbb{I}-\tau\mathbb{A}_\tau^{-1}\mathbb{B}$ is invertible except for, at most, a discrete set of values $\tau\in\mathbb{C}$ that can only accumulate at infinity.

Assuming for the time being that $\mathsf{p}_*\geqslant 1\geqslant \mathsf{P}$ i.e. $\mathcal{D}_\rho>0$, let $\boldsymbol{v}\in H^2_0(D)$ satisfying (14), which employing $\boldsymbol{\varphi}=\boldsymbol{v}$ and the divergence theorem yields

$$\int_D \mathcal{D}_\rho|\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{v}]+\rho_*\omega^2\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{x} \;+\; \tau\int_D \big(\boldsymbol{\nabla}\boldsymbol{v}:\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\bar{\boldsymbol{v}}-\rho_*\tau|\boldsymbol{v}|^2\big)\;\mathrm{d}\boldsymbol{x}=0. \tag{27}$$ ppsivanish1

Whenever the second integral is non-negative, one must clearly have $\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{v}]+\rho_*\,\tau\boldsymbol{v}=\boldsymbol{0}$ in D. Since $\boldsymbol{v}=\boldsymbol{0}$ and $\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{v}:\boldsymbol{n}=\boldsymbol{0}$ on ∂D for $\boldsymbol{v}\in H^2_0(D)$, it follows that $\boldsymbol{v}$ must also vanish in D by virtue of the Holmgren's uniqueness theorem (see [26] for a discussion in the context of elasticity). Due to (1) and Courant-Fischer min-max formulae [29], on the other hand, the Rayleigh quotient of elastic tensor $\boldsymbol{\mathcal{C}}$ is found to be bounded from below as

$$\inf_{\boldsymbol{v}\in H^2_0(D)}\frac{\displaystyle\int_D \boldsymbol{\nabla}\boldsymbol{v}:\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}}{\displaystyle\int_D|\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{x}} \;\geqslant\; \mathsf{c}\inf_{\boldsymbol{v}\in H^1_0(D)}\frac{\displaystyle\int_D|\boldsymbol{\nabla}\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{x}}{\displaystyle\int_D|\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{x}} \;\geqslant\; \mathsf{c}\lambda_1(D), \tag{28}$$

so that

$$\int_D\big(\boldsymbol{\nabla}\boldsymbol{v}:\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\bar{\boldsymbol{v}}-\rho_*\tau|\boldsymbol{v}|^2\big)\;\mathrm{d}\boldsymbol{x}\;\geqslant\;\|\boldsymbol{v}\|^2_{L^2(D)}\,(\mathsf{c}\lambda_1(D)-\tau\mathsf{P}_*) \tag{29}$$

As a result, the last integral in (27) is necessarily non-negative whenever $\omega^2=\tau\leqslant\lambda_1(D)\mathsf{c}/\mathsf{P}_*$, whereby no eigenvalues can exist within interval $(0,\lambda_1(D)\mathsf{c}/\mathsf{P}_*]$. The companion claim (when $\mathsf{p}\geqslant 1\geqslant\mathsf{P}_*$ and $\mathcal{D}_\rho\neq 0$) can be established by interchanging the roles of ρ and ρ_*. □

THexist1 **Theorem 3.** *If either $\mathsf{p}_*\geqslant 1\geqslant\mathsf{P}$ or $\mathsf{p}\geqslant 1\geqslant\mathsf{P}_*$ while $\boldsymbol{\mathcal{D}_{\mathcal{C}}}=\boldsymbol{0}$ and $\mathcal{D}_\rho\neq 0$ hold almost everywhere in D, there exists a countable set of transmission eigenvalues affiliated with (5).*

Proof. The proof of the theorem relies on the existence of a countable set of transmission eigenvalues for the spherically-symmetric case of homogeneous isotropic elastic bodies examined in Section 2.2. Suppose that $\mathsf{p}_*\geqslant 1\geqslant\mathsf{P}$ and that $\mathcal{D}_\rho\neq 0$ holds almost everywhere in D. Then by virtue of Lemma 2, operators $\mathbb{A}_\tau$ and $\mathbb{B}$ satisfy the hypotheses of Theorem 1 with $W:=H^2_0(D)$. In this case, inequalities (20)–(22) of Lemma 1 further ensure the existence of a real-valued constant $\beta'>0$ such that

$$\big\langle\mathbb{A}_\tau\boldsymbol{v},\boldsymbol{v}\big\rangle_{H^2_0(D)}\;\geqslant\;\beta'\,\|\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{v}]\|^2_{L^2(D)}, \tag{30}$$

for all $\boldsymbol{v}\in H^2_0(D)$. Moreover since $\boldsymbol{n}\cdot\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{v}=\boldsymbol{0}$ on ∂D, one finds from (1), the major symmetry of $\boldsymbol{\mathcal{C}}$, and application of the Poincaré inequality as in [29] that

$$\big\langle\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{v},\boldsymbol{\nabla}\boldsymbol{v}\big\rangle_{L^2(D)}\;\leqslant\;\frac{1}{\mathsf{c}}\big\langle\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{v},\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{v}\big\rangle_{L^2(D)}\;\leqslant\;\frac{1}{\mathsf{c}\lambda_1(D)}\|\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{v}]\|^2_{L^2(D)}, \tag{31}$$

whereby

$$\big\langle(\mathbb{A}_\tau - \tau\mathbb{B})\boldsymbol{v},\boldsymbol{v}\big\rangle_{H^2_0(D)} \;\geqslant\; \left(\beta' - \frac{\tau}{\mathsf{c}\lambda_1(D)}\right)\|\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{v}]\|^2_{L^2(D)}. \tag{32}$$

Accordingly when $0<\tau_0<\mathsf{c}\lambda_1(D)\beta'$, operator $\mathbb{A}_{\tau_0}-\tau_0\mathbb{B}$ is positive on $H^2_0(D)$ and thus meets Assumption 1 of Theorem 1.

Next, from the results in Section 2.2 it follows that interior transmission problem (5) with $\boldsymbol{\mathcal{C}}=\boldsymbol{\mathcal{C}}_*$, formulated for a ball $B_r\subset D$ of radius r with *constant* material parameters $\hat{\boldsymbol{\mathcal{C}}}=\hat{\boldsymbol{\mathcal{C}}}_*$, $\hat\rho:=\mathsf{P}$ and $\hat\rho_*:=\mathsf{p}_*$, is affiliated with a countable set of transmission eigenvalues. To help establish the claim of the theorem, let $\hat\tau$ be one such eigenvalue and let $\hat{\boldsymbol{v}}\in H^2_0(B_r)$ be the corresponding eigenfunction. In particular, $\hat{\boldsymbol{v}}$ satisfies $\big\langle(\hat{\mathbb{A}}_{\hat\tau}-\hat\tau\hat{\mathbb{B}})\hat{\boldsymbol{v}},\boldsymbol{\varphi}\big\rangle_{H^2_0(B_r)}=0$ with featured operators corresponding to the assumed (constant) material parameters. Accordingly, by taking $\boldsymbol{\varphi}=\hat{\boldsymbol{v}}$ and integrating by parts, one finds

$$\mathsf{p}_*\mathsf{P}\,\hat\tau^2\|\hat{\boldsymbol{v}}\|^2_{L^2(B_r)} \;=\; -\|\boldsymbol{\nabla}\cdot[\hat{\boldsymbol{\mathcal{C}}}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{v}}]\|^2_{L^2(B_r)} \;+\; (\mathsf{p}_*+\mathsf{P})\,\hat\tau\int_{B_r}\boldsymbol{\nabla}\hat{\boldsymbol{v}}:\hat{\boldsymbol{\mathcal{C}}}:\boldsymbol{\nabla}\bar{\hat{\boldsymbol{v}}}\,\mathrm{d}\boldsymbol{x}. \tag{33}$$

infTE1

Moreover, if $\tilde{\boldsymbol{v}}\in H^2_0(D)$ denotes the extension of $\hat{\boldsymbol{v}}$ by zero to the whole of D one has

$$\begin{aligned}\big\langle(\mathbb{A}_{\hat\tau}-\hat\tau\mathbb{B})\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\big\rangle_{H^2_0(D)} \leqslant{}& \left(\frac{1+\mathsf{P}-\mathsf{p}}{\mathsf{p}_*-\mathsf{P}}\right)\|\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\hat{\boldsymbol{v}}]\|^2_{L^2(B_r)} - \left(\frac{\mathsf{p}_*+\mathsf{P}}{\mathsf{p}_*-\mathsf{P}}\right)\hat\tau\int_{B_r}\boldsymbol{\nabla}\hat{\boldsymbol{v}}:\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\bar{\hat{\boldsymbol{v}}}\,\mathrm{d}\boldsymbol{x}\\ &+\left(\frac{\mathsf{P}\mathsf{P}_*}{\mathsf{p}_*-\mathsf{P}}\right)\hat\tau^2\|\hat{\boldsymbol{v}}\|^2_{L^2(B_r)},\end{aligned} \tag{34}$$

infTE2

where $\mathbb{A}_{\hat\tau}$ and $\mathbb{B}$ are given by (24) assuming $\tau=\hat\tau$ and the original set of material parameters in terms of distributions $(\boldsymbol{\mathcal{C}},\rho)$ and $(\boldsymbol{\mathcal{C}}_*,\rho_*)$ over D. A substitution of (33) into (34) yields

$$\begin{aligned}\big\langle(\mathbb{A}_{\hat\tau}-\hat\tau\mathbb{B})\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\big\rangle_{H^2_0(D)} \;\leqslant\;& \frac{1}{\mathsf{p}_*-\mathsf{P}}\left\{(1+\mathsf{P}-\mathsf{p})\|\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\hat{\boldsymbol{v}}]\|^2_{L^2(B_r)} - \frac{\mathsf{P}_*}{\mathsf{p}_*}\|\boldsymbol{\nabla}\cdot[\hat{\boldsymbol{\mathcal{C}}}:\boldsymbol{\nabla}\hat{\boldsymbol{v}}]\|^2_{L^2(B_r)}\right\}\\ &+\left(\frac{\mathsf{p}_*+\mathsf{P}}{\mathsf{p}_*-\mathsf{P}}\right)\hat\tau\int_{B_r}\boldsymbol{\nabla}\hat{\boldsymbol{v}}:\left[\frac{\mathsf{P}_*}{\mathsf{p}_*}\hat{\boldsymbol{\mathcal{C}}}-\boldsymbol{\mathcal{C}}\right]:\boldsymbol{\nabla}\bar{\hat{\boldsymbol{v}}}\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{35}$$

infTE3

On choosing the maximum eigenvalue, $\hat{\mathsf{C}}$, of elastic tensor $\hat{\boldsymbol{\mathcal{C}}}$ such that

$$\hat{\mathsf{C}} \;<\; \frac{\mathsf{p}_*}{\mathsf{P}_*}\mathsf{c}, \tag{36}$$

CondC

inequality (35) demonstrates that for sufficiently large $\hat\tau=\tau_1$, operator $\mathbb{A}_{\tau_1}-\tau_1\mathbb{B}$ is non-positive on the subspace of $H^2_0(D)$ spanned by $\tilde{\boldsymbol{v}}$ – a result which constitutes Assumption 2 of Theorem 1. As a consequence, one concludes from Theorem 1 that there is at least one transmission eigenvalue within interval $[\tau_0,\tau_1]$ located on the positive real axis, where $0<\tau_0<\mathsf{c}\lambda_1(D)\beta'$ as examined earlier.

Next, consider $\varepsilon>0$ such that D contains $m\geqslant 1$ disjoint balls $B^1_\varepsilon, B^2_\varepsilon,\dots B^m_\varepsilon$ of radius εr, whence $\overline{B^i_\varepsilon}\subset D$ for $i=1,\dots,m$ and $\overline{B^i_\varepsilon}\cap\overline{B^j_\varepsilon}=\emptyset$ for $i\neq j$. By the scaling argument, $\hat\tau_\varepsilon=\hat\tau/\varepsilon^2$ is a transmission eigenvalue for each of these balls associated with the interior transmission problem formulated assuming mass densities $\hat\rho=\mathsf{P}$ and $\hat\rho_*=\mathsf{p}_*$, and homogeneous isotropic elastic tensor $\hat{\boldsymbol{\mathcal{C}}}$ verifying (36). Thus, if $\hat{\boldsymbol{v}}^i\in H^2_0(B^i_\varepsilon)$ is an eigenfunction corresponding to $\hat\tau_\varepsilon$ for all $i=1,\dots,m$ whose extension by zero to the whole of D is denoted by $\tilde{\boldsymbol{v}}^i\in H^2_0(D)$, vectors $\{\tilde{\boldsymbol{v}}^1,\tilde{\boldsymbol{v}}^2,\dots,\tilde{\boldsymbol{v}}^m\}$ are linearly independent and orthogonal in $H^2_0(D)$ since they have disjoint supports. With reference to (35) and (36), on the other hand, operator

$\mathbb{A}_{\hat{\tau}_{\varepsilon 1}} - \tau_{\varepsilon 1}\mathbb{B}$ is non-positive on the m-dimensional subspace of $H^2_0(D)$ spanned by $\{\tilde{\boldsymbol{v}}^1, \tilde{\boldsymbol{v}}^2, \ldots, \tilde{\boldsymbol{v}}^m\}$ for sufficiently large $\tau_{\varepsilon 1} = \tau_1/\varepsilon^2$. By virtue of Theorem 1, there exist at least m transmission eigenvalues within interval $[\tau_0, \tau_{\varepsilon 1}]$, counting their multiplicity. By letting $\varepsilon \to 0$ and $m \to \infty$, one concludes that the set of transmission eigenvalues characterizing problem (5) when $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_*$ is countable with infinity being the only possible point of accumulation.

The case when $\mathsf{p} \geqslant 1 \geqslant \mathsf{P}_*$ and $\mathcal{D}_\rho \neq 0$ almost everywhere in D can be treated by the same argument due to the symmetry in ρ and ρ_* of the formulation employed. $\square$

The above analysis allows one to establish *implicit bounds* on ρ, ρ_* and $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_*$ in terms of the first transmission eigenvalue (see Corollary 2.6 in [13] for detailed proof). To this end, denote by B_r the largest ball of radius r such that $B_r \subset D$, and by B_R the smallest ball of radius R such that $D \subset B_R$. Further, let $\hat{\boldsymbol{\mathcal{C}}}$ be a constant elastic tensor satisfying (36), and let $\omega_1^{\text{ball}}(r, \hat{\boldsymbol{\mathcal{C}}}, \mathsf{P}, \mathsf{p}_*)$ and $\omega_1^{\text{ball}}(R, \hat{\boldsymbol{\mathcal{C}}}, \mathsf{p}, \mathsf{P}_*)$ denote respectively the first transmission eigenvalue of (5) for ball B_r with material parameters $\hat{\boldsymbol{\mathcal{C}}}_* = \hat{\boldsymbol{\mathcal{C}}}$, $\hat{\rho} := \mathsf{P}$ and $\hat{\rho}_* := \mathsf{p}_*$, and ball B_R with material parameters $\hat{\boldsymbol{\mathcal{C}}}_* = \hat{\boldsymbol{\mathcal{C}}}$, $\hat{\rho} := \mathsf{p}$ and $\hat{\rho}_* := \mathsf{P}_*$.

cor **Corollary 1.** *Assume that $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_*$, and let ρ and ρ_* satisfy $\mathsf{p}_* \geqslant 1 \geqslant \mathsf{P}$. Then the first transmission eigenvalue ω_1 affiliated with (5) is such that*

$$\max\left(\omega_1^{ball}(R, \hat{\boldsymbol{\mathcal{C}}}, \mathsf{P}, \mathsf{p}_*), \sqrt{\mathsf{c}\,\frac{\lambda_1(D)}{\mathsf{P}_*}}\right) \;\leqslant\; \omega_1 \;\leqslant\; \omega_1^{ball}(r, \hat{\boldsymbol{\mathcal{C}}}, \mathsf{p}, \mathsf{P}_*). \tag{37}$$ est1

where c is defined in (1), $\hat{\boldsymbol{\mathcal{C}}}$ satisfies (36), and $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D. For completeness, it is noted that the analogous bounds when $\mathsf{p} \geqslant 1 \geqslant \mathsf{P}_*$ can be obtained from (37) by reversing the roles of ρ and ρ_* due to symmetry of the problem.

VanishR

3.2 Equal mass densities

This section deals with the case when $\mathcal{D}_\rho$ vanishes (i.e. $\rho = \rho_*$), while $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} \neq \boldsymbol{0}$ almost everywhere in D following (4). With such premise, consider the pair $(\boldsymbol{u}, \boldsymbol{u}_*) \in H^1(D) \times H^1(D)$ satisfying (5) with $\rho = \rho_*$ and introduce the Sobolev spaces of symmetric second-order tensor fields

$$\begin{aligned} \mathcal{V}(D) &:= \big\{\boldsymbol{\Phi} \in L^2(D):\; \boldsymbol{\Phi} = \boldsymbol{\Phi}^{\text{T}},\; \boldsymbol{\nabla}\cdot\boldsymbol{\Phi} \in L^2(D)\big\}, \\ \mathcal{V}_0(D) &:= \{\boldsymbol{\Phi} \in \mathcal{V}(D):\; \boldsymbol{n}\cdot\boldsymbol{\Phi} = \boldsymbol{0} \;\text{ on }\; \partial D\}, \end{aligned} \tag{38}$$ Sobolev2

and

$$\begin{aligned} \mathcal{W}(D) &= \big\{\boldsymbol{\Phi} \in \mathcal{V}(D):\; \boldsymbol{\Phi} = \boldsymbol{\Phi}^{\text{T}},\; \boldsymbol{\nabla}\cdot\boldsymbol{\Phi} \in H^1(D)\big\}, \\ \mathcal{W}_0(D) &= \big\{\boldsymbol{\Phi} \in \mathcal{V}_0(D):\; \boldsymbol{\nabla}\cdot\boldsymbol{\Phi} \in H^1_0(D)\big\}, \end{aligned} \tag{39}$$ Sobolev3

equipped with the inner product $\big\langle \boldsymbol{\Phi}, \boldsymbol{\Psi} \big\rangle_{\mathcal{W}(D)} = \big\langle \boldsymbol{\Phi}, \boldsymbol{\Psi} \big\rangle_{L^2(D)} + \big\langle \boldsymbol{\nabla}\cdot\boldsymbol{\Phi}, \boldsymbol{\nabla}\cdot\boldsymbol{\Psi} \big\rangle_{H^1(D)}$.

To facilitate the ensuing developments, one may recall that any vector field $\boldsymbol{\varphi} \in H^1(D)$ and second-order tensor field $\boldsymbol{\Phi} \in \mathcal{V}(D)$ satisfy the relationship

$$\int_D (\boldsymbol{\nabla}\cdot\boldsymbol{\Phi})\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_{\partial D} \boldsymbol{n}\cdot\boldsymbol{\Phi}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} \;-\; \int_D \boldsymbol{\Phi} : \boldsymbol{\nabla}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}, \tag{40}$$ IntPart

and note that $\boldsymbol{\Phi} \in \mathcal{W}_0(D)$ verifies $\boldsymbol{n}\cdot\boldsymbol{\Phi} = \boldsymbol{0}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} = \boldsymbol{0}$ on ∂D. In this setting, one may take the gradient of the field equations in (5) and reformulate the problem in terms of $\boldsymbol{\mathcal{U}} := \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}\boldsymbol{u} \in \mathcal{W}(D)$ and $\boldsymbol{\mathcal{U}}_* := \boldsymbol{\mathcal{C}}_* : \boldsymbol{\nabla}\boldsymbol{u}_* \in \mathcal{W}(D)$ as

$$
\begin{aligned}
&\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{U}} + \rho\,\omega^2\boldsymbol{\mathcal{C}}^{-1}\!:\boldsymbol{\mathcal{U}} = \mathbf{0} \quad \text{in } D, \qquad \boldsymbol{\nabla}\cdot(\boldsymbol{\mathcal{U}} - \boldsymbol{\mathcal{U}}_*) = \mathbf{0} \quad \text{on } \partial D,\\
&\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{U}}_* + \rho\,\omega^2\boldsymbol{\mathcal{C}}_*^{-1}\!:\boldsymbol{\mathcal{U}}_* = \mathbf{0} \quad \text{in } D, \qquad \boldsymbol{n}\cdot(\boldsymbol{\mathcal{U}} - \boldsymbol{\mathcal{U}}_*) = \mathbf{0} \quad \text{on } \partial D.
\end{aligned}
\tag{41}
$$

ITP21

Following the developments in Section 3.1, one finds that the featured solution difference $\boldsymbol{\mathcal{V}} := \boldsymbol{\mathcal{U}} - \boldsymbol{\mathcal{U}}_*$ satisfies $\boldsymbol{\mathcal{V}} \in \mathcal{W}_0(D)$ and meets the fourth-order differential equation

$$
\big(\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot + \rho\,\omega^2\boldsymbol{\mathcal{C}}^{-1}\!:\big)\,\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} : \big(\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot + \rho\,\omega^2\boldsymbol{\mathcal{C}}_*^{-1}\!:\big)\,\boldsymbol{\mathcal{V}} = \mathbf{0} \quad \text{in } D,
\tag{42}
$$

Fourth2

when $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} \neq \mathbf{0}$, $\rho > 0$ and $\omega > 0$. By virtue of (40), the variational formulation of (42) can be posed as the task of finding $\boldsymbol{\mathcal{V}} \in \mathcal{W}_0(D)$ such that

$$
\int_D \big(\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{V}} + \rho\,\omega^2\boldsymbol{\mathcal{C}}_*^{-1}\!:\,\boldsymbol{\mathcal{V}}\big) : \boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} : \big(\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Phi}} + \rho\,\omega^2\boldsymbol{\mathcal{C}}^{-1}\!:\,\bar{\boldsymbol{\Phi}}\big)\,\text{d}\boldsymbol{x} = 0 \quad \forall\boldsymbol{\Phi} \in \mathcal{W}_0(D).
\tag{43}
$$

Var2

To aid the treatment of the featured variational problem, one may introduce the auxiliary sesquilinear forms on $\mathcal{W}_0(D) \times \mathcal{W}_0(D)$ as

$$
\begin{aligned}
\mathcal{F}_\tau(\boldsymbol{\Phi},\boldsymbol{\Psi}) &= \big\langle \boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} : \big(\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} + \rho\,\tau\boldsymbol{\mathcal{C}}^{-1}\!:\boldsymbol{\Phi}\big)\,,\,\big(\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\Psi} + \rho\,\tau\boldsymbol{\mathcal{C}}^{-1}\!:\boldsymbol{\Psi}\big)\big\rangle_{L^2(D)} + \tau^2\big\langle \rho^2\boldsymbol{\mathcal{C}}^{-1}\!:\boldsymbol{\Phi},\boldsymbol{\Psi}\big\rangle_{L^2(D)},\\
\mathcal{G}(\boldsymbol{\Phi},\boldsymbol{\Psi}) &= \big\langle \rho\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}, \boldsymbol{\nabla}\cdot\boldsymbol{\Psi}\big\rangle_{L^2(D)},
\end{aligned}
\tag{44}
$$

SesF2

where again the inner product between two nth-order tensors is understood in the sense of n-tuple contraction. With such definitions, (43) can be restated as

$$
\mathcal{F}_\tau(\boldsymbol{\mathcal{V}},\boldsymbol{\Phi}) \;-\; \tau\,\mathcal{G}(\boldsymbol{\mathcal{V}},\boldsymbol{\Phi}) = 0 \qquad \forall\boldsymbol{\Phi} \in \mathcal{W}_0(D).
\tag{45}
$$

vv1

By virtue of the symmetry of elastic tensors $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$, $\mathcal{F}_\tau$ and $\mathcal{F}_\tau^*$ can be conveniently rewritten as

$$
\begin{aligned}
\mathcal{F}_\tau(\boldsymbol{\Phi},\boldsymbol{\Psi}) = &\big\langle \boldsymbol{\mathcal{C}}^{-1}\!:\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} : (\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} + \rho\,\tau\boldsymbol{\Phi})\,,\,(\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\Psi} + \rho\,\tau\boldsymbol{\Psi})\big\rangle_{L^2(D)}\\
&+ \big\langle (\boldsymbol{\mathcal{I}}_4 - \boldsymbol{\mathcal{C}}^{-1}) : \boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} : \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\Psi}\big\rangle_{L^2(D)} + \tau^2\big\langle \rho^2(\boldsymbol{\mathcal{C}}_*^{-1} - \boldsymbol{\mathcal{I}}_4) : \boldsymbol{\mathcal{C}}^{-1} : \boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} : \boldsymbol{\Phi}, \boldsymbol{\Psi}\big\rangle_{L^2(D)},
\end{aligned}
\tag{46}
$$

SesF2bis

to help expose the conditions for their ellipticity, where $\boldsymbol{\mathcal{I}}_4$ is the symmetric fourth-order identity tensor.

rmk2 **Remark 2.** *When $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} \neq \mathbf{0}$ and $\mathcal{D}_\rho = 0$ almost everywhere in D, condition $\mathsf{c} \geqslant 1 \geqslant \mathsf{C}_*$ implies the existence of real-valued constants $\alpha \geqslant 0$, $\alpha_* \geqslant 0$ and $\gamma > 0$ such that for all complex-valued second-order tensors $\boldsymbol{\xi}$*

$$
\boldsymbol{\xi} : (\boldsymbol{\mathcal{I}}_4 - \boldsymbol{\mathcal{C}}^{-1}) : \bar{\boldsymbol{\xi}} \;\geqslant\; \alpha|\boldsymbol{\xi}|^2, \qquad \boldsymbol{\xi} : (\boldsymbol{\mathcal{C}}_*^{-1} - \boldsymbol{\mathcal{I}}_4) : \bar{\boldsymbol{\xi}} \;\geqslant\; \alpha_*|\boldsymbol{\xi}|^2, \qquad \boldsymbol{\xi} : \boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} : \bar{\boldsymbol{\xi}} \;\geqslant\; \gamma|\boldsymbol{\xi}|^2.
\tag{47}
$$

rem11

LemCoer2 **Lemma 3.** *Assuming $\rho = \rho_*$ and restrictions on the contrast in elastic tensors $\mathsf{c} \geqslant 1 \geqslant \mathsf{C}_*$ and $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} \neq \mathbf{0}$, then $\mathcal{F}_\tau$ is a coercive sesquilinear form on $\mathcal{W}_0(D) \times \mathcal{W}_0(D)$.*

Proof. On the basis of (1) and Remark 2, one accordingly has

$$
\mathcal{F}_\tau(\boldsymbol{\Phi},\boldsymbol{\Phi}) \;\geqslant\; \frac{\gamma}{\mathsf{C}}\,\big\{(1+\alpha\mathsf{C})\,x^2 + (1+\alpha_*)\,y^2 - 2xy\big\}
\tag{48}
$$

Coercive2

for all $\boldsymbol{\Phi} \in \mathcal{W}_0(D)$, where $x = \|\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\|_{L^2(D)}$ and $y = \tau\|\rho\boldsymbol{\Phi}\|_{L^2(D)}$. Depending on the sign of α and α_*, one further has

$$\mathcal{F}_\tau(\boldsymbol{\Phi},\boldsymbol{\Phi}) \;\geqslant\; \frac{\gamma}{\mathsf{C}}\,\big\{\alpha\mathsf{C}x^2 + \alpha_* y^2 + (x-y)^2\big\}, \qquad \text{when} \qquad \begin{matrix}\alpha > 0,\\ \alpha_* > 0,\end{matrix} \tag{49}$$

coer21

$$\mathcal{F}_\tau(\boldsymbol{\Phi},\boldsymbol{\Phi}) \;\geqslant\; \frac{\gamma}{\mathsf{C}}\left\{\left(1-\frac{1}{\delta_*}\right)x^2 + (1+\alpha_*-\delta_*)\,y^2 + \delta_*\left(y-\frac{x}{\delta_*}\right)^2\right\}, \qquad \text{when} \qquad \begin{matrix}\alpha = 0,\\ \alpha_* > 0,\end{matrix} \tag{50}$$

coer22

assuming $\delta_* \in (1, 1{+}\alpha_*)$, and

$$\mathcal{F}_\tau(\boldsymbol{\Phi},\boldsymbol{\Phi}) \;\geqslant\; \frac{\gamma}{\mathsf{C}}\left\{(1+\alpha\mathsf{C}-\delta)\,x^2 + \left(1-\frac{1}{\delta}\right)y^2 + \delta\left(x-\frac{y}{\delta}\right)^2\right\}, \qquad \text{when} \qquad \begin{matrix}\alpha > 0,\\ \alpha_* = 0,\end{matrix} \tag{51}$$

coer23

where $\delta \in (1, 1{+}\alpha\mathsf{C})$. Moreover since $\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} \in H^1_0(D)$ the Poincaré inequality holds, i.e. there exists a constant $C_P > 0$, dependent only on D, such that

$$\|\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\|_{L^2(D)} \leqslant C_P\|\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\|_{L^2(D)}. \tag{52}$$

Poincare

On dropping the squared-difference terms in (49)–(51) and recalling (1) which guarantees that ρ is bounded, one concludes that there is a constant $C'_\tau > 0$ such that $\mathcal{F}_\tau(\boldsymbol{\Phi},\boldsymbol{\Phi}) \geqslant C'_\tau\,\|\boldsymbol{\Phi}\|^2_{\mathcal{W}(D)}$ which concludes the proof. $\square$

With reference to (44), the Riesz representation theorem ensures the existence of bounded linear operators $\mathbb{F}_\tau, \mathbb{F}^*_\tau, \mathbb{G} : \mathcal{W}_0(D) \to \mathcal{W}_0(D)$ such that for all $(\boldsymbol{\Phi},\boldsymbol{\Psi}) \in \mathcal{W}_0(D)\times\mathcal{W}_0(D)$

$$\big\langle \mathbb{F}_\tau\boldsymbol{\Phi},\boldsymbol{\Psi}\big\rangle_{\mathcal{W}_0(D)} = \mathcal{F}_\tau(\boldsymbol{\Phi},\boldsymbol{\Psi}), \qquad \big\langle \mathbb{G}\boldsymbol{\Phi},\boldsymbol{\Psi}\big\rangle_{\mathcal{W}_0(D)} = \mathcal{G}(\boldsymbol{\Phi},\boldsymbol{\Psi}), \tag{53}$$

LinF2

which permits (45) to be rewritten as $\big\langle(\mathbb{F}_\tau - \tau\mathbb{G})\boldsymbol{\mathcal{V}},\boldsymbol{\Phi}\big\rangle_{\mathcal{W}_0(D)} = 0$ for all $\boldsymbol{\Phi}\in\mathcal{W}_0(D)$. Here it is again noted, analogous to the observation made in Section 3.1, that $\tau = \omega^2$ is a transmission eigenvalue associated with (5) when $\rho = \rho_*$ if $\ker(\mathbb{F}_\tau - \tau\mathbb{G}) \neq \{\mathbf{0}\}$.

PropGen2 **Lemma 4.** *Assuming $\rho = \rho_*$, linear operator $\mathbb{F}_\tau : \mathcal{W}_0(D) \to \mathcal{W}_0(D)$ is positive definite, self-adjoint and depends continuously on $\tau > 0$ when $\mathsf{c} \geqslant 1 \geqslant \mathsf{C}_*$ and $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} \neq \mathbf{0}$ holds almost everywhere in D. Further, linear operator $\mathbb{G} : \mathcal{W}_0(D) \to \mathcal{W}_0(D)$ is self-adjoint, positive, and compact.*

Proof. Linear operators $\mathbb{F}_\tau$ and $\mathbb{G}$ are self-adjoint since ρ, $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ are real-valued functions; the positivity of $\mathbb{F}_\tau$ is a direct consequence of Lemma 3, while the positivity of $\mathbb{G}$ is implied by the fact that ρ is positive according to (1).

Next, let $\boldsymbol{\Phi}_n$ denote a bounded sequence in $\mathcal{W}_0(D)$ whose subsequence, $\tilde{\boldsymbol{\Phi}}_n$, converges weakly with respect to the $\mathcal{W}_0(D)$-norm to $\boldsymbol{\Phi}_\text{o} \in \mathcal{W}_0(D)$. Since $\tilde{\boldsymbol{\Phi}}_n \in \mathcal{W}_0(D)$, one has by (39) that $\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{\Phi}}_n \in H^1(D)$ which is compactly embedded in $L^2(D)$, whereby $\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{\Phi}}_n$ converges strongly to $\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_\text{o}$ in $L^2(D)$. Accordingly, one has

$$\|\mathbb{G}(\tilde{\boldsymbol{\Phi}}_n - \boldsymbol{\Phi}_\text{o})\|_{\mathcal{W}_0(D)} \;\leqslant\; \mathsf{P}\|\boldsymbol{\nabla}\cdot(\tilde{\boldsymbol{\Phi}}_n - \boldsymbol{\Phi}_\text{o})\|_{L^2(D)}, \tag{54}$$

which ensures the strong convergence of $\mathbb{G}\tilde{\boldsymbol{\Phi}}_n$ in the $\mathcal{W}_0(D)$-norm sense to $\mathbb{G}\boldsymbol{\Phi}_\text{o}$, and thus the compactness of $\mathbb{G}$. $\square$

Following the path established in Section 3.1, the ensuing theorem provides a lower bound for possible transmission eigenvalues when $\rho = \rho_*$. To this end consider the linear operator $-\boldsymbol{\nabla\nabla}\cdot$, which is known to possess an increasing sequence of positive eigenvalues $\tilde{\lambda}_n(D)$ and associated (second-order) eigentensors $\boldsymbol{\Phi}_n$ [1, 3] such that

$$-\boldsymbol{\nabla\nabla}\cdot\boldsymbol{\Phi}_n = \tilde{\lambda}_n(D)\boldsymbol{\Phi}_n \quad \text{in } D, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_n = \boldsymbol{0} \quad \text{on } \partial D. \tag{55}$$

GradDiv

Alternatively, (55) can be written in terms of the sequence of first-order tensors $\boldsymbol{\varphi}_n := \boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_n$ as

$$-\Delta\boldsymbol{\varphi}_n = \tilde{\lambda}_n(D)\boldsymbol{\varphi}_n \quad \text{in } D, \qquad \boldsymbol{\varphi}_n = \boldsymbol{0} \quad \text{on } \partial D, \tag{56}$$

GradDivbis

where $(\tilde{\lambda}_n(D), \boldsymbol{\varphi}_n)$ are the solutions of the Laplace eigenvalue problem over D assuming Dirichlet boundary conditions. Thus if $\lambda_1(D)$ denotes the first Dirichlet eigenvalue of the negative Laplace operator, one has that $\tilde{\lambda}_1(D) \geqslant \lambda_1(D)$.

Theorem 4. *If either* $\mathsf{c} \geqslant 1 \geqslant \mathsf{C}_*$ *or* $\mathsf{c}_* \geqslant 1 \geqslant \mathsf{C}$ *while* $\boldsymbol{\mathcal{D}_\mathcal{C}} \neq \boldsymbol{0}$ *and* $\mathcal{D}_\rho = 0$ *hold almost everywhere in* D*, the set of transmission eigenvalues associated with (5) is discrete, with infinity being the only possible accumulation point. Further, every feasible transmission eigenvalue* ω^2 *is such that*

$$\omega^2 \; > \; \lambda_1(D)\,\frac{\min(\mathsf{c},\mathsf{c}_*)}{\mathsf{P}}.$$

Proof. Under the premises of the theorem when assumed $\mathsf{c} \geqslant 1 \geqslant \mathsf{C}_*$, then $\mathbb{F}_\tau$ is invertible owing to Lemma 4 and, since $\mathbb{G}$ is a compact operator, so is ${\mathbb{F}_\tau}^{-1}\mathbb{G}$. The Fredholm alternative then ensures that $\mathbb{I} - \tau\mathbb{F}_\tau^{-1}\mathbb{G}$ is invertible except for, at most, a discrete set of values $\tau \in \mathbb{C}$ that can only accumulate at infinity.

Whereby $\boldsymbol{\xi} : \boldsymbol{\mathcal{D}_\mathcal{C}} : \bar{\boldsymbol{\xi}} \geqslant \gamma|\boldsymbol{\xi}|^2$ for some $\gamma > 0$ due to (47), next let $\boldsymbol{\mathcal{V}} \in \mathcal{W}_0(D)$ verifying (43), which with $\boldsymbol{\Phi} = \boldsymbol{\mathcal{V}}$ and integration by parts implies

$$\begin{aligned} &\int_D \big(\boldsymbol{\nabla\nabla}\cdot\boldsymbol{\mathcal{V}} + \tau\rho\,\boldsymbol{\mathcal{C}}_*^{-1}\!:\!\boldsymbol{\mathcal{V}}\big) : \boldsymbol{\mathcal{D}_\mathcal{C}} : \big(\boldsymbol{\nabla\nabla}\cdot\bar{\boldsymbol{\mathcal{V}}} + \tau\rho\,\boldsymbol{\mathcal{C}}_*^{-1}\!:\!\bar{\boldsymbol{\mathcal{V}}}\big)\, \mathrm{d}\boldsymbol{x} \\ &\qquad\qquad + \int_D \big(\tau\rho\,(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{V}})\cdot\big(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\mathcal{V}}}\big) - \tau^2\rho^2\,\boldsymbol{\mathcal{V}} : \boldsymbol{\mathcal{C}}_*^{-1} : \bar{\boldsymbol{\mathcal{V}}}\big)\, \mathrm{d}\boldsymbol{x} = 0. \end{aligned} \tag{57}$$

Whenever the second integral is non-negative, one finds that $\boldsymbol{\nabla\nabla}\cdot\boldsymbol{\mathcal{V}} + \rho\,\tau\boldsymbol{\mathcal{C}}_*^{-1}\!:\!\boldsymbol{\mathcal{V}} = \boldsymbol{0}$ in D. However, since $\boldsymbol{n}\cdot\boldsymbol{\mathcal{V}} = \boldsymbol{0}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{V}} = \boldsymbol{0}$ on ∂D, one must also have $\boldsymbol{\mathcal{V}} = \boldsymbol{0}$ in D due to Holmgren's uniqueness theorem. From an application of the Courant-Fischer min-max formulae [29], on the other hand, one has

$$\inf_{\boldsymbol{\mathcal{V}} \in \mathcal{W}_0(D)} \frac{\displaystyle\int_D (\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{V}})\cdot\big(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\mathcal{V}}}\big)\, \mathrm{d}\boldsymbol{x}}{\displaystyle\int_D |\boldsymbol{\mathcal{V}}|^2\, \mathrm{d}\boldsymbol{x}} \geqslant \inf_{\substack{\boldsymbol{\mathcal{V}} \in \mathcal{W}(D) \\ \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{V}} = \boldsymbol{0} \text{ on } \partial D}} \frac{\displaystyle\int_D (\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{V}})\cdot\big(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\mathcal{V}}}\big)\, \mathrm{d}\boldsymbol{x}}{\displaystyle\int_D |\boldsymbol{\mathcal{V}}|^2\, \mathrm{d}\boldsymbol{x}} \geqslant \lambda_1(D) \tag{58}$$

and, owing to the bounds on $\boldsymbol{\mathcal{C}}$ and ρ as in (1),

$$\int_D \big(\rho\,(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{V}})\cdot\big(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\mathcal{V}}}\big) - \tau\rho^2\boldsymbol{\mathcal{V}} : \boldsymbol{\mathcal{C}}_*^{-1}\!:\bar{\boldsymbol{\mathcal{V}}}\big)\, \mathrm{d}\boldsymbol{x} \;\geqslant\; \mathsf{p}\,\|\boldsymbol{\mathcal{V}}\|^2_{L^2(D)}\big(\lambda_1(D) - \tau\mathsf{P}\mathsf{c}_*^{-1}\big), \tag{59}$$

whereby $\tau \leqslant \lambda_1(D)\mathsf{c}_*/\mathsf{P}$ clearly cannot be a transmission eigenvalue. Then one can conclude owing to the fact that the roles of $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ are interchangeable. $\square$

THexist2 **Theorem 5.** *If either* $\mathsf{c} \geqslant 1 \geqslant \mathsf{C}_*$ *or* $\mathsf{c}_* \geqslant 1 \geqslant \mathsf{C}$ *while* $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} \neq \mathbf{0}$ *and* $\mathcal{D}_\rho = 0$ *hold almost everywhere in* D*, there exists a countable set of transmission eigenvalues affiliated with (5).*

Proof. The proof in this case follows the ideas developed in the context of Theorem 3. Suppose that $\mathsf{c} \geqslant 1 \geqslant \mathsf{C}_*$ and that $\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{C}}} \neq \mathbf{0}$ holds almost everywhere in D, so that operators $\mathbb{F}_\tau$ and $\mathbb{G}$ satisfy the hypotheses of Theorem 1 with $W \equiv \mathcal{W}_0(D)$.

With reference to the proof of Lemma 3 and inequalities (49)–(51), there exists a constant $\beta'' > 0$ such that for all $\boldsymbol{\mathcal{V}} \in \mathcal{W}_0(D)$

$$\big\langle \mathbb{F}_\tau \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}} \big\rangle_{\mathcal{W}_0(D)} \;\geqslant\; \beta'' \|\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{V}}\|^2_{L^2(D)}, \tag{60}$$

which together with Poincaré inequality (52) ensures that

$$\big\langle (\mathbb{F}_\tau - \tau\mathbb{G})\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}} \big\rangle_{\mathcal{W}_0(D)} \geqslant (\beta'' - \tau \mathsf{P} C_P)\, \|\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{V}}\|^2_{L^2(D)}. \tag{61}$$ sf

From (61), one concludes that $\mathbb{F}_{\tau_0} - \tau_0\mathbb{G}$ is positive on $\mathcal{W}_0(D)$ for $0 < \tau_0 < \beta''/(\mathsf{P}C_P)$, which meets Assumption 1 of Theorem 1.

Next, consider the interior transmission problem (5) when $\rho = \rho_*$ for a ball $B_r \subset D$ of radius r with constant mass densities $\hat{\rho} = \hat{\rho}_* = const.$ and homogeneous isotropic elastic tensors $\hat{\boldsymbol{\mathcal{C}}}$ and $\hat{\boldsymbol{\mathcal{C}}}_*$ given by their eigenvalues

$$\hat{\mathsf{C}} = \mathsf{C}, \qquad \hat{\mathsf{c}} = \mathsf{c}, \qquad \hat{\mathsf{C}}_* = \mathsf{C}_*, \qquad \hat{\mathsf{c}}_* = \mathsf{c}_*. \tag{62}$$ ULbounds

From the analytical solution in Section 2.2, it is known that there exists an infinite set of transmission eigenvalues for this problem. To help establish the claim of the theorem, let $\hat{\tau}$ be one such eigenvalue and let $\hat{\boldsymbol{\mathcal{V}}} \in \mathcal{W}_0(B_r)$ be the corresponding eigenfunction. Accordingly, $\hat{\boldsymbol{\mathcal{V}}}$ satisfies $\big\langle (\hat{\mathbb{F}}_{\hat{\tau}} - \hat{\tau}\hat{\mathbb{G}})\hat{\boldsymbol{\mathcal{V}}}, \boldsymbol{\Phi} \big\rangle_{\mathcal{W}_0(B_r)} = 0$ with the featured operators taken as those corresponding to assumed (constant) material parameters. Accordingly by taking $\boldsymbol{\Phi} = \hat{\boldsymbol{\mathcal{V}}}$, recalling that $\mathsf{c}_*^{-1} > \mathsf{C}^{-1}$, and integrating by parts, one finds that

$$\mathsf{C}^{-1}\mathsf{C}_*^{-1}\hat{\rho}^2\hat{\tau}^2\|\hat{\boldsymbol{\mathcal{V}}}\|^2_{L^2(B_r)} \leqslant -\big(1 + \mathsf{C}^{-1} - \mathsf{c}^{-1}\big)\, \|\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\mathcal{V}}}\|^2_{L^2(B_r)} + \big(\mathsf{c}_*^{-1} + \mathsf{C}^{-1}\big)\, \hat{\rho}\hat{\tau}\|\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\mathcal{V}}}\|^2_{L^2(B_r)}. \tag{63}$$ infTE4

If $\tilde{\boldsymbol{\mathcal{V}}} \in \mathcal{W}_0(D)$ is the extension of $\hat{\boldsymbol{\mathcal{V}}}$ by zero to the whole D, then

$$\begin{aligned} \big\langle (\mathbb{F}_{\hat{\tau}} - \hat{\tau}\mathbb{G})\tilde{\boldsymbol{\mathcal{V}}}, \tilde{\boldsymbol{\mathcal{V}}} \big\rangle_{\mathcal{W}_0(D)} \leqslant & \left(\frac{1 + \mathsf{c}^{-1} - \mathsf{C}^{-1}}{\mathsf{C}_*^{-1} - \mathsf{c}^{-1}}\right) \|\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\mathcal{V}}}\|^2_{L^2(B_r)} - \left(\frac{\mathsf{C}_*^{-1} + \mathsf{c}^{-1}}{\mathsf{C}_*^{-1} - \mathsf{c}^{-1}}\right) \mathsf{p}\hat{\tau}\|\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\mathcal{V}}}\|^2_{L^2(B_r)} \\ & + \left(\frac{\mathsf{c}^{-1}\mathsf{c}_*^{-1}}{\mathsf{C}_*^{-1} - \mathsf{c}^{-1}}\right) \mathsf{P}^2\hat{\tau}^2\|\hat{\boldsymbol{\mathcal{V}}}\|^2_{L^2(B_r)}, \end{aligned} \tag{64}$$ infTE5

where $\mathbb{F}_{\hat{\tau}}$ and $\mathbb{G}$ are given by (53) assuming $\tau = \hat{\tau}$ and the original set of material parameters in terms of distributions $(\boldsymbol{\mathcal{C}}, \rho)$ and $(\boldsymbol{\mathcal{C}}_*, \rho_*)$ over D. A substitution of (63) into (64) yields

$$\begin{aligned} \big\langle (\mathbb{F}_{\hat{\tau}} - \hat{\tau}\mathbb{G})\tilde{\boldsymbol{\mathcal{V}}}, \tilde{\boldsymbol{\mathcal{V}}} \big\rangle_{\mathcal{W}_0(D)} \leqslant & \left\{ \left(\frac{1 + \mathsf{c}^{-1} - \mathsf{C}^{-1}}{\mathsf{C}_*^{-1} - \mathsf{c}^{-1}}\right) - \frac{\mathsf{c}^{-1}\mathsf{c}_*^{-1}}{\mathsf{C}^{-1}\mathsf{C}_*^{-1}} \left(\frac{1 + \mathsf{C}^{-1} - \mathsf{c}^{-1}}{\mathsf{C}_*^{-1} - \mathsf{c}^{-1}}\right) \frac{\mathsf{P}^2}{\hat{\rho}^2} \right\} \|\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\mathcal{V}}}\|^2_{L^2(B_r)} \\ & + \left\{ \frac{\mathsf{c}^{-1}\mathsf{c}_*^{-1}}{\mathsf{C}^{-1}\mathsf{C}_*^{-1}} \left(\frac{\mathsf{c}_*^{-1} + \mathsf{C}^{-1}}{\mathsf{C}_*^{-1} - \mathsf{c}^{-1}}\right) \frac{\mathsf{P}^2}{\hat{\rho}} - \left(\frac{\mathsf{C}_*^{-1} + \mathsf{c}^{-1}}{\mathsf{C}_*^{-1} - \mathsf{c}^{-1}}\right) \mathsf{p} \right\} \hat{\tau}^2\|\hat{\boldsymbol{\mathcal{V}}}\|^2_{L^2(B_r)}. \end{aligned} \tag{65}$$ infTE6

Recalling further that $\mathsf{C}_*^{-1} > \mathsf{c}^{-1}$ and choosing the constant mass density $\hat{\rho} > 0$ such that

$$\hat{\rho} > \frac{\mathsf{c}^{-1}\mathsf{c}_*^{-1}}{\mathsf{C}^{-1}\mathsf{C}_*^{-1}} \left(\frac{\mathsf{c}_*^{-1} + \mathsf{C}^{-1}}{\mathsf{C}_*^{-1} + \mathsf{c}^{-1}} \right) \frac{\mathsf{P}^2}{\mathsf{p}}, \tag{66}$$

one finds from (65) that for sufficiently large $\hat{\tau} = \tau_1$, operator $\mathbb{F}_{\tau_1} - \tau_1 \mathbb{G}$ is non-positive on the subspace of $\mathcal{W}_0(D)$ spanned by $\tilde{\boldsymbol{\mathcal{V}}}$ – a result which meets Assumption 2 of Theorem 1. As a result, one finds from the latter theorem that there is at least one transmission eigenvalue of B_r within interval $[\tau_0, \tau_1]$, where $\tau_0 < \beta''/(\mathsf{P}C_P)$. The reminder of the proof mimics that in Theorem 3 and is omitted for brevity. □

Note again that the above analysis allows one to establish implicit estimates on the extreme eigenvalues of $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ in terms of the first transmission eigenvalue, ω_1, of (5) with $\rho = \rho_*$ in a way analogous to that in Corollary 1.

NonVanish 4 Configurations without material similitude

For a comprehensive treatment of the subject, this section assumes that the mass density and elasticity contrasts between the two solids, $\Delta_\rho := \rho_* - \rho$ and $\boldsymbol{\Delta_{\mathcal{C}}} := \boldsymbol{\mathcal{C}}_* - \boldsymbol{\mathcal{C}}$, are *both* non-zero almost everywhere in D. The difficulty in the treatment of such class of configurations stems from the imposed "dual" boundary condition in (5). In particular, if one attempts to apply the methods of analysis established in Section 3, the fact that $\Delta_\rho \neq 0$ and $\boldsymbol{\Delta_{\mathcal{C}}} \neq \boldsymbol{0}$ simultaneously makes it impossible to deploy the featured functional spaces which postulate homogeneous boundary conditions over ∂D. To deal with the impediment, the ensuing analysis pursues an alternate route by generalizing upon the developments in [5] and [17].

To help establish the necessary framework, one may recast the interior transmission problem (5) in a variational setting as

$$\int_D \big(\boldsymbol{\nabla u} \!:\! \boldsymbol{\mathcal{C}} \!:\! \boldsymbol{\nabla} \bar{\boldsymbol{\varphi}} - \rho\, \omega^2 \boldsymbol{u} \!\cdot\! \bar{\boldsymbol{\varphi}} \big) \, \text{d}\boldsymbol{x} = 0 \qquad \forall \boldsymbol{\varphi} \in H_0^1(D), \tag{67}$$ ITPweak1

and

$$\int_D \big(\boldsymbol{\nabla u} \!:\! \boldsymbol{\mathcal{C}} \!:\! \boldsymbol{\nabla} \bar{\boldsymbol{\varphi}} \, - \, \rho\, \omega^2 \boldsymbol{u} \!\cdot\! \bar{\boldsymbol{\varphi}} \big) \, \text{d}\boldsymbol{x} = \int_D \big(\boldsymbol{\nabla u}_* \!:\! \boldsymbol{\mathcal{C}}_* \!:\! \boldsymbol{\nabla} \bar{\boldsymbol{\varphi}} - \rho_* \omega^2 \boldsymbol{u}_* \!\cdot\! \bar{\boldsymbol{\varphi}} \big) \, \text{d}\boldsymbol{x} \qquad \forall \boldsymbol{\varphi} \in H^1(D), \tag{68}$$ ITPweak2

where $H_0^1(D)$ denotes the Hilbert space of all $\boldsymbol{\varphi} \in H^1(D)$ such that $\boldsymbol{\varphi} = \boldsymbol{0}$ on ∂D. As a result, if $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{u}_*$ then clearly $\boldsymbol{v} \in H_0^1(D)$ and from (68) it follows that

$$\int_D \big(\boldsymbol{\nabla u} \!:\! \boldsymbol{\Delta_{\mathcal{C}}} \!:\! \boldsymbol{\nabla} \bar{\boldsymbol{\varphi}} \, - \, \Delta_\rho \omega^2 \boldsymbol{u} \!\cdot\! \bar{\boldsymbol{\varphi}} \big) \, \text{d}\boldsymbol{x} \; = \int_D \big(\boldsymbol{\nabla v} \!:\! \boldsymbol{\mathcal{C}}_* \!:\! \boldsymbol{\nabla} \bar{\boldsymbol{\varphi}} \, - \, \rho_* \omega^2 \boldsymbol{v} \!\cdot\! \bar{\boldsymbol{\varphi}} \big) \, \text{d}\boldsymbol{x} \qquad \forall \boldsymbol{\varphi} \in H^1(D). \tag{69}$$ ITPdiffW1

4.1 4.1 Elasticity and mass density contrasts of opposite sign

To examine the issues of discreteness and existence of the transmission eigenvalues characterizing (5) that have, for this class of material configurations, eluded earlier studies [5], set $\tau = \omega^2$ and let $\mathcal{M}_\tau$ be the bilinear form on $H^1(D)\times H^1(D)$ and, for given $\boldsymbol{v}\in H_0^1(D)$, $\mathcal{N}_{\tau,\boldsymbol{v}}$ be the linear form on $H^1(D)$ defined by

$$\begin{aligned}\mathcal{M}_\tau(\boldsymbol{\eta},\boldsymbol{\psi}) &:= \big\langle \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} : \boldsymbol{\nabla}\boldsymbol{\eta}, \boldsymbol{\nabla}\boldsymbol{\psi}\big\rangle_{L^2(D)} \,-\, \tau\big\langle \Delta_\rho \boldsymbol{\eta}, \boldsymbol{\psi}\big\rangle_{L^2(D)},\\ \mathcal{N}_{\tau,\boldsymbol{v}}(\boldsymbol{\eta}) &:= \big\langle \boldsymbol{\mathcal{C}}_* : \boldsymbol{\nabla}\boldsymbol{v}, \boldsymbol{\nabla}\boldsymbol{\eta}\big\rangle_{L^2(D)} \,-\, \tau\big\langle \rho_*\,\boldsymbol{v}, \boldsymbol{\eta}\big\rangle_{L^2(D)},\end{aligned} \qquad (70)$$ DefBilin

for all $(\boldsymbol{\eta},\boldsymbol{\psi})\in H^1(D)\times H^1(D)$. With such definitions, variational problem (69) consists in finding $\boldsymbol{u}\in H^1(D)$ such that

$$\mathcal{M}_\tau(\boldsymbol{u},\boldsymbol{\varphi}) = \mathcal{N}_{\tau,\boldsymbol{v}}(\boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi}\in H^1(D). \qquad (71)$$ DefVarPb1

Mform **Lemma 5.** *For every $\boldsymbol{v}\in H_0^1(D)$ and $\tau\in\mathbb{C}$ such that $\Re(\tau)>-\delta$ for some $\delta>0$, there exists unique $\boldsymbol{u}\in H^1(D)$ satisfying (69) when $\mathsf{P}_*<\mathsf{p}$ and $\mathsf{c}_*>\mathsf{C}$. Further, the linear operator $\mathbb{M}_\tau : H_0^1(D)\to H^1(D)$ constructed such that $\mathbb{M}_\tau\boldsymbol{v}=\boldsymbol{u}$ is solution of (69) is bounded and depends analytically on $\tau\in\{z\in\mathbb{C} : \Re(z)>-\delta\}$.*

Proof. The proof assumes $\mathsf{P}_*<\mathsf{p}$ and $\mathsf{c}_*>\mathsf{C}$. Next, assuming $\boldsymbol{v}\in H_0^1(D)$ and setting $\boldsymbol{\varphi}$ in (71) to be a constant vector, one finds that

$$\int_D \Delta_\rho\,\boldsymbol{u}\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} \;=\; \int_D \rho_*\,\boldsymbol{v}\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x}.$$

As a result, the solution $\boldsymbol{u}\in H^1(D)$ of (71) when $\tau=0$ is unique up to a constant vector which can be chosen such that the above equality holds for three linearly independent constant vectors $\boldsymbol{\varphi}$. In light of this result, the solution for $\tau\in\mathbb{C}$ can be conveniently sought as $\boldsymbol{u}=\tilde{\boldsymbol{u}}+\mathbf{k}$, where $\mathbf{k}$ is a constant vector and $\tilde{\boldsymbol{u}}\in\tilde{H}^1(D)$ belongs to the space of "zero-weighted-mean" functions

$$\tilde{H}^1(D) := \left\{\boldsymbol{\psi}\in H^1(D) : \int_D \Delta_\rho\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x} = \mathbf{0}\right\}, \qquad (72)$$ SoblevZWM

equipped with the usual $H^1(D)$ norm. On selecting $\mathbf{k}$ independent of τ as

$$\mathbf{k} = \left(\int_D \Delta_\rho\,\mathrm{d}\boldsymbol{x}\right)^{-1}\int_D \rho_*\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},$$

one finds from (70)-(71) that $\tilde{\boldsymbol{u}}$ satisfies the same equation as $\boldsymbol{u}$. By the standard arguments for $\boldsymbol{\psi}\in\tilde{H}^1(D)$, it also follows that $\|\boldsymbol{\nabla}\boldsymbol{\psi}\|^2_{L^2(D)}$ is an equivalent norm in $\tilde{H}^1(D)$ since

$$\frac{\mu}{\mu+1}\|\boldsymbol{\psi}\|^2_{H^1(D)} \leqslant \|\boldsymbol{\nabla}\boldsymbol{\psi}\|^2_{L^2(D)} \leqslant \|\boldsymbol{\psi}\|^2_{H^1(D)}, \qquad (73)$$ equi

where $\mu>0$ is the unique minimizer

$$\mu = \inf_{\boldsymbol{\psi}\in\tilde{H}^1(D)} \frac{\|\boldsymbol{\nabla}\boldsymbol{\psi}\|^2_{L^2(D)}}{\|\boldsymbol{\psi}\|^2_{L^2(D)}}.$$

When $\mathsf{c}_*>\mathsf{C}$ and $\mathsf{P}_*<\mathsf{p}$, it follows from (70a) and (73) that for sufficiently small $\delta>0$ one has

$$\Re\left(\mathcal{M}_\tau(\boldsymbol{\varphi},\boldsymbol{\varphi})\right) \;\geqslant\; (\mathsf{c}_*-\mathsf{C})\|\boldsymbol{\nabla}\boldsymbol{\varphi}\|^2_{L^2(D)} - \delta\,(\mathsf{P}-\mathsf{p}_*)\|\boldsymbol{\varphi}\|^2_{L^2(D)} \;\geqslant\; C''\|\boldsymbol{\varphi}\|^2_{H^1(D)}, \qquad (74)$$

for all $\boldsymbol{\varphi}\in\tilde{H}^1(D)$ and some positive constant C'' independent of $\tau\in\{z\in\mathbb{C} : \Re(z)>-\delta\}$, whereby $\mathcal{M}_\tau$

is coercive in $\tilde{H}^1(D)$. Since $\mathcal{M}_\tau$ and $\mathcal{N}_{\tau,\boldsymbol{v}}$ are also continuous, application of the Lax-Milgram theorem ensures the existence of a unique $\tilde{\boldsymbol{u}}$ that solves (71) and depends continuously on $\boldsymbol{v}$. Furthermore $\boldsymbol{u} = \tilde{\boldsymbol{u}} + \mathbf{k}$ also satisfies (71) by the definition of $\mathbf{k}$. As a result, one concludes that bounded linear operator $\mathbb{M}_\tau$, which maps $\boldsymbol{v}$ to a unique solution $\boldsymbol{u}$ of (71), is well defined and depends analytically on $\tau \in \{z \in \mathbb{C}\colon \Re(z) > -\delta\}$. ☐

On recalling (67) and making reference to the relationship $\boldsymbol{u} = \mathbb{M}_\tau \boldsymbol{v}$ where $\boldsymbol{v} \in H^1_0(D)$, one can define the corresponding linear form on $H^1_0(D)$ as

$$\mathcal{L}_\tau(\boldsymbol{\varphi}) := \big\langle \boldsymbol{\mathcal{C}} \colon \boldsymbol{\nabla} \boldsymbol{u}, \boldsymbol{\nabla} \boldsymbol{\varphi} \big\rangle_{L^2(D)} - \tau \big\langle \rho \boldsymbol{u}, \boldsymbol{\varphi} \big\rangle_{L^2(D)}, \tag{75}$$ DefLf

such that, in light of Lemma 5 and the Riesz representation theorem, there exists a bounded linear operator $\mathbb{L}_\tau$ from $H^1_0(D)$ into $H^1_0(D)$ such that for all $\boldsymbol{\varphi} \in H^1_0(D)$ one has $\big\langle \mathbb{L}_\tau \boldsymbol{v}, \boldsymbol{\varphi} \big\rangle_{H^1_0(D)} = \mathcal{L}_\tau(\boldsymbol{\varphi})$. Thus if $\mathsf{P}_* < \mathsf{p}$ and $\mathsf{c}_* > \mathsf{C}$ and $\tau = \omega^2$ is a transmission eigenvalue of (5) associated with eigenfunction pair $(\boldsymbol{u}, \boldsymbol{u}_*) \in H^1(D) \times H^1(D)$, then $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_* \in H^1_0(D)$ verifies $\boldsymbol{v} \neq \boldsymbol{0}$ and $\boldsymbol{v} \in \ker(\mathbb{L}_\tau)$. Conversely, if $\boldsymbol{v} \in \ker(\mathbb{L}_\tau) \backslash \{\boldsymbol{0}\}$, then $\boldsymbol{u} = \mathbb{M}_\tau \boldsymbol{v}$ and $\boldsymbol{u}_* = \boldsymbol{u} - \boldsymbol{v}$ solve (67) and (68) as a consequence of (69). Thus, $(\boldsymbol{u}, \boldsymbol{u}_*)$ defines a set of transmission eigenfunctions in $H^1(D) \times H^1(D)$ in each case. Note that, owing to Lemma 5, $\mathbb{L}_\tau$ depends analytically on $\tau \in \{z \in \mathbb{C} : \Re(z) > -\delta\}$.

LemCoer3 **Lemma 6.** *Linear operator $\mathbb{L}_0 : H^1_0(D) \to H^1_0(D)$ is coercive if $\mathsf{P}_* < \mathsf{p}$ and $\mathsf{c}_* > \mathsf{C}$.*

Proof. With reference to (75), one finds by setting $\tau = 0$ that

$$\big\langle \mathbb{L}_0 \boldsymbol{v}, \boldsymbol{v} \big\rangle_{H^1_0(D)} = \int_D \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\boldsymbol{v}} \, \mathrm{d}\boldsymbol{x}, \tag{76}$$ coerint1

where $\boldsymbol{v} \in H^1_0(D)$ and $\boldsymbol{u} = \mathbb{M}_0 \boldsymbol{v}$ also satisfies (69) due to Lemma 5. On substituting $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{u}_*$ in (69) and (76), one further has

$$\big\langle \mathbb{L}_0 \boldsymbol{v}, \boldsymbol{v} \big\rangle_{H^1_0(D)} = \int_D \big(\boldsymbol{\nabla} \boldsymbol{v} \colon \boldsymbol{\mathcal{C}} \colon \boldsymbol{\nabla} \bar{\boldsymbol{v}} + \boldsymbol{\nabla} \boldsymbol{u}_* \colon \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \boldsymbol{\nabla} \bar{\boldsymbol{u}}_* \big) \, \mathrm{d}\boldsymbol{x} \tag{77}$$

and, due to the bounds in (1) on elastic tensors,

$$\big\langle \mathbb{L}_0 \boldsymbol{v}, \boldsymbol{v} \big\rangle_{H^1_0(D)} \; \geqslant \; \mathsf{c} \| \boldsymbol{\nabla} \boldsymbol{v} \|^2_{L^2(D)} \; + \; (\mathsf{c}_* - \mathsf{C}) \| \boldsymbol{\nabla} \boldsymbol{u}_* \|^2_{L^2(D)}. \tag{78}$$

Finally, since $\boldsymbol{v} \in H^1_0(D)$ one finally concludes from the Poincaré inequality that there exists a constant $C > 0$ such that $\big\langle \mathbb{L}_0 \boldsymbol{v}, \boldsymbol{v} \big\rangle_{H^1_0(D)} \geqslant C \| \boldsymbol{v} \|^2_{H^1_0(D)}$ whereby $\mathbb{L}_0$ is coercive on $H^1_0(D)$. ☐

PropLr **Lemma 7.** *Linear operator $\mathbb{L}_\tau$ from $H^1_0(D)$ into $H^1_0(D)$ is self-adjoint and has the property that $\mathbb{L}_\tau - \mathbb{L}_0$ is compact on $H^1_0(D)$, if $\mathsf{P}_* < \mathsf{p}$ and $\mathsf{c}_* > \mathsf{C}$.*

Proof. Let $(\boldsymbol{v}, \boldsymbol{v}') \in H^1_0(D) \times H^1_0(D)$, then due to Lemma 5, $\boldsymbol{u} = \mathbb{M}_\tau \boldsymbol{v}$ and $\boldsymbol{u}' = \mathbb{M}_\tau \boldsymbol{v}'$ each satisfy (69). With reference to (75), one has

$$\begin{aligned} \big\langle \mathbb{L}_\tau \boldsymbol{v}, \boldsymbol{v}' \big\rangle_{H^1_0(D)} &= \int_D \big(\boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\boldsymbol{v}}' - \rho \, \tau \boldsymbol{u} \cdot \bar{\boldsymbol{v}}' \big) \, \mathrm{d}\boldsymbol{x} \\ &= -\int_D \big(\boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} : \boldsymbol{\nabla} \bar{\boldsymbol{v}}' - \Delta_\rho \tau \boldsymbol{u} \cdot \bar{\boldsymbol{v}}' \big) \, \mathrm{d}\boldsymbol{x} + \int_D \big(\boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\mathcal{C}}_* : \boldsymbol{\nabla} \bar{\boldsymbol{v}}' - \rho_* \tau \boldsymbol{u} \cdot \bar{\boldsymbol{v}}' \big) \, \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{79}$$ lvv

which by applying (69) twice, yields

$$\big\langle \mathbb{L}_\tau \boldsymbol{v}, \boldsymbol{v}' \big\rangle_{H^1_0(D)} = -\int_D (\boldsymbol{\nabla v} : \boldsymbol{\mathcal{C}}_* : \boldsymbol{\nabla} \bar{\boldsymbol{v}}' - \rho_* \tau \boldsymbol{v} \cdot \bar{\boldsymbol{v}}') \,\mathrm{d}\boldsymbol{x} + \int_D (\boldsymbol{\nabla u}' : \boldsymbol{\Delta_{\mathcal{C}}} : \boldsymbol{\nabla} \bar{\boldsymbol{u}} - \Delta_\rho \tau \boldsymbol{u}' \cdot \bar{\boldsymbol{u}}) \,\mathrm{d}\boldsymbol{x}. \quad (80)$$

As a result, $\big\langle \mathbb{L}_\tau \boldsymbol{v}, \boldsymbol{v}' \big\rangle_{H^1_0(D)} = \overline{\big\langle \mathbb{L}_\tau \boldsymbol{v}', \boldsymbol{v} \big\rangle}_{H^1_0(D)}$ i.e. $\mathbb{L}_\tau$ is self-adjoint.

To establish the compactness of $\mathbb{L}_\tau - \mathbb{L}_0$, consider a bounded sequence $\boldsymbol{v}_n$ in $H^1_0(D)$ for which there exists a subsequence $\tilde{\boldsymbol{v}}_n$ that weakly converges with respect to the $H^1_0(D)$-norm to $\boldsymbol{v} \in H^1_0(D)$. Since $H^1_0(D)$ is compactly embedded in $L^2(D)$, $\tilde{\boldsymbol{v}}_n$ converges strongly to $\boldsymbol{v}$ with respect to the $L^2(D)$-norm and, due to Lem ma 5, sequences $\tilde{\boldsymbol{u}}_n := \mathbb{M}_\tau \tilde{\boldsymbol{v}}_n$ and $\tilde{\boldsymbol{u}}^0_n := \mathbb{M}_0 \tilde{\boldsymbol{v}}_n$ converge strongly in $L^2(D)$ to $\boldsymbol{u}$ and $\boldsymbol{u}^0$, respectively. On the basis of (75), the Cauchy-Schwarz inequality, and the bounds on $\boldsymbol{\mathcal{C}}$ and ρ as in (1), on the other hand, one has

$$\|(\mathbb{L}_\tau - \mathbb{L}_0)(\tilde{\boldsymbol{v}}_n - \boldsymbol{v})\|_{H^1_0(D)} \leqslant \mathsf{C} \big\{ \|\boldsymbol{\nabla}(\tilde{\boldsymbol{u}}_n - \boldsymbol{u})\|_{L^2(D)} + \|\boldsymbol{\nabla}(\tilde{\boldsymbol{u}}^0_n - \boldsymbol{u}^0)\|_{L^2(D)} \big\} + \mathsf{P}\tau \|\tilde{\boldsymbol{u}}_n - \boldsymbol{u}\|_{L^2(D)}, \quad (81)$$

which guarantees that $(\mathbb{L}_\tau - \mathbb{L}_0)\tilde{\boldsymbol{v}}_n$ converges strongly to $(\mathbb{L}_\tau - \mathbb{L}_0)\boldsymbol{v}$ with respect to the $H^1_0(D)$-norm, i.e. that $\mathbb{L}_\tau - \mathbb{L}_0$ is compact. □

main1 **Theorem 6.** *If either $\mathsf{P}_* < \mathsf{p}$ and $\mathsf{c}_* > \mathsf{C}$ or $\mathsf{p}_* > \mathsf{P}$ and $\mathsf{C}_* < \mathsf{c}$, the set of transmission eigenvalues associated with (5) is discrete, with infinity being the only possible accumulation point. Further, every feasible transmission eigenvalue ω^2 is such that*

$$\omega^2 \;\geqslant\; \lambda_1(D)\, \frac{\min(\mathsf{c}, \mathsf{c}_*)}{\max(\mathsf{P}, \mathsf{P}_*)}.$$

Proof. The discreteness of the set of transmission eigenvalues is a direct consequence of Lemmas 5, 6 and 7. Indeed, under the hypothesis that $\mathsf{P}_* < \mathsf{p}$ and $\mathsf{c}_* > \mathsf{C}$, one has that $\mathbb{L}_0$ is invertible and that $\mathbb{L}_\tau - \mathbb{L}_0$ is compact, while $\mathbb{L}_\tau$depends analytically on τ in a neighborhood of the real axis. On employing the decomposition $\mathbb{L}_\tau = \mathbb{L}_0 + (\mathbb{L}_\tau - \mathbb{L}_0)$, it follows from the analytic Fredholm theory [24] that compact operator $\mathbb{I} + \mathbb{L}_0^{-1}(\mathbb{L}_\tau - \mathbb{L}_0)$ is invertible except for a discrete set of values $\tau \in \mathbb{C}$ that can only accumulate at infinity.

Next, assuming $\mathsf{P}_* < \mathsf{p}$ and $\mathsf{c}_* > \mathsf{C}$, and let $\boldsymbol{v} \in H^1_0(D)$ such that $\boldsymbol{v} \in \ker(\mathbb{L}_\tau)$. On recalling that $\boldsymbol{u} = \mathbb{M}_\tau \boldsymbol{v}$ and $\boldsymbol{u}_* = \boldsymbol{u} - \boldsymbol{v}$, one finds from (67) and (69) that

$$\int_D (\boldsymbol{\nabla u}_* : \boldsymbol{\Delta_{\mathcal{C}}} : \boldsymbol{\nabla} \bar{\boldsymbol{u}}_* - \Delta_\rho \tau\, \boldsymbol{u}_* \cdot \bar{\boldsymbol{u}}_*) \,\mathrm{d}\boldsymbol{x} \;+\; \int_D (\boldsymbol{\nabla v} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\boldsymbol{v}} - \rho\, \tau\, \boldsymbol{v} \cdot \bar{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x} = 0. \quad (82)$$ eq1

But

$$\int_D (\boldsymbol{\nabla u}_* : \boldsymbol{\Delta_{\mathcal{C}}} : \boldsymbol{\nabla} \bar{\boldsymbol{u}}_* - \Delta_\rho \tau\, \boldsymbol{u}_* \cdot \bar{\boldsymbol{u}}_*) \,\mathrm{d}\boldsymbol{x} \;\geqslant\; (\mathsf{c}_* - \mathsf{C}) \|\boldsymbol{\nabla u}_*\|^2_{L^2(D)} + (\mathsf{p} - \mathsf{P}_*)\tau \|\boldsymbol{u}_*\|^2_{L^2(D)} \;\geqslant\; 0, \quad (83)$$ int1

and since $\boldsymbol{v} \in H^1_0(D)$ one has

$$\int_D (\boldsymbol{\nabla v} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\boldsymbol{v}} - \rho\, \tau\, \boldsymbol{v} \cdot \bar{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x} \;\geqslant\; \|\boldsymbol{v}\|^2_{L^2(D)} \,(\lambda_1(D)\mathsf{c} - \tau \mathsf{P})\,, \quad (84)$$ int2

due to (1) and Courant-Fischer min-max formulae. As a result, one finds from (82)–(84) assuming $\tau < \lambda_1(D)\mathsf{c}/\mathsf{P}$ that $\|\boldsymbol{v}\|_{L^2(D)} = \|\boldsymbol{u}_*\|_{L^2(D)} = 0$ and consequently that $\boldsymbol{u} = \boldsymbol{u}_* = \boldsymbol{0}$, whereby such τ cannot be a transmission eigenvalue. The remainder of the claim is established owing to the material symmetry. $\square$

The last step of the analysis is to demonstrate the *existence* of a countable set of (real-valued) transmission eigenvalues associated with (5) assuming that Δ_ρ and $\boldsymbol{\Delta_{\mathcal{C}}}$ are both non-zero almost everywhere in D. In what follows, this is accomplished by employing the methodology proposed in [17] for scalar problems and making an *additional restriction* that the medium represented by $(\boldsymbol{\mathcal{C}},\rho)$ is homogeneous and isotropic, i.e. that

$$\rho = \mathsf{p} = \mathsf{P} \qquad \text{and} \qquad \begin{cases} \boldsymbol{\mathcal{C}} \;=\; \frac{1}{3}(\mathsf{c}-\mathsf{C})\,\boldsymbol{\mathcal{I}}_2\otimes\boldsymbol{\mathcal{I}}_2 \,+\, \mathsf{C}\,\boldsymbol{\mathcal{I}}_4 \quad \text{for} \quad \nu\in(-1,0], \\ \boldsymbol{\mathcal{C}} \;=\; \frac{1}{3}(\mathsf{C}-\mathsf{c})\,\boldsymbol{\mathcal{I}}_2\otimes\boldsymbol{\mathcal{I}}_2 \,+\, \mathsf{c}\,\boldsymbol{\mathcal{I}}_4 \quad \text{for} \quad \nu\in[0,\frac{1}{2}), \end{cases} \tag{85}$$ isohom

where $\otimes$ signifies the (outer) tensor product, and $\boldsymbol{\mathcal{I}}_n$ is the symmetric nth-order identity tensor. In this setting one may first invoke the result of Lemma 6 and note, assuming $\mathsf{P}_*<\rho$ and $\mathsf{c}_*>\mathsf{C}$, that the kernel of $\mathbb{L}_\tau$ coincides with that of $\mathbb{I}+(\mathbb{L}_0)^{-1/2}\mathbb{C}_\tau(\mathbb{L}_0)^{-1/2}$, $\mathbb{C}_\tau:=(\mathbb{L}_\tau-\mathbb{L}_0)$ owing to the fact that operator $\mathbb{L}_0: H^1_0(D)\to H^1_0(D)$ is positive definite (recall that $\mathbb{C}_\tau$ is compact by virtue of Lemma 7). As a result, the multiplicity of any given transmission eigenvalue is finite for τ is a transmission eigenvalue of (5) if and only if 1 is an eigenvalue of the compact self-adjoint operator $-(\mathbb{L}_0)^{-1/2}\mathbb{C}_\tau(\mathbb{L}_0)^{-1/2}$. Here it is noted that operator $\mathbb{T}_\tau := (\mathbb{L}_0)^{-1/2}\mathbb{C}_\tau(\mathbb{L}_0)^{-1/2}$, being compact and self-adjoint, is characterized by an infinite sequence of eigenvalues $\mu_j(\tau)$ accumulating at $+\infty$. Owing to the Courant-Fischer min-max principle, one can further deduce that $\mu_j(\tau)$ are continuous in τ.

Making use of the above hypotheses and discussion, the existence of transmission eigenvalues characterizing (5) in situations where the elasticity and mass density contrasts are of opposite sign can be established by way of the following theorem proven in [38], which plays a similar role as Theorem 1 in Section 3.

ThRef2 **Theorem 7.** *Assume that $\mathsf{P}_*<\rho$ and $\mathsf{c}_*>\mathsf{C}$, and let $\tau\to\mathbb{L}_\tau$ be a continuous mapping from $[0,+\infty)$ to the set of linear self-adjoint operators $H^1_0(D)\to H^1_0(D)$ with property that $\mathbb{L}_0$ is coercive and $\mathbb{L}_\tau-\mathbb{L}_0$ is compact. Provided that there are two non-negative constants $\tau_0\geqslant 0$ and $\tau_1>\tau_0$ such that*

1. *$\mathbb{L}_{\tau_0}$ is positive on $H^1_0(D)$,*
2. *$\mathbb{L}_{\tau_1}$ is non-positive on an m-dimensional subspace of $H^1_0(D)$,*

operator $\mathbb{L}_\tau$ possesses m transmission eigenvalues (counting multiplicity) within interval $[\tau_0,\tau_1]$, i.e. m values of τ for which $\ker(\mathbb{L}_\tau)\neq\{\boldsymbol{0}\}$.

With the above result in place, the next theorem establishes the existence of an infinite set of transmission eigenvalues.

main **Theorem 8.** *Assume that the medium represented by $(\boldsymbol{\mathcal{C}},\rho)$ is homogeneous and isotropic as in (85), and let either $\mathsf{P}_*<\rho$ and $\mathsf{c}_*>\mathsf{C}$, or $\mathsf{p}_*>\rho$ and $\mathsf{C}_*<\mathsf{c}$. Then there exists an infinite sequence of transmission eigenvalues $\tau_j=\omega_j^2$ associated with (5) with $+\infty$ as their only accumulation point.*

Proof. The proof is essentially the same in the two cases owing to the material symmetry, and is shown here for $\mathsf{P}_*<\rho$ and $\mathsf{c}_*>\mathsf{C}$. Without loss of generality, it is also assumed that the Poisson's ratio ν affiliated with the homogeneous background solid, see (85), is non-negative. First recall that, by virtue of Lemma 6, the first assumption of Theorem 7 is satisfied for $\tau_0=0$. From Theorem 6, self-adjoint operator $\mathbb{L}_{\tau_0}$ (see Lemma 7) is thus positive on $H^1_0(D)$ for all sufficiently small $\tau_0\geqslant 0$. Next, from (79) and the fact that $\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{u}_*$ one finds

$$
\begin{aligned}
\big\langle \mathbb{L}_\tau \boldsymbol{v}, \boldsymbol{v} \big\rangle_{H^1_0(D)} &= \int_D \left(\boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\boldsymbol{v}} - \rho\, \tau\, \boldsymbol{u} \cdot \bar{\boldsymbol{v}} \right) \, \mathrm{d}\boldsymbol{x} \\
&= \int_D \left(\boldsymbol{\nabla} \boldsymbol{u}_* : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\boldsymbol{v}} - \rho\, \tau\, \boldsymbol{u}_* \cdot \bar{\boldsymbol{v}} + \boldsymbol{\nabla} \boldsymbol{v} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\boldsymbol{v}} - \rho\, \tau\, \boldsymbol{v} \cdot \bar{\boldsymbol{v}} \right) \, \mathrm{d}\boldsymbol{x}
\end{aligned} \tag{86}
$$

which, combined with (69) where $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{u}_*$ and when $\boldsymbol{\varphi} = \boldsymbol{u}_*$, yields

$$
\big\langle \mathbb{L}_\tau \boldsymbol{v}, \boldsymbol{v} \big\rangle_{H^1_0(D)} = \int_D \left(\boldsymbol{\nabla} \boldsymbol{u}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} : \boldsymbol{\nabla} \bar{\boldsymbol{u}}_* - \tau\, \Delta_\rho |\boldsymbol{u}_*|^2 + \boldsymbol{\nabla} \boldsymbol{v} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\boldsymbol{v}} - \rho\, \tau |\boldsymbol{v}|^2 \right) \, \mathrm{d}\boldsymbol{x} \tag{87}
$$

ll

due to major symmetry of the elastic tensor. To facilitate the application of (87), let $B_r \subset D$ be an arbitrary ball of radius r included in D, and let $\hat{\tau}$ be a transmission eigenvalue corresponding to ball B_r, see Section 2.2, affiliated with two sets of *constant* material properties $(\hat{\boldsymbol{\mathcal{C}}}, \hat{\rho}) := (\boldsymbol{\mathcal{C}}, \rho)$ and $(\hat{\boldsymbol{\mathcal{C}}}_*, \hat{\rho}_*) := (\mathsf{c}_* \boldsymbol{\mathcal{I}}_4, \mathsf{P}_*)$, where $\boldsymbol{\mathcal{C}}$ and ρ are given by (85). Note that the hypothesis of a fourth-order elasticity tensor $\hat{\boldsymbol{\mathcal{C}}}_*$ having only one distinct eigenvalue amounts to the assumption that $\hat{\boldsymbol{\mathcal{C}}}_*$ is isotropic with trivial Poisson's ratio. Recalling an earlier assumption that $\nu \geqslant 0$, such configuration in particular implies that

$$
\boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} = \hat{\boldsymbol{\mathcal{C}}}_* - \hat{\boldsymbol{\mathcal{C}}} = \frac{1}{3} \big[(\mathsf{c}_* - \mathsf{C}) - (\mathsf{c}_* - \mathsf{c}) \big] \, \boldsymbol{\mathcal{I}}_2 \otimes \boldsymbol{\mathcal{I}}_2 \, + \, (\mathsf{c}_* - \mathsf{c}) \, \boldsymbol{\mathcal{I}}_4 \tag{88}
$$

deltac

which is, in of itself, an isotropic elastic tensor whose maximum and minimum eigenvalue are given respectively by $\mathsf{c}_* - \mathsf{c} > 0$ and $\mathsf{c}_* - \mathsf{C} > 0$ (compare with the expression for $\boldsymbol{\mathcal{C}}$ in (85) for negative Poisson's ratio). Hereon, the nontrivial solutions corresponding to $\hat{\tau}$ are denoted by $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{u}}_*$, and their difference by $\hat{\boldsymbol{v}} = \hat{\boldsymbol{u}} - \hat{\boldsymbol{u}}_*$ which is clearly in $H^1_0(B_r)$. If $\hat{\mathbb{L}}_\tau$ is the corresponding operator constructed from $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{u}}$ by the same procedure as in Lemma 5, one has

$$
\big\langle \hat{\mathbb{L}}_{\hat{\tau}} \hat{\boldsymbol{v}}, \hat{\boldsymbol{v}} \big\rangle_{H^1_0(B_r)} = \int_{B_r} \left(\boldsymbol{\nabla} \hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} : \boldsymbol{\nabla} \bar{\hat{\boldsymbol{u}}}_* - \hat{\tau}\, \Delta_{\hat{\rho}} |\hat{\boldsymbol{u}}_*|^2 + \boldsymbol{\nabla} \hat{\boldsymbol{v}} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\hat{\boldsymbol{v}}} - \rho\, \hat{\tau} |\hat{\boldsymbol{v}}|^2 \right) \, \mathrm{d}\boldsymbol{x} = 0. \tag{89}
$$

ll-hat

Next, letting $\tilde{\boldsymbol{v}} \in H^1_0(D)$ be the extension by zero of $\hat{\boldsymbol{v}} \in H^1_0(B_r)$ to the whole of D, and taking the corresponding unique solution of (69) as $\tilde{\boldsymbol{u}} := \mathbb{M}_{\hat{\tau}} \tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{u}}_* := \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{v}}$, sequential application of (69) where $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{u}_*$, to pairs $(\tilde{\boldsymbol{u}}_*, \tilde{\boldsymbol{v}})$ and $(\hat{\boldsymbol{u}}_*, \hat{\boldsymbol{v}})$ yields

$$
\begin{aligned}
\int_D \left(\boldsymbol{\nabla} \tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} : \boldsymbol{\nabla} \bar{\boldsymbol{\varphi}} - \hat{\tau}\, \Delta_\rho \tilde{\boldsymbol{u}}_* \cdot \bar{\boldsymbol{\varphi}} \right) \, \mathrm{d}\boldsymbol{x} &= \int_D \left(\boldsymbol{\nabla} \tilde{\boldsymbol{v}} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\boldsymbol{\varphi}} - \rho\, \hat{\tau}\, \tilde{\boldsymbol{v}} \cdot \bar{\boldsymbol{\varphi}} \right) \, \mathrm{d}\boldsymbol{x} \\
= \int_{B_r} \left(\boldsymbol{\nabla} \hat{\boldsymbol{v}} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla} \bar{\boldsymbol{\varphi}} - \rho\, \hat{\tau}\, \hat{\boldsymbol{v}} \cdot \bar{\boldsymbol{\varphi}} \right) \, \mathrm{d}\boldsymbol{x} &= \int_{B_r} \left(\boldsymbol{\nabla} \hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} : \boldsymbol{\nabla} \bar{\boldsymbol{\varphi}} - \hat{\tau}\, \Delta_{\hat{\rho}} \hat{\boldsymbol{u}}_* \cdot \bar{\boldsymbol{\varphi}} \right) \, \mathrm{d}\boldsymbol{x}
\end{aligned} \tag{90}
$$

kot

for all $\boldsymbol{\varphi} \in H^1(D)$. Since $\boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}}$ is positive definite, see (88) while $\Delta_{\hat{\rho}} < 0$, the last integral in (90) is positive for $\boldsymbol{\varphi} = \tilde{\boldsymbol{u}}_*$. With the latter restriction on the trial function, one accordingly finds from (88) and (90) via the

Cauchy-Schwarz inequality that

$$
\begin{aligned}
&\int_D \big(\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \boldsymbol{\nabla}\bar{\tilde{\boldsymbol{u}}}_* - \hat{\tau}\,\Delta_\rho|\tilde{\boldsymbol{u}}_*|^2\big)\,\mathrm{d}\boldsymbol{x} = \int_{B_r} \big(\boldsymbol{\nabla}\hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \boldsymbol{\nabla}\bar{\tilde{\boldsymbol{u}}}_* - \hat{\tau}\,\Delta_{\hat{\rho}}\hat{\boldsymbol{u}}_* \cdot \bar{\tilde{\boldsymbol{u}}}_*\big)\,\mathrm{d}\boldsymbol{x} \\
&\leqslant \left[\int_{B_r} \big(\boldsymbol{\nabla}\hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \boldsymbol{\nabla}\bar{\hat{\boldsymbol{u}}}_* - \hat{\tau}\,\Delta_{\hat{\rho}}|\hat{\boldsymbol{u}}_*|^2\big)\,\mathrm{d}\boldsymbol{x}\right]^{1/2} \left[\int_{B_r} \big(\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \boldsymbol{\nabla}\bar{\tilde{\boldsymbol{u}}}_* - \hat{\tau}\,\Delta_{\hat{\rho}}|\tilde{\boldsymbol{u}}_*|^2\big)\,\mathrm{d}\boldsymbol{x}\right]^{1/2} \qquad (91) \quad \boxed{\texttt{CS}} \\
&\leqslant \left[\int_{B_r} \big(\boldsymbol{\nabla}\hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \boldsymbol{\nabla}\bar{\hat{\boldsymbol{u}}}_* - \hat{\tau}\,\Delta_{\hat{\rho}}|\hat{\boldsymbol{u}}_*|^2\big)\,\mathrm{d}\boldsymbol{x}\right]^{1/2} \left[\int_{D} \big(\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \boldsymbol{\nabla}\bar{\tilde{\boldsymbol{u}}}_* - \hat{\tau}\,\Delta_{\rho}|\tilde{\boldsymbol{u}}_*|^2\big)\,\mathrm{d}\boldsymbol{x}\right]^{1/2}
\end{aligned}
$$

since $\boldsymbol{\xi} \colon \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \bar{\boldsymbol{\xi}} = \boldsymbol{\xi} \colon (\hat{\boldsymbol{\mathcal{C}}}_* - \hat{\boldsymbol{\mathcal{C}}}) \colon \bar{\boldsymbol{\xi}} \leqslant \boldsymbol{\xi} \colon (\boldsymbol{\mathcal{C}}_* - \boldsymbol{\mathcal{C}}) \colon \bar{\boldsymbol{\xi}} = \boldsymbol{\xi} \colon \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \bar{\boldsymbol{\xi}}$ and $-\Delta_{\hat{\rho}} = \rho - \mathsf{P}_* \leqslant \rho - \rho_* = -\Delta_\rho$. As a result, one has

$$
\int_D \big(\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \boldsymbol{\nabla}\bar{\tilde{\boldsymbol{u}}}_* - \hat{\tau}\,\Delta_\rho|\tilde{\boldsymbol{u}}_*|^2\big)\,\mathrm{d}\boldsymbol{x} \;\leqslant\; \int_{B_r} \big(\boldsymbol{\nabla}\hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \boldsymbol{\nabla}\bar{\hat{\boldsymbol{u}}}_* - \hat{\tau}\,\Delta_{\hat{\rho}}|\hat{\boldsymbol{u}}_*|^2\big)\,\mathrm{d}\boldsymbol{x}.
$$

A substitution of this result into (87) with $\tau = \hat{\tau}$ and $\boldsymbol{v} = \tilde{\boldsymbol{v}}$, followed by the use of (89), yields

$$
\begin{aligned}
\big\langle \mathbb{L}_{\hat{\tau}}\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}} \big\rangle_{H_0^1(D)} &= \int_D \big(\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \boldsymbol{\nabla}\bar{\tilde{\boldsymbol{u}}}_* - \hat{\tau}\,\Delta_\rho|\tilde{\boldsymbol{u}}_*|^2 + \boldsymbol{\nabla}\tilde{\boldsymbol{v}} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}\bar{\tilde{\boldsymbol{v}}} - \rho\,\hat{\tau}|\tilde{\boldsymbol{v}}|^2\big)\,\mathrm{d}\boldsymbol{x} \\
&\leqslant \int_{B_r} \big(\boldsymbol{\nabla}\hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \boldsymbol{\nabla}\bar{\hat{\boldsymbol{u}}}_* - \hat{\tau}\,\Delta_{\hat{\rho}}|\hat{\boldsymbol{u}}_*|^2 + \boldsymbol{\nabla}\hat{\boldsymbol{v}} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}\bar{\hat{\boldsymbol{v}}} - \rho\,\hat{\tau}|\hat{\boldsymbol{v}}|^2\big)\,\mathrm{d}\boldsymbol{x} = 0.
\end{aligned}
$$

By making reference to Theorem 7, one concludes that there exists at least one transmission eigenvalue within interval $(0, \hat{\tau}]$. Finally, by arguing in exactly the same way as in the last part of the proof of Theorem 3, it is possible to demonstrate that in fact there exists a countable set of transmission eigenvalues affiliated with (5). $\square$

Remark 3. *As a consequence of the proof of Theorem, 8 one obtains an upper bound for the first transmission eigenvalue* ω_1*. More specifically, consider* $B_r \subset D$ *as the largest ball contained in* D*. If* $\mathsf{P}_* < \rho$ *and* $\mathsf{c}_* > \mathsf{C}$*, then the first eigenvalue associated with (5) is not larger than the first transmission eigenvalue corresponding to* B_r *endowed with a pair of constant material properties* $(\hat{\boldsymbol{\mathcal{C}}}, \hat{\rho}) := (\boldsymbol{\mathcal{C}}, \rho)$ *and* $(\hat{\boldsymbol{\mathcal{C}}}_*, \hat{\rho}_*) := (\mathsf{c}_*\boldsymbol{\mathcal{I}}_4, \mathsf{P}_*)$*, where* $\boldsymbol{\mathcal{C}}$ *and* ρ *are given by (85). Conversely if* $\mathsf{p}_* > \rho$ *and* $\mathsf{C}_* < \mathsf{c}$*, then the first eigenvalue affiliated with (5) is not larger than the first transmission eigenvalue corresponding to* B_r *endowed with* $(\hat{\boldsymbol{\mathcal{C}}}, \hat{\rho}) := (\boldsymbol{\mathcal{C}}, \rho)$ *and* $(\hat{\boldsymbol{\mathcal{C}}}_*, \hat{\rho}_*) := (\mathsf{C}_*\boldsymbol{\mathcal{I}}_4, \mathsf{p}_*)$*.*

4.2 Elasticity and mass density contrasts of the same sign

The methodology proposed in [8, 28], together with its extensions to the elasticity case [5, 18], allow one to deal with situations where (5) involves contrasts in material parameters that are of the same sign, namely when either $\mathsf{p}_* > \mathsf{P}$ and $\mathsf{c}_* > \mathsf{C}$, or $\mathsf{p} > \mathsf{P}_*$ and $\mathsf{c} > \mathsf{C}_*$. To facilitate the discussion, one may introduce the space of first-order tensors

$$
\mathscr{H}(D) := \{(\boldsymbol{\varphi}, \boldsymbol{\varphi}_*) \in H^1(D) \times H^1(D) \colon\; \boldsymbol{\nabla} \cdot [\boldsymbol{\mathcal{C}} \colon \boldsymbol{\nabla}\boldsymbol{\varphi}] \in L^2(D),\; \boldsymbol{\nabla} \cdot [\boldsymbol{\mathcal{C}}_* \colon \boldsymbol{\nabla}\boldsymbol{\varphi}_*] \in L^2(D)\}, \qquad (92) \quad \boxed{\texttt{elas1}}
$$

together with the pair of (linear) differential-trace operators $\mathbb{P}, \mathbb{Q} : \mathscr{H}(D) \to L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$
\begin{aligned}
&\mathbb{P}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_*) := \big(\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\varphi}]-\rho\boldsymbol{\varphi},\ \boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{C}}_*:\boldsymbol{\nabla}\boldsymbol{\varphi}_*]-\rho_*\boldsymbol{\varphi}_*,\ (\boldsymbol{\varphi}-\boldsymbol{\varphi}_*)_{|\partial D},\ \boldsymbol{n}\cdot(\boldsymbol{\mathcal{C}}:\boldsymbol{\nabla}\boldsymbol{\varphi}-\boldsymbol{\mathcal{C}}_*:\boldsymbol{\nabla}\boldsymbol{\varphi}_*)_{|\partial D}\big)\,,\\
&\mathbb{Q}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_*) := (\rho\boldsymbol{\varphi},\ \rho_*\boldsymbol{\varphi}_*,\ \boldsymbol{0},\ \boldsymbol{0}).
\end{aligned}
\tag{93}
$$

elas2

for all $(\boldsymbol{\varphi}, \boldsymbol{\varphi}_*) \in \mathscr{H}(D)$. On the basis of (92) and (93), the interior transmission problem (5) can be recast as a task of finding $(\boldsymbol{u}, \boldsymbol{u}_*) \in \mathscr{H}(D)$ such that

$$
\mathbb{P}(\boldsymbol{u}, \boldsymbol{u}_*) \ +\ (1+\tau)\mathbb{Q}(\boldsymbol{u}, \boldsymbol{u}_*) = \boldsymbol{0}. \tag{94}
$$

ITPequiv

Next, it is useful to define the auxiliary space of symmetric second-order tensors

$$
\mathscr{W}(D) := \{\boldsymbol{\Phi} \in L^2(D):\ \boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathrm{T}},\ \boldsymbol{\nabla}\cdot\boldsymbol{\Phi} \in L^2(D),\ \boldsymbol{\nabla}\times[\boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\Phi}] = \boldsymbol{0}\}, \tag{95}
$$

and introduce a bounded bilinear form, $\mathcal{R}$, on $\mathscr{K}(D){:=}\mathscr{W}(D)\times H^1(D)$ so that

$$
\begin{aligned}
\mathcal{R}\big((\boldsymbol{\Phi}, \boldsymbol{\varphi}_*), (\boldsymbol{\Psi}, \boldsymbol{\psi}_*)\big) :=& \big\langle \rho^{-1}\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}, \boldsymbol{\nabla}\cdot\boldsymbol{\Psi}\big\rangle_{L^2(D)} + \big\langle \boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\Phi}, \boldsymbol{\Psi}\big\rangle_{L^2(D)} + \big\langle \boldsymbol{\mathcal{C}}_*:\boldsymbol{\nabla}\boldsymbol{\varphi}_*, \boldsymbol{\nabla}\boldsymbol{\psi}_*\big\rangle_{L^2(D)}\\
&+ \big\langle \rho_*\boldsymbol{\varphi}_*, \boldsymbol{\psi}_*\big\rangle_{L^2(D)} - \big\langle \boldsymbol{\varphi}_*, \boldsymbol{\Psi}\cdot\boldsymbol{n}\big\rangle_{L^2(\partial D)} - \big\langle \boldsymbol{\Phi}\cdot\boldsymbol{n}, \boldsymbol{\psi}_*\big\rangle_{L^2(\partial D)},
\end{aligned}
\tag{96}
$$

SesF31

for all $(\boldsymbol{\Phi}, \boldsymbol{\varphi}_*)$ and $(\boldsymbol{\Psi}, \boldsymbol{\psi}_*)$ in $\mathscr{K}(D)$. With reference to (96), the Riesz representation theorem guarantees the existence of a linear operator $\mathbb{R} : \mathscr{K}(D) \to \mathscr{K}(D)$ such that for all $(\boldsymbol{\Phi}, \boldsymbol{\varphi}_*)$ and $(\boldsymbol{\Psi}, \boldsymbol{\psi}_*)$ in $\mathscr{K}(D)$

$$
\big\langle \mathbb{R}(\boldsymbol{\Phi}, \boldsymbol{\varphi}_*), (\boldsymbol{\Psi}, \boldsymbol{\psi}_*)\big\rangle_{\mathscr{K}(D)} = \mathcal{R}((\boldsymbol{\Phi}, \boldsymbol{\varphi}_*), (\boldsymbol{\Psi}, \boldsymbol{\psi}_*)). \tag{97}
$$

With the above notation in place, one is in position to state the key results from [5] that are essential for the treatment of the problem at hand.

EquivSol **Lemma 8.** *Operator $\mathbb{P}$ is bijective if and only if operator $\mathbb{R}$ is bijective.*

LemCoer4 **Lemma 9.** *Operator $\mathbb{R}{:}\ \mathscr{K}(D)\to\mathscr{K}(D)$ is self-adjoint and positive definite if $\mathsf{P}<\mathsf{p}_*$ and $\mathsf{C}<\mathsf{c}_*$. Further, linear operator $\mathbb{Q}{:}\ \mathscr{H}(D)\to L^2(D)\times L^2(D)\times H^{\frac{1}{2}}(\partial D)\times H^{-\frac{1}{2}}(\partial D)$ is self-adjoint, positive and compact.*

For further reference, it is also recalled that that the negative Laplace operator, $-\Delta$, admits an increasing sequence of positive Neumann eigenvalues $\mu_n(D)$ and associated (first-order) eigentensors $\boldsymbol{\psi}_n$ [29] satisfying

$$
-\Delta\boldsymbol{\psi}_n = \mu_n(D)\boldsymbol{\psi}_n \quad \text{in } D, \qquad \boldsymbol{\nabla}\boldsymbol{\psi}_n\cdot\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \partial D. \tag{98}
$$

neum

Due to the fact that the first eigenvalue in (98) is $\mu_1 = 0$, μ_2 denotes the smallest non-zero Neumann eigenvalue of the negative Laplace operator.

Theorem 9. *If either $\mathsf{P} < \mathsf{p}_*$ and $\mathsf{C} < \mathsf{c}_*$ or $\mathsf{P}_* < \mathsf{p}$ and $\mathsf{C}_* < \mathsf{c}$, the set of transmission eigenvalues associated with (5) is discrete, with infinity being the only possible accumulation point. Moreover, every feasible transmission eigenvalue ω^2 is such that*

$$
\omega^2 \ \geqslant\ \min\left[\lambda_1(D)\min(\mathsf{c},\mathsf{c}_*)\left(\frac{1}{\min(\mathsf{P},\mathsf{P}_*)} - \frac{1}{\max(\mathsf{p},\mathsf{p}_*)}\right),\ \mu_2(D)\frac{\max(\mathsf{c},\mathsf{c}_*)-\min(\mathsf{C},\mathsf{C}_*)}{\max(\mathsf{P},\mathsf{P}_*)-\min(\mathsf{p},\mathsf{p}_*)}\right]. \tag{99}
$$

lowb

Proof. The first part of the theorem is a direct consequence of Lemmas 8 and 9. Under the hypothesis that $\mathbb{R}$ is positive definite (which is ensured by the featured restriction on material contrasts), the use of the Lax-Milgram theorem demonstrates that $\mathbb{P}$ is invertible [5]. In light of the "operator" formulation (94) of the interior transmission problem (5), on the other hand, the Fredholm alternative applied to compact operator $\mathbb{I}+(1+\tau)\mathbb{P}^{-1}\mathbb{Q}$ (where $\mathbb{I}$ is the relevant identity operator) affirms the claim regarding the nature of the set of transmission eigenvalues.

To establish the lower bound (99) on the transmission eigenvalues, assume that $\mathsf{P} < \mathsf{p}_*$ and $\mathsf{C} < \mathsf{c}_*$. The combination of (67) and (69) then yields

$$-\int_D (\boldsymbol{\nabla}\boldsymbol{u}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} : \boldsymbol{\nabla}\bar{\boldsymbol{u}}_* - \Delta_\rho \tau \boldsymbol{u}_* \cdot \bar{\boldsymbol{u}}_*) \, \mathrm{d}\boldsymbol{x} = \int_D (\boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}\bar{\boldsymbol{v}} - \rho\tau \boldsymbol{v}\cdot\bar{\boldsymbol{v}}) \, \mathrm{d}\boldsymbol{x}, \tag{100}$$

ITPdiffW21

see also (82). If $\boldsymbol{u}_*$ is next decomposed as $\boldsymbol{u}_* = \tilde{\boldsymbol{u}}_* + \boldsymbol{c}_*$, where $\tilde{\boldsymbol{u}}_* \in \tilde{H}^1(D)$ (recall that $\tilde{H}^1(D)$ is the Sobolev space of weighted zero-mean functions defined by (72)) and $\boldsymbol{c}_*$ is a complex-valued vector constant. Then (69) where $\boldsymbol{u} = \boldsymbol{u}_* + \boldsymbol{v}$ and $\boldsymbol{\varphi} = \boldsymbol{1}$ provides the value of the constant as

$$\boldsymbol{c}_* = \left(\int_D \Delta_\rho \, \mathrm{d}\boldsymbol{x} \right)^{-1} \int_D \rho \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}, \tag{101}$$

which permits (100) to be rewritten as

$$-\int_D (\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} : \boldsymbol{\nabla}\bar{\tilde{\boldsymbol{u}}}_* - \Delta_\rho \tau \tilde{\boldsymbol{u}}_* \cdot \bar{\tilde{\boldsymbol{u}}}_*) \, \mathrm{d}\boldsymbol{x} = \int_D (\boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}\bar{\boldsymbol{v}} - \rho\tau \boldsymbol{v}\cdot\bar{\boldsymbol{v}}) \, \mathrm{d}\boldsymbol{x} \;-\; \tau |\boldsymbol{c}_*|^2 \int_D \Delta_\rho \, \mathrm{d}\boldsymbol{x}. \tag{102}$$

ITPdiffW22

Here the application of relationship $\boldsymbol{u}_* = \tilde{\boldsymbol{u}}_* + \boldsymbol{c}_*$ and Courant-Fischer min-max formulae [29] yield

$$\inf_{\tilde{\boldsymbol{u}}_* \in \tilde{H}^1(D)} \frac{\displaystyle\int_D |\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_*|^2 \, \mathrm{d}\boldsymbol{x}}{\displaystyle\int_D |\tilde{\boldsymbol{u}}_*|^2 \, \mathrm{d}\boldsymbol{x}} \;\geqslant\; \inf_{\substack{\boldsymbol{u}_* \in H^1(D) \\ \int_D \boldsymbol{u}_* \, \mathrm{d}\boldsymbol{x} = \boldsymbol{0}}} \frac{\displaystyle\int_D |\boldsymbol{\nabla}\boldsymbol{u}_*|^2 \, \mathrm{d}\boldsymbol{x}}{\displaystyle\int_D |\boldsymbol{u}_*|^2 \, \mathrm{d}\boldsymbol{x}} \;\geqslant\; \mu_2(D), \tag{103}$$

while (1) requires that $\sup_D \Delta_\rho = \mathsf{P}_* - \mathsf{p} > 0$ and $\inf_D \inf_{\boldsymbol{\xi}} \boldsymbol{\xi} : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} : \bar{\boldsymbol{\xi}} = (\mathsf{c}_* - \mathsf{C})|\boldsymbol{\xi}|^2 \geqslant 0$ for all complex-valued vectors $\boldsymbol{\xi}$. As a result, the left-hand side of (102) can be shown to be bounded from above as

$$-\int_D (\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} : \boldsymbol{\nabla}\bar{\tilde{\boldsymbol{u}}}_* - \Delta_\rho \tau \tilde{\boldsymbol{u}}_* \cdot \bar{\tilde{\boldsymbol{u}}}_*) \, \mathrm{d}\boldsymbol{x} \;\leqslant\; \big(-\mu_2(D)(\mathsf{c}_* - \mathsf{C}) + \tau(\mathsf{P}_* - \mathsf{p}) \big) \, \|\tilde{\boldsymbol{u}}_*\|^2_{L^2(D)}. \tag{104}$$

Ineq21

On recalling that $\boldsymbol{v} \in H^1_0(D)$, a similar treatment of the right-hand side yields

$$\int_D (\boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}\bar{\boldsymbol{v}} - \rho\tau \boldsymbol{v}\cdot\bar{\boldsymbol{v}}) \, \mathrm{d}\boldsymbol{x} - \tau|\boldsymbol{c}_*|^2 \int_D \Delta_\rho \, \mathrm{d}\boldsymbol{x} \;\geqslant\; \left(\mathsf{c}\lambda_1(D) - \tau \frac{\mathsf{p}_* \mathsf{P}}{\mathsf{p}_* - \mathsf{P}} \right) \|\boldsymbol{v}\|^2_{L^2(D)}. \tag{105}$$

Ineq22

As a result, when τ is such that

$$\tau \;<\; \mu_2(D) \frac{\mathsf{c}_* - \mathsf{C}}{\mathsf{P}_* - \mathsf{p}} \qquad \text{and} \qquad \tau \;<\; \mathsf{c}\lambda_1(D) \left(\frac{1}{\mathsf{P}} - \frac{1}{\mathsf{p}_*} \right), \tag{106}$$

Cond2

substitution of (104) and (105) into (102) guarantees that $\tilde{\boldsymbol{u}}_* = \boldsymbol{v} = \boldsymbol{0}$ and consequently $\boldsymbol{u}_* = \boldsymbol{u} = \boldsymbol{0}$, whereby such τ cannot be a transmission eigenvalue. Finally, the combination of condition (106) and similar reasoning using the material symmetry recovers (99) and thus completes the proof. $\square$

To establish the existence of the transmission eigenvalues in situations where the elasticity and mass density contrasts are of the same sign, it is possible to adapt the methodology developed in Section 4.1. To this end, it is again assumed that the background medium is *homogeneous and isotropic*, whereby $\boldsymbol{\mathcal{C}}$ and ρ are given by (85). For brevity, the ensuing discussion assumes that $\rho < \mathsf{p}_*$ and $\mathsf{C} < \mathsf{c}_*$, noting that the case when $\mathsf{P}_* < \rho$ and $\mathsf{C}_* < \mathsf{c}$ can be handled in exactly the same way. To avoid repetition, the focus is made on the differences between the current treatment and that in Section 4.1.

The main difficulty in dealing with the problem at hand resides in solving (69), or equivalently (71), due to general lack of coercivity of the bilinear form $\mathcal{M}_\tau$ given by (70). To deal with the impediment, let $B_r \subset D$ be a ball of radius r contained in D, and let $\hat{\tau}$ be the first transmission eigenvalue corresponding to B_r endowed with two sets of constant material properties $(\hat{\boldsymbol{\mathcal{C}}}, \hat{\rho}) := (\boldsymbol{\mathcal{C}}, \rho)$ and $(\hat{\boldsymbol{\mathcal{C}}}_*, \hat{\rho}_*) := (\boldsymbol{\mathcal{C}} + \frac{\mathsf{c}_* - \mathsf{C}}{2} \boldsymbol{\mathcal{I}}_4, \rho)$, where $\boldsymbol{\mathcal{C}}$ and ρ are given by (85). In this setting, it is further required that

$$
(\mathsf{P}_* - \rho) \;<\; \frac{\mu}{2\hat{\tau}} (\mathsf{c}_* - \mathsf{C}), \tag{107}
$$

rest

where μ is the unique minimizer defined via (73). With reference to the analytical framework developed in Lemma 5, for $\boldsymbol{\varphi} \in \tilde{H}^1(D)$ and $\tau \in \{z \in \mathbb{C} \colon \Re(z) \leqslant \hat{\tau}\}$ one now has

$$
\begin{aligned}
\Re\left(\mathcal{M}_\tau(\boldsymbol{\varphi}, \boldsymbol{\varphi})\right) \;&=\; \Re\Big(\big\langle \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \nabla \boldsymbol{\varphi}, \nabla \boldsymbol{\varphi} \big\rangle_{L^2(D)} \;-\; \tau \big\langle \Delta_\rho \boldsymbol{\varphi}, \boldsymbol{\varphi} \big\rangle_{L^2(D)} \Big) \\
&\geqslant\; (\mathsf{c}_* - \mathsf{C}) \|\nabla \boldsymbol{\varphi}\|^2_{L^2(D)} \;-\; \hat{\tau}\, (\mathsf{P}_* - \rho) \|\boldsymbol{\varphi}\|^2_{L^2(D)} \\
&\geqslant\; \left[(\mathsf{c}_* - \mathsf{C}) - \frac{\hat{\tau}}{\mu} (\mathsf{P}_* - \rho) \right] \|\nabla \boldsymbol{\varphi}\|^2_{L^2(D)} \;\geqslant\; \frac{(\mathsf{c}_* - \mathsf{C})}{2} \frac{\mu}{\mu + 1} \|\boldsymbol{\varphi}\|^2_{H^1(D)},
\end{aligned}
$$

which ensures the coercivity of $\mathcal{M}_\tau$ in $\tilde{H}^1(D)$ under the featured set of of restrictions. Following the proof of Lemma 5, one can consequently construct a linear operator $\mathbb{M}_\tau \colon H^1_0(D) \to H^1(D)$ such that $\mathbb{M}_\tau \boldsymbol{v} = \boldsymbol{u}$. This construction leads to the definition of operator $\mathbb{L}_\tau : H^1_0(D) \to H^1_0(D)$ associated with (75) thanks to Riesz representation theorem. By mimicking the proofs of Lemma 6 and Lemma 7, one can next show that $\mathbb{L}_0$ is coercive, that $\mathbb{L}_\tau$ is self-adjoint, and that $\mathbb{L}_\tau - \mathbb{L}_0$ is compact. On recalling the first transmission eigenvalue $\hat{\tau}$ for ball $B_r \subset D$ described earlier and denoting the corresponding nonzero solutions as $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{u}}_*$ so that $\hat{\boldsymbol{v}} = \hat{\boldsymbol{u}} - \hat{\boldsymbol{u}}_* \in H^1_0(B_r)$, it follows that (89) also holds for $\hat{\mathbb{L}}_{\hat{\tau}}$ in the present case. Further, if $\tilde{\boldsymbol{v}} \in H^1_0(D)$ is the extension by zero of $\hat{\boldsymbol{v}} \in H^1_0(B_r)$ to the whole of D, one finds by taking $\tilde{\boldsymbol{u}} := \mathbb{M}_{\hat{\tau}} \tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{u}}_* := \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{v}}$, and performing similar calculations as in (90) and (91) that

$$
\begin{aligned}
&\int_D \big(\nabla \tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \nabla \bar{\tilde{\boldsymbol{u}}}_* - \hat{\tau}\, \Delta_\rho |\tilde{\boldsymbol{u}}_*|^2 \big) \, \mathrm{d}\boldsymbol{x} = \int_{B_r} \big(\nabla \hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \nabla \bar{\hat{\boldsymbol{u}}}_* \big) \, \mathrm{d}\boldsymbol{x} \\
&\quad \leqslant \left[\int_{B_r} \big(\nabla \hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \nabla \bar{\hat{\boldsymbol{u}}}_* \big) \, \mathrm{d}\boldsymbol{x} \right]^{1/2} \left[\int_{B_r} \big(\nabla \tilde{\boldsymbol{u}}_* : 2\boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \nabla \bar{\tilde{\boldsymbol{u}}}_* - \nabla \tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \nabla \bar{\tilde{\boldsymbol{u}}}_* \big) \, \mathrm{d}\boldsymbol{x} \right]^{1/2} \\
&\quad \leqslant \left[\int_{B_r} \big(\nabla \hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \nabla \bar{\hat{\boldsymbol{u}}}_* \big) \, \mathrm{d}\boldsymbol{x} \right]^{1/2} \left[\int_D \big(\nabla \tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \nabla \bar{\tilde{\boldsymbol{u}}}_* - \tfrac{1}{2} (\mathsf{c}_* - \mathsf{C}) |\nabla \tilde{\boldsymbol{u}}_*|^2 \big) \, \mathrm{d}\boldsymbol{x} \right]^{1/2} \\
&\quad \leqslant \left[\int_{B_r} \big(\nabla \hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \nabla \bar{\hat{\boldsymbol{u}}}_* \big) \, \mathrm{d}\boldsymbol{x} \right]^{1/2} \left[\int_D \big(\nabla \tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \nabla \bar{\tilde{\boldsymbol{u}}}_* - \hat{\tau}\, \Delta_\rho |\tilde{\boldsymbol{u}}_*|^2 \big) \, \mathrm{d}\boldsymbol{x} \right]^{1/2},
\end{aligned}
$$

due to (107) and relationships $\boldsymbol{\xi} : 2\boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} : \bar{\boldsymbol{\xi}} = \boldsymbol{\xi} : 2(\hat{\boldsymbol{\mathcal{C}}}_* - \hat{\boldsymbol{\mathcal{C}}}) : \bar{\boldsymbol{\xi}} \leqslant \boldsymbol{\xi} : (\boldsymbol{\mathcal{C}}_* - \boldsymbol{\mathcal{C}}) : \bar{\boldsymbol{\xi}} = \boldsymbol{\xi} : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} : \bar{\boldsymbol{\xi}}$ and $\mathsf{P}_* - \rho \geqslant \rho_* - \rho = \Delta_\rho$. As a result,

$$\int_D \big(\nabla \tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \nabla \bar{\tilde{\boldsymbol{u}}}_* - \hat{\tau}\, \Delta_\rho |\tilde{\boldsymbol{u}}_*|^2\big) \,\mathrm{d}\boldsymbol{x} \;\leqslant\; \int_{B_r} \big(\nabla \hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \nabla \bar{\hat{\boldsymbol{u}}}_*\big) \,\mathrm{d}\boldsymbol{x}.$$

On substituting this result into (87) when $\tau = \hat{\tau}$ and $\boldsymbol{v} = \tilde{\boldsymbol{v}}$, it follows by virtue of (89) that

$$\begin{aligned} \big\langle \mathbb{L}_{\hat{\tau}} \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}} \big\rangle_{H^1_0(D)} &= \int_D \big(\nabla \tilde{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\boldsymbol{\mathcal{C}}} \colon \nabla \bar{\tilde{\boldsymbol{u}}}_* - \hat{\tau}\, \Delta_\rho |\tilde{\boldsymbol{u}}_*|^2 + \nabla \tilde{\boldsymbol{v}} : \boldsymbol{\mathcal{C}} : \nabla \bar{\tilde{\boldsymbol{v}}} - \rho\, \hat{\tau} |\tilde{\boldsymbol{v}}|^2\big) \,\mathrm{d}\boldsymbol{x} \\ &\leqslant \int_{B_r} \big(\nabla \hat{\boldsymbol{u}}_* : \boldsymbol{\Delta}_{\hat{\boldsymbol{\mathcal{C}}}} \colon \nabla \bar{\hat{\boldsymbol{u}}}_* + \nabla \hat{\boldsymbol{v}} : \boldsymbol{\mathcal{C}} : \nabla \bar{\hat{\boldsymbol{v}}} - \rho\, \hat{\tau} |\hat{\boldsymbol{v}}|^2\big) \,\mathrm{d}\boldsymbol{x} = 0, \end{aligned}$$

which implies, via Theorem 7, that there exists at least one transmission eigenvalue within interval $(0, \hat{\tau}]$. The above analysis establishes the following result on the existence of transmission eigenvalues for the case where the elasticity and mass density contrasts share the same sign.

main2 **Theorem 10.** *Assume that the medium represented by $(\boldsymbol{\mathcal{C}}, \rho)$ is homogeneous and isotropic as in (85). If either*

1. *$\rho < \mathsf{p}_*$ and $\mathsf{C} < \mathsf{c}_*$ such that*
$$(\mathsf{P}_* - \rho) \;<\; \frac{\mu}{2\hat{\tau}}(\mathsf{c}_* - \mathsf{C})$$
where $\hat{\tau}$ is the first transmission eigenvalue corresponding to ball $B_r \subset D$ endowed with constant material properties $(\hat{\boldsymbol{\mathcal{C}}}, \hat{\rho}) := (\boldsymbol{\mathcal{C}}, \rho)$ and $(\hat{\boldsymbol{\mathcal{C}}}_*, \hat{\rho}_*) := (\boldsymbol{\mathcal{C}} + \frac{\mathsf{c}_* - \mathsf{C}}{2} \boldsymbol{\mathcal{I}}_4, \rho)$, or
2. *$\mathsf{P}_* < \rho$ and $\mathsf{C}_* < \mathsf{c}$ such that*
$$(\rho - \mathsf{p}_*) \;<\; \frac{\mu}{2\hat{\tau}}(\mathsf{c} - \mathsf{C}_*)$$
where $\hat{\tau}$ the first transmission eigenvalue corresponding to $B_r \subset D$ endowed with constant material properties $(\hat{\boldsymbol{\mathcal{C}}}, \hat{\rho}) := (\boldsymbol{\mathcal{C}}, \rho)$ and $(\hat{\boldsymbol{\mathcal{C}}}_*, \hat{\rho}_*) := (\boldsymbol{\mathcal{C}} + \frac{\mathsf{C}_* - \mathsf{c}}{2} \boldsymbol{\mathcal{I}}_4, \rho)$,

there exists at least one transmission eigenvalue associated with (5) within interval $(0, \hat{\tau}]$.

Remark 4. *The foregoing developments, catering for the case where the elasticity and mass density contrasts are of the same sign, unfortunately can not be carried further along the lines of the proof of Theorem 3 to establish the existence of infinitely many eigenvalues since the linear operator $\mathbb{L}_\tau$ has the required properties only for $\tau \leqslant \hat{\tau}$, where $\hat{\tau}$ is bounded by (107). However, if the mass density contrast is sufficiently small so that (107) is met for $r > 0$ such that $m > 1$ balls of radius r can be fitted in D (see the proof of Theorem 3), one can show that there are $m > 1$ transmission eigenvalues within interval $(0, \hat{\tau}]$ counting multiplicity.*

5 Conclusions

In this study, the existence and structure of the transmission eigenvalues for heterogeneous and anisotropic elastic bodies is considered for a wide class of mass density and elasticity contrasts between the two solids featured by the interior transmission problem. When no external excitation is present, the latter boundary value problem entails two body-force-free equations of (anisotropic, inhomogeneous) linear elasticity in a bounded domain $D \subset \mathbb{R}^3$, with shared Cauchy data over ∂D. In the context of the inverse scattering theory, these two equations model respectively penetrable obstacle D and background medium occupying region D. The resulting eigenvalue problem turns out to be nonlinear and may, at best, be transformed into a linear eigenvalue problem for a non-self-adjoint compact operator. For generality, the interior transmission eigenvalue problem is investigated for a wide class of material contrasts between the obstacle and the background, namely those with material similitude in terms of equal elastic tensors or equal mass densities, and configurations without material similitude where the mass density and elasticity contrast are each sign-definite throughout D. For configurations involving either equal elastic tensor distributions or equal mass density distributions over D it is shown, via a suitable variational formulation of the interior transmission problem for heterogeneous anisotropic solids, that the latter is necessarily characterized by a countable set of (positive) transmission eigenvalues that accumulate only at infinity. For configurations without material similitude, on the other hand, a further distinction is made between the situations where the elasticity and mass density contrasts are of the same sign, and those where the two are of the opposite sign. In the latter case the discreteness of transmission eigenvalues is again established for a general case involving anisotropic heterogeneous solids, while the existence of a countable set of transmission eigenvalues is proven under an additional restriction that either the background or the obstacle is homogeneous and isotropic. In situations where the elasticity and mass density contrasts share the sign over D, an earlier result on the discreteness of the transmission eigenspectrum [5] is complemented by the proof of its non-emptiness, requiring again that either the background or the obstacle be homogeneous and isotropic. Necessitated by the breadth of material configurations studied, the above claims are established through the development of a suite of variational techniques, each customized to meet the needs of a particular class of eigenvalue problems. As a secondary result, the lower and upper bounds on the first transmission eigenvalue are obtained in terms of the elasticity and mass density contrasts between the obstacle and the background. Given the fact that the transmission eigenvalues are computable from the observations of the scattered field, such estimates may have significant potential toward exposing the nature (e.g. compliance) of penetrable scatterers in elasticity, see [10] for a discussion in the context of scalar problems.

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References

Ahusborde [1] E. Ahusborde, M. Azaïez, M.O. Deville, and E.H. Mund. Legendre spectral methods for the -**grad**(div) operator. *Comput. Methods Appl. Mech. Engrg.*, 196:4538–4547, 2007.

Arens [2] T. Arens. Linear sampling methods for 2D inverse elastic wave scattering. *Inverse Problems*, 17:1445–1464, 2001.

Azaiez [3] M. Azaïez, R. Gruber, M.O. Deville, and E.H. Mund. On a stable spectral method for the grad(div) eigenvalue problem. *Journal of Scientific Computing*, 27, 2006.

Bag2006 [4] K. Baganas, B. B. Guzina, A. Charalambopoulos, and G. D. Manolis. A linear sampling method for the inverse transmission problem in near-field elastodynamics. *Inverse Problems*, 22:1835–1853, 2006.

Bellis [5] C. Bellis and B. B. Guzina. On the existence and uniqueness of a solution to the interior transmission problem for piecewise-homogeneous solids. *Journal of Elasticity*, 101:29–57, 2010.

CakoCayo [6] F. Cakoni, M. Çayören, and D. Colton. Transmission eigenvalues and the nondestructive testing of dielectrics. *Inverse Problems*, 24:065016, 2008.

CakoniBook [7] F. Cakoni and D. Colton. *Qualitative methods in inverse scattering theory*. Springer-Verlag, Berlin, 2006.

CakoniAniso [8] F. Cakoni, D. Colton, and H. Haddar. The linear sampling method for anisotropic media. *J. Comput. Appl. Math.*, 146:285–299, 2002.

CakoniBound [9] F. Cakoni, D. Colton, and H. Haddar. The computation of lower bounds for the norm of the index of refraction in an anisotropic media from far field data. *J. Integral Equations Appl.*, 21:203–227, 2009.

CakoniBlow [10] F. Cakoni, D. Colton, and H. Haddar. On the determination of dirichlet and transmission eigenvalues from far field data. *Comptes Rendus Mathematique*, 348:379–383, 2010.

Cako2007 [11] F. Cakoni, D. Colton, and P. Monk. On the use of transmission eigenvalues to estimate the index of refraction from far field data. *Inverse Problems*, 23:507–522, 2007.

CakoNew [12] F. Cakoni and D. Gintides. New results on transmission eigenvalues. *Inverse Problems and Imaging*, 4:39–48, 2010.

CakoniSIAM [13] F. Cakoni, D. Gintides, and H. Haddar. The existence of an infinite discrete set of transmission eigenvalues. *SIAM J. Math. Analysis*, 42:237–255, 2010.

CakoCav [14] F. Cakoni, D. Gintides, and H. Haddar. The interior transmission problem for regions with cavities. *SIAM J. Math. Anal.*, 42:145–162, 2010.

CakoniMax [15] F. Cakoni and H. Haddar. A variational approach for the solution of electromagnetic interior transmission problem for anisotropic media. *Inverse Problems and Imaging*, 1:443–456, 2007.

CakoniIHM [16] F. Cakoni and H. Haddar. On the existence of transmission eigenvalues in an inhomogeneous medium. *Applicable Analysis*, 88:475–493, 2009.

CakoniTEP [17] F. Cakoni and A. Kirsch. On the interior transmission eigenvalue problem. *Int. J. Comput. Sci. Math.*, 3:142–167, 2010.

CharaITP [18] A. Charalambopoulos. On the interior transmission problem in nondissipative, inhomogeneous, anisotropic elasticity. *J. Elasticity*, 67:149–170, 2002.

CharaITP2 [19] A. Charalambopoulos and K. A. Anagnostopoulos. On the spectrum of the interior transmission problem in isotropic elasticity. *J. Elasticity*, 90:295–313, 2008.

CharaITP1 [20] A. Charalambopoulos, D. Gintides, and K. Kiriaki. The linear sampling method for the transmission problem in three-dimensional linear elasticity. *Inverse Problems*, 18:547–558, 2002.

Cha2007 [21] A. Charalambopoulos, A. Kirsch, K. A. Anagnostopoulos, D. Gintides, and K. Kiriaki. The factorization method in inverse elastic scattering from penetrable bodies. *Inverse Problems*, 23:27–51, 2007.

ColtonReview [22] D. Colton, J. Coyle, and P. Monk. Recent developments in inverse acoustic scattering theory. *SIAM Review*, 42:369–414, 2000.

ColtonFF [23] D. Colton, A. Kirsch, and L. Päivärinta. Far-field patterns for acoustic waves in an inhomogeneous medium. *SIAM J. Math. Anal.*, 20:1472–1483, 1989.

ColtonBook [24] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*. Springer-Verlag, 1992.

Col2007 [25] D. Colton, L. Päivärinta, and J. Sylvester. The interior transmission problem. *Inverse Problems and Imaging*, 1:13–28, 2007.

Guz2007 [26] B. B. Guzina and A. I. Madyarov. A linear sampling approach to inverse elastic scattering in piecewise-homogeneous domains. *Inverse Problems*, 23:1467–1493, 2007.

HaddarMax [27] H. Haddar. The interior transmission problem for anisotropic maxwell's equations and its applications to the inverse problem. *Math. Methods Applied Sciences*, 18:2111–2129, 2004.

Hahner [28] P. Hähner. On the uniqueness of the shape of a penetrable, anisotropic obstacle. *J. Comput. Appl. Math.*, 116:167–180, 2000.

Henrot [29] A. Henrot. *Extremum problems for eigenvalues of elliptic operators*. Birkhäuser Verlag, 2006.

Lassi [30] M. Hitrik, K. Ksupchyk, P. Ola, and L. Päivärinta. The transmission eigenvalues for operators with constant coeficients. *SIAM J. Math Analysis*, to appear.

Katznelson [31] Y. Katznelson. *An introduction to harmonic analysis*. Dover, New York, 1976.

Kirsch [32] A. Kirsch. An integral equation approach and the interior transmission problem for maxwell's equations. *Inverse Problems and Imaging*, 1:107–127, 2007.

KirschTE [33] A. Kirsch. On the existence of transmission eigenvalues. *Inverse Problems and Imaging*, 3:155–172, 2009.

KirschBook [34] A. Kirsch and N. Grinberg. *The Factorization Method for Inverse Problems*. Oxford University Press, New York, 2008.

Knowles [35] J. K. Knowles. On the representation of the elasticity tensor for isotropic media. *J. Elasticity*, 39:175–180, 1995.

Marsden [36] J. E. Marsden and T. J. R. Hughes. *Mathematical foundations of elasticity*. Dover, 1994.

boj:04 [37] S. Nintcheu Fata and B. B. Guzina. A linear sampling method for near-field inverse problems in elastodynamics. *Inverse Problems*, 20:713–736, 2004.

PaivarintaTE [38] L. Päivärinta and J. Sylvester. Transmission eigenvalues. *SIAM J. Math. Anal.*, 40:738–753, 2008.

Rynne [39] B. P. Rynne and Sleeman B. D. The interior transmission problem and inverse scattering from inhomogeneous media. *SIAM J. Math. Anal.*, 22:1755–1762, 1991.

Yosida [40] K. Yosida. *Functional Analysis*. Springer-Verlag, 1980.