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On the boundary integral equation method for a mixed boundary value problem of the biharmonic equation§

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This article is concerned with weak solution of a mixed boundary value problem for the biharmonic equation in the plane. Using Green's formula, the problem is converted into a system of Fredholm integral equations for the unknown data on different parts of the boundary. Existence and uniqueness of the solutions of the system of boundary integral equations are established in appropriate Sobolev spaces.

Keywords: Biharmonic equation; Mixed boundary value problems; Boundary integral equations; Variational formulations

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1. Introduction

In the article [1], a mixed boundary value problem for the two-dimensional Laplace equation is considered. Using Green's formula the problem is converted into a system of Fredholm integral equations for the missing part of Cauchy data on different parts of the boundary. One of these boundary integral equations has a principal part of the second kind, whereas the other is of the first kind. However, the crucial point of the approach there is that the derived system of integral equations can be

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§Dedicated to Heinrich Begehr on the occasion of his 65th birthday.

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interpreted as a strongly elliptic system of pseudodifferential equations. Hence it can be solved constructively by Galerkin's method.

The purpose of the present article is to see the feasibility of extending the approach in [1] for the Laplace equation to the biharmonic equation. Clearly for the latter, it is much more involved; there is double the amount of Cauchy data and 16 boundary operators needed to be considered. However, as will be seen, the recent systematic characterization of the Calderón projector in [2] has simplified the approach in the same manner as in the case for the Laplace equation.

The article is organized as follows: in section 1, we formulate the mixed boundary problem and present some preliminary results for the weak solutions of the boundary value problems for the biharmonic equation. Section 2 contains the core materials for the four basic boundary integral equations. Theorems 3.3, 3.4 and 3.5 in section 3 are the main results concerning existence and uniqueness of the system of boundary integral equations in section 2. Finally, in the last section, we conclude the article by a brief discussion on the regularity results of the solutions of boundary integral equations, and these can be served as the mathematical foundations for the augmented Galerkin method in the same manner as in the case of Laplace equation (see [1]).

2. Formulation of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected region with $C^{1,1}$- boundary Γ. We assume that the boundary Γ has a dissection $\Gamma = \Gamma_D \cup \Gamma_c \cup \Gamma_N$, where Γ_D and Γ_N are disjoint, relatively open subsets of Γ, having Γ_c as their common boundary points in Γ. We denote by $n = (n_1, n_2)$ the unit outward normal vector to Γ.

Now let a thin plate in elastostatic equilibrium occupy the region Ω. We assume that the part Γ_D of the boundary is clamped while the part Γ_N is free. If we denote by u the equilibrium state of the plate, we obtain the following mixed boundary value problems for the biharmonic equation

$$\Delta^2 u = 0 \qquad \text{in } \Omega \tag{1}$$

$$u = f \quad \text{and} \quad \frac{\partial u}{\partial n} = g \qquad \text{on } \Gamma_D \tag{2}$$

$$Mu = p \quad \text{and} \quad Nu = q \qquad \text{on } \Gamma_N, \tag{3}$$

where the boundary operators $M|_{\Gamma_D}$ and $N|_{\Gamma_N}$ are the restrictions to Γ_D and Γ_N respectively, of the following boundary differential operators

$$Mu = \nu \Delta u + (1 - \nu) M_0 u \tag{4}$$

and

$$Nu = -\frac{\partial}{\partial n} \Delta u - (1 - \nu) \frac{\partial}{\partial s} N_0 u. \tag{5}$$

Here ν is the Poisson ratio, a real constant and in application (especially in the theory of elasticity) we have $0 \le \nu < 1$. The normal and tangential derivatives are given by

$$\frac{\partial}{\partial n} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} \qquad \text{and} \qquad \frac{\partial}{\partial s} = -n_1 \frac{\partial}{\partial x_2} + n_2 \frac{\partial}{\partial x_1},$$

while the boundary operators $M_0 u$ and $N_0 u$ are defined by

$$M_0 u := \frac{\partial^2 u}{\partial x_1^2} n_1^2 + 2 \frac{\partial^2 u}{\partial x_1 \, \partial x_2} n_1 n_2 + \frac{\partial^2 u}{\partial x_2^2} n_2^2$$

and

$$N_0 u := -\left\{ \left(\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} \right) n_1 n_2 - \frac{\partial^2 u}{\partial x_1 \, \partial x_2} \left(n_1^2 - n_2^2 \right) \right\}.$$

Physically, Mu is the bending moment and Nu is the transverse force consisting of the shear force and twisting moment [3]. The mixed conditions (2) and (3) may be interpreted that the plate is clamped on Γ_D and has a free edge on Γ_N. We are interested in the weak solutions of the mixed boundary value problem, (1), (2), and (3).

Our solution space for the biharmonic equation (1) is the standard Sobolev space $H^2(\Omega)$ of distributions that are square integrable and have square integrable derivatives up to the second order. We first observe that since the boundary Γ is $C^{1,1}$, the trace spaces $H^{3/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ are well defined [4] and moreover for $u \in H^2(\Omega)$ we have that $u|_\Gamma \in H^{3/2}(\Gamma)$ and $(\partial u/\partial n)|_\Gamma \in H^{1/2}(\Gamma)$. To discuss the boundary value problem for (1), it is best to begin with the Green formula for (1) in Ω. By using integration by parts formulas, one can obtain

$$\int_\Omega (\Delta^2 u) v \, dx = a(u, v) - \int_\Gamma \left[(Mu) \frac{\partial v}{\partial n} + (Nu) v \right] ds, \tag{6}$$

for smooth functions, where the bilinear form $a(u, v)$ is defined by

$$\begin{aligned} a_\Omega(u, v) := & \int_\Omega \nu \Delta u \Delta v \, dx \\ & + \int_\Omega (1 - \nu) \left(\frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} + 2 \frac{\partial^2 u}{\partial x_1 \, \partial x_2} \frac{\partial^2 v}{\partial x_1 \, \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_2^2} \right) dx. \end{aligned} \tag{7}$$

We note that the bilinear form in (7) is well defined for functions in $H^2(\Omega)$. Now let $u \in H^2(\Omega, \Delta^2)$ where

$$H^2(\Omega, \Delta^2) := \{ u \in H^2(\Omega) : \ \Delta^2 u \in \widetilde{H}^{-2}(\Omega) \}$$

with $\widetilde{H}^{-2}(\Omega)$ denoting the dual space of $H^2(\Omega)$ and choose $v \in H^2(\Omega)$. Then the above Green formula holds and by a duality argument one shows that $Mu \in H^{-1/2}(\Gamma)$ and $Nu \in H^{-3/2}(\Gamma)$ are well defined, where $H^{-1/2}(\Gamma)$ and $H^{-3/2}(\Gamma)$ are the dual spaces of $H^{1/2}(\Gamma)$ and $H^{3/2}(\Gamma)$, respectively.

For later use we recall the following classical result by Agmon [5].

LEMMA 2.1 *Assume that* $-3 < \nu < 1$. *Then the bilinear form* $a(\cdot,\cdot)$ *given by* (7) *satisfies a Gårding inequality in the form*

$$a(v, v)_\Omega \geq c_0 \|v\|^2_{H^2(\Omega)} - \lambda_0 \|v\|^2_{L^2(\Omega)}$$

for all $v \in H^2(\Omega)$, *where* $c_0 > 0$ *and* $\lambda_0 \geq 0$ *are constant*.

In order to formulate the mixed boundary value problem (1)–(3) more precisely we need to define the following trace spaces on an open arc $\Gamma_0 \subset \Gamma$. For any $s \in \mathbb{R}$, $s \geq 0$, we define

$$H^s(\Gamma_0) := \{u|_{\Gamma_0} : \quad u \in H^s(\Gamma)\},$$

$$\tilde{H}^s(\Gamma_0) := \{u \in H^s(\Gamma) : \operatorname{supp} u \subseteq \overline{\Gamma}_0\}$$

(see, e.g., [1,6]). For $s > 0$ we denote by $H^{-s}(\Gamma_0)$ the dual space of $\tilde{H}^s(\Gamma_0)$ and by $\tilde{H}^{-s}(\Gamma_0)$ the dual space of $H^s(\Gamma_0)$ with $L^2(\Gamma_0)$ as the pivot space. Note that $\tilde{H}^{-s}(\Gamma_0)$ can also be identified with the space of distributions in $H^{-s}(\Gamma)$ supported in $\overline{\Gamma}_0$. The following chain of continuous embeddings holds

$$\tilde{H}^s(\Gamma_0) \subset H^s(\Gamma_0) \subset L^2(\Gamma_0) \subset \tilde{H}^{-s}(\Gamma_0) \subset H^{-s}(\Gamma_0) \quad s > 0.$$

Now we are ready to formulate precisely the mixed boundary value problem for the biharmonic equation: *Given* $f \in H^{3/2}(\Gamma_D)$, $g \in H^{1/2}(\Gamma_D)$, $p \in H^{-1/2}(\Gamma_N)$, *and* $q \in H^{-3/2}(\Gamma_N)$, *find* $u \in H^2(\Omega)$ *that satisfies* (1)–(3). We will refer to this problem as (MBP).

THEOREM 2.2 *The mixed boundary value problem MBP has at most one solution for* $0 \leq \nu \leq 1$.

Proof Let u be the solution of MBP with $f = g = p = q = 0$. Then an application of Green's formula to u and $\bar{u}$ yields

$$\int_\Omega \left[\nu|\Delta u|^2 + (1-\nu)\left(\left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 + 2\left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 u}{\partial x_2^2}\right|^2 \right) \right] dx = 0.$$

Hence for $0 \leq \nu < 1$ we have $(\partial^2 u/\partial x_1^2) = (\partial^2 u/\partial x_2^2) = 0$ in Ω which implies $u = ax_1 + bx_2 + c$ and form the boundary conditions $u|_{\Gamma_D} = 0$, $\partial u/\partial n$, $\Gamma_D = 0$, we conclude that $u = 0$ in Ω.

In the case of $\nu = 1$ we obtain that $\Delta u = 0$ in Ω and $u = (\partial u/\partial n) = 0$ on Γ_D. Now let B_ρ be a ball of radius ρ with center on Γ_D such that $\overline{B}_\rho \cap \Gamma_N = 0$ and define $v = u$

in $\Omega \cap B_\rho$, $v = 0$ in $(\mathbb{R}^2 \setminus \overline{\Omega}) \cap B_\rho$. Then v satisfies $\Delta v = 0$ in B_ρ and hence real-analytic in B_ρ. We can now conclude that $u \equiv 0$ in B_ρ and thus $u \equiv 0$ in Ω.

3. Boundary integral equations

In order to prove the existence of a solution to MBP and provide a solution formula we will reformulate the problem as a system of boundary integral equations of the first kind. We start with the Green representation formula of a weak solution in $H^2(\Omega)$ [2]

$$u(x) = \mathcal{V}(Mu, Nu)(x) - \mathcal{W}\left(u, \frac{\partial u}{\partial n}\right)(x), \qquad x \in \Omega \tag{8}$$

in terms of simple- and double-layer potentials. Here

$$\mathcal{V}\colon H^{-1/2}(\Gamma) \times H^{-3/2}(\Gamma) \longrightarrow H^2(\Omega), \quad \text{and} \quad \mathcal{W}\colon H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \longrightarrow H^2(\Omega)$$

are continuous operators defined by

$$\mathcal{V}(\sigma_1, \sigma_2)(x) := \int_\Gamma \left\{ E(x, y)\sigma_2(y) + \frac{\partial E(x, y)}{\partial n_y}\sigma_1 \right\} ds_y, \quad x \in \mathbb{R}^2 \setminus \Gamma,$$

$$\mathcal{W}(\phi_1, \phi_2)(x) := \int_\Gamma \left\{ M_y E(x, y)\phi_2(y) + N_y E(x, y)\phi_1(y) \right\} ds_y, \quad x \in \mathbb{R}^2 \setminus \Gamma$$

where

$$E(x, y) := \frac{1}{8\pi}|x - y|^2 \log|x - y|$$

is the fundamental solution of the biharmonic equation. Letting $x \to \Gamma$ from inside Ω, and following the standard procedure in potential theory involving jump relations, we obtain the following integral equations on Γ,

$$\begin{aligned} u(x)|_\Gamma := & \int_\Gamma \left\{ E(x, y)Nu(y) + \frac{\partial E(x, y)}{\partial n_y} Mu(y) \right\} ds_y \\ & - \int_\Gamma M_y E(x, y) \frac{\partial u}{\partial n}(y) ds_y + \left[\frac{1}{2}u(x) - \int_\Gamma N_y E(x, y)u(y) ds_y \right] \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\partial u}{\partial n}(x)|_\Gamma := & \int_\Gamma \left\{ \frac{\partial E(x, y)}{\partial n_x} Nu(y) + \frac{\partial^2 E(x, y)}{\partial n_x \partial n_y} Mu(y) \right\} ds_y \\ & + \left[\frac{1}{2}\frac{\partial u}{\partial n}(x) - \int_\Gamma \frac{\partial}{\partial n_x} M_y E(x, y) \frac{\partial u}{\partial n_y}(y) ds_y \right] - \int_\Gamma \frac{\partial}{\partial n_x} N_y E(x, y)u(y) ds_y \end{aligned} \tag{10}$$

$$Mu(x)|_\Gamma := \int_\Gamma M_x E(x,y) Nu(y) ds_y + \left[\frac{1}{2} Mu(x) - \int_\Gamma M_x \frac{\partial E(x,y)}{\partial n_y} Mu(y) ds_y\right] - \int_\Gamma \left\{ M_x M_y E(x,y) \frac{\partial u}{\partial n_y}(y) + M_x N_y E(x,y) u(y)\right\} ds_y \tag{11}$$

$$N(x)|_\Gamma := \left[\frac{1}{2} Nu(x) + \int_\Gamma N_x E(x,y) Nu(y) ds_y\right] + \int_\Gamma N_x \frac{\partial E(x,y)}{\partial n_y} Mu(y) ds_y - \int_\Gamma \left\{ N_x M_y E(x,y) \frac{\partial u}{\partial n_y}(y) + N_x N_y E(x,y) u(y)\right\} ds_y. \tag{12}$$

In order to understand the mapping properties of the above 16 boundary integral operators we may rewrite (9)–(12) in the form

$$\begin{bmatrix} u \\ \partial u/\partial n \\ Mu \\ Nu \end{bmatrix}_{|\Gamma} = \begin{bmatrix} (1/2) I - K_{11} & V_{12} & V_{13} & V_{14} \\ D_{21} & (1/2) I + K_{22} & V_{23} & V_{24} \\ D_{31} & D_{32} & (1/2) I - K_{33} & V_{34} \\ D_{41} & D_{42} & D_{43} & (1/2) I + K_{44} \end{bmatrix} \begin{bmatrix} u \\ \partial u/\partial n \\ Mu \\ Nu \end{bmatrix}_{|\Gamma},$$

where the operators are defined in an obvious manner. This matrix of integral operators is the corresponding Calderón projector for the biharmonic equation with respect to domain Ω and will be denoted by $C_\Omega := ((C_\Omega^{i,j}))_{4\times 4}$. This Calderón projector in fact consists of pseudodifferential operators on Γ and has been studied in detail in [2]. In particular it maps $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{-3/2}(\Gamma)$ into itself continuously. The mapping properties of each of the operators appearing in C_Ω can be easily obtained from its principal symbol as

$$\mathrm{Ord}\,(C_\Omega) := \begin{bmatrix} 0 & -1 & -3 & -3 \\ +1 & 0 & -1 & -3 \\ +1 & +1 & 0 & -1 \\ +3 & +1 & +1 & 0 \end{bmatrix}.$$

We note that in particular the order of each operator $C_\Omega^{i,j}$ in the matrix C_Ω can be computed from the difference of its index $i-j$ (or $i-j-1$, when $|i-j|=2$). The operators of negative order are smoothing operators while the operators of positive order are singular and the absolute value of the order shows how much regularity we gain or loose by mapping. For instance the operator $C_\Omega^{13} = V_{1,3}$ defined by

$$(V_{13}\varphi)(x) := \int_\Gamma \frac{\partial E(x,y)}{\partial n_y} \varphi(y)\, ds_y$$

is of order -2 (in fact in this case is of order $-2-1$) and hence maps continuously for $\Gamma \in C^\infty$ from $H^{-1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$ ($H^{5/2}(\Gamma)$), while the operator $C_\Omega^{42} = D_{42}$,

$$(D_{42}\varphi)(x) := \int_\Gamma N_x M_y E(x,y) \varphi(y)\, ds$$

is of order 2 (or rather $2-1$) and maps from $H^{1/2}(\Gamma)$ to $H^{-3/2}(\Gamma)$ ($H^{(-1/2)}(\Gamma)$) continuously.

Now we return to the mixed boundary value problem (1)–(3). We denote by $\tilde{f} \in H^{3/2}(\Gamma)$, $\tilde{g} \in H^{1/2}(\Gamma)$, $\tilde{p} \in H^{-1/2}(\Gamma)$ and $\tilde{q} \in H^{-3/2}(\Gamma)$ bounded extensions to the whole of Γ of the corresponding boundary data f, g, p and q. Then we write

$$u|_\Gamma = \phi_N + \tilde{f} \qquad \left.\frac{\partial u}{\partial n}\right|_\Gamma = \psi_N + \tilde{g} \tag{13}$$

$$Mu|_\Gamma = \sigma_D + \tilde{p} \qquad Nu|_\Gamma = \tau_D + \tilde{q}. \tag{14}$$

Obviously $\phi_N \in \tilde{H}^{3/2}(\Gamma_N)$, $\psi_N \in \tilde{H}^{1/2}(\Gamma_N)$, $\sigma_D \in \tilde{H}^{-1/2}(\Gamma_D)$ and $\tau_D \in \tilde{H}^{-3/2}(\Gamma_D)$ since $\phi_N = \psi_N = 0$ on Γ_D and $\sigma_D = \tau_D = 0$ on Γ_N.

By now restricting (9) and (10) to Γ_D and (11) and (12) to Γ_N we obtain the following system of boundary integral equations of the first kind for $\phi_N, \psi_N, \sigma_D, \tau_D$

$$\int_{\Gamma_D} E(x,y)\tau_D(y)ds + \int_{\Gamma_D} \frac{\partial E(x,y)}{\partial n_y}\sigma_D(y)ds - \int_{\Gamma_N} M_y E(x,y)\psi_N(y)ds - \int_{\Gamma_N} N_y E(x,y)\phi_N(y)ds = F_1(x), \qquad x \in \Gamma_D \tag{15}$$

$$\int_{\Gamma_D} \frac{\partial E(x,y)}{\partial n_y}\tau_D(y)\,ds + \int_{\Gamma_D} \frac{\partial^2 E(x,y)}{\partial n_x \partial n_y}\sigma_D(y)\,ds - \int_{\Gamma_N} \frac{\partial}{\partial n_x} M_y E(x,y)\psi_N(y)\,ds - \int_{\Gamma_N} \frac{\partial}{\partial n_x} N_y E(x,y)\phi_N(y)\,ds = F_2(x), \qquad x \in \Gamma_D \tag{16}$$

$$\int_{\Gamma_D} M_x E(x,y)\tau_D(y)ds + \int_{\Gamma_D} M_x \frac{\partial E(x,y)}{\partial n_y}\sigma_D(y)ds - \int_{\Gamma_N} M_x M_y E(x,y)\psi_N(y)ds - \int_{\Gamma_N} M_x N_y E(x,y)\phi_N(y)ds = F_3(x), \qquad x \in \Gamma_N \tag{17}$$

$$\int_{\Gamma_D} N_x E(x,y)\tau_D(y)ds + \int_{\Gamma_D} N_x \frac{\partial E(x,y)}{\partial n_y}\sigma_D(y)ds - \int_{\Gamma_N} N_x M_y E(x,y)\psi_N(y)ds - \int_{\Gamma_N} N_x N_y E(x,y)\phi_N(y)ds = F_4(x), \qquad x \in \Gamma_N \tag{18}$$

where

$$
\begin{aligned}
F_1(x) &= -\int_\Gamma E(x,y)\tilde{q}(y)ds - \int_\Gamma \frac{\partial E(x,y)}{\partial n_y}\tilde{p}(y)ds \\
&\quad + \int_\Gamma M_y E(x,y)\tilde{g}(y)ds + \left[\frac{1}{2}\tilde{f}(x) + \int_\Gamma N_y E(x,y)\tilde{f}(y)ds\right], \quad x \in \Gamma_D \\
F_2(x) &= -\int_\Gamma \frac{\partial E(x,y)}{\partial n_y}\tilde{q}(y)ds - \int_{\Gamma_D} \frac{\partial^2 E(x,y)}{\partial n_x \partial n_y}\tilde{p}(y)ds \\
&\quad + \left[\frac{1}{2}\tilde{g}(x) + \int_\Gamma \frac{\partial}{\partial n_x} M_y E(x,y)\tilde{g}(y)ds\right] + \int_\Gamma \frac{\partial}{\partial n_x} N_y E(x,y)\tilde{f}(y)ds, \quad x \in \Gamma_D \\
F_3(x) &= -\int_\Gamma M_x E(x,y)\tilde{q}(y)ds\left[\frac{1}{2}\tilde{p}(x) - \int_\Gamma M_x \frac{\partial E(x,y)}{\partial n_y}\tilde{p}(y)ds\right] \\
&\quad + \int_\Gamma M_x M_y E(x,y)\tilde{g}(y)ds + \int_\Gamma M_x N_y E(x,y)\tilde{f}(y)ds, \quad x \in \Gamma_N \\
F_4(x) &= \left[\frac{1}{2}\tilde{q}(x) - \int_\Gamma N_x E(x,y)\tilde{q}(y)ds\right] - \int_\Gamma N_x \frac{\partial E(x,y)}{\partial n_y}\tilde{p}(y)ds \\
&\quad + \int_\Gamma N_x M_y E(x,y)\tilde{g}(y)ds + \int_\Gamma N_x N_y E(x,y)\tilde{f}(y)ds, \quad x \in \Gamma_N
\end{aligned}
$$

The system of equations (15)–(18) can be written in a matrix form as follows

$$
\mathcal{A}\begin{bmatrix} \tau_D \\ \sigma_D \\ \psi_N \\ \phi_N \end{bmatrix} := \begin{bmatrix} V_{14}^{DD} & V_{13}^{DD} & V_{12}^{DN} & -K_{11}^{DN} \\ V_{24}^{DD} & V_{23}^{DD} & K_{22}^{DN} & D_{21}^{DN} \\ V_{34}^{ND} & -K_{33}^{ND} & D_{32}^{NN} & D_{31}^{NN} \\ K_{44}^{ND} & D_{43}^{ND} & D_{42}^{NN} & D_{41}^{NN} \end{bmatrix}\begin{bmatrix} \tau_D \\ \sigma_D \\ \psi_N \\ \phi_N \end{bmatrix} = \mathbf{F} \tag{19}
$$

with $\mathbf{F} = [F_1, F_2, F_3, F_4]^\top$. Here $V_{ij}, D_{i,j}, K_{ii}$, $i = 1, \ldots, 4$, $j = 1, \ldots, 4$ are the operators that appear in the Calderón operator, and V_{12}^{DN} stands for the operator V_{24} applied to a function with support in Γ_N and evaluated on Γ_D, with analogous definitions for the other operators. From the mapping properties of the Calderón operator, one can see that the operator $\mathcal{A}$ defines a continuous mapping $\mathcal{A}: \mathcal{H} \to \mathcal{H}^*$ with $\mathcal{H} := \tilde{H}^{-3/2}(\Gamma_D) \times \tilde{H}^{-1/2}(\Gamma_D) \times \tilde{H}^{1/2}(\Gamma_N) \times \tilde{H}^{3/2}(\Gamma_N)$ and $\mathcal{H}^* := H^{3/2}(\Gamma_D) \times H^{1/2}(\Gamma_D) \times H^{-1/2}(\Gamma_N) \times H^{-3/2}(\Gamma_N)$ the dual space of $\mathcal{H}$.

We remark that if $\phi_N, \psi_N, \sigma_D, \tau_D$ satisfy (19) then, after defining $u, (\partial u/\partial n), Mu, Nu$ on Γ by (13) and (14), the representation formula (8) gives a solution to MBP which from Theorem 2.2 is the unique solution. Hence we need to study the solvability

of the system of integral equations of the first kind (19). For this purpose, let us first introduce the matrix operator $V: H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ defined by

$$V := \begin{bmatrix} V_{14} & V_{13} \\ V_{24} & V_{23} \end{bmatrix}. \tag{20}$$

Then the following lemma holds.

LEMMA 3.1 *There exists a compact operator* $C_V: H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ *such that*

$$\langle (V + C_V)\Theta, \Theta \rangle \geq C \|\Theta\|^2_{H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)} \quad \textit{for} \quad \Theta \in H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$$

where the bracket $\langle \cdot, \cdot \rangle$ *denotes the* $L_2(\Gamma)$*-duality pairing between* $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ *and* $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$.

Proof The proof follows Hsiao and Wendland [6] and Costabel and Wendland [15]. For any $\Theta = (\theta_1, \theta_2) \in H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$, let

$$u(x) = \int_\Gamma \left\{ E(x, y)\theta_1(y) + \frac{\partial E(x, y)}{\partial n_y} \theta_2 \right\} ds_y \qquad x \in \mathbb{R}^2 \setminus \Gamma.$$

Then $u \in H^2(\Omega, \Delta^2)$, $u \in H^2_{\text{loc}}(\Omega_c, \Delta^2)$ where $\Omega_c := \mathbb{R}^2 \setminus \overline{\Omega}$. Moreover (9)–(12) yield

$$[u]_\Gamma = 0, \quad \left[\frac{\partial u}{\partial n}\right]_\Gamma = 0, \quad [Mu]_\Gamma = \theta_2, \quad [Nu]_\Gamma = \theta_1$$

where $[\cdot]_\Gamma$ denotes the jump across the boundary Γ. Next we introduce a fixed $C_0^\infty(\mathbb{R}^2)$ cut-off function χ with $\chi|_{\overline{\Omega}} = 1$. Hence from the jump properties, (9)–(12) and the Green formula (6) we can write

$$\langle V\Theta, \Theta \rangle = \int_\Gamma \left(\frac{\partial u}{\partial n} [Mu] + u [Nu] \right) ds = a_\Omega(u, u) + a_{\Omega_c}(\chi u, \chi u),$$

where $a_\Omega(\cdot, \cdot)$ and $a_{\Omega_c}(\cdot, \cdot)$ are the bilinear forms (7) corresponding to Ω and Ω_c. Note that $a_{\Omega_c}(\chi u, \chi u)$ is well defined since χu has compact support. Then the result follows from Lemma 2.1, the fact that H^2 is compactly imbedded in L^2 and the Riesz representation theorem. It is important to notice that $\chi u = u$ in a neighborhood of Γ and therefore a Gårding inequality of the type of Lemma 2.1 holds also for $a_{\Omega_c}(\cdot, \cdot)$ since the integrals containing χ are simply compact perturbations (for details see the proof of Theorem 3.7 in [8] for the case of the Laplace equation).

Next, let $D: H^{1/2}(\Gamma) \times H^{3/2}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{-3/2}(\Gamma)$ be the continuous mapping defined by

$$D := \begin{bmatrix} D_{32} & D_{31} \\ D_{42} & D_{41} \end{bmatrix}. \tag{21}$$

Similarly, we have the following lemma.

LEMMA 3.2 *There exists a compact operator* $C_D: H^{1/2}(\Gamma) \times H^{3/2}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{-3/2}(\Gamma)$ *such that*

$$\big\langle (D + C_D)\Psi, \Psi \big\rangle \geq C\|\Psi\|^2_{H^{1/2}(\Gamma)\times H^{3/2}(\Gamma)} \quad \textit{for} \quad \Psi \in H^{1/2}(\Gamma) \times H^{3/2}(\Gamma)$$

where the bracket $\langle \cdot, \cdot \rangle$ *denotes the* $L_2(\Gamma)$*-duality pairing between* $H^{1/2}(\Gamma) \times H^{3/2}(\Gamma)$ *and* $H^{-1/2}(\Gamma) \times H^{-3/2}(\Gamma)$.

Proof For any $\Psi = (\psi_1, \psi_2) \in H^{1/2}(\Gamma) \times H^{3/2}(\Gamma)$, let

$$u(x) = \int_\Gamma \big\{ M_y E(x, y)\psi_1(y) + N_y E(x, y)\psi_2 \big\} ds_y \qquad x \in \mathbb{R}^2 \setminus \Gamma.$$

Then $u \in H^2(\Omega, \Delta^2)$, $u \in H^2_{\text{loc}}(\Omega_e, \Delta^2)$ and from (9)–(12) we have that

$$[u]_\Gamma = \psi_2, \quad \left[\frac{\partial u}{\partial n}\right]_\Gamma = \psi_1, \quad [Mu]_\Gamma = 0, \quad [Nu]_\Gamma = 0.$$

Then we can write

$$\begin{aligned} \langle D\Psi, \Psi \rangle &= \int_\Gamma \left(N_x [u] + M_x u \left[\frac{\partial u}{\partial n}\right] \right) ds \\ &= a_\Omega(u, u) + a_{\Omega_e}(\chi u, \chi u) \end{aligned}$$

and the result is obtain by the same argument as in Lemma 3.1 where χ is the cut-off function introduced in Lemma 3.1.

THEOREM 3.3 *Let* $\mathcal{H} := \tilde{H}^{-3/2}(\Gamma_D) \times \tilde{H}^{-1/2}(\Gamma_D) \times \tilde{H}^{1/2}(\Gamma_N) \times \tilde{H}^{3/2}(\Gamma_N)$ *and its dual* $\mathcal{H}^* := H^{3/2}(\Gamma_D) \times H^{1/2}(\Gamma_D) \times H^{-1/2}(\Gamma_N) \times H^{-3/2}(\Gamma_N)$. *Then the operator* $\mathcal{A} : \mathcal{H} \to \mathcal{H}^*$ *is Fredholm with index zero.*

Proof From Lemma 3.1 and 3.2, let $V_0 = V + C_V$ and $D_0 = D + C_D$, where V and D are defined by (20) and (21). Then we know that V_0 and D_0 are bounded below and positive. Since $\Xi := (\tau_D, \sigma_D, \psi_N, \phi_N) \in \tilde{H}^{-3/2}(\Gamma_D) \times \tilde{H}^{-1/2}(\Gamma_D) \times \tilde{H}^{1/2}(\Gamma_N) \times \tilde{H}^{3/2}(\Gamma_N)$ it can be extended by zero to functions $\tilde{\Xi} := (\tilde{\tau}_D, \tilde{\sigma}_D, \tilde{\psi}_N, \tilde{\phi}_N)$ in $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{3/2}(\Gamma)$. Therefore we can write $\mathcal{A}$ in the form

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{C}_\mathcal{A} := \begin{bmatrix} V_0 & M_{DN} \\ M_{ND} & D_0 \end{bmatrix} + \mathcal{C}_\mathcal{A}$$

where $\mathcal{C}_\mathcal{A}$ is compact, and

$$M_{DN} := \begin{bmatrix} V_{12}^{DN} & -K_{11}^{DN} \\ K_{22}^{DN} & D_{21}^{DN} \end{bmatrix} \quad \text{and} \quad M_{ND} := \begin{bmatrix} V_{34}^{ND} & -K_{33}^{ND} \\ K_{44}^{ND} & D_{43}^{DN} \end{bmatrix}.$$

Furthermore from (15)–(18) we have

$$\begin{aligned}
\langle V_{12}^{DN}\psi_N, \tau_D\rangle &= \int_{\Gamma_D} \bar{\tau}_D(x) \int_{\Gamma_N} M_y\, E(x,y)\psi_N(y)\, ds(y)\, ds(x) \\
&= \int_{\Gamma} \tau_D(x) \int_{\Gamma} M_y\, E(x,y)\tilde{\psi}_N(y)\, ds(y)\, ds(x) \\
&= -\int_{\Gamma} \tilde{\psi}_N(y) \int_{\Gamma} M_x\, E(x,y)\tilde{\tau}_D(x)\, ds(x)\, ds(y) \\
&= -\int_{\Gamma_N} \psi_N(y) \int_{\Gamma_D} M_x\, E(x,y)\tau_D(x)\, ds(x)\, ds(y) \\
&= -\langle V_{34}^{ND}\tau_D, \psi_N\rangle.
\end{aligned}$$

In the same manner one can show that

$$\begin{aligned}
\langle K_{11}^{DN}\phi_N, \tau_D\rangle &= \langle K_{44}^{ND}\tau_D, \phi_N\rangle \\
\langle K_{22}^{DN}\psi_N, \sigma_D\rangle &= \langle K_{33}^{ND}\sigma_D, \psi_N\rangle
\end{aligned}$$

and

$$\langle D_{21}^{DN}\phi_N, \sigma_D\rangle = -\langle D_{43}^{ND}\sigma_D, \phi_N\rangle.$$

Finally, combining these equations and using Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}
\langle \mathcal{A}_0\Xi, \Xi\rangle_{\mathcal{H},\mathcal{H}^*} &= \langle V_0(\tau_D,\sigma_D),(\tau_D,\sigma_D)\rangle + \mathrm{Re}\langle D_0(\psi_N,\phi_N),(\psi_N,\phi_N)\rangle \\
&\geq c_1\|(\tau_D,\sigma_D)\|^2_{H^{-3/2}\times H^{-1/2}} + c_2\|(\psi_N,\phi_N)\|^2_{H^{1/2}\times H^{3/2}} \\
&\geq c\|\Xi\|^2_{\mathcal{H}}
\end{aligned}$$

for any $\Xi \in \mathcal{H}$ where $c > 0$ is a constant. Hence $\mathcal{A}$ is a Fredholm operator with index zero. In particular the uniqueness of (19) implies the existence of the solution to (19).

The next theorem establishes the uniqueness of (19).

THEOREM 3.4 *The kernel of the operator* $\mathcal{A}: \mathcal{H} \to \mathcal{H}^*$ *is zero.*

Let $\Xi := (\tau_D, \sigma_D, \psi_N, \phi_N) \in \tilde{H}^{-3/2}(\Gamma_D) \times \tilde{H}^{-1/2}(\Gamma_D) \times \tilde{H}^{1/2}(\Gamma_N) \times \tilde{H}^{3/2}(\Gamma_N)$ be a solution to the homogeneous equation $\mathcal{A}\Xi = 0$, and $\tilde{\Xi} := (\tilde{\tau}_D, \tilde{\sigma}_D, \tilde{\psi}_N, \tilde{\phi}_N)$ in

$H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{3/2}(\Gamma)$ be the extension by zero. The potential w defined by

$$\begin{aligned} w(x) &= \mathcal{V}(\tilde{\sigma}_D, \tilde{\tau}_D)(x) - \mathcal{W}(\tilde{\phi}_N, \tilde{\psi}_N)(x) \qquad x \in \mathbb{R}^2 \setminus \Gamma \\ &= \int_\Gamma \left\{ E(x,y)\tilde{\tau}_D + \frac{\partial E(x,y)}{\partial n_y}\tilde{\sigma}_D \right\} ds_y \\ &\quad - \int_\Gamma \left\{ M_y E(x,y)\tilde{\psi}_N(y) + N_y E(x,y)\tilde{\phi}_N(y) \right\} ds_y \end{aligned} \tag{22}$$

is in $H^2(\Omega, \Delta^2)$ and $H^2_{\text{loc}}(\Omega_c)$ and satisfies the biharmonic equation. Now let $x \to \Gamma$ from inside Ω, using the jump relations (9)–(10) we obtain:

$$\begin{aligned} w|_\Gamma &= V_{14}\tilde{\tau}_D + V_{13}\tilde{\sigma}_D + V_{12}\tilde{\psi}_N + \left[\frac{1}{2}\tilde{\phi}_N - K_{11}\tilde{\phi}_N\right] \\ \left.\frac{\partial w}{\partial n}\right|_\Gamma &= V_{24}\tilde{\tau}_D + V_{23}\tilde{\sigma}_D + \left[\frac{1}{2}\tilde{\psi}_N + K_{22}\tilde{\psi}_N\right] + D_{21}\tilde{\phi}_N \\ Mw|_\Gamma &= V_{34}\tilde{\tau}_D + \left[\frac{1}{2}\tilde{\sigma}_D - K_{33}\tilde{\sigma}_D\right] + D_{32}\tilde{\psi}_N + D_{31}\tilde{\phi}_N \\ Nw|_\Gamma &= \left[\frac{1}{2}\tilde{\tau}_D + K_{44}\tilde{\tau}_D\right] + D_{43}\tilde{\sigma}_D + D_{42}\tilde{\psi}_N + D_{41}\tilde{\phi}_N. \end{aligned}$$

Using the fact that $\operatorname{supp}\sigma_D$, $\operatorname{supp}\tau_D$ are in Γ_D and $\operatorname{supp}\phi_N$, $\operatorname{supp}\psi_N$ are in Γ_N, the integral equation $\mathcal{A}\Xi = 0$ implies that

$$w|_{\Gamma_D} = 0, \quad \left.\frac{\partial w}{\partial n}\right|_{\Gamma_D} = 0, \quad Mw|_{\Gamma_N} = 0, \quad Nw|_{\Gamma_N} = 0.$$

The latter means that (22) is a weak solution to homogeneous interior mixed boundary value for biharmonic equation, and hence, from Theorem 2.2, $w = 0$ in Ω.

Now, using Green's representation formula (8) for $w \in H^2(\Omega)$ we have

$$0 = \mathcal{V}(Mw, Nw)(x) - \mathcal{W}(w, \frac{\partial w}{\partial n})(x), \quad x \in \Omega, \tag{23}$$

whence from the above boundary conditions $\Xi := (\tau_D, \sigma_D, \psi_N, \phi_N) \in \tilde{H}^{-3/2}(\Gamma_D) \times \tilde{H}^{-1/2}(\Gamma_D) \times \tilde{H}^{1/2}(\Gamma_N) \times \tilde{H}^{3/2}(\Gamma_N)$ in the kernel of $\mathcal{A}$ satisfies

$$\begin{aligned} 0 &= \int_{\Gamma_D} \left\{ E(x,y)\tau_D(y) + \frac{\partial E(x,y)}{\partial n_y}\sigma_D \right\} ds_y \\ &\quad - \int_{\Gamma_N} \left\{ M_y E(x,y)\psi_N(y) + N_y E(x,y)\phi_N(y) \right\} ds_y, \quad x \in \Omega. \end{aligned} \tag{24}$$

Let us denote by

$$h(x) = \int_{\Gamma_N} \{M_y E(x,y)\psi_N(y) + N_y E(x,y)\phi_N(y)\} ds_y. \tag{25}$$

From the mapping properties of the potentials, we have that $h \in H^2(\mathbb{R}^2 \setminus \Gamma)$ and from the asymptotic behavior of the biharmonic double layer potential $h(x) = O(r)$ as $r = |x| \to \infty$. Therefore from the uniqueness of the exterior Dirichlet problem for the biharmonic equation with the required growth conditions (see [2]), we can represent h in the form of a simple potential

$$h(x) = \int_{\Gamma} \left\{ E(x,y)\kappa(y) + \frac{\partial E(x,y)}{\partial n_y} \xi(y) \right\} ds_y \tag{26}$$

with densities $\xi \in H^{-1/2}(\Gamma)$ and $\kappa \in H^{-3/2}(\Gamma)$ subject to the constraints

$$\int_{\Gamma} \kappa(y) ds_y = 0 \quad \int_{\Gamma} y_i\, \kappa(y) + n_i\, \xi(y) ds_y = 0 \qquad i = 1, 2. \tag{27}$$

The jump relations imply that

$$\xi = Mh^+ - Mh^-, \qquad \kappa = Nh^+ - Nh^- \qquad \text{on } \Gamma$$

where the $\mp$ signs correspond to the interior and the exterior domain, respectively. From the definition of h given by (25) we see that $\xi = 0$ and $\kappa = 0$ on Γ_D, i.e., their support is included in $\overline{\Gamma}_N$. Inserting (26) into (24) gives

$$\mathcal{V}(\xi, \kappa) = -\mathcal{V}(\tilde{\sigma}_D, \tilde{\tau}_D) \quad \text{in } \Omega$$

which implies that

$$\xi = -\tilde{\sigma}_D \quad \text{and} \quad \kappa = -\tilde{\tau}_D \quad \text{on } \Gamma$$

are the unique solutions because of (27) (see [2]) , where $\tilde{\tau}_D \in H^{-3/2}(\Gamma)$ and $\tilde{\sigma}_D \in H^{-1/2}(\Gamma)$ are the extension by zero to the whole Γ of τ_D and σ_D. But since the support of $\tilde{\sigma}_D$ and $\tilde{\tau}_D$ intersect the support of ξ and κ only in a finite set of boundary points we can conclude that

$$\tilde{\sigma}_D = \tilde{\tau}_D = \xi = \kappa = 0.$$

Hence

$$h(x) = \int_{\Gamma_N} \{M_y E(x,y)\psi_N(y) + N_y E(x,y)\phi_N(y)\} ds_y = 0 \qquad \text{in } \Omega.$$

The jump relation and the fact that $\mathcal{A}\tilde{\Xi} = 0$, $\tilde{\Xi} := (\tilde{\tau}_D, \tilde{\sigma}_D, \tilde{\psi}_N, \tilde{\phi}_N) = (0, 0, \tilde{\psi}_N, \tilde{\phi}_N)$ now imply

$$0 = V_{12}\tilde{\psi}_N + \left(\frac{1}{2}\tilde{\phi}_N - K_{11}\tilde{\phi}_N\right) = \frac{1}{2}\tilde{\phi}_N$$

$$0 = \left(\frac{1}{2}\tilde{\psi}_N + K_{22}\tilde{\psi}_N\right) + D_{21}\tilde{\phi}_N = \frac{1}{2}\tilde{\psi}_N.$$

So, we have shown that the kernel of $\mathcal{A}$ is zero, which proves the theorem.

Summarizing the above analysis we have proved the following result.

THEOREM 3.5 *Assume that* $0 < \nu < 1$ *and let* $f \in H^{3/2}(\Gamma_D)$, $g \in H^{1/2}(\Gamma_D)$, $p \in H^{-1/2}(\Gamma_N)$, *and* $q \in H^{-3/2}(\Gamma_N)$ *be given. Then the mixed boundary value problem* (1)–(3) *has a weak solution in* $H^2(\Omega, \Delta^2)$. *Moreover the solution satisfies the estimate*

$$\|u\|_{H^2(\Omega)} \leq c(\|f\|_{H^{3/2}(\Gamma_D)} + \|g\|_{H^{1/2}(\Gamma_D)} + \|p\|_{H^{-1/2}(\Gamma_N)} + \|q\|_{H^{-3/2}(\Gamma_N)})$$

with c a positive constant.

Remark In the same way one can treat other type of mixed boundary value problems for the biharmonic equation that corresponds to different physical combinations of the boundary conditions.

4. Remarks on the regularity of the solution

To conclude the article, we now discuss briefly the regularity of the solution of the boundary integral equation. In general, the change of the boundary condition causes the solution of the mixed boundary value problem for the biharmonic equation (1)–(3) to be singular in the neighborhood of the boundary interface points in Γ_c even for C^∞-data. From Kondratjev's theory (see e.g., [9]) we know that in suitable polar coordinates with the origin at a point in Γ_c (where the boundary conditions change) the most singular part of the solution is in general of the form $r^{3/2}F(\theta)$.

However, our main concern in this section is to discuss the singularity of the solution of the boundary integral equation of the first kind $\mathcal{A}\Xi = \mathbf{F}$ near the interface point. Ultimately our solution formula leads to a boundary element method and an understanding of the singularities of the solution of the integral equation is necessary to improve the numerical scheme. In particular, one can employ an augmented Galerkin method where the approximating space is enriched by singular functions that have the same singularity as the solution of the integral equation (see [10] for the Laplace equation).

For sufficiently smooth boundary it can be shown [2,3] that the Calderón operator C_Ω maps $H^{(3/2)+s}(\Gamma) \times H^{(1/2)+s}(\Gamma) \times H^{(-1/2)+s}(\Gamma) \times H^{(-3/2)+s}(\Gamma)$ into itself for $s > 0$. Hence, assuming smooth boundary and smooth boundary data f, g, p, q the right-hand side $\mathbf{F}$ of the integral equation $\mathcal{A}\Xi = \mathbf{F}$ is sufficiently smooth. In order to obtain the local behavior of the solution near the interface point, one may use the localization procedure of Eskin [11] and apply near the interface point in Γ_c the Wiener–Hopf technique which is based on the factorization of the homogeneous elliptic

symbols in half line. Applications of this technique to the boundary integral equations for singular boundary value problem for Helmholtz and Maxwell equations are very well developed by Stephan [12] and [13] and Costabel and Stephan [14].

First we note that one needs to treat only the case when Ω coincides with $y > 0$, Γ_D with $R_- := \{x\colon\ x < 0\}$ and Γ_N with $R_+ := \{x\colon\ x > 0\}$ (here (0, 0) is the interface point). It suffices to consider the following two 2×2 systems

$$p_- V\Theta = \mathbf{h}_1 \text{ on } R_- \quad \text{and} \quad p_+ D\Psi = \mathbf{h}_2 \text{ on } R_+$$

where $p_{(\pm)}$ denotes the restriction to the half line $R_{(\pm)}$, the operators V and D are given by (20) and (21), respectively, and $\mathbf{h}_1$ and $\mathbf{h_2}$ are the respective restrictions of F_1, F_2 and F_3, F_4. Roughly speaking, the original integral equation is separated into two first kind integral equations on half line. Note that $p_- V\Theta = \mathbf{h}_1$ on R_- corresponds to the crack problem on Γ_D with Dirichlet boundary conditions while $p_+ D\Psi = \mathbf{h}_2$ on R_+ corresponds to the crack problem on Γ_N with Neumann boundary condition. By using one-dimensional Fourier transform, straightforward but long calculations [2] show that the above pseudodifferential operators are essentially similar to those in [12,13] whence are suitable to perform a Wiener–Hopf factorization. For instance the principal symbol $\sigma(V)$ of V is given by

$$\sigma(V) = \begin{pmatrix} 1/(4|\xi_1|^3) & 0 \\ 0 & -1/(4|\xi_1|) \end{pmatrix} \qquad (\xi_1, 0) \in \partial\mathbb{R}^2_+$$

where ξ_1 is the dual variable of x of the Fourier transform.

In the following we only state regularity results providing that the boundary Γ is sufficiently smooth. Let $P_1 \in \Gamma_c$ be a point where the boundary condition change and let $f \in H^{(5/2)+\eta}(\Gamma_D)$, $g \in H^{(3/2)+\eta}(\Gamma_D)$, $p \in H^{(1/2)+\eta}(\Gamma_N)$ and $q \in H^{(-1/2)+\eta}(\Gamma_N)$ where $0 < \eta < \frac{1}{2} - \epsilon$ with some $\epsilon > 0$. Then the solution $\Xi = (\tau_D, \sigma_D, \psi_N, \phi_N)$ of the integral equation

$$\mathcal{A}\Xi = \mathbf{F}$$

is such that

$$\tau_D = c_1 \rho^{-3/2} \chi(\rho) + \tau^{\text{reg}} \quad \text{on } \Gamma_D \quad \tau^{\text{reg}} \in \tilde{H}^{-(1/2)+\eta'}(\Gamma_D)$$

$$\sigma_D = c_2 \rho^{-1/2} \chi(\rho) + \sigma^{\text{reg}} \quad \text{on } \Gamma_D \quad \sigma^{\text{reg}} \in \tilde{H}^{(1/2)+\eta'}(\Gamma_D)$$

$$\psi = c_3 \rho^{1/2} \chi(\rho) + \psi^{\text{reg}} \quad \text{on } \Gamma_N \quad \psi^{\text{reg}} \in \tilde{H}^{(3/2)+\eta'}(\Gamma_N)$$

$$\phi = c_4 \rho^{3/2} \chi(\rho) + \phi^{\text{reg}} \quad \text{on } \Gamma_N \quad \phi^{\text{reg}} \in \tilde{H}^{(5/2)+\eta'}(\Gamma_N)$$

where $0 < \eta' < \eta$, ρ is the distance to P_1, χ is a C^∞ cut-off function with $\chi = 1$ for $|\rho| < 1/2$ and $\chi = 0$ for $|\rho| > 1$ and c_1, c_2, c_3, c_4 are constants. The above expression provides a decomposition into regular and singular parts of the solution of the boundary integral equations. In particular, even for C^∞ boundary data we have that the solution exhibits singularities around the interface points in Γ_c which needs to be

taken into consideration in the boundary element approximations if higher order convergence is desired.

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