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The Born transmission eigenvalue problem

Fioralba Cakoni[1,3], **David Colton**[2] and **Jacob D Rezac**[2]

[1] Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
[2] Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

E-mail: fc292@math.rutgers.edu, colton@udel.edu and jrezac@udel.edu

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Abstract

In this paper we study the distribution of transmission eigenvalues in the complex plane for obstacles whose contrast is small in magnitude. We use a first order approximation of the refractive index to derive and study an approximate interior transmission problem. In the case of spherically stratified media, we prove existence and discreteness of transmission eigenvalues and derive a condition under which the complex part of transmission eigenvalues cannot lie in a strip parallel to the real axis. For obstacles with general shape, we demonstrate that if transmission eigenvalues exist then they form a discrete set.

Keywords: Born approximation, transmission eigenvalues, inverse scattering, inhomogeneous media

(Some figures may appear in colour only in the online journal)

1. Introduction

The scattering of a time-harmonic acoustic plane wave by an inhomogeneous medium can be modeled by

$$\begin{aligned}&\Delta u^{\mathrm{s}} + k^2 n(x) u^{\mathrm{s}} = -k^2 (n(x) - 1) u^{\mathrm{i}}, \qquad x \in \mathbb{R}^3\\ &u^{\mathrm{i}}(x) = \mathrm{e}^{\mathrm{i}kx\cdot d},\\ &\lim_{r\to\infty} r\left(\frac{\partial u^{\mathrm{s}}}{\partial \nu} - \mathrm{i}ku^{\mathrm{s}}\right) = 0,\end{aligned}$$

where $d \in \mathbb{R}^3$ with $|d| = 1$, $r = |x|$, and k is the wave number. The squared refractive index of a penetrable obstacle is represented by the (possibly complex-valued) function $n \in L^\infty(D)$,

[3] Author to whom any correspondence should be addressed.

$\mathrm{Re}(n) > 0$, $\mathrm{Im}(n) \geqslant 0$, and $D \subset \mathbb{R}^3$ is a bounded domain with connected complement defined by $D = \mathrm{supp}(n - 1)$. Denote the boundary of D by ∂D which we assume to be $\mathcal{C}^1$ and let ν be the outward unit normal vector to ∂D. We call u^{s} the scattered field and u^{i} the incident field which is an entire solution to the Helmholtz equation (otherwise one could consider point sources also located outside D).

There is a large amount of interest in inverse problem of finding properties of n based on the measured scattered field. One approach to this problem is the linear sampling method [1]. The linear sampling method is a technique for finding qualitative information about the location and shape of D based on scattered far field data. Central to its justification is the interior transmission problem which consists of finding v, $w \in L^2(D)$, and nonzero $k \in \mathbb{C}$ which satisfy

$$\begin{aligned} \Delta w + k^2 n w = 0 \quad & \text{in } D \\ \Delta v + k^2 v = 0 \quad & \text{in } D \\ w = v,\ \frac{\partial w}{\partial \nu} = \frac{\partial w}{\partial \nu} \quad & \text{on } \partial D. \end{aligned} \tag{1}$$

We call k a transmission eigenvalue if there exists a nontrivial solution to (1).

The implementation of the linear sampling method requires that the set of associated transmission eigenvalues be discrete. The location of real transmission eigenvalues can be determined from scattered far field data [2] and this knowledge can be used to obtain information about the properties of n [3, 10]. Though typically discussed for time-harmonic problems, the linear sampling method can be justified for transient problems in the time domain if the transmission eigenvalues in the frequency domain form a discrete set and lie in a strip parallel to the real axis [11]. Conditions under which the latter is true are known only for the eigenvalues with spherically symmetric eigenfunctions for spherically stratified media [7]. However, in a more general context the problem is open. The best result on the location of transmission eigenvalues known in the general case is in [13, 21], where it is shown that, except for a finite set, all transmission eigenvalues lie in an arbitrary small wedge about the real axis. This, unfortunately, does not suffice to justify the linear sampling method in the time domain. The question then becomes, can one do better? In this paper we show by an example that at least in the Born regime the answer is no. As follow up to [12], an investigation of the linear sampling method with quasi-back scattering data in the time domain is initiated in [5] for weak scattering media. The question of the discreteness and location in the complex plane of transmission eigenvalues for the Born approximation model, which is central to the justification of the linear sampling method in the time domain, motivated the current investigation. For the justification of the Born approximation in the time domain we refer the reader to [17] and the references therein, whereas for the mathematical understanding of inverse scattering theory in the Born regime see [14, 15, 18] (and the references therein).

In this paper we study an approximation to (1) in the case that $n - 1$ is small in magnitude. In particular, assume $n(x) = 1 + \epsilon m(x)$ for $\epsilon \ll 1$ and $\|m\|_\infty = 1$ and that solutions take the form $w(x) = w_0(x) + \epsilon u(x)$. The function $u = \frac{\partial w}{\partial \epsilon}\Big|_{\epsilon=0}$ is the first term in the well-known Born approximation. When v is of the same order as w_0, (1) becomes

$$\Delta u + k^2 u = -k^2\, m v \quad \text{in } D \tag{2}$$

$$\Delta v + k^2 v = 0 \quad \text{in } D \tag{3}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D. \tag{4}$$

In what follows we always take $m(x) = m_1(x) + \frac{\mathrm{i}}{k} m_2(x)$ for real-valued $m_1, m_2 \in L^\infty(D)$ with $m_2 \geqslant 0$ in D. When m_2 is not identically zero, this corresponds to scattering from an absorbing medium embedded in free space with speed of sound normalized to one. Note that this form of contrast appears after taking the Fourier–Laplace transform of the wave equation with damping term. We will always assume that there is some $\alpha > 0$ so that $k \in \mathbb{C}_{|z|>\alpha} := \{z \in \mathbb{C} : \ |z| > \alpha > 0\}$.

Definition 1. All $k \in \mathbb{C}_{|z|>\alpha}$ such that there exists a nontrivial solution $u \in H_0^2(D)$, $v \in L^2(D)$ to (2) are called Born transmission eigenvalues.

Real Born transmission eigenvalues do not exist for (2) so long as either m_1 or m_2 are strictly positive. Indeed, we follow the argument from [8] and assume there are u, v, and $k \in \mathbb{R}$ which satisfy (2). Multiplying (2) by its complex conjugate, dividing by m, and integrating over D yields

$$\begin{aligned} \int_D \frac{\overline{m}}{|m|^2}\, |(\Delta u + k^2 u)|^2 \,\mathrm{d}V &= -k^2 \int_D \overline{(\Delta u + k^2 u)} v \,\mathrm{d}V \\ &= -k^2 \int_D \overline{u}\,(\Delta v + k^2 v)\,\mathrm{d}V - k^2 \int_{\partial D} \overline{u}\frac{\partial v}{\partial \nu} - v\overline{\frac{\partial u}{\partial \nu}}\,\mathrm{d}s = 0. \end{aligned} \tag{5}$$

The last line vanishes from (3) and (4). By the assumptions on m, u satisfies $\Delta u + k^2 u = 0$. Along with vanishing Cauchy data on ∂D, this gives $u \equiv 0$ and hence by (2), $v \equiv 0$. Note that, since $v \in L^2(D)$ with $\Delta v \in L^2(D)$ and $u \in H^2(D)$, the boundary terms in (5) are understood in the sense of the duality $H^{3/2}(\partial D)$, $H^{-3/2}(\partial D)$ and $H^{1/2}(\partial D)$, $H^{-1/2}(\partial D)$, respectively. Indeed if $v \in L^2(D)$ with $\Delta v \in L^2(D)$ then its trace $v \in H^{-1/2}(\partial D)$ is defined by duality using the identity

$$\langle v, \tau \rangle_{H^{-1/2}(\partial D), H^{1/2}(\partial D)} = \int_D (v\Delta w - w\Delta v)\mathrm{d}x,$$

where $w \in H^2(D)$ is such that $w = 0$ and $\partial w/\partial \nu = \tau$ on ∂D. Similarly, the trace of $\partial v/\partial \nu \in H^{-3/2}(\partial D)$ is defined by duality using the identity

$$\left\langle \frac{\partial v}{\partial \nu}, \tau \right\rangle_{H^{-3/2}(\partial D), H^{3/2}(\partial D)} = -\int_D (v\Delta w - w\Delta v)\mathrm{d}x,$$

where $w \in H^2(D)$ is such that $w = \tau$ and $\partial w/\partial \nu = 0$ on ∂D.

With the exception of the above argument, the existence, discreteness, and distribution in the complex plane of Born transmission eigenvalues for (2) has not been studied. We study each of these properties in this paper. In section 2 we give results for spherically stratified media. We show that there are contrasts whose associated eigenvalues are real. We also show that, except for special cases, the complex parts of these eigenvalues are unbounded. In section 3, we discuss the case when the media is no longer spherically stratified. We show that eigenvalues to (2) are discrete when they exist and have no finite accumulation points.

2. Spherically stratified media

In this section we consider D to be a ball and m to be spherically symmetric. Under these assumptions, we are able to obtain considerable information about Born transmission eigenvalues. We begin our study by deriving a function whose zeros coincide with Born transmission eigenvalues. By defining $w = \frac{1}{k^2}u + v$, we can write (2)–(4) as

$$\Delta w + k^2 w = -mv \quad \text{in } D, \tag{6}$$

$$\Delta v + k^2 v = 0 \quad \text{in } D, \tag{7}$$

$$v = w,\ \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \quad \text{on } \partial D. \tag{8}$$

We will use this formulation in this section.

Theorem 1. *Let $R > 0$ be fixed and assume $D = B_R(0) \subset \mathbb{R}^3$ and $m = m(|x|)$. Let $j_\ell(r)$ be a spherical Bessel function of the first kind of order ℓ. Then the zeros of the function*

$$d_\ell(k) \coloneqq -\frac{k}{R^2}\int_0^R \rho^2 m(\rho) j_\ell(k\rho)^2\,\mathrm{d}\rho, \quad \ell = 0, 1, 2, \ldots$$

are Born transmission eigenvalues $k \in \mathbb{C}_{|z|>\alpha}$.

Proof. Let $r = |x| > 0$ and $\hat{x} = x/|x| \in \mathbb{S}^2 \coloneqq \{x \in \mathbb{R}^3\colon\ |x| = 1\}$. Then there exist unique $\alpha_\ell^m \in \mathbb{C}$, $|m| \leqslant \ell$ such that (3) has solutions of the form [6]

$$v(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \alpha_\ell^m j_\ell(kr) Y_\ell^m(\hat{x}), \quad 0 \leqslant r \leqslant R,\ \hat{x} \in \mathbb{S}^2,$$

where $Y_\ell^m(\hat{x})$ are spherical harmonics of degree ℓ and order m. Solutions to (2) take the form

$$w(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \beta_\ell^m z_\ell(r) Y_\ell^m(\hat{x}), \quad 0 \leqslant r \leqslant R,\ \hat{x} \in \mathbb{S}^2$$

for unknown constants β_ℓ^m and functions $z_\ell(r)$, where $\lim_{r\to 0} z_\ell(r) < \infty$. The orthogonality of spherical harmonics reduces the defining equation for each $z_\ell(r)$ to

$$z_\ell''(r) + \frac{2}{r}z_\ell'(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)z_\ell(r) = -\frac{\alpha_\ell^m}{\beta_\ell^m}m(r) j_\ell(kr). \tag{9}$$

We proceed by the method of variation of parameters. Let $y_\ell(r)$ be a spherical Bessel function of the second kind and of order ℓ. Then since the set of linearly independent solutions to the homogeneous equation, $\{j_\ell(kr), y_\ell(kr)\}$, has Wronskian equal to $k^{-1}r^{-2}$

$$\begin{aligned} z_\ell(r) = &\left(c_1 + \frac{\alpha_\ell^m}{\beta_\ell^m}\left(\int_0^r k\rho^2 m(\rho) j_\ell(k\rho) y_\ell(k\rho)\,\mathrm{d}\rho\right)\right) j_\ell(kr) \\ &+ \left(c_2 - \frac{\alpha_\ell^m}{\beta_\ell^m}\left(\int_0^r k\rho^2 m(\rho) j_\ell^2(k\rho)\,\mathrm{d}\rho\right)\right) y_\ell(kr). \end{aligned}$$

We can ensure $\lim_{r\to 0} z_\ell(r) < \infty$ by taking $c_2 = 0$. Define f_ℓ and g_ℓ by

$$f_\ell(r) = \int_0^r k\rho^2 m(\rho) j_\ell(k\rho) y_\ell(k\rho)\,\mathrm{d}\rho \quad \text{and} \quad g_\ell(r) = \int_0^r k\rho^2 m(\rho) j_\ell^2(k\rho)\,\mathrm{d}\rho.$$

Using the identity $f_\ell'(R) j_\ell(kR) - g_\ell'(R) y_\ell(kR) = 0$, the boundary conditions (8) imply that there is a nontrivial solution to (6) if and only if

$$d_\ell(k) = \det\begin{pmatrix} j_\ell(kR) & (f_\ell(R) - 1) j_\ell(kR) - g_\ell(R) y_\ell(kR) \\ k j_\ell'(kR) & k(f_\ell(R) - 1) j_\ell'(kR) - k g_\ell(R) y_\ell'(kR) \end{pmatrix} = 0.$$

Expanding this determinant shows that Born transmission eigenvalues correspond to the zeros of

$$d_\ell(k) = -k \int_0^R \rho^2 m(\rho) j_\ell^2(k\rho)\,\mathrm{d}\rho.$$

□

Two examples suggest that the behavior of transmission eigenvalues differs when m changes sign compared to when it does not. In what follows, we always take $R = 1$ for simplicity.

Example 1. We first consider an example where m_1 is strictly positive. Let $m = 1 + \frac{\mathrm{i}}{k}$. We claim that the complex part of the zeros of $d_0(k)$ are unbounded as $|k| \to \infty$. In the case of the full interior transmission problem without the Born approximation, we know of no examples of transmission eigenvalues with unbounded complex part.

Since $j_0(r) = \frac{\sin r}{r}$

$$d_0(k) = -(k + \mathrm{i}) \int_0^1 \frac{\sin^2(k\rho)}{k^2}\,\mathrm{d}\rho = \frac{(k + \mathrm{i})}{2k^2}\left(\frac{\sin(2k)}{2k} - 1\right).$$

Hence, there are no non-zero real solutions to $d_0(k) = 0$. Let $k = x + \mathrm{i}y$ for $x, y \in \mathbb{R}$. Some manipulation of the equation $d_0(k) = 0$ reveals that $d_0(x + \mathrm{i}y) = 0$ if and only if either $k = -\mathrm{i}$ or

$$x = \sin(x)\cosh(y) \qquad \text{and} \qquad y = \cos(x)\sinh(y).$$

Taking the modulus of the first equation shows $|x| \leqslant |\cosh(y)|$, which is possible only if $|y| \to \infty$ as $|x| \to \infty$. We show below that there are infinitely many solutions to $d_0(k) = 0$ which form a discrete set in the complex plane. Hence, there are Born transmission eigenvalues with arbitrarily large imaginary part.

Example 2. When m changes sign in D, the behavior of transmission eigenvalues also changes. For instance, real eigenvalues can exist when m_1 is no longer bounded away from zero. Let $m(r) = 1 - 2r$. Then,

$$d_0(k) = \frac{\sin(k)(\sin(k) - k\cos(k))}{2k^3}.$$

Both $\sin(k) = 0$ and $\sin(k) = k\cos(k)$ have infinitely many solutions, all of which are real. As such there are infinitely many real transmission eigenvalues associated with m. Numerical experiments suggest that $d_\ell(k) = 0$ has both real and complex solutions when $\ell > 0$.

We now turn to a general theory to better explain these examples. We begin by demonstrating that for spherically stratified media, there are always infinitely many transmission eigenvalues.

Theorem 2. *If $D = B_R(0)$ and the contrast is radially dependent then $d_0(k)$ has infinitely many zeros.*

Proof. We claim that $d_0(k)$ is an entire function of k of order at most one and finite type. Indeed, expanding into a Taylor series about $k = 0$

$$d_0(k) = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} \left(k^{2n-1} \int_0^1 m_1(\rho)\rho^{2n}\, \mathrm{d}\rho + \mathrm{i}k^{2n-2} \int_0^1 m_2(\rho)\rho^{2n}\, \mathrm{d}\rho \right).$$

This series converges for all k since $\|m_1\|_{L^\infty}$, $\|m_2\|_{L^\infty}$ are bounded and so $d_0(k)$ is entire. Moreover, it is easily verified that there exists a positive constant A such that

$$\max_{|z|=r} \ |d_0(z)| \leqslant A \cosh 2|z|$$

for r sufficiently large. This corresponds to $d_0(k)$ being of at most order one and type two (see, e.g., [9]).

Assume now to the contrary that $d_0(k)$ has only $0 \leqslant M < \infty$ zeros (not necessarily distinct). Since $d_0(k)$ is an entire function of at most order one and type two, the Hadamard factorization theorem [9] give

$$d_0(k) = k^n \mathrm{e}^{ak+b} \prod_{j=1}^{M} \left(1 - \frac{k}{k_j}\right),$$

where n is an integer, a, b are constants, and k_j, $j = 0, 1, \ldots, M$, are the zeros of $d_0(k)$ (if there are no zeros, the finite product is replaced by the constant one). This factorization implies that $d_0(k)$ does not tend to zero for both $k \to \infty$ and $k \to -\infty$ on the real axis. However

$$d_0(k) = -\int_0^1 k\rho^2 m(\rho) j_0^2(k\rho)\, \mathrm{d}\rho = -\frac{1}{k}\int_0^1 m(\rho)\sin^2(k\rho)\, \mathrm{d}\rho.$$

Since $m = m_1 + \frac{\mathrm{i}}{k} m_2$, $d_0(k) \to 0$ as $k \to \pm\infty$ on the real axis. This contradiction implies that $d_0(k)$ has infinitely many zeros. □

As demonstrated by the above examples, the location in the complex plane of Born transmission eigenvalues appears to depend on whether the contrast m changes sign or not. Two results explain these examples more precisely.

Theorem 3. *Assume $m_1 \in C^1([0, 1])$. If the zeros of $d_0(k)$, $k \in \mathbb{C}_{|z|>\alpha}$, are contained in a horizontal strip of the complex plane then $\int_0^1 m_1(\rho)\, \mathrm{d}\rho = 0$.*

Proof. Let $\alpha = \int_0^1 m_1(\rho)\, \mathrm{d}\rho$, $\beta = \int_0^1 m_2(\rho)\, \mathrm{d}\rho$, and $k = x + \mathrm{i}y$ for $x, y \in \mathbb{R}$. From the identity $\sin(x + \mathrm{i}y) = \sin(x)\cosh(y) + \mathrm{i}\cos(x)\sinh(y)$ we have that for $k \in \mathbb{C}_{|z|>\alpha}$

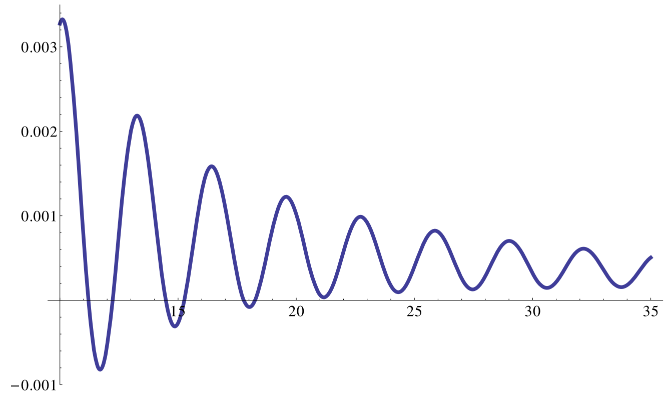

Figure 1. Plot of the function $d_0(k)$ associated with $m = 1 - 1.95r$. When $d_0(k)$ crosses the real axis, k is a transmission eigenvalue.

$$
\begin{aligned}
-2\,\mathrm{Re}(kd_0(k)) &= \alpha - \int_0^1 m_1(\rho)\cos(2x\rho)\cosh(2y\rho)\,\mathrm{d}\rho \\
&- \frac{1}{x^2+y^2}\left(\beta y - \int_0^1 m_2(\rho)(x\sin(2x\rho)\sinh(2y\rho) + y\cos(2x\rho)\cosh(2y\rho))\,\mathrm{d}\rho\right).
\end{aligned} \tag{10}
$$

Assume that the zeros of d_0 lie in a horizontal strip of the complex plane so that if $d_0(x+iy) = 0$ then $y \in [a, b]$ for some $-\infty < a < b < \infty$. From (10), if $y \in [a, b]$ then there is a constant $C > 0$ depending only on a and b so that

$$
|2\,\mathrm{Re}((x+\mathrm{i}y)d_0(x+\mathrm{i}y)) + \alpha| \leqslant \frac{C}{|x|}\left(\int_0^1 |m_1(\rho)|\,\mathrm{d}\rho + |m_1(1)| + \int_0^1 |m_2(\rho)|\,\mathrm{d}\rho\right).
$$

This demonstrates that as k grows on the real axis, $\mathrm{Re}(kd_0(k))$ becomes arbitrarily close to $-\alpha/2$. However, since there exist infinitely many transmission eigenvalues, there exist transmission eigenvalues with arbitrarily large real part and complex part in $[a, b]$ so that $\mathrm{Re}(kd_0(k)) = 0$. Since $\mathrm{Re}(kd_0(k))$ is a continuous function, this is a contradiction unless $\alpha = 0$. □

Theorem 4. *If* $m_2 \equiv 0$, $m_1 \in C^2([0, 1])$, $m_1(1) \neq 0$, *and* $\int_0^1 m_1(r)\,\mathrm{d}r = 0$, *then* $d_0(k)$ *has infinitely many real zeros.*

Proof. Integrating-by-parts twice and using the assumption that $m_1'(\rho)$ and $m_1''(\rho)$ are bounded on $0 \leqslant \rho \leqslant 1$, we have that for $k > 0$

$$
4k^2 d_0(k) = 2k\int_0^1 m_1(\rho)\cos(2k\rho)\,\mathrm{d}\rho = m_1(1)\sin(2k) + O\left(\frac{1}{k}\right).
$$

Therefore, for large-enough $k > 0$, $4k^2 d_0(k)$—and hence $d_0(k)$ itself—has infinitely many real zeros. □

Numerical examples suggest that (possibly finitely many) real zeros exist when $\int_0^1 m_1(\rho)\,\mathrm{d}\rho \neq 0$ as well. As an example, figure 1 shows $d_0(k)$ on the real axis for $m(\rho) = 1 - 1.95\rho$. The intersections of $d_0(k)$ with the real k-axis correspond to the location of real Born transmission eigenvalues.

3. Transmission eigenvalues for general shapes and contrasts

In this section, we return to the case of more general contrasts such that $D = \mathrm{supp}(n - 1)$ is a $\mathcal{C}^1$ domain. Make the change of variables $u \mapsto \frac{1}{k^2}u$ in (2). Then we look for nontrivial solutions $(u, v) \in H_0^2(D) \times L^2(D)$ to

$$\Delta u + k^2 u = -mv \quad \text{in } D, \tag{11}$$

$$\Delta v + k^2 v = 0 \quad \text{in } D, \tag{12}$$

where $k \in \mathbb{C}_{|z|>\alpha}$ and $m(x) = m_1(x) + \frac{\mathrm{i}}{k}m_2$ for real-valued $m_1, m_2 \in L^\infty(D)$. If there exists a nontrivial solution to (11)–(12) then k is a Born transmission eigenvalue as defined in definition 1 and conversely.

We now demonstrate that Born transmission eigenvalues form at most a discrete set in $\mathbb{C}_{|z|>\alpha}$. Here, we adapt to the case of absorbing weak scattering media the approach developed by Kirsch in [16] which revisits [20]. To this end, we must impose restrictions on the contrast in a neighborhood of the boundary ∂D. In what follows, let $\mathcal{N} \subset D$ be a neighborhood of ∂D so that $\partial D \subset \overline{\mathcal{N}}$. We are interested in $m(x)$ which have the same sign for all $x \in \mathcal{N}$. Define

$$m_* = \inf_{x\in\mathcal{N}} m_1(x) \quad \text{and} \quad m^* = \sup_{x\in\mathcal{N}} m_1(x).$$

In the following theorems, we always take $0 < m_* < 1$. However, under the change of variables $u \mapsto -u$, all results also hold for $-1 < m^* < 0$.

Define the Hilbert space $X(D) = H_0^2(D) \times L^2(D)$ equipped with the norm $\|(u, v)\|_{X(D)} = \|u\|_{H^2(D)} + \|v\|_{L^2(D)}$ and corresponding inner product $\langle\cdot, \cdot\rangle_{X(D)}$. The variational form of (11)–(12) is to find $(u, v) \in X(D)$ such that

$$\int_D (\Delta\overline{\varphi} + k^2\overline{\varphi})v\,\mathrm{d}x + \int_D (\Delta u + k^2 u)\overline{\psi} + mv\overline{\psi}\,\mathrm{d}x = 0 \quad \text{for all } (\varphi, \psi) \in X(D).$$

For each $k \in \mathbb{C}_{|z|>\alpha}$, define the sesquilinear form $\mathcal{B}_k : X(D) \times X(D) \to \mathbb{C}$ by

$$\mathcal{B}_k(u, v; \varphi, \psi) := \int_D (\Delta\overline{\varphi} + k^2\overline{\varphi})v\,\mathrm{d}x + \int_D (\Delta u + k^2 u)\overline{\psi} + mv\overline{\psi}\,\mathrm{d}x.$$

From the Riesz representation theorem, there exists a bounded linear operator $B_k : X(D) \to X(D)$ which satisfies

$$\mathcal{B}_k(u, v; \varphi, \psi) = \langle B_k(u, v), (\varphi, \psi)\rangle_{X(D)}.$$

Born transmission eigenvalues are the values $k \in \mathbb{C}_{|z|>\alpha}$ such that $B_k(u, v) = 0$ does not have a unique solution $(u, v) \in X(D)$.

We decompose the above sesquilinear form as $\mathcal{B}_k(u, v; \varphi, \psi) = \mathcal{A}_k(u, v; \varphi, \psi) + \mathcal{C}_k(v, \psi)$, where

$$\mathcal{A}_k(u, v; \varphi, \psi) = \int_D (\Delta\overline{\varphi} + k^2\overline{\varphi})v\,\mathrm{d}x + \int_D (\Delta u + k^2 u)\overline{\psi} + m_1 v\overline{\psi}\,\mathrm{d}x$$

and

$$\mathcal{C}_k(v;\,\psi) = \frac{\mathrm{i}}{k}\int_D m_2 v\overline{\psi}\,\mathrm{d}x.$$

Again by the Riesz representation theorem, there are bounded linear operators $A_k : X(D) \to X(D)$ and $C_k : L^2(D) \to L^2(D)$ so that

$$\mathcal{A}_k(u,\,v;\,\varphi,\,\psi) = \langle A_k(u,\,v),\,(\varphi,\,\psi)\rangle_{X(D)} \quad \text{and} \quad \mathcal{C}_k(v;\,\psi) = \langle C_k(v),\,\psi\rangle_{L^2(D)}.$$

We will make use of the analytic Fredholm theory to determine that if transmission eigenvalues exist then they are discrete in the complex plane.

Theorem 5. *For any $k_1,\,k_2 \in \mathbb{C}$, the operator $A_{k_1} - A_{k_2}$ is compact.*

Proof. We will show that $A_{k_1} - A_{k_2}$ maps weakly convergent sequences in $X(D)$ to strongly convergent sequences in $X(D)$. Assume that $(u_j,\,v_j)$ converges weakly in $X(D)$ to $(0,\,0)$ and let $(\varphi,\,\psi) \in X(D)$. Then

$$(\mathcal{A}_{k_1} - \mathcal{A}_{k_2})(u_j,\,v_j;\,\varphi,\,\psi) = (k_1^2 - k_2^2)\left(\int_D v_j\overline{\varphi} + u_j\overline{\psi}\,\mathrm{d}x\right)\mathrm{d}x.$$

We bound each of these terms separately. First, by the Cauchy–Schwarz inequality

$$\left|\int_D u_j\overline{\psi}\,\mathrm{d}x\right| \leqslant \|u_j\|_{L^2(D)}\,\|\psi\|_{L^2(D)}.$$

Next, define $z_j \in H^1(D)$ to satisfy $\Delta z_j = v_j$ in D and $z_j = 0$ on ∂D. By Green's second identity and the Cauchy–Schwarz inequality

$$\left|\int_D v_j\overline{\varphi}\,\mathrm{d}x\right| = \left|\int_D \Delta z_j\overline{\varphi}\,\mathrm{d}x\right| = \left|\int_D z_j\Delta\overline{\varphi}\,\mathrm{d}x\right| \leqslant \|z_j\|_{L^2(D)}\,\|\varphi\|_{H^2(D)}.$$

Therefore

$$\begin{aligned}\|A_{k_1} - A_{k_2}\|_{X(D)} &= \sup_{\substack{0\neq(\varphi,\psi)\in X(D)\\ |\varphi,\psi|=1}} |(\mathcal{A}_{k_1} - \mathcal{A}_{k_2})(u,\,v;\,\varphi,\,\psi)|\\ &\leqslant C\,(\|u_j\|_{L^2(D)} + \|z_j\|_{L^2(D)}),\end{aligned}$$

where C depends on k_1 and k_2. Since $u_j \rightharpoonup 0$ in $H_0^2(D)$, Rellich's compact embedding theorem gives that $u_j \to 0$ in $L^2(D)$. Moreover, $z_j \rightharpoonup 0$ in $H^1(D)$ so $z_j \to 0$ in $L^2(D)$. Hence, $(A_{k_1} - A_{k_2})(u,\,v)$ converges strongly to zero in $X(D)$ proving compactness of the operator. □

Before continuing, we need a technical lemma from [16].

Lemma 1. *Assume $m \in L^\infty(D)$ is such that either $0 < m_* < 1$ and $m_2(x) = 0$ for all $x \in \mathcal{N}$. Then there exist constants $\gamma > 0$ and $c > 0$ so that for all $k = \mathrm{i}\kappa_0$, $\kappa_0 > 0$*

$$\int_{D\setminus\mathcal{N}} |v|^2\,\mathrm{d}x \leqslant c\mathrm{e}^{-\gamma\kappa_0}\int_{\mathcal{N}} m_1|v|^2\,\mathrm{d}x$$

for all solutions $v \in L^2(D)$ of $\Delta v - \kappa_0^2 v = 0$ in D.

We now prove a Babuška–Brezzi inf-sup condition which will be used to establish that there is a $\kappa_0 \in \mathbb{C}$ so that $(A_{\kappa_0} + C_k)$ is invertible with bounded inverse for all $k \in \mathbb{C}_{|z|>\alpha}$ (see [16]). We say that a sesquilinear form $\mathcal{L} : X(D) \times X(D) \to \mathbb{C}$ satisfies the inf-sup condition if there exists a $c > 0$ so that for all $(u, v) \in X(D)$

$$\sup_{0 \neq (\varphi,\psi) \in X(D)} \frac{|\mathcal{L}(u, v; \varphi, \psi)|}{\|(\varphi, \psi)\|_{X(D)}} \geqslant c\|(u, v)\|_{X(D)}. \tag{13}$$

Theorem 6. *Assume that $m \in L^\infty(D)$ and $0 < m_* < 1$. Then for all $k \in \mathbb{C}_{|z|>\alpha}$:*

(i) If $m_2(x) \equiv 0$ for $x \in \mathcal{N}$ then there is a $\kappa_* > 0$ so that for all $\kappa_0 > \kappa_*$, $\mathcal{A}_{i\kappa_0} + \mathcal{C}_k$ satisfies the inf-sup condition (13).

(ii) If $\operatorname{Im}(k) > 0$ and there is some $\beta > 0$ so that $m_2(x) > \beta > 0$ for $x \in D$ then for every $0 < \kappa_0 < \sqrt{\lambda_0(D)}$, $\mathcal{A}_{\kappa_0} + \mathcal{C}_k$ satisfies the inf-sup condition (13), where $\lambda_0(D) > 0$ is the smallest Dirichlet eigenvalue for the negative Laplacian on the domain D.

Proof. In both cases, we assume by contradiction that there does not exist a constant $c > 0$ such that (13) holds. Then there is a sequence $\{(u_j, v_j)\} \in X(D)$ with $\|(u_j, v_j)\|_{X(D)} = 1$ such that

$$\sup_{0 \neq (\varphi,\psi) \in X(D)} \frac{|(\mathcal{A}_\kappa + \mathcal{C}_k)(u, v; \varphi, \psi)|}{\|(\varphi, \psi)\|_{X(D)}} \to 0 \quad \text{as } j \to \infty, \tag{14}$$

where $\kappa = \mathrm{i}\kappa_0$ in Case 1 and $\kappa = \kappa_0$ for $0 < \kappa_0 < \sqrt{\lambda_0(D)}$ in Case 2. There is a weakly convergent subsequence, still denoted by $\{(u_j, v_j)\}$, such that $u_j \rightharpoonup u$ and $v_j \rightharpoonup v$ for some $(u, v) \in X(D)$. By (14), we have that $\Delta u + \kappa^2 u = -mv$ and $\Delta v + \kappa^2 v = 0$ in D.

- We first prove Case 1. We claim that $(u, v) = (0, 0)$ in D. Indeed

$$0 = \operatorname{Re}((\mathcal{A}_{i\kappa_0} + \mathcal{C}_k)(u, v; -u, v)) = \int_D \left(m_1 + \frac{\operatorname{Im}(k)}{|k|^2} m_2\right)|v|^2 \,\mathrm{d}x. \tag{15}$$

Since $0 < m_* < 1$ and $m_2 \equiv 0$ in $\mathcal{N}$, lemma 1 and equation (15) imply that there is a sufficiently large $\kappa_* \in \mathbb{R}$ so that if $\kappa_0 > \kappa_*$ and $k \in \mathbb{C}_{|z|>\alpha}$ then

$$\begin{aligned} \int_{\mathcal{N}} m_1|v|^2 \,\mathrm{d}x &= \left| \int_{D\setminus\mathcal{N}} \left(m_1(x) + \frac{\operatorname{Im}(k)}{|k|^2} m_2\right)|v|^2 \,\mathrm{d}x \right| \\ &\leqslant \|m\|_{L^\infty(D)} \int_{D\setminus\mathcal{N}} |v|^2 \,\mathrm{d}x \leqslant \frac{1}{2}\int_{\mathcal{N}} m_1|v|^2 \,\mathrm{d}x. \end{aligned}$$

Thus, $v \equiv 0$ in $\mathcal{N}$ and hence by unique continuation $v \equiv 0$ in D. Since $0 = -(\mathcal{A}_{i\kappa_0} + \mathcal{C}_k)(u, 0; 0, u) = \int_D |\nabla u|^2 + \kappa_0^2|u|^2 \,\mathrm{d}x$, we have that $u \equiv 0$ in D as well. We continue by proving a contradiction with $\|(u, v)\|_{X(D)} = 1$. To this end, define $\mathcal{N}'$ to be a neighborhood of ∂D such that $\overline{\mathcal{N}'} \subset \mathcal{N} \cup \partial D$. Choose a nonnegative cutoff function $\eta \in \mathcal{C}^\infty(D)$ so that $\eta = 0$ in $D\setminus\mathcal{N}$ and $\eta = 1$ in $\mathcal{N}'$. Since $\{(\eta u_j, -\eta v_j)\}$ is bounded in $X(D)$ and $m_2 \equiv 0$ in $\mathcal{N}$, assumption (14) yields

$$(\mathcal{A}_{i\kappa_0} + \mathcal{C}_k)(u_j, v_j; \eta u_j, -\eta v_j)$$
$$= \int_{\mathcal{N}} (\Delta(\eta\overline{u}_j) - \kappa_0^2\eta\overline{u}_j)v_j - \eta(\Delta u_j - \kappa_0^2 u_j)\overline{v}_j - m_1\eta|v_j|^2 \, \mathrm{d}x \to 0$$

as $j \to \infty$. Taking the real part of the above

$$\mathrm{Re}\left(\int_{\mathcal{N}} \overline{u}_j v_j \Delta\eta + 2v_j\nabla\eta \cdot \nabla\overline{u}_j - m_1\eta|v_j|^2 \, \mathrm{d}x\right) \to 0.$$

From the compact embedding of $H^2(D)$ in $H^1(D)$, $u_j \rightharpoonup 0$ in $H_0^2(D)$ implies that $\|u_j\|_{H^1(D)} \to 0$ as well. Applying the Cauchy–Schwarz inequality then yields

$$\int_{\mathcal{N}} m_1\eta|v_j|^2 \, \mathrm{d}x \to 0, \qquad \text{as } j \to \infty. \tag{16}$$

Since m_1 is strictly positive in $\mathcal{N}$ and $m_1\eta \geqslant \delta > 0$ in $\mathcal{N}'$ for some $\delta > 0$, we have that $v_j \to 0$ in $L^2(\mathcal{N}')$.

Now pick another neighborhood of ∂D called $\mathcal{N}''$ with $\overline{\mathcal{N}''} \subset \mathcal{N}' \cup \partial D$. Define a new nonnegative cutoff function with the same name, $\eta \in \mathcal{C}^\infty(D)$ with $\eta = 0$ in $\mathcal{N}''$ and $\eta = 1$ in $D\backslash\mathcal{N}'$. There is a $z_j \in H^2(D)$ which is a solution to $\Delta z_j - \kappa_0^2 z_j = v_j$ in D and $z_j = 0$ on ∂D. Since $\{(\eta z_j, 0)\}$ is bounded in $X(D)$, (14) gives

$$(\mathcal{A}_{i\kappa_0} + \mathcal{C}_k)(u_j, v_j; \eta z_j, 0) = \int_{D\backslash\mathcal{N}''} (\Delta(\eta\overline{z}_j) - \kappa_0^2\eta\overline{z}_j)v_j \, \mathrm{d}x$$
$$= \int_{D\backslash\mathcal{N}''} \eta|v_j|^2 + 2v_j\nabla\eta \cdot \nabla\overline{z}_j + v_j\overline{z}_j\Delta\eta \, \mathrm{d}x \to 0$$

as $j \to \infty$. Since $v_j \rightharpoonup 0$ in $L^2(D)$, $z_j \rightharpoonup 0$ in $H^2(D)$ and so $z_j \to 0$ in $H^1(D)$. By applying the Cauchy–Schwarz inequality and using the definition of η, we conclude that $v_j \to 0$ on $L^2(D\backslash\mathcal{N}'')$. Along with the previous argument, $v_j \to 0$ in $L^2(D)$.

Finally, letting $\varphi = 0$ and $\psi = \Delta u_j + k_0^2 u_j$ in (14) yields

$$\frac{1}{\|\Delta u_j - \kappa_0^2 u_j\|_{L^2(D)}}\left(\int_D |u_j - \kappa_0^2 u_j|^2 + m_1 v_j(\Delta\overline{u}_j - \kappa_0^2\overline{u}_j) \, \mathrm{d}x\right)$$
$$= \|\Delta u_j - \kappa_0^2 u_j\|_{L^2(D)} + \int_D m_1 v_j\left(\frac{\Delta\overline{u}_j - \kappa_0^2\overline{u}_j}{\|\Delta u_j - \kappa_0^2 u_j\|_{L^2(D)}}\right)\mathrm{d}x \to 0.$$

Since $v_j \to 0$ in $L^2(D)$, we have from Cauchy–Schwarz that $\Delta u_j - \kappa_0^2 u_j \to 0$ in $L^2(D)$ as well. Since $u_j \rightharpoonup 0$ in $H_0^2(D)$, $u \to 0$ in $L^2(D)$. Hence, $\Delta u_j \to 0$ in $L^2(D)$. As $\|\Delta u_j\|_{L^2(D)}$ is equivalent to $\|u_j\|_{H^2(D)}$ for $u_j \in H_0^2(D)$, we have that $u_j \to 0$ in $H^2(D)$. Therefore, $(u_j, v_j) \to (u, v) = (0, 0)$ in $X(D)$, contradicting the claim that $\|(u, v)\|_{X(D)} = 1$.

- Now consider Case 2. We again first show that $(u, v) = (0, 0)$. Since $\kappa_0 \in \mathbb{R}$,

$$0 = \mathrm{Im}((\mathcal{A}_{\kappa_0} + \mathcal{C}_k))(u, v; u, v) = \frac{1}{\kappa_0}\int m_2|v|^2 \, \mathrm{d}x.$$

Since $m_2(x) > 0$ for all $x \in D$, we conclude that $v \equiv 0$ in D. Next, integrating-by-parts and using the Poincaré inequality for functions in $H_0^2(D)$

$$0 = (\mathcal{A}_\kappa + \mathcal{C}_k)(u, 0; 0, u) = \int_D (\Delta u + \kappa_0^2 u)\overline{u} \, \mathrm{d}x$$
$$= \int_D -|\nabla u|^2 + \kappa_0^2|u|^2 \, \mathrm{d}x \leqslant (\kappa_0^2 - \lambda_0(D))\|u\|^2_{L^2(D)}.$$

The assumption that $0 < \kappa_0 < \sqrt{\lambda_0(D)}$ gives $u = 0$ in D. After replacing $\mathrm{i}\kappa_0$ by κ_0, nearly the same arguments as in the first part of this proof give that $(u_j, v_j) \to (u, v) = (0, 0)$ in $X(D)$ which contradicts the assumption $\|(u, v)\|_{X(D)} = 1$. In particular, (16) is replaced by

$$\int_{\mathcal{N}} \left(m_1 + \frac{\mathrm{Im}(k)}{|k|^2} m_2\right) \eta |v_j|^2 \,\mathrm{d}x \to 0$$

which gives that $v_j \to 0$ in $L^2(\mathcal{N}')$ since for all $\mathrm{Im}(k) > 0$ there is an $\alpha \in \mathbb{R}$ so that $\left(m_1(x) + \frac{\mathrm{Im}(k)}{|k|^2} m_2(x)\right) \geqslant \alpha > 0$ for all $x \in \mathcal{N}$. Moreover, since κ_0 is not a Dirichlet eigenvalue, the equation $\Delta z_j + \kappa_0^2 z_j = v_j$ in D with $z_j = 0$ on ∂D has a solution $z_j \in H^2(D)$. Following the proof from Case 1 gives the desired result. □

To finish, we must show that there are $k \in \mathbb{C}_{|z|>\alpha}$ which are *not* Born transmission eigenvalues.

Theorem 7. *Assume that* $m \in L^\infty(D)$, $0 < m_* < 1$, *and* $m_2(x) \equiv 0$ *for* $x \in \mathcal{N}$. *Then for sufficiently large* $\kappa_0 > 0$ *the operator* $B_{i\kappa_0} : X(D) \to X(D)$ *is invertible with bounded inverse.*

Proof. Let $\kappa_0 > 0$. If $m_2 \equiv 0$, $B_{i\kappa_0} = A_{i\kappa_0}$. By theorem 6, $A_{i\kappa_0}$ is invertible with bounded inverse (see, e.g., [16]) which gives the result. Assume then that m_2 is not identically zero in D. For $0 < \kappa_1 \neq \kappa_0$

$$B_{i\kappa_0} = (A_{i\kappa_1} + C_{i\kappa_0}) + (A_{i\kappa_0} - A_{i\kappa_1})$$

which, by theorems 5 and 6, is an invertible operator plus a compact operator when κ_1 is sufficiently large. By the Fredholm theory, it is sufficient to show that $B_{i\kappa_0}$ is injective. Assume by contradiction that there is a sequence $\kappa_{0,j} \to \infty$ and functions $(u_j, v_j) \in X(D)$ with $\|(u_j, v_j)\|_{X(D)} = 1$ so that $B_{i\kappa_{0,j}}(u_j, v_j) = 0$.

Let $\alpha_j = c\|m\|_{L^\infty(D)} \,\mathrm{e}^{-\gamma\kappa_{0,j}}$, where $c, \gamma > 0$ are the constants in lemma 1. By this lemma

$$\begin{aligned}\int_D m_1 |v_j|^2 \,\mathrm{d}x &\geqslant \int_{\mathcal{N}} m_1 |v_j|^2 \,\mathrm{d}x - \int_{D\setminus\mathcal{N}} |m_1| |v_j|^2 \,\mathrm{d}x \\ &\geqslant (1 - \alpha_j) \int_{\mathcal{N}} m_1 |v_j|^2 \,\mathrm{d}x > 0\end{aligned}$$

since $0 < m_* < 1$.

On the other hand, since $B_{i\kappa_{0,j}}(u_j, v_j) = 0$

$$0 = \mathrm{Re}(\mathcal{B}_{i\kappa_{0,j}}(u_j, v_j; -u_j, v_j)) = \int_D \left(m_1 + \frac{1}{\kappa_{0,j}} m_2\right) |v_j|^2 \,\mathrm{d}x.$$

Since m_2 is not identically zero, this implies that $\int_D m_1 |v_j|^2 \,\mathrm{d}x < 0$ which is a contradiction. □

With this in hand, the analytic Fredholm theory gives the following.

Theorem 8. *Assume that* $m = m_1 + \frac{i}{k}m_2 \in L^\infty(D)$ *for* $k \in \mathbb{C}$ *and real-valued* $m_1, m_2 \in L^\infty(D)$ *with* $m_2(x) \geqslant 0$ *for all* $x \in D$. *Assume further that there is a neighborhood* $\mathcal{N} \subset D$ *of* ∂D *with* $\partial D \subset \overline{\mathcal{N}}$ *so that* $0 < \inf_{x\in\mathcal{N}} m_1(x) < 1$. *If:*

(i) $m_2(x) \equiv 0$ *for* $x \in \mathcal{N}$ *then there is at most a discrete set of Born transmission eigenvalues* $k \in \mathbb{C}_{|z|>\alpha}$ *which do not have an accumulation point with finite modulus.*
(ii) There is some $\beta > 0$ *so that* $m_2(x) > \beta > 0$ *for* $x \in D$ *then there is at most a discrete set of Born transmission eigenvalues* $k \in \mathbb{C}_{|z|>\alpha}$ *with* $\operatorname{Im}(k) > 0$ *which do not have an accumulation point with finite modulus.*

Proof. By theorem 7, if Hypothesis 1 holds and $\kappa_1 > 0$ is sufficiently large then $B_{i\kappa_1}$ is invertible. Moreover, from the arguments in the introduction, if Hypothesis 2 holds then there are no real Born transmission eigenvalues, so if $k \in \mathbb{R}$ then B_k is invertible. As such, there exist $\kappa \in \mathbb{C}_{|z|>\alpha}$ which are not transmission eigenvalues.

Under either hypothesis, theorem 6 gives that for some $\kappa_0 \in \mathbb{C}$ and $k \in \mathbb{C}$, $A_{\kappa_0} + C_k$ is invertible with bounded inverse. In particular, under Hypothesis 1, this is true for any $k \in \mathbb{C}_{|z|>\alpha}$ and for $\kappa_0 > \kappa_* > 0$ with κ_* sufficiently large. Under Hypothesis 2, it is true for $k \in \mathbb{C}_{|z|>\alpha}$ with $\operatorname{Im}(k) > 0$ and $0 < \kappa_0 < \sqrt{\lambda_0(D)}$, where $\lambda_0(D)$ is the smallest Dirichlet eigenvalue for the negative Laplacian on D. Then for all $(u, v) \in X(D)$ and for appropriate k, κ_0

$$(A_{\kappa_0} + C_k)^{-1}B_k(u, v) = (u, v) + (A_{\kappa_0} + C_k)^{-1}(A_k - A_{k_0})(u, v).$$

Applying theorem 5 shows that this is of the form of an identity operator plus a compact operator. Since there are values of k which are not transmission eigenvalues, the analytic Fredholm theory then gives the result. □

Remark 1. In this paper we do not discuss the existence of transmission eigenvalues. The approach taken in [4] provides monotonicity of transmission eigenvalues in terms of the refractive index, but only works for real eigenvalues and one-sign contrast. This is precisely the case when real Born transmission eigenvalues do not exist. In principle the approach of Robbiano [19] can be applied to the Born transmission eigenvalue problem in order to obtain a general spectral theorem for a $\mathcal{C}^\infty$ domain D and contrast m. For constant contrast m (possibly complex valued) the Born transmission eigenvalue problem becomes

$$(\Delta + \lambda)^2 u = 0, \quad u \in H_0^2(D),$$

a quadratic eigenvalue problem independent of the contrast where we let $k^2 \coloneqq \lambda$. The investigation of this interesting eigenvalue problem is the subject of a forthcoming study.

Remark 2. It is rather surprising that there is little correlation between the transmission eigenvalues with and without the Born approximation. For example, without the Born approximation there are an infinite number of real transmission eigenvalues whereas with the Born approximation there are no real eigenvalues. In the case when $n > 1$, the first real transmission eigenvalue is proportional to the L^∞-norm of $(n-1)^{-1}$, and hence it goes to $+\infty$ as the contrast $\|n-1\|_{L^\infty}$ approaches 0. In the case of complex eigenvalues, there also appears to be minimal correlation, although in our opinion the information is too limited to make any quantitative statement.

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