Inverse Problems and Imaging
Volume 1, No. 3, 2007, 443–456

Web site: http://www.aimSciences.org

A VARIATIONAL APPROACH FOR THE SOLUTION OF THE ELECTROMAGNETIC INTERIOR TRANSMISSION PROBLEM FOR ANISOTROPIC MEDIA

Fioralba Cakoni

Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19716-2553, USA

Houssem Haddar

INRIA, Domaine de Voluceau
Rocquencourt, B.P. 105
78153 Le Chesnay Cedex, France

(Communicated by David Colton)

Abstract. The interior transmission problem plays a basic role in the study of inverse scattering problems for inhomogeneous medium. In this paper we study the interior transmission problem for the Maxwell equations in the electromagnetic scattering problem for an anisotropic inhomogeneous object. We use a variational approach which extends the method developed in [15] to the case when the index of refraction is less than one as well as for partially coated scatterers. In addition, we also describe the structure of the transmission eigenvalues.

1. Introduction

The electromagnetic scattering problem for anisotropic media presents difficulties that are not present in the isotropic case. These difficulties are all connected to the fact that the (tensor) index of refraction is not uniquely determined from the scattering data and hence the basis inverse scattering problem to be considered is different from the corresponding isotropic case. In particular, it has been shown that only the support of the inhomogeneous media can be uniquely determined [3] and this fact has led to the problem of deriving reconstruction algorithms to recover the support from the measured scattering data [2], [7], [8], [15]. Central to the derivation of both uniqueness theorems and reconstruction algorithms has been the interior transmission problem and a better understanding of the behavior of solutions to this problem is basic to further developments in the inverse scattering problem for anisotropic media. Since all materials exhibit some degree of anisotropy and many, such as human tissue, to a large degree such problems in inverse scattering are not only of considerable mathematical interest but also of also of central importance in numerous applications.

The known results on the electromagnetic interior transmission problem for anisotropic media are contained in Haddar [15]. In this paper it was shown that

2000 *Mathematics Subject Classification:* Primary: 35R30, 35Q60, 35J40, 78A25.
Key words and phrases: interior transmission problem; anisotropic media.
The research of F. Cakoni was supported in part by the U.S. Air Force Office of Scientific Research under Grant FA9550-05-1-0127.

if the real part of the index of refraction is positive definite and greater then one then in an appropriate function space there exists a unique solution to the interior transmission problem provided the wave number is not a transmission eigenvalue. The crucial question of whether or not the set of transmission eigenvalues is discrete was not answered nor was the situation considered when the real part of the index of refraction is less then one. The class of problems when the anisotropic media is partially coated by a thin highly conducting layer was also not considered, although such problems arise in various areas of application [10]. The purpose of this paper is to continue where Haddar [15] left off and to address the above problems.

In the next section of our paper we formulate the direct scattering problem for time-harmonic electromagnetic waves in an anisotropic medium including the case when the scattering object may be coated by a thin conducting layer. We then show how the interior transmission problem arises when one considers the corresponding inverse scattering problem from the far field data. In Section 3 we consider the interior transmission problem in a homogenous background and develop a variational approach when the index of refraction is greater or less then one and show that this variational approach leads to a Fredholm equation. Then in both cases we show that the set of transmission eigenvalues is discrete. In Section 4 we extend our approach to the case of anisotropic scattering objects partially coated by a thin conducting layer. In this case additional difficulties arise due to need to consider non-standard function spaces associated with the conducting boundary conditions. We conclude our paper by briefly considering the case when the scattering object is situated in a known (possibly anisotropic) inhomogenous background.

The analysis of the interior transmission problem is more complicated than the analysis for the Helmholtz equation [2, 4] due to the lack of compactness properties. Earlier results on this problem in an isotopic medium under stronger assumptions on the regularity of the index of refraction can be found in [9] and [14]. We also note the recent paper by Kirsch [12] where he considers the interior transmission problem for Maxwell's equations in the isotropic case by using an integral equation approach and assuming that the relative permeability and permeability are greater than one. The results presented in our paper are optimal in the sense that in general the interior transmission problem has only L^2 solutions.

2. Inverse scattering and interior transmission problem

We formulate here the direct scattering problem for electromagnetic waves and a corresponding inverse problem which lead to the interior transmission problem considered in this paper (see [11] for more about the role of the interior transmission problem in inverse scattering). Let $D \subset \mathbb{R}^3$ be a bounded open set having a Lipshitz boundary ∂D such that the exterior domain $D_e := \mathbb{R}^3 \setminus \overline{D}$ is connected. The unit normal vector to ∂D directed into the exterior of D is denoted by ν. We assume that the boundary $\partial D = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ is split in two open disjoint parts Γ_1 and Γ_2. The domain D is the support of an anisotropic (possibly disconnected object) that is partially coated on a portion Γ_2 of the boundary by a very thin layer of a highly conductive material and the incident field is a time-harmonic electromagnetic plane wave with frequency ω (Γ_2 may be the empty set!). The exterior electric and magnetic fields $\tilde{\mathbf{E}}^{ext}$, $\tilde{\mathbf{H}}^{ext}$ and the interior electric and magnetic fields $\tilde{\mathbf{E}}^{int}$, $\tilde{\mathbf{H}}^{int}$, satisfy

$$(1) \qquad \begin{cases} \operatorname{curl}\tilde{\mathbf{E}}^{ext} - i\omega\mu_0\tilde{\mathbf{H}}^{ext} = 0 \\ \operatorname{curl}\tilde{\mathbf{H}}^{ext} + i\omega\epsilon_0\tilde{\mathbf{E}}^{ext} = 0 \end{cases} \quad \text{in } D_e$$

$$(2) \qquad \begin{cases} \operatorname{curl}\tilde{\mathbf{E}}^{int} - i\omega\mu_0\tilde{\mathbf{H}}^{int} = 0 \\ \operatorname{curl}\tilde{\mathbf{H}}^{int} + (i\omega\epsilon(x) - \sigma(x))\tilde{\mathbf{E}}^{int} = 0 \end{cases} \quad \text{in } D$$

and on the boundary ∂D

$$(3) \qquad \tilde{\mathbf{E}}^{ext} \times \nu - \tilde{\mathbf{E}}^{int} \times \nu = 0 \quad \text{on} \quad \partial D$$

$$(4) \qquad \tilde{\mathbf{H}}^{ext} \times \nu - \tilde{\mathbf{H}}^{int} \times \nu = 0 \quad \text{on} \quad \Gamma_1$$

$$(5) \qquad \tilde{\mathbf{H}}^{ext} \times \nu - \tilde{\mathbf{H}}^{int} \times \nu = \tilde{\eta}(\mathbf{x})\nu \times (\tilde{\mathbf{E}}^{ext} \times \nu) \quad \text{on} \quad \Gamma_2.$$

The electric permittivity ϵ_0 and magnetic permeability μ_0 of the exterior dielectric medium are positive constants whereas the scatterer has the same magnetic permeability μ_0 as the exterior medium but the electric permittivity ϵ and conductivity σ are real 3×3 matrix valued functions. The function $\tilde{\eta} > 0$, defined on the portion Γ_2 of the boundary, describes the physical properties of the thin coating layer [1] and ω denotes the frequency. If we define $\tilde{\mathbf{E}}^{(ext,int)} = \frac{1}{\sqrt{\epsilon_0}}\mathbf{E}^{(ext,int)}$, $\tilde{\mathbf{H}}^{(ext,int)} = \frac{1}{\sqrt{\mu_0}}\mathbf{H}^{(ext,int)}$, $k^2 = \epsilon_0\mu_0\omega^2$, $N(\mathbf{x}) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right)$, and $\eta(x) = \sqrt{\frac{\mu_0}{\epsilon_0}}\,\tilde{\eta}(\mathbf{x})$ and express $\mathbf{H}$-fields in terms of $\mathbf{E}$-fields we obtain the transmission problem

$$(6) \qquad \begin{aligned} &\operatorname{curl}\operatorname{curl}\mathbf{E}^{ext} - k^2\mathbf{E}^{ext} = 0 && \text{in} \quad D_e \\ &\operatorname{curl}\operatorname{curl}\mathbf{E}^{int} - k^2 N(\mathbf{x})\mathbf{E}^{int} = 0 && \text{in} \quad D \\ &\mathbf{E}^{ext} \times \nu - \mathbf{E}^{int} \times \nu = 0 && \text{on} \quad \partial D \\ &\operatorname{curl}\mathbf{E}^{ext} \times \nu - \operatorname{curl}\mathbf{E}^{int} \times \nu = 0 && \text{on} \quad \Gamma_1 \\ &\operatorname{curl}\mathbf{E}^{ext} \times \nu - \operatorname{curl}\mathbf{E}^{int} \times \nu - ik\eta(\mathbf{x})\,\nu \times (\mathbf{E}^{ext} \times \nu) = 0 && \text{on} \quad \Gamma_2 \end{aligned}$$

where

$$\mathbf{E}^{ext} = \mathbf{E}^s + \mathbf{E}^i,$$

the scattered field $\mathbf{E}^s$ satisfies the Silver Müller radiation condition

$$(7) \qquad \lim_{r \to \infty}(\operatorname{curl}\mathbf{E}^s \times \mathbf{x} - ikr\mathbf{E}^s) = 0$$

uniformly in $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, $r = |\mathbf{x}|$ and the incident field $\mathbf{E}^i$ is (for sake of presentation) the electric field of time harmonic electromagnetic plane waves given by

$$\mathbf{E}^i(\mathbf{x}) := \frac{i}{k}\operatorname{curl}\operatorname{curl}\mathbf{p}e^{ik\mathbf{x}\cdot\mathbf{d}}$$

where $\mathbf{d}$ is a unit vector giving the direction of propagation and $\mathbf{p}$ is the polarization vector. The scattered electric field $\mathbf{E}^s$ has the asymptotic behavior [9]

$$\mathbf{E}^s(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\left\{\mathbf{E}_\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{p}) + O\left(\frac{1}{|\mathbf{x}|}\right)\right\}$$

as $|\mathbf{x}| \to \infty$, where $\mathbf{E}_\infty$ is a tangential vector field defined on the unit sphere Ω and known as the *electric far field pattern.* The *inverse scattering problem* is to determine D and η from a knowledge of $\mathbf{E}_\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{p})$ for $\hat{\mathbf{x}} \in \Omega_0 \subset \Omega$, $\mathbf{d} \in \Omega_1 \subset \Omega$ and three linearly independent polarization. The uniqueness of this inverse problem is proven in [3] (we remind the reader that N is not uniquely determined by the given data). A solution method, the so-called the linear sampling method, for solving this inverse problem [15], is based on the study of the far field equation

$$(8)\quad (\mathcal{F}\mathbf{g})(\hat{\mathbf{x}}) := \int_{\Omega_1} \mathbf{E}_\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{g}(\mathbf{d}))ds(\mathbf{d}) = \mathbf{E}_{e,\infty}(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{q}), \quad \mathbf{g} \in L^2_t(\Omega), \quad \hat{x} \in \Omega_0,$$

where

$$\mathbf{E}_{e,\infty}(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{q}) = \frac{ik}{4\pi}(\hat{\mathbf{x}} \times \mathbf{q}) \times \hat{\mathbf{x}}\, e^{-ik\hat{\mathbf{x}}\cdot\mathbf{z}}$$

is the far field of the electric dipole $\mathbf{E}_e := \mathbf{E}_e(\cdot, \mathbf{z}, \mathbf{q})$ given by

$$\mathbf{E}_e := \frac{i}{k}\mathrm{curl}_x\, \mathrm{curl}_x\, q\, \frac{1}{4\pi}\frac{e^{ik|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|}.$$

It is easily verified that (8) is solvable if and only if $\mathbf{z} \in D$ and $\mathbf{E}$ and $\mathbf{E}_0$ solve the *interior transmission problem*

$$(9)\qquad \begin{cases} \mathrm{curl}\,\mathrm{curl}\,\mathbf{E} - k^2\, N\,\mathbf{E} = 0 & \text{in} \quad D \\ \mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_0 - k^2\,\mathbf{E}_0 = 0 & \text{in} \quad D \end{cases}$$

$$(10)\qquad \begin{cases} \mathbf{E} \times \nu - (\mathbf{E}_0 + \mathbf{E}_e) \times \nu = 0 & \text{on} \quad \partial D \\ \mathrm{curl}\,\mathbf{E} \times \nu - \mathrm{curl}\,(\mathbf{E}_0 + \mathbf{E}_e) \times \nu = 0 & \text{on} \quad \Gamma_1 \\ \mathrm{curl}\,\mathbf{E} \times \nu - \mathrm{curl}\,(\mathbf{E}_0 + \mathbf{E}_e) \times \nu - ik\eta\nu \times [(\mathbf{E}_0 + \mathbf{E}_e) \times \nu] = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

such that $\mathbf{E}_0$ is a Herglotz function i.e. a solution $\mathbf{E}_g$ of the Maxwell equations of the form

$$\mathbf{E}_g(\mathbf{x}) = \int_\Omega \mathbf{g}(\mathbf{d})e^{ik\mathbf{x}\cdot\mathbf{d}}ds(\mathbf{d}).$$

In particular, the far field operator $(\mathcal{F}\mathbf{g})$ in (8) is injective if and only if the interior transmission problem (9)-(10) with $\mathbf{E}_e = 0$ has the only trivial solution. Hence the study of (9)-(10) is essential for solving the inverse problem.

3. Interior transmission problem

3.1. The case $\eta = 0$. In this section we consider the interior transmission problem (9)-(10) with $\Gamma_2 = \emptyset$, which is related to the scattering problem for an (uncoated) anisotropic inhomogeneity. In this section we complement the results of [15] by considering the case of $N < I$ (where I denotes the identity matrix) and prove in all cases that transmission eigenvalues form a discrete set.

We assume here that $D \subset \mathbb{R}^3$ is a Lipshitz bounded domain with a unit outward normal denoted by ν. We denote by $(\cdot, \cdot)_D$ the $L^2(D)$ scalar product and consider the Hilbert spaces

$$H(\mathrm{curl}\,, D) := \{\mathbf{u} \in L^2(D)^3 :\ \mathrm{curl}\,\mathbf{u} \in L^2(D)^3\},$$
$$H_0(\mathrm{curl}\,, D) := \{\mathbf{u} \in H(\mathrm{curl}\,, D) :\ \mathbf{u} \times \nu = 0 \text{ on } \partial D\},$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathrm{curl}} = (\mathbf{u}, \mathbf{v})_D + (\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v})_D$ and the corresponding norm $\|\cdot\|_{\mathrm{curl}}$, and define

$$\mathcal{U}(D) := \{\mathbf{u} \in H(\mathrm{curl}\,, D) :\ \mathrm{curl}\,\mathbf{u} \in H(\mathrm{curl}\,, D)\},$$
$$\mathcal{U}_0(D) := \{\mathbf{u} \in H_0(\mathrm{curl}\,, D) :\ \mathrm{curl}\,\mathbf{u} \in H_0(\mathrm{curl}\,, D)\},$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathcal{U}} = (\mathbf{u}, \mathbf{v})_{\mathrm{curl}} + (\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v})_{\mathrm{curl}}$ and the corresponding norm $\|\cdot\|_{\mathcal{U}}$. We recall that $C_0^\infty(D)$ is dense in $\mathcal{U}_0(D)$ (see Appendix of [15]).

Let $\mathbf{F}$ and $\mathbf{F}_0$ be two vector valued functions on D and $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ two tangential vector fields on ∂D. Then, more generally, the *interior transmission problem* (ITP) is formulated as the problem of finding two vector valued functions $\mathbf{E}$ and $\mathbf{E}_0$ such that

$$\begin{cases} (i) & \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\, N\,\mathbf{E} = \mathbf{F} \quad \text{in } D, \\ (ii) & \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\,\mathbf{E}_0 = \mathbf{F}_0 \quad \text{in } D, \end{cases} \tag{11}$$

$$\begin{cases} (\mathbf{E}-\mathbf{E}_0)\times\nu = \boldsymbol{\varphi} & \text{on } \partial D, \\ (\operatorname{curl}\mathbf{E} - \operatorname{curl}\mathbf{E}_0)\times\nu = \boldsymbol{\psi} & \text{on } \partial D. \end{cases} \tag{12}$$

The existence of solutions to this problem will be studied for data that satisfy the following assumption

Assumption 3.1. *The data* $\mathbf{F}$*,* $\mathbf{F}_0$*,* $\boldsymbol{\varphi}$ *and* $\boldsymbol{\psi}$ *are such that:*

(i) $\mathbf{F}$ *and* $\mathbf{F}_0$ *are in* $L^2(D)^3$*.*
(ii) $\boldsymbol{\varphi}$ *and* $\boldsymbol{\psi}$ *tangential functions defined on* ∂D *such that there exists a function* $\mathbf{w}$ *in* $\mathcal{U}(D)$ *such that*

$$\mathbf{w}\times\nu = \boldsymbol{\varphi} \text{ and } (\operatorname{curl}\mathbf{w})\times\nu = \boldsymbol{\psi} \text{ on } \partial D.$$

Let us denote by $Y(\partial D)$ the set of $(\boldsymbol{\varphi},\boldsymbol{\psi})$ satisfying (ii) equipped with the norm

$$\|(\boldsymbol{\varphi},\boldsymbol{\psi})\|_{Y(\partial D)} := \inf_{\mathbf{w} \text{ as in } (ii)} \|\mathbf{w}\|_{\mathcal{U}(D)}.$$

It is proved in [15] that if ∂D is a C^3 boundary then $H_t^{3/2}(\partial D)\times H_t^{1/2}(\partial D)$ is continuously embedded into $Y(\partial D)$, where $H_t^{3/2}(\partial D)$ and $H_t^{1/2}(\partial D)$ are the spaces of tangential vectors that component-wise are $H^{3/2}(\partial D)$ and $H^{1/2}(\partial D)$, respectively. In the applications to inverse problems $\mathbf{w}$ can be easily constructed from the fundamental solution $\mathbf{E}_e$ and a suitable cut-off function.

Definition 3.1. A strong solution to (ITP) is a pair $(\mathbf{E},\mathbf{E}_0)\in L^2(D)^3$ that satisfies (11) in the sense of distributions such that $\mathbf{E}-\mathbf{E}_0\in\mathcal{U}(D)$, and $\mathbf{E}-\mathbf{E}_0$ satisfies (12).

We remark that the solutions to this problem do not belong to $H(\operatorname{curl}, D)$ in general. Example of such solutions can be easily constructed by taking

$$\mathbf{E} = \mathbf{E}_0 = \mathbf{h}$$

where $\mathbf{h}$ is a function of $L^2(D)^3$ such that $\operatorname{curl}\operatorname{curl}\mathbf{h} = 0$ in D and $\operatorname{curl}\mathbf{h}\notin L^2(D)^3$. In cylindrical coordinates (r,θ,z) and for D a bounded domain where the z axis is tangent to ∂D and do not intersect D, one can take

$$\mathbf{h}(r,\theta,z) = r^{-\alpha}\cos(\alpha\,\theta)\mathbf{e}_z$$

with $0<\alpha<1$, where $\mathbf{e}_z$ denotes a vector in the z direction.

In addition to the study of the existence and uniqueness of solutions to (ITP) we also describe the set of wave numbers k for which (ITP) may not have unique solutions.

Definition 3.2. A wave number $k>0$ is said to be a transmission eigenvalue if (ITP) has nontrivial strong solutions when $\mathbf{F}=\mathbf{F}_0=0$ and $\boldsymbol{\psi}=\boldsymbol{\varphi}=0$.

Note that in the cases where $\sigma \neq 0$ (see Section 1) $\Im(N)$ depends on k as well.

To study the existence and uniqueness of solutions to (ITP), we rewrite (11-12) as a fourth order boundary value problem. For that purpose we need to assume that $N - I$ is invertible a.e. in D.

Setting

$$\mathbf{u} = \mathbf{E} - \mathbf{E}_0, \quad \mathbf{v} = N\mathbf{E} - \mathbf{E}_0, \tag{13}$$

we obtain that

$$\mathbf{E} = (N - I)^{-1}(\mathbf{v} - \mathbf{u}), \quad \mathbf{E}_0 = (N - I)^{-1}(N\mathbf{u} - \mathbf{v}). \tag{14}$$

Taking the difference between two equations in (11) we get

$$\operatorname{curl}\operatorname{curl}\mathbf{u} = k^2\mathbf{v} + (\mathbf{F} - \mathbf{F}_0) \ \text{ in } \ D. \tag{15}$$

In particular,

$$\mathbf{E} = (N - I)^{-1}(k^{-2}(\operatorname{curl}\operatorname{curl}\mathbf{u} - (\mathbf{F} - \mathbf{F}_0)) - \mathbf{u}). \tag{16}$$

Substituting for $\mathbf{E}$ in (11) one obtains the following fourth order partial differential equation satisfied by $\mathbf{u}$

$$\begin{aligned}(\operatorname{curl}\operatorname{curl} - k^2N)(N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}) = \\ \operatorname{curl}\operatorname{curl}(N - I)^{-1}(\mathbf{F} - \mathbf{F}_0) + k^2(N - I)^{-1}(N\mathbf{F}_0 - \mathbf{F})) \quad \text{in } \ D.\end{aligned} \tag{17}$$

In addition from (12), one obtains that

$$\mathbf{u} \times \nu = \boldsymbol{\varphi}, \ \ (\operatorname{curl}\mathbf{u}) \times \nu = \boldsymbol{\psi} \ \text{ on } \ \partial D. \tag{18}$$

Hence, based on (13-15) we can state the following result.

Theorem 3.1. *Assume that $(N - I)^{-1}$ is a bounded matrix field in D and that the data satisfies Assumption 3.1. Then the existence and uniqueness of strong solutions to (ITP) is equivalent to the existence and uniqueness of $\mathbf{u} \in \mathcal{U}(D)$ and $\mathbf{v} \in L^2(D)^3$ satisfying (15) and (17-18).*

Variational formulations. The study of (17-18) will be done using a variational framework. Using the denseness in $\mathcal{U}_0(D)$ of regular functions with compact support in D (see [15]), one can easily see that $\mathbf{u} \in \mathcal{U}(D)$ satisfies (17) if and only if

$$\begin{aligned}&\left((N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}), (\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2\bar{N}\mathbf{u}')\right)_D \\ &\quad = \left((N - I)^{-1}(\mathbf{F} - \mathbf{F}_0), (\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2\mathbf{u}')\right)_D + k^2\left(\mathbf{F}_0, \mathbf{u}'\right)_D\end{aligned} \tag{19}$$

for all $\mathbf{u}' \in \mathcal{U}_0(D)$. Let us set

$$\ell(\mathbf{u}') = \left((N - I)^{-1}(\mathbf{F} - \mathbf{F}_0), (\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2\mathbf{u}')\right)_D + k^2\left(\mathbf{F}_0, \mathbf{u}'\right)_D$$

which defines an antilinear form on $\mathcal{U}(D)$. Using the identity $N(N - I)^{-1} = I + (N - I)^{-1}$, one can rewrite (19) in one the following equivalent forms

$$\mathcal{A}_k(\mathbf{u}, \mathbf{u}') - k^2\mathcal{B}(\mathbf{u}, \mathbf{u}') = \ell(\mathbf{u}') \ \text{ for all } \mathbf{u}' \in \mathcal{U}_0(D), \tag{20}$$

or

$$-\tilde{\mathcal{A}}_k(\mathbf{u}, \mathbf{u}') + k^2\mathcal{B}(\mathbf{u}, \mathbf{u}') = \ell(\mathbf{u}') \ \text{ for all } \mathbf{u}' \in \mathcal{U}_0(D), \tag{21}$$

where $\mathcal{A}_k$, $\tilde{\mathcal{A}}_k$ and $\mathcal{B}$ are sesquilinear forms on $\mathcal{U}(D) \times \mathcal{U}(D)$, respectively defined by

$$\mathcal{A}_k(\mathbf{u}, \mathbf{u}') = \left((N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}),\, (\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2\mathbf{u}')\right)_D + k^4\,(\mathbf{u},\, \mathbf{u}')_D$$

$$\tilde{\mathcal{A}}_k(\mathbf{u}, \mathbf{u}') = \left((I-N)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2 N\mathbf{u}),\, (\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2\bar{N}\mathbf{u}')\right)_D + k^4\,(N\mathbf{u},\, \mathbf{u}')_D \tag{22}$$

and

$$\mathcal{B}(\mathbf{u}, \mathbf{u}') = (\operatorname{curl}\mathbf{u},\, \operatorname{curl}\mathbf{u}')_D\,, \tag{23}$$

where the expression of $\mathcal{B}$ is obtained after using the identity

$$(\operatorname{curl}\operatorname{curl}\mathbf{u},\, \mathbf{u}')_D = (\operatorname{curl}\mathbf{u},\, \operatorname{curl}\mathbf{u}')_D$$

for all $(\mathbf{u}, \mathbf{u}') \in \mathcal{U}(D) \times \mathcal{U}_0(D)$.

Our goal now is to establish the existence and uniqueness of $\mathbf{u} \in \mathcal{U}(D)$ that satisfies (19) and (18) by proving that (20) and (21) form a Fredholm set of equations given suitable assumptions on N. For the study of (21) it is more convenient to use the following equivalent expression of $\tilde{\mathcal{A}}_k$:

$$\begin{aligned}\tilde{\mathcal{A}}_k(\mathbf{u}, \mathbf{u}') \;=\; & \left(N(I-N)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}),\, (\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2\mathbf{u}')\right)_D \\ & + (\operatorname{curl}\operatorname{curl}\mathbf{u},\, \operatorname{curl}\operatorname{curl}\mathbf{u}')_D\end{aligned} \tag{24}$$

Lemma 3.1. *Assume that there exists a constant $\gamma > 0$ such that,*

$$\Re(N(I-N)^{-1}\boldsymbol{\xi}, \boldsymbol{\xi}) \geq \gamma\,|\boldsymbol{\xi}|^2, \;\; \forall \boldsymbol{\xi} \in \mathbb{C}^3 \quad \text{and a.e. in } D, \tag{25}$$

$$(\textit{respectively,} \quad \Re((N-I)^{-1}\boldsymbol{\xi}, \boldsymbol{\xi}) \geq \gamma\,|\boldsymbol{\xi}|^2, \;\; \forall \boldsymbol{\xi} \in \mathbb{C}^3 \quad \text{and a.e. in } D). \tag{26}$$

Then $\tilde{\mathcal{A}}_k$ (respectively $\mathcal{A}_k$) is a coercive sesquilinear form on $\mathcal{U}_0(D) \times \mathcal{U}_0(D)$.

Proof. Let us prove first the result for $\tilde{\mathcal{A}}_k$. Using (25) and (24) yields

$$\Re(\tilde{\mathcal{A}}_k(\mathbf{u}_0, \mathbf{u}_0)) \geq \gamma \left\|\operatorname{curl}\operatorname{curl}\mathbf{u}_0 - k^2\mathbf{u}_0\right\|^2_{L^2(D)} + \left\|\operatorname{curl}\operatorname{curl}\mathbf{u}_0\right\|^2_{L^2(D)}.$$

Setting $X = \|\operatorname{curl}\operatorname{curl}\mathbf{u}_0\|_{L^2(D)}$ and $Y = k^2\,\|\mathbf{u}_0\|_{L^2(D)}$, one has

$$\left\|\operatorname{curl}\operatorname{curl}\mathbf{u}_0 - k^2\mathbf{u}_0\right\|^2_{L^2(D)} \geq X^2 - 2XY + Y^2$$

and therefore

$$\Re(\tilde{\mathcal{A}}_k(\mathbf{u}_0, \mathbf{u}_0)) \geq (1+\gamma)X^2 - 2\gamma XY + \gamma Y^2. \tag{27}$$

Using the identity

$$(1+\gamma)X^2 - 2\gamma XY + \gamma Y^2 = (\gamma + \tfrac{1}{2})\left(X - \frac{\gamma}{\gamma+\frac{1}{2}}Y\right)^2 + \tfrac{1}{2}X^2 + \frac{\gamma}{1+2\gamma}Y^2,$$

one concludes that

$$\Re(\tilde{\mathcal{A}}_k(\mathbf{u}_0, \mathbf{u}_0)) \geq \frac{\gamma}{1+2\gamma}\left(X^2 + Y^2\right). \tag{28}$$

Integrating by parts, one has the following equality valid for $\mathbf{u}_0 \in \mathcal{U}_0(D)$:

$$\left\|\operatorname{curl}\operatorname{curl}\mathbf{u}_0 - k^2\mathbf{u}_0\right\|^2_{L^2(D)} = \|\operatorname{curl}\operatorname{curl}\mathbf{u}_0\|^2_{L^2(D)} - 2k^2\,\|\operatorname{curl}\mathbf{u}_0\|^2_{L^2(D)} + k^4\,\|\mathbf{u}_0\|^2_{L^2(D)} \tag{29}$$

Therefore

$$2k^2\,\|\operatorname{curl}\mathbf{u}_0\|^2_{L^2(D)} \leq X^2 + Y^2,$$

which combined with (28) yields the existence of a constant c_k0 (independent of $\mathbf{u}_0$ and γ) such that

$$|\tilde{\mathcal{A}}_k(\mathbf{u}_0, \mathbf{u}_0)| \geq c_k \frac{\gamma}{1+2\gamma} \|\mathbf{u}_0\|^2_{\mathcal{U}}. \tag{30}$$

The sesquilinear form $\mathcal{A}_k$ also satisfies (30) under condition (26), since (as one can easily check)

$$\Re(\mathcal{A}_k(\mathbf{u}_0, \mathbf{u}_0)) \geq \gamma X^2 - 2\gamma XY + (\gamma+1)Y^2$$

(see also [15]). □

Based on the Riesz representation theorem let us define the operator $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ by

$$(B\mathbf{u}_0, \mathbf{u}')_{\mathcal{U}} = \mathcal{B}(\mathbf{u}_0\,, \mathbf{u}') \ \ \forall\, \mathbf{u}' \in \mathcal{U}_0(D).$$

As shown in [15], $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is a compact operator. This result is also a special case of Lemma 3.4 proved in next subsection.

Based on this result and Lemma 3.1 we are in position to prove the first main theorem of this section.We first need to make precise the definition of bounded positive definite matrix fields.

Definition 3.3. A matrix field K is said to be bounded positive definite on D if $K \in L^\infty(D, \mathbb{C}^3)^{3\times 3}$ and if there exists a constant $\gamma > 0$ such that

$$\Re(K\boldsymbol{\xi}, \boldsymbol{\xi}) \geq \gamma\, |\boldsymbol{\xi}|^2, \ \ \forall \boldsymbol{\xi} \in \mathbb{C}^3 \quad \text{and a.e. in } D. \tag{31}$$

Theorem 3.2. *Assume that $(N-I)^{-1}$ or $N(I-N)^{-1}$ is a bounded positive definite matrix field on D and that k is not a transmission eigenvalue. Then for all data $(\mathbf{F}, \mathbf{F}_0, \boldsymbol{\varphi}, \boldsymbol{\psi})$ satisfying Assumption 3.1 there exists a unique solution $\mathbf{u} \in \mathcal{U}(D)$ to (18)-(19) such that*

$$\|\mathbf{u}\|_{\mathcal{U}(D)} \leq C(\|\mathbf{F}\|_{L^2(D)} + \|\mathbf{F}_0\|_{L^2(D)} + \|(\boldsymbol{\varphi}, \boldsymbol{\psi})\|_{Y(\partial D)},$$

where $C > 0$ is a constant independent of $\mathbf{u}$ and $(\mathbf{F}, \mathbf{F}_0, \boldsymbol{\varphi}, \boldsymbol{\psi})$.

Proof. Let us first prove this theorem in the case where $N(I-N)^{-1}$ is a bounded positive definite matrix field on D. In this case, one can easily see that $\tilde{\mathcal{A}}_k$ is a continuous sesquilinear form on $\mathcal{U}(D) \times \mathcal{U}(D)$. Based on the Riesz representation theorem, one can therefore define continuous operator $\tilde{A}_k : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ such that

$$(\tilde{A}_k\mathbf{u}_0, \mathbf{u}')_{\mathcal{U}} = \tilde{\mathcal{A}}_k(\mathbf{u}_0\,, \mathbf{u}') \ \ \forall\, \mathbf{u}' \in \mathcal{U}_0(D).$$

Lemma 3.1 and the Lax-Milgram theorem prove that $\tilde{A}_k : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is a bijective operator. The identity $(N-I)^{-1} = N(N-I)^{-1} - I$ implies that the antilinear form ℓ is continuous on $\mathcal{U}_0(D)$. We denote by $\boldsymbol{\ell} \in \mathcal{U}_0(D)$ the Riesz representative of ℓ in $\mathcal{U}_0(D)$. Let $\mathbf{w}$ be as in Assumption 3.1 and define $\mathbf{t}$ such that

$$-\tilde{\mathcal{A}}_k(\mathbf{w}\,, \mathbf{u}') + k^2\mathcal{B}(\mathbf{u}_0\,, \mathbf{u}') = (\mathbf{t}, \mathbf{u}')_{\mathcal{U}} \ \ \forall\, \mathbf{u}' \in \mathcal{U}_0(D).$$

Then (18)-(19) is equivalent to $\mathbf{u} = \mathbf{w} + \mathbf{u}_0$ where $\mathbf{u}_0 \in \mathcal{U}_0(D)$ is the solution of

$$-\tilde{A}_k\mathbf{u}_0 + k^2 B\mathbf{u}_0 = \mathbf{t} + \boldsymbol{\ell} \quad \text{in } \mathcal{U}_0(D). \tag{32}$$

Since $\tilde{A}_k$ is an isomorphism and B is compact, the Fredholm alternative can be applied to (32). Hence, assuming that k is not a transmission eigenvalue implies the existence and uniqueness of a solution $\mathbf{u}_0$ to (32) satisfying the a priori estimate.

The proof in the case $(N-I)^{-1}$ is a bounded positive definite matrix carries over in the same way by replacing $-\tilde{A}_k + k^2 B$ by $A_k + k^2 B$ where $\tilde{A}_k : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is defined by

$$(A_k \mathbf{u}_0, \mathbf{u}')_{\mathcal{U}} = \mathcal{A}_k(\mathbf{u}_0\,, \mathbf{u}') \;\; \forall\, \mathbf{u}' \in \mathcal{U}_0(D). \tag{33}$$

□

Theorem 3.3. *Assume that $(N-I)^{-1}$ or $N(I-N)^{-1}$ is a bounded positive definite matrix field on D. Then*

(i) *The set of transmission eigenvalues is discrete and does not accumulate at* 0.
(ii) *If $\Im(N\boldsymbol{\xi}, \boldsymbol{\xi}) > 0, \;\; \forall\, \boldsymbol{\xi} \in \mathbb{C}^3 \setminus \{0\}$ and a.e. in D then the set of transmission eigenvalues is empty.*

Proof. The proof of part (i) is based on the use of the analytic Fredholm theory. For sake of presentation we consider only the case when $(N-I)^{-1}$ is a bounded and positive definite, and therefore use formulation (20). We first prove that A_k^{-1} is analytic for $k \in \mathbb{C}$ in a neighborhood of the positive real axis, where A_k is defined by (33). Let $k_0 > 0$. Then there exists a positive constant C independent of k such that

$$\|(A_k - A_{k_0})\mathbf{u}_0\| \leq C\left(|k^2-k_0^2|\|\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0\|_{L^2(D)}\|\mathbf{u}_0\|_{L^2(D)} + |k^4-k_0^4|\|\mathbf{u}_0\|^2_{L^2(D)}\right).$$

Hence, A_k is a bijective operator for $|k-k_0|$ sufficiently small. Moreover, since $k \mapsto A_k$ is analytic, then $k \mapsto A_k^{-1}$ is analytic in a neighborhood of k_0.

It suffices to show that for $k > 0$ small enough, the operator $A_k - B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is an isomorphism, in other words sufficiently small positive k are not transmission eigenvalues. To this end let $\mathbf{u}_0 \in \mathcal{U}_0(D)$ be such that

$$\mathcal{A}_k(\mathbf{u}_0, \mathbf{u}') - k^2\mathcal{B}(\mathbf{u}_0, \mathbf{u}') = 0 \;\text{ for all } \mathbf{u}' \in \mathcal{U}_0(D).$$

First we observe that since $\mathbf{u}_0 \times \nu = 0$ on ∂D, then

$$\mathrm{curl}\,\mathbf{u}_0 \cdot \nu = 0 \text{ on } \partial D.$$

This works for Lipshitz boundaries by interpreting the relationship $\mathrm{curl}\,\mathbf{u}_0 \cdot \nu = \mathrm{div}_{\partial D}(\mathbf{u}_0 \times \nu)$ in the weak sense [16]. On the other hand, the continuous embedding of

$$\{\mathbf{u} \in H_0(\mathrm{curl}\,, D) : \; \mathrm{div}\,\mathbf{u} = 0 \text{ in } D\}$$

into $H^1(D)^3$ implies that $\mathrm{curl}\,\mathbf{u}_0 \in H^1(D)^3$. The Poincaré inequality now implies the existence of a constant $C > 0$ such that

$$\|\mathrm{curl}\,\mathbf{u}_0\|^2 \leq C\|\nabla \mathrm{curl}\,\mathbf{u}_0\|^2_{L^2(D)}.$$

Let $\tilde{\mathbf{v}}_0$ be the extension of $\mathrm{curl}\,\mathbf{u}_0$ by 0 outside D. Then

$$\begin{aligned}\|\nabla \mathrm{curl}\,\mathbf{u}_0\|^2_{L^2(D)} = \|\nabla \tilde{\mathbf{v}}_0\|^2_{L^2(\mathbb{R}^3)} &= \|\mathrm{curl}\,\tilde{\mathbf{v}}_0\|^2_{L^2(\mathbb{R}^3)} + \|\mathrm{div}\,\tilde{\mathbf{v}}_0\|^2_{L^2(\mathbb{R}^3)} \\ &= \|\mathrm{curl}\,\tilde{\mathbf{v}}_0\|^2_{L^2(D)} + \|\mathrm{div}\,\tilde{\mathbf{v}}_0\|^2_{L^2(D)}.\end{aligned}$$

We therefore obtain that

$$\|\mathrm{curl}\,\mathbf{u}_0\|^2_{L^2(D)} \leq C\|\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0\|^2_{L^2(D)}.$$

From inequality (28) (satisfied here by $\mathcal{A}_k$) we now obtain that

$$\begin{aligned}\Re(\mathcal{A}_k(\mathbf{u}_0, \mathbf{u}_0) - k^2\mathcal{B}(\mathbf{u}_0, \mathbf{u}_0)) \;\;\geq\;\; & \tfrac{\gamma}{1+2\gamma}\left(\|\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0\|^2_{L^2(D)} + k^4\|\mathbf{u}_0\|^2_{L^2(D)}\right) \\ & -Ck^2\|\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0\|^2_{L^2(D)}.\end{aligned}$$

Therefore there are no eigenvalues such that $k^2 \leq \frac{\gamma}{C(1+2\gamma)}$.

Part (ii) does not require the assumption on the positive definite property of the corresponding matrices. Note that $\Im(N\boldsymbol{\xi},\boldsymbol{\xi}) > 0$ implies $\Im((N-I)^{-1}\boldsymbol{\xi},\boldsymbol{\xi}) < 0$. Now assume that $\mathbf{u}_0$ is a solution of

$$\mathcal{A}_k(\mathbf{u}_0,\mathbf{u}') - k^2\mathcal{B}(\mathbf{u}_0,\mathbf{u}') = 0 \text{ for all } \mathbf{u}' \in \mathcal{U}_0(D).$$

Taking the imaginary part one deduces that

$$\operatorname{curl}\operatorname{curl}\mathbf{u}_0 - k^2\mathbf{u}_0 = 0 \quad \in D.$$

Since $\mathbf{u}_0 \times \nu = 0$ and $\operatorname{curl}\mathbf{u}_0 \times \nu = 0$ on ∂D, the extension of $\mathbf{u}_0$ by 0 outside D by 0 gives an outgoing solution to Maxwell's equation in $\mathbb{R}^3$ with vanishing far field. This implies that this function is 0 in $\mathbb{R}^3$, and therefore $\mathbf{u}_0 = 0$. □

Remark 3.1. Under sufficient regularity on N that allows the use of the unique continuation principle for $\operatorname{curl}\operatorname{curl} - k^2N$, one can prove that if If $\Im(N\boldsymbol{\xi},\boldsymbol{\xi}) > 0$, for all $\boldsymbol{\xi} \in \mathbb{C}^3 \setminus \{0\}$ and a.e. in D_0 where $D_0 \subset D$ and $|D_0| \neq 0$, then the set of eigenvalues is empty.

3.2. ITP FOR CONDUCTING BOUNDARY CONDITION. We now generalize the results of the previous section to the case where we have a conductive boundary condition on a portion of ∂D. More precisely we assume that the surface conductivity satisfies $\eta(x) \geq \eta_0 > 0$ on Γ_2. Also, for technical reasons related to the conducting boundary condition (see discussion following Definition 3.4), we shall assume that the boundary ∂D is of class C^3.

The interior transmission problem (ITP-η) is now formulated as the problem of finding two vectorial functions $\mathbf{E}$ and $\mathbf{E}_0$ such that

$$\left\{\begin{array}{lll} (i) & \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\,N\,\mathbf{E} = \mathbf{F} & \text{in } D, \\ (ii) & \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\,\mathbf{E}_0 = \mathbf{F}_0 & \text{in } D, \end{array}\right. \tag{34}$$

$$\left\{\begin{array}{ll} (\mathbf{E}-\mathbf{E}_0)\times\nu = \boldsymbol{\varphi} & \text{on } \partial D, \\ (\operatorname{curl}\mathbf{E} - \operatorname{curl}\mathbf{E}_0)\times\nu = \boldsymbol{\psi} & \text{on } \Gamma_1, \\ (\operatorname{curl}\mathbf{E} - \operatorname{curl}\mathbf{E}_0)\times\nu - i\frac{k}{\eta}\nu\times(\mathbf{E}\times\nu) = \boldsymbol{\psi}+\boldsymbol{\tau} & \text{on } \Gamma_2. \end{array}\right. \tag{35}$$

In order to define the functional setting for this set of equations we introduce

$$\mathcal{U}(D,\Gamma_2) := \{\mathbf{u} \in \mathcal{U}(D) : \ (\operatorname{curl}\mathbf{u}\times\nu)|_{\Gamma_2} \in L^2_t(\Gamma_2)\},$$

which is a Hilbert space equipped with the norm

$$\|\mathbf{u}\|^2_{\mathcal{U}(D,\Gamma_2)} = \|\mathbf{u}\|^2_{\mathcal{U}(D)} + \|\operatorname{curl}\mathbf{u}\times\nu\|^2_{L^2(\Gamma_2)}.$$

Then we define the subspace

$$\mathcal{U}_0(D,\Gamma_2) := \{\mathbf{u} \in \mathcal{U}(D,\Gamma_2) : \ (\mathbf{u}\times\nu)|_{\partial D} = 0, \ (\operatorname{curl}\mathbf{u}\times\nu)|_{\Gamma_1} = 0\}.$$

The existence of a solution to this problem will be studied for data $\mathbf{F}$, $\mathbf{F}_0$, $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ that satisfies Assumption 3.1 and $\boldsymbol{\tau} \in L^2_t(\Gamma_2)$. We denote by $\mathbf{w}_0 \in \mathcal{U}(D)$ a lifting associated with $(\boldsymbol{\varphi},\boldsymbol{\psi})$.

Definition 3.4. A strong solution to (ITP-η) is a pair $(\mathbf{E},\mathbf{E}_0) \in L^2(D)^3$ that satisfies (34) in the sense of distributions such that $(\mathbf{E}-\mathbf{E}_0-\mathbf{w}_0) \in \mathcal{U}_0(D,\Gamma_2)$ and $(\mathbf{E},\mathbf{E}_0)$ satisfies (35).

This definition of strong solution requires that one can define the trace of the tangential components of $\mathbf{E}$ on ∂D when $\mathbf{E} \in L^2(D)^3$ and that $\mathbf{E}$ satisfies the first equation in (34) which means $\operatorname{curl}\operatorname{curl}\mathbf{E} \in L^2(D)^3$ and therefore $\mathbf{E}$ is in the space

$$\Lambda(D) = \{\mathbf{E} \in L^2(D)^3, \text{ such that, } \operatorname{curl}\operatorname{curl}\mathbf{E} \in L^2(D)^3\}.$$

We equip this space with the norm $\|\mathbf{E}\|^2_{\Lambda(D)} = \|\mathbf{E}\|^2_{L^2(D)} + \|\operatorname{curl}\operatorname{curl}\mathbf{E}\|^2_{L^2(D)}$. By classical duality arguments, one can prove that the tangential trace of $\mathbf{E}$ and $\operatorname{curl}\mathbf{E}$ is defined for $\mathbf{E} \in \Lambda(D)$. To this end we first note that proceeding along the lines of the proof of Theorem 3.26 and Theorem 3.25 in [17] (see also [13]), replacing the curl operator (respectively $H(\operatorname{curl}, D)$) with the curl curl operator (respectively $\Lambda(D)$), one can check that $C^\infty(\overline{D})$ is dense in $\Lambda(D)$. From the Stokes' formula one has

(36)
$$\langle \nu \times (\mathbf{E} \times \nu), (\operatorname{curl}\mathbf{w} \times \nu)\rangle_{H_t^{-1/2}, H_t^{1/2}} + \langle \nu \times (\operatorname{curl}\mathbf{E} \times \nu), (\mathbf{w} \times \nu)\rangle_{H_t^{-3/2}, H_t^{3/2}}$$
$$= (\operatorname{curl}\operatorname{curl}\mathbf{E}, \mathbf{w})_D - (\mathbf{E}, \operatorname{curl}\operatorname{curl}\mathbf{w})_D$$

for all $\mathbf{E} \in C^\infty(\overline{D})$ and $\mathbf{w} \in H^2(D)$. From Lemma 3.1 of [15] we have that the mapping

$$\mathbf{w} \mapsto \{(\mathbf{w} \times \nu)_{|\partial D}, (\operatorname{curl}\mathbf{w} \times \nu)_{|\partial D}\}$$

from $H^2(D)$ into $H_t^{3/2}(\partial D) \times H_t^{1/2}(\partial D)$ is surjective. Therefore one can conclude from (36), by using a density argument, that the mapping

$$\mathbf{E} \mapsto \{\nu \times (\mathbf{E} \times \nu)_{|\partial D}, \nu \times (\operatorname{curl}\mathbf{E} \times \nu)_{|\partial D}\}$$

extends to a continuous mapping from $\Lambda(D)$ into $H_t^{-1/2}(\partial D) \times H_t^{-3/2}(\partial D)$.

In the present case, it is more convenient to directly derive the variational formulation satisfied by $\mathbf{u} = \mathbf{E} - \mathbf{E}_0$ from (34)-(35) rather than just writing the equations for $\mathbf{u}$. The reason is that we need more regularity assumptions on the data so that the boundary condition for $\mathbf{u}$ can be expressed in the sense of functions.

To derive this variational formulation we need to justify the use of (36) for $\mathbf{w} \in \mathcal{U}_0(D, \Gamma_2)$. This is a conseqence of following density result.

Lemma 3.2. *$C^\infty(\overline{D})$ is dense in $\mathcal{U}_0(D, \Gamma_2)$.*

Proof. We first observe that if $\mathbf{w} \in \mathcal{U}_0(D, \Gamma_2)$ then $\operatorname{curl}\mathbf{w} \cdot \nu = 0$ on ∂D. Therefore, since D is a regular domain, $\operatorname{curl}\mathbf{w} \in H^1(D)$ (see [13]) and $\|\cdot\|_{\mathcal{U}(D)}$ is an equivalent norm on $\mathcal{U}_0(D, \Gamma_2)$. Combining this result with equality (29) that remains valid on $\mathcal{U}_0(D, \Gamma_2)$ we conclude that $\|\cdot\|_{\Lambda(D)}$ is also an equivalent norm on $\mathcal{U}_0(D, \Gamma_2)$. The statement of the present Lemma then follows from the density of $C^\infty(\overline{D})$ functions in $\Lambda(D)$. ☐

Variational formulation: We shall now explain how to derive the variational formulation for $\mathbf{u}$. Consider equation (34)-(i) multiplied by $\mathbf{u}' \in \mathcal{U}_0(D, \Gamma_2)$ and apply (36). After using the boundary conditions satisfied by $\mathbf{u}'$ we obtain that

$$\int_D \mathbf{E}\cdot(\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2 N\mathbf{u}')\, dx + \int_{\Gamma_2} (\operatorname{curl}\mathbf{u}' \times \nu)\cdot(\nu \times (\mathbf{E} \times \nu))\, ds = \int_D \mathbf{F}\cdot\mathbf{u}'\, dx \quad (37)$$

Next, setting $\lambda = 1/\eta$ for convenience and using the conducting boundary condition (35), we arrive at

$$\begin{aligned}\int_{\Gamma_2} (\operatorname{curl}\mathbf{u}'\times\nu)\cdot(\nu\times(\mathbf{E}\times\nu)) &= -\frac{i}{k}\int_{\Gamma_2}\lambda(\operatorname{curl}\mathbf{u}\times\nu)\cdot(\operatorname{curl}\mathbf{u}'\times\nu)\,ds\\ &\quad+\frac{i}{k}\int_{\Gamma_2}\lambda(\boldsymbol{\psi}+\boldsymbol{\tau})\cdot(\operatorname{curl}\mathbf{u}'\times\nu)ds,\end{aligned}$$

where the term in $\boldsymbol{\psi}$ needs to be interpreted as a duality pairing. We now recall the expression of $\mathbf{E}$ in terms of $\mathbf{u} := \mathbf{E}-\mathbf{E}_0$,

$$\mathbf{E} = \frac{1}{k^2}(N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u} - (\mathbf{F}-\mathbf{F}_0)).$$

Therefore, after multiplying (37) by $-k^2$, replacing $\mathbf{u}'$ by $\bar{\mathbf{u}}'$, and substituting for $\mathbf{E}$ one can easily check that $\mathbf{u}_0 = \mathbf{u}-\mathbf{w}_0 \in \mathcal{U}_0(D,\Gamma_2)$, and satisfies

(38)

$$\begin{aligned}&\big((N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u}_0 - k^2\mathbf{u}_0),\,(\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2\bar{N}\mathbf{u}')\big)_D\\ &\qquad -ik\,(\lambda(\operatorname{curl}\mathbf{u}_0\times\nu),\,(\operatorname{curl}\mathbf{u}'\times\nu))_{\Gamma_2}\\ &= \big((N-I)^{-1}(\mathbf{F}-\mathbf{F}_0),\,(\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2\bar{N}\mathbf{u}')\big)_D\\ &\qquad +k^2\,(\mathbf{F},\,\mathbf{u}')_D - ik\,(\lambda\boldsymbol{\tau},\,(\operatorname{curl}\mathbf{u}'\times\nu))_{\Gamma_2}\\ &\qquad -\big((N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{w}_0 - k^2\mathbf{w}_0),\,(\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2\bar{N}\mathbf{u}')\big)_D\end{aligned}$$

for all $\mathbf{u}' \in \mathcal{U}_0(D,\Gamma_2)$.

Theorem 3.4. *Assume that N and $(N-I)^{-1}$ are bounded matrix fields in D. Then the existence and uniqueness of strong solutions to (ITP-η) is equivalent to the existence and uniqueness of $\mathbf{u}_0 = (\mathbf{E}-\mathbf{E}_0-\mathbf{w}_0) \in \mathcal{U}_0(D,\Gamma_2)$ satisfying (38).*

Proof. As in Section 3.1 the proof is based on the use of formulas (13) and (14). From the above considerations we only have to check that the solution $\mathbf{u}_0 = (\mathbf{u} - \mathbf{w}_0) \in \mathcal{U}_0(D,\Gamma_2)$ defines a strong solution to (ITP-η). Taking $\mathbf{u}' \in \mathcal{U}_0(D)$ we obviously obtain that $\mathbf{u} = \mathbf{u}_0 + \mathbf{w}_0$ satisfies equation (17) in the distribution sense. Therefore $\mathbf{E}$ and $\mathbf{E}_0$ as defined by (14) are in $L^2(D)^3$ and satisfy (34). On the other hand, the variational formulation implies that $\mathbf{E}$ satisfy

$$\begin{aligned}&\big(\mathbf{E},\,(\operatorname{curl}\operatorname{curl}\mathbf{u}' - k^2\bar{N}\mathbf{u}')\big)_D - ik\,(\lambda(\operatorname{curl}\mathbf{u}_0\times\nu),\,(\operatorname{curl}\mathbf{u}'\times\nu))_{\Gamma_2}\\ &\qquad = (\mathbf{F},\,\mathbf{u}')_D - ik\,(\lambda\boldsymbol{\tau},\,(\operatorname{curl}\mathbf{u}'\times\nu))_{\Gamma_2}\end{aligned}$$

for all $\mathbf{u}' \in \mathcal{U}_0(D,\Gamma_2)$. One then obtains the conducting boundary condition by taking $\mathbf{u}' \in \mathcal{U}_0(D,\Gamma_2)\cap C^\infty(\overline{D})$ and using formulas (36). □

The remainder of this section is devoted to the study of the variational formulation (38). Our proofs hold for Lipshitz domains D (and therefoe does not need the regularity result used in Lemma 3.2).

Let us denote by $\ell(\mathbf{u}')$ the right hand side of (38) which defines a continuous antilinear form on $\mathcal{U}_0(D,\Gamma_2)$. As in the previous section depending on the sign of $N-I$ we will use one the following equivalent arrangements of (38)

(39) $$\mathcal{A}_k(\mathbf{u}_0,\mathbf{u}') - ik\mathcal{T}(\mathbf{u}_0,\mathbf{u}') - k^2\mathcal{B}(\mathbf{u}_0,\mathbf{u}') = \ell(\mathbf{u}') \text{ for all } \mathbf{u}' \in \mathcal{U}_0(D,\Gamma_2),$$

or

(40) $$-\tilde{\mathcal{A}}_k(\mathbf{u}_0,\mathbf{u}') - ik\mathcal{T}(\mathbf{u}_0,\mathbf{u}') + k^2\mathcal{B}(\mathbf{u}_0,\mathbf{u}') = \ell(\mathbf{u}') \text{ for all } \mathbf{u}' \in \mathcal{U}_0(D,\Gamma_2),$$

where $\mathcal{A}_k$, $\tilde{\mathcal{A}}_k$ and $\mathcal{B}$ have the same expression as in the previous section but are now defined on $\mathcal{U}_0(D,\Gamma_2)\times\mathcal{U}_0(D,\Gamma_2)$, and where $\mathcal{T}$ is defined on $\mathcal{U}_0(D,\Gamma_2)\times\mathcal{U}_0(D,\Gamma_2)$ by

$$\mathcal{T}(\mathbf{u}_0,\mathbf{u}') = (\lambda(\operatorname{curl}\mathbf{u}_0\times\nu),\,(\operatorname{curl}\mathbf{u}'\times\nu))_{\Gamma_2}\,. \tag{41}$$

Lemma 3.3. *Assume that* $(N-I)^{-1}$ *(respectively,* $N(I-N)^{-1}$*) is a bounded positive definite matrix field on* D *such that* $\Im(N)\geq 0$. *Then* $\mathcal{A}_k - ik\mathcal{T}$ *(respectively,* $\tilde{\mathcal{A}}_k + ik\mathcal{T}$*) is a continuous, coercive sesquilinear form on* $\mathcal{U}_0(D,\Gamma_2)\times\mathcal{U}_0(D,\Gamma_2)$.

Proof. The proof relies on the observations that

$$\Re(\mathcal{A}_k(\mathbf{u}_0,\mathbf{u}_0) - ik\mathcal{T}(\mathbf{u}_0,\mathbf{u}_0)) = \Re(\mathcal{A}_k(\mathbf{u}_0,\mathbf{u}_0))$$

and

$$-\Im(\mathcal{A}_k(\mathbf{u}_0,\mathbf{u}_0) - ik\mathcal{T}(\mathbf{u}_0,\mathbf{u}_0)) \geq k\Im(\mathcal{T}(\mathbf{u}_0,\mathbf{u}_0)).$$

The coerciveness is then a straightforward consequence of the estimate (28), the equality (29) that remains valid for $\mathbf{u}_0\in\mathcal{U}_0(D,\Gamma_2)$, and the obvious inequality

$$\Im(\mathcal{T}(\mathbf{u}_0,\mathbf{u}_0)) \leq \inf|\lambda|\|\mathbf{u}_0\times\nu\|^2_{L^2(\Gamma_2)}.$$

□

Based on the Riesz representation theorem let us define the operator $B:\mathcal{U}_0(D,\Gamma_2)\to\mathcal{U}_0(D,\Gamma_2)$ by

$$(B\mathbf{u}_0,\,\mathbf{u}')_{\mathcal{U}(D,\Gamma_2)} = \mathcal{B}(\mathbf{u}_0\,,\mathbf{u}')\ \ \forall\,\mathbf{u}'\in\mathcal{U}_0(D,\Gamma_2).$$

Lemma 3.4. *The operator* $B:\mathcal{U}_0(D,\Gamma_2)\longrightarrow\mathcal{U}_0(D,\Gamma_2)$ *is compact.*

Proof. Using the definition of $\mathcal{B}$, one obtains that

$$\|B\mathbf{u}_0\|_{\mathcal{U}(D,\Gamma_2)} \leq \|\operatorname{curl}\mathbf{u}_0\|_{L^2(D)}\quad\forall\,\mathbf{u}_0\in\mathcal{U}_0(D,\Gamma_2).$$

Now observe that if $\mathbf{u}_0\in\mathcal{U}_0(D,\Gamma_2)$ then $\operatorname{curl}\mathbf{u}_0$ belongs to

$$\{\mathbf{u}\in H(\operatorname{curl},D)\ /\ \operatorname{div}\mathbf{u}=0 \text{ in } D,\ \text{and } (\mathbf{u}\times\nu)_{|\partial D}\in L^2_t(D)\}$$

which is compactly embedded into $L^2(D)^3$ (see [17] for instance). □

Combining the results of Lemma 3.3 and Lemma 3.4, one concludes using the same arguments as in the proof of Theorem 3.2 that the following theorem holds.

Theorem 3.5. *Assume that* $(N-I)^{-1}$ *or* $N(I-N)^{-1}$ *is a bounded positive definite matrix field in* D *and* $\Im(N)\geq 0$. *If* k *is not a transmission eigenvalue, then there exists a unique solution* $\mathbf{u}_0\in\mathcal{U}_0(D,\Gamma_2)$ *to (38) such that*

$$\|\mathbf{u}_0\|_{\mathcal{U}(D,\Gamma_2)} \leq C(\|\mathbf{F}\|_{L^2(D)} + \|\mathbf{F}_0\|_{L^2(D)} + \|\boldsymbol{\tau}\|_{L^2(\Gamma)},$$

where $C>0$ *is a constant independent of* $\mathbf{u}_0$ *and* $(\mathbf{F},\mathbf{F}_0,\boldsymbol{\varphi},\boldsymbol{\psi})$.

We also can state

Theorem 3.6. *Assume that* $(N-I)^{-1}$ *or* $N(I-N)^{-1}$ *is a bounded positive definite matrix field on* D. *Then*

(i) *The set of transmission eigenvalues for (ITP-η) is discrete and does not accumulate at* 0.
(ii) *If* $\Im(N\boldsymbol{\xi},\boldsymbol{\xi})>0,\ \ \forall\,\boldsymbol{\xi}\in\mathbb{C}^3\setminus\{0\}$ *and a.e. in* D *then the set of transmission eigenvalues is empty.*

The proof of Theorem 3.6 follows from the simple observation that the set of transmission eigenvalues for (ITP-η) is a subset of the transmission eigenvalues for (ITP) [7] and Theorem 3.3.

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Received for publication December 2006.

E-mail address: cakoni@math.udel.edu
E-mail address: Houssem.Haddar@inria.fr