The Inverse Electromagnetic Scattering Problem for Anisotropic Media ‡

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Abstract. The inverse electromagnetic scattering problem for anisotropic media plays a special role in inverse scattering theory due to the fact that the (matrix) index of refraction is not uniquely determined from the far field pattern of the scattered field even if multi-frequency data is available. In this paper we describe how transmission eigenvalues can be determined from the far field pattern and be used to obtain upper and lower bounds on the norm of the index of refraction. Numerical examples will be given for the case when the scattering object is an infinite cylinder and the inhomogeneous medium is orthotropic.

1. Introduction

The inverse electromagnetic scattering problem for anisotropic media plays a special role in inverse scattering theory. This is due to the fact that the (matrix) index of refraction is not uniquely determined from the far field pattern of the scattered field even if multi-frequency data is available. In particular, it has been shown that only the support of the inhomogeneous media can be uniquely determined [2], [12] and this fact has led to the derivation of the linear sampling method for recovering the support of the scattering obstacle from the measured scattering data [13]. Although the material properties of the scattering obstacle cannot be uniquely determined from the far field data, there remains the possibility of obtaining upper and lower bounds for quantities of physical interest. This problem was investigated in [5] and [6] where bounds for the smallest and largest eigenvalues of the (matrix) index of refraction were obtained in terms of the support of the scattering obstacle and the first transmission eigenvalue of the anisotropic media. Since transmission eigenvalues can be determined from the far field pattern of the scattered wave these results, together with the linear sampling method for

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determining the support, provide new methods for studying the inverse electromagnetic scattering problem for anisotropic media. Indeed, unless severe a priori assumptions are made on the permittivity tensor (e.g. that the permittivity is a constant tensor), to our knowledge sampling methods coupled with the use of transmission eigenvalues is the only available method to date for obtaining information on the permittivity tensor from a knowledge of the far field pattern of the scattered wave. The purpose of this paper is to review these new methods and to provide numerical examples showing the practicality of this new approach for the case of simple two dimensional problems having an anisotropic structure.

The plan of our paper is as follows. In the next section we will formulate the direct scattering problem for time harmonic electromagnetic waves in an anisotropic medium and present recent results on the existence and countability of transmission eigenvalues. We then turn our attention to the inverse scattering problem and discuss the issues of uniqueness and the use of the linear sampling method to determine the support of the scattering obstacle. This is followed by an explanation of how transmission eigenvalues (which can be determined from a knowledge of the far field pattern of the scattered wave) can be used to provide estimates for the largest and smallest eigenvalue of the matrix index of refraction. We will conclude our survey by providing numerical examples in the case when the scattering object is an infinite cylinder and the inhomogeneous medium is orthotropic. As well be seen from the discussion in this paper, new research is needed from both a theoretical and numerical point of view in order to bring this new approach in inverse scattering to fruition!

2. Transmission Eigenvalues

Let $D \subset \mathbb{R}^3$ be a bounded, simply connected open set having piecewise smooth boundary ∂D. The unit normal vector to ∂D directed into the exterior of D is denoted by ν. We assume that the domain D is the support of an anisotropic dielectric object and the incident field is a time-harmonic electromagnetic plane wave with frequency ω. The exterior electric and magnetic fields $\tilde{E}^{ext}, \tilde{H}^{ext}$ and the interior electric and magnetic fields $\tilde{E}^{int}, \tilde{H}^{int}$ satisfy

$$\left.\begin{aligned} \nabla \times \tilde{E}^{ext} - i\omega\mu_0 \tilde{H}^{ext} &= 0 \\ \nabla \times \tilde{H}^{ext} + i\omega\epsilon \tilde{E}^{ext} &= 0 \end{aligned}\right\} \qquad \text{in} \quad \mathbb{R}^3 \backslash \overline{D} \tag{1}$$

$$\left.\begin{aligned} \nabla \times \tilde{E}^{int} - i\omega\mu_0 \tilde{H}^{int} &= 0 \\ \nabla \times \tilde{H}^{int} + i\omega\epsilon(x) \tilde{E}^{int} &= 0 \end{aligned}\right\} \qquad \text{in} \quad D \tag{2}$$

and on the boundary ∂D we assume the continuity of the tangential component of both fields, i.e.

$$\left.\begin{aligned} \tilde{E}^{ext} \times \nu - \tilde{E}^{int} \times \nu &= 0 \\ \tilde{H}^{ext} \times \nu - \tilde{H}^{int} \times \nu &= 0 \end{aligned}\right\} \qquad \text{on} \quad \partial D \tag{3}$$

The electric permittivity ϵ and magnetic permeability μ_0 of the exterior dielectric medium are positive constants whereas the dielectric scatterer has the same magnetic permeability μ_0 as the exterior medium but the (continuous) electric permittivity ϵ is a real 3×3 symmetric matrix valued function. If we define $\tilde{E}^{(ext,int)} = \frac{1}{\sqrt{\epsilon}} E^{(ext,int)}$, $\tilde{H}^{(ext,int)} = \frac{1}{\sqrt{\mu_0}} H^{(ext,int)}, k^2 = \epsilon\mu_0\omega^2$ and $N(x) = \epsilon(x)/\epsilon$, the direct scattering problem for an anisotropic dielectric medium reads

$$\left.\begin{aligned} \nabla \times E^{ext} - ikH^{ext} &= 0 \\ \nabla \times H^{ext} + ikE^{ext} &= 0 \end{aligned}\right\} \quad \text{in} \quad \mathbb{R}^3 \backslash \overline{D} \tag{4}$$

$$\left.\begin{aligned} \nabla \times E^{int} - ikH^{int} &= 0 \\ \nabla \times H^{int} + ikN(x)E^{int} &= 0 \end{aligned}\right\} \quad \text{in} \quad D \tag{5}$$

$$\left.\begin{aligned} E^{ext} \times \nu - E^{int} \times \nu &= 0 \\ H^{ext} \times \nu - H^{int} \times \nu &= 0 \end{aligned}\right\} \quad \text{on} \quad \partial D \tag{6}$$

where

$$E^{ext} = E^s + E^i, \qquad H^{ext} = H^s + H^i \tag{7}$$

and the scattered electric and magnetic fields E^s and H^s satisfy the Silver-Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{8}$$

uniformly in $\hat{x} = x/|x|, r = |x|$. The incident electric field E^i and incident magnetic field H^i are time harmonic plane waves given by

$$E^i(x) := \frac{i}{k} \nabla \times \nabla \times pe^{ikx\cdot d} \;\text{ and }\; H^i(x) := \frac{1}{k^2} \nabla \times pe^{idx\cdot d} \tag{9}$$

where d is a unit vector giving the direction of propagation and p is the polarization vector.

In terms of the electric fields, (4)-(8) become

$$\begin{aligned} \nabla \times \nabla \times E^{ext} - k^2 E^{ext} &= 0 \qquad \text{in} \quad \mathbb{R}^3 \backslash \overline{D} \\ \nabla \times \nabla \times E^{int} - k^2 N(x) E^{int} &= 0 \qquad \text{in} \quad D \end{aligned} \tag{10}$$

$$\left.\begin{aligned} E^{ext} \times \nu - E^{int} \times \nu &= 0 \\ \nabla \times E^{ext} \times \nu - \nabla \times E^{int} \times \nu &= 0 \end{aligned}\right\} \text{on} \quad \partial D \tag{11}$$

$$E^{ext} = E^s + E^i \tag{12}$$

$$\lim_{r\to\infty} (\nabla \times E^s \times x - ikrE^s) = 0 \tag{13}$$

where E^i is given by (9). In [16] it is shown that under the assumption that $N(x)$ is continuous for $x \in \bar{D}$ then (10)-(13) has a unique solution in $H_{loc}(\text{curl}\,, \mathbb{R}^3)$. Moreover, the scattered electric field has the asymptotic behavior [10]

$$E^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right) \right\}$$

as $|x| \to \infty$ where E_∞ is a tangential vector field defined for $\hat{x}$ on the unit sphere Ω and is known as the *electric far field pattern*. Note that $E_\infty(\hat{x}, d, p)$ depends linearly on the polarization p. We define the *electric far field operator* $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))ds(d) \quad , \quad \hat{x} \in \Omega \tag{14}$$

for $g \in L^2_t(\Omega)$ where $L^2_t(\Omega)$ is the space of square integrable tangential vector valued functions defined on Ω. Note that F depends linearly on g and F is injective with dense range provided k is not a *transmission eigenvalue* [6], [14], i.e. a value of k for which the *interior transmission problem*

$$\left.\begin{array}{rcl} \nabla\times\nabla\times E - k^2 N(x)E &=& 0 \\ \nabla\times\nabla\times E_0 - k^2 E_0 &=& 0 \end{array}\right\} \quad \text{in} \quad D \tag{15}$$

$$\left.\begin{array}{r} \nu\times E = \nu\times E_0 \\ \nu\times\nabla\times E = \nu\times\nabla\times E_0 \end{array}\right\} \quad \text{on} \quad \partial D \tag{16}$$

has a nontrivial solution E, E_0 where $E, E_0 \in L^2(D)$ and $E - E_0 \in \mathcal{U}_0(D)$ where

$$\mathcal{U}_0(D) := \{u \in H_0(\mathrm{curl}\,, D) : \nabla\times u \in H_0(\mathrm{curl}\,, D)\}$$

equipped with the inner product

$$(u,v)_{\mathcal{U}_0} = (u,v)_{L^2(D)} + (\nabla\times u, \nabla\times v)_{L^2(D)} + (\nabla\times\nabla\times u, \nabla\times\nabla\times v)_{L^2(D)}$$

If we further assume that N, N^{-1} and either $(N-I)^{-1}$ or $(I-N)^{-1}$ are bounded positive definite real matrix valued functions on D, then as is shown in [7], (15)-(16) is equivalent to finding $u = E - E_0 \in \mathcal{U}_0(D)$ such that

$$(\nabla\times\nabla\times -k^2 N)(N-I)^{-1}(\nabla\times\nabla\times u - k^2 u) = 0 \tag{17}$$

which in variational form can be written as

$$\int_D (N-I)^{-1}(\nabla\times\nabla\times u - k^2 u)\cdot(\nabla\times\nabla\times \overline{v} - k^2 N\overline{v})\,dx = 0 \tag{18}$$

for all $\overline{v} \in \mathcal{U}_0(D)$. Note that the above assumptions on N guarantee that either $1+\alpha \le (\bar{\xi}\cdot N(x)\xi) < \infty$ or $0 < (\bar{\xi}\cdot N(x)\xi) < 1-\beta$ for $x \in \bar{D}$, every $\xi \in \mathbb{C}^3$ such that $||\xi|| = 1$ and some positive constants α and β (recall that $N(x)$ is continuous in $\bar{D}$).

In [6] and [8] it is shown that there exists an infinite discrete set of transmission eigenvalues with $+\infty$ as the only accumulation point. Let us denote by $\mathcal{W}_0(D) := \mathcal{U}_0(D) \cap H_0(\mathrm{div}0, D)$ where

$$H_0(\mathrm{div}0, D) := \left\{u \in L^2(D)^3 :\ \mathrm{div}\, u = 0 \text{ and } \nu\cdot u = 0\right\}.$$

The following decomposition is orthogonal with respect to $L^2(D)^3$-inner product

$$\mathcal{U}_0(D) = \mathcal{W}_0(D) \oplus \left\{u :\ u = \nabla\varphi,\ \varphi \in H^1(D)\right\}.$$

In particular, in [6] and [8] it is shown that the first transmission eigenvalue $k_1 > 0$ is the smallest solution to the algebraic equation

$$\lambda_1(\tau, D, N)) - \tau = 0 \qquad \tau := k^2 \tag{19}$$

where

$$\lambda_1(\tau, D, N) = \inf_{u \in \mathcal{W}_0(D)} \frac{(\mathbb{A}_\tau u, u)_{\mathcal{U}}}{(\mathbb{B}u, u)_{\mathcal{U}}} \tag{20}$$

with $\mathbb{B} : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ and $\mathbb{A}_\tau : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ being the bounded linear operators defined by means of Riesz representation theorem associated with the sesquilinear forms $\mathcal{B}$ and $\mathcal{A}_\tau$ if $1 + \alpha \leq (\bar{\xi} \cdot N(x)\xi) < \infty$, (and $\mathcal{B}$ and $\tilde{\mathcal{A}}_\tau$ if $0 < (\bar{\xi} \cdot N(x)\xi) < 1 - \beta$), where

$$\begin{aligned}
\mathcal{B}(u, v) \;&= \int_D (\nabla \times u) \cdot (\nabla \times \overline{v})\, dx, \\
\mathcal{A}_\tau(u, v) &= \int_D (N - I)^{-1}(\nabla \times \nabla \times u - \tau u) \cdot (\nabla \times \nabla \times \overline{v} - \tau \overline{v})\, dx + \tau^2 \int_D u \cdot \overline{v}\, dx \\
\tilde{\mathcal{A}}_\tau(u, v) &= \int_D N(I - N)^{-1}(\nabla \times \nabla \times u - \tau u) \cdot (\nabla \times \nabla \times \overline{v} - \tau \overline{v})\, dx \\
&\quad + \int_D (\nabla \times \nabla \times u) \cdot (\nabla \times \nabla \times \overline{v})\, dx.
\end{aligned}$$

Note that

$$\text{kernel of } \mathbb{B} = \left\{ u \in \mathcal{U}_0(D) \quad \text{such that } u := \nabla \varphi,\ \varphi \in H^1(D) \right\}.$$

As in the scalar case [1], [4], we expect that the norm of the (regularized) solution to the *far field equation*

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q), \quad \hat{x} \in \Omega,\ z \in D,\ q \in \mathbb{R}^3 \tag{21}$$

where

$$E_{e,\infty}(\hat{x}, z, q) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x} e^{-ik\hat{x}\cdot z}$$

is the far field pattern of the electric field E_e of the electric dipole defined by

$$E_e(x, z, q) := \frac{i}{k} \nabla_x \times \nabla_x \times q\, \Phi(x, z)$$

with

$$\Phi(x, z) := \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x - z|} \qquad x \neq z,$$

should be large if k is a transmission eigenvalue, thus providing us with a method for determining transmission eigenvalues from far field data.

3. The Inverse Scattering Problem

The inverse scattering problem that we are interested in is to determine D and information about $N(x)$ from a knowledge of $E_\infty(\hat{x}, d, p)$ for all $\hat{x}, d \in \Omega$ and three linearly independent polarizations $p_1, p_2, p_3 \in \mathbb{R}^3$. In [2] it is shown that with this knowledge D is uniquely determined by $E_\infty(\hat{x}, d, p)$. However, as in the scalar case, it is expected that this information on $E_\infty(\hat{x}, d, p)$ is not sufficient to uniquely determine the matrix N even if this data is known for an interval of values of k. The determination of D from $E_\infty(\hat{x}, d, p)$ can be obtained by using the *linear sampling method* [13], [15]. In particular from [13] we have the following theorem where $||\cdot||_2$ denotes the Euclidean norm:

Theorem 3.1 *Let α and β be positive constants and assume that either $||N(x)||_2 \geq 1+\alpha$ for all x in $\overline{D}$ or $0 < ||N(x)||_2 \leq 1-\beta$ for all x in $\overline{D}$. Let $q \in \mathbb{R}^3\backslash\{0\}$ be a fixed vector and ϵ a given positive number. Then the following statements are true where E_g is the Herglotz wave function defined by*

$$E_g(x) := \int_\Omega e^{ikx\cdot d} g(d)\,ds(d).$$

(i) Let $z \in D$. Then there exists $g_\epsilon \in L^2_t(\Omega)$ such that

$$||Fg_\epsilon - E_{e,\infty}(\cdot, z, q)|| \leq \epsilon$$

and $\lim_{\epsilon\to 0} ||E_{g_\epsilon}||_{L^2(D)} < \infty$.

(ii) Let $z \in \mathbb{R}^3\backslash\bar{D}$. Then for every $g_\epsilon \in L^2_t(\Omega)$ such that

$$||Fg_\epsilon - E_{e,\infty}(\cdot, z, q)|| \leq \epsilon$$

we have $\lim_{\epsilon\to 0} ||E_{g_\epsilon}||_{L^2(D)} = \infty$.

The linear sampling method is based on keeping the wave number k fixed and determining the support D of the scatterer by "sampling" a region containing D by the point z. On the other hand, if $z \in D$ is kept fixed and k is varied we can use the far field equation (21) to determine the smallest transmission eigenvalue, i.e. the regularized solution of the far field equation will have large norm when k is a transmission eigenvalue (c.f. [1] and [4] for the scalar case). In this case if the Euclidean norm of $N(x)$ is greater than one we can determine a lower bound for the largest eigenvalue $\sigma^*(x)$ for $N(x)$ [5].

Theorem 3.2 *If $||N(x)||_2 \geq 1+\alpha$ for all $x \in D$ and some positive constant α then*

$$\sup_D \sigma^*(x) \geq \frac{\lambda(D)}{k^2} \tag{22}$$

where k is a transmission eigenvalue for (15)-(16) and $\lambda(D)$ is the first eigenvalue of $-\Delta$ in D.

Cakoni, Gintides and Haddar [6] have recently shown how the estimate in Theorem 3.2 can potentially be improved as well as providing estimates in the case when the Euclidean norm of $N(x)$ is less than one. In particular let $\sigma^*(x)$ and $\sigma_*(x)$ be the largest and smallest eigenvalues respectively of the matrix $N(x)$ and define n^* and n_* by $n^* = \sup_D \sigma^*(x)$ and $n_* = \inf_D \sigma_*(x)$ respectively. Let $n > 0$ be a constant such that $n \neq 1$ and let $k_{1,D,n}$ denote the first transmission eigenvalue corresponding to

$$\left.\begin{array}{rcl} \nabla \times \nabla \times W - k^2 n\, W &=& 0 \\ \nabla \times \nabla \times V - k^2 V &=& 0 \end{array}\right\} \quad \text{in} \quad D \tag{23}$$

$$\left.\begin{array}{rcl} W \times \nu &=& V \times \nu \\ \nabla \times W \times \nu &=& \nabla \times V \times \nu \end{array}\right\} \quad \text{on} \quad \partial D \tag{24}$$

which exists according to [6]. The possibility of improving the bound given in Theorem 3.2 rests on the following theorem [6]:

Theorem 3.3 *Let $k_{1,D,N(x)}$ be the first transmission eigenvalue for (15)-(16) and let α and β be positive constants. Denote by k_{1,D,n_*} and k_{1,D,n^*} the first transmission eigenvalue of (23)-(24) for $n = n_*$ and $n = n^*$ respectively.*

(i) If $||N(x)||_2 \geq \alpha > 1$ then $0 < k_{1,D,n^*} \leq k_{1,D,N(x)} \leq k_{1,D,n_*}$

(ii) If $0 < ||N(x)||_2 \leq 1 - \beta$ then $0 < k_{1,D,n_*} \leq k_{1,D,N(x)} \leq k_{1,D,n^*}$.

Proof. We sketch the proof for the case of $||N(x)||_2 \geq \alpha > 1$. Obviously for any $u \in \mathcal{U}_0(D)$ we have

$$\frac{\frac{1}{n^*-1}\|\nabla \times \nabla \times u - \tau u\|_D^2 + \tau^2\|u\|_D^2}{\|\nabla \times u\|_D^2} \tag{25}$$

$$\leq \frac{((N-I)^{-1}(\nabla \times \nabla \times u - \tau u), (\nabla \times \nabla \times u - \tau u))_D + \tau^2\|u\|_D^2}{\|\nabla \times u\|_D^2}$$

$$\leq \frac{\frac{1}{n_*-1}\|\nabla \times \nabla \times u - \tau u\|_D^2 + \tau^2\|u\|_D^2}{\|\nabla \times u\|_D^2}. \tag{26}$$

Therefore we have that for an arbitrary $\tau > 0$

$$\lambda_1(\tau, D, n^*) - \tau \leq \lambda(\tau, D, N(x)) - \tau \leq \lambda_1(\tau, D, n_*) - \tau \tag{27}$$

where $\lambda_1(\tau, D, n^*)$, $\lambda(\tau, D, N(x))$ and $\lambda_1(\tau, D, n_*)$ are given by (20) corresponding to the index of refraction n^*, $N(x)$ and n_*, respectively. Now using (27) for $\tau_1 := k^2_{1,D,n^*}$ we have that $\lambda(\tau_1, D, N(x)) - \tau_1 \geq 0$. Again using (27) for $\tau_2 := k^2_{1,D,n_*}$ we have that $\lambda(\tau_2, D, N(x)) - \tau_2 \leq 0$. Then by continuity of of the mapping $\tau \rightarrow \lambda_1(\tau, D, N(x))$ there is an eigenvalue corresponding to $D, N(x)$ between k_{1,D,n^*} and k_{1,D,n_*}. To complete the proof we need to show that this is the first eigenvalue for $D, N(x)$. Indeed, if $k_{1,D,N(x)} < k_{1,D,n^*}$ then from (27) $\lambda_1(\tau_3, D, n^*) - \tau_3 \leq 0$ for $\tau_3 := k^2_{1,D,N(x)}$. On the other hand from (22) (see also [8]) for $\tau_0 > 0$ sufficiently small we have $\lambda_1(\tau_0, D, n^*) - \tau_0 \geq 0$ which means that there is a transmission eigenvalue for D, n^* less then the first one, which is a contradiction. The theorem now follows. $\square$

Recalling that $k_{1,D,N(x)}$ can be computed from the far field measurements, our approach to estimating n_* and n^* is based on computing a constant n such that $k_{1,D,N(x)}$ is the first transmission eigenvalue corresponding to (23)-(24) for this n. From the above theorem, which shows that transmission eigenvalues for n constant are monotonically decreasing with respect to n, we have that $n_* \le n \le n^*$. To fully justify this idea we need to show that for constant index of refraction the first transmission eigenvalue depends continuously on n. To this end, we assume that $n > 1$ (the case of $0 < n < 1$ can be treated the same way). Then we can re-write (20) as

$$\begin{aligned}\lambda_1(\tau, D, n) &= \inf_{u\in\mathcal{W}_0(D)} \frac{\frac{1}{n-1}\|\nabla\times\nabla\times u - \tau u\|^2_{L^2(D)} + \tau^2\|u\|^2_{L^2(D)}}{\|\nabla\times u\|^2_{L^2(D)}} \\ &= \frac{1}{n-1}\inf_{u\in\mathcal{W}_0(D)} \frac{\|\nabla\times\nabla\times u\|^2_{L^2(D)} + n\tau^2\|u\|^2_{L^2(D)}}{\|\nabla\times u\|^2_{L^2(D)}} - \frac{2\tau}{n-1}.\end{aligned}$$

Hence, from (19) the first transmission eigenvalue $k > 0$ is the smallest zero $\tau := k^2$ of

$$f(\tau, n) := \mu_1(n\tau^2) - (n+1)\tau = 0 \tag{28}$$

where

$$\mu_1(\kappa) = \inf_{\substack{u\in\mathcal{W}_0(D)\\ \|\nabla\times u\|^2_{L^2(D)}=1}} \left(\|\nabla\times\nabla\times u\|^2_{L^2(D)} + \kappa\|u\|^2_{L^2(D)}\right)$$

Lemma 1 *The continuous function $\mu_1 : (0, +\infty) \to (0, +\infty)$ is differentiable and $\mu_1'(\kappa) = \|u_\kappa\|^2_{L^2(D)}$ where $u_\kappa \in \mathcal{W}_0(D)$ satisfies*

$$\|\nabla\times\nabla\times u_\kappa\|^2_{L^2(D)} + \kappa\|u_\kappa\|^2_{L^2(D)} = \mu_1(\kappa).$$

Proof. Take $h \in (-\epsilon, \epsilon)$ and for a fixed $\kappa \in (0, +\infty)$ we have

$$\begin{aligned}\mu_1(\kappa+h) - \mu_1(\kappa) \le &\left(\|\nabla\times\nabla\times u_\kappa\|^2_{L^2(D)} + (\kappa+h)\|u_\kappa\|^2_{L^2(D)}\right) \\ &- \left(\|\nabla\times\nabla\times u_\kappa\|^2_{L^2(D)} + (\kappa+h)\|u_\kappa\|^2_{L^2(D)}\right) \le h\|u_\kappa\|^2_{L^2(D)}\end{aligned}$$

and

$$\begin{aligned}\mu_1(\kappa+h) - \mu_1(\kappa) \ge &\left(\|\nabla\times\nabla\times u_{\kappa+h}\|^2_{L^2(D)} + (\kappa+h)\|u_{\kappa+h}\|^2_{L^2(D)}\right) \\ &- \left(\|\nabla\times\nabla\times u_{\kappa+k}\|^2_{L^2(D)} + (\kappa+h)\|u_{\kappa+h}\|^2_{L^2(D)}\right) \le h\|u_{\kappa+h}\|^2_{L^2(D)}.\end{aligned}$$

Thus we have

$$\|u_{\kappa+h}\|^2_{L^2(D)} \le \frac{\mu_1(\kappa+h) - \mu_1(\kappa)}{h} \le \|u_\kappa\|^2_{L^2(D)}.$$

Now, we want to show that $u_{\kappa+h}$ converges to u_κ in the $L^2(D)$ norm as $h \to 0$. To this end, from the above we have that $\|u_{\kappa+h}\|_{L^2(D)} \le \|u_0\|_{L^2(D)}$ which means that $u_{\kappa+h}$ is bounded in the $L^2(D)$ norm. Furthermore, from

$$\|\nabla\times\nabla\times u_{\kappa+h}\|^2_{L^2(D)} + (\kappa+h)\|u_{\kappa+h}\|^2_{L^2(D)} = \mu_1(\kappa+h),$$

continuity of μ_1 and the fact that $\|\nabla\times u_{\kappa+h}\|_{L^2(D)} = 1$ we conclude that $u_{\kappa+h} \in \mathcal{W}_0(D)$ is bounded with respect to the $\mathcal{U}_0(D)$ in terms of h. Since $\mathcal{W}_0(D)$ is compactly embedded in $H_0(\mathrm{curl}\,, D)$ and in $L^2(D)$ we have that $u_{\kappa+h}$ converges to $\tilde{u}$, weakly in the $\mathcal{U}_0(D)$-norm and strongly in the $H_0(\mathrm{curl}\,, D)$-norm and in the $L^2(D)$-norm. But

$$\int_D (\nabla\times\nabla\times u_{\kappa+h})\cdot(\nabla\times\nabla\times\psi)\,dx + (b+h)\int_D u_{\kappa+h}\cdot\psi\,dx$$
$$= \mu(\kappa+h)\int_D (\nabla\times u_{\kappa+h})\cdot(\nabla\times\psi)\,dx \quad \text{for all} \quad \psi\in\mathcal{W}_0(D). \qquad (29)$$

Letting $h\to 0$ in (29), we obtain that $\tilde{u}$ satisfies

$$\int_D (\nabla\times\nabla\times\tilde{u})\cdot(\nabla\times\nabla\times\psi)\,dx + b\int_D \tilde{u}\cdot\psi\,dx = \mu(\kappa)\int_D (\nabla\times\tilde{u})\cdot(\nabla\times\psi)\,dx$$

for all $\psi\in\mathcal{W}_0(D)$, whence $\tilde{u} = u_\kappa$ and the strong $L^2(D)$ convergence yields the result. □

To deduce that the first transmission eigenvalue $k_1(n)$ is a continuous function of n for $n > 1$ we can now apply the implicit function theorem to (28) in a neighborhood of $(\tau_1(n), n)$ where $\tau_1(n) = k_1^2(n)$, provided that

$$\frac{\partial f}{\partial\tau} := 2n\tau_1\|u_{n,\tau_1^2}\|^2_{L^2(D)} - (n+1) \neq 0.$$

In particular, since the divergence of u_{n,τ_1^2} is zero, from the Poincarè inequality and our normalization of the eigenfunction we have that $\|u_{n,\tau_1^2}\|^2_{L^2(D)} \leq 1/\lambda(D)\|\nabla\times u_{n,\tau_1^2}\|^2_{L^2(D)} = 1/\lambda(D)$, where $\lambda(D)$ is the first Dirichlet eigenvalue of the negative Laplasian in D. Hence $\frac{\partial f}{\partial\tau} < 0$ provided that $\tau_1 < \frac{n+1}{2n}\lambda(D)$.

In the next section of this paper we will use the above three theorems to obtain estimates on the eigenvalues of $N(x)$ in the case of scattering by an orthotropic infinite cylinder.

4. Numerical Experiments

In this section we present some numerical results that illustrate the use of the techniques outlined in the previous section. We will not show examples of reconstructing the domain D since this has been investigated exhaustively in the past [3]. Instead we shall assume that D is known and examine the problem of estimating the anisotropic permittivity using the technique described after the proof of Theorem 3.3.

We restrict ourselves here to the two dimensional problem of scattering from an orthotropic medium. Suppose D is an infinite cylinder with cross section D_2 in the (x_1, x_2) plane (in this section $x = (x_1, x_2)$) and suppose the magnetic field is transverse by which we mean that $H^{ext} = (0, 0, u^{ext}(x))$, $H^s = (0, 0, u^s(x))$ and $H^i = (0, 0, u^i(x))$

where $u^i(x) = \exp(ik(d_1x_1 + d_2x_2))$, $d_1^2 + d_2^2 = 1$. If, in addition, the relative electric permittivity is assumed to be orthotropic so that $N(x)$ given by

$$N(x) = \begin{pmatrix} n_{1,1} & n_{1,2} & 0 \\ n_{1,2} & n_{2,2} & 0 \\ 0 & 0 & n_{3,3} \end{pmatrix}, \quad \det \begin{vmatrix} n_{1,1} & n_{1,2} \\ n_{1,2} & n_{2,2} \end{vmatrix} \neq 0,$$

then it is shown in Section 5 of [3] that u^s and u^{int} satisfy the Helmholtz system

$$\nabla \cdot A\nabla u^{int} + k^2 u^{int} = 0 \text{ in } D_2, \tag{30}$$

$$\Delta u^s + k^2 u^s = 0 \text{ in } D_{2,e} := \mathbb{R}^2 \setminus \overline{D}_2, \tag{31}$$

where

$$A = \frac{1}{n_{1,1}n_{2,2} - n_{1,2}^2} \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{1,2} & n_{2,2} \end{pmatrix}.$$

In addition the transmission conditions (11) become

$$\frac{\partial}{\partial \nu}(u^s + u^i) = \frac{\partial}{\partial \nu_A} u^{int} \text{ on } \Gamma_2 := \partial D_2, \tag{32}$$

$$u^i + u^s = u^{ext} \text{ on } \Gamma_2, \tag{33}$$

where ν now denotes the two dimensional unit outward normal to D_2 and $\partial u^{int}/\partial \nu_A = \nu \cdot (A\nabla u^{int})$. In addition u^s must satisfy the standard Sommerfeld radiation condition.

Given a suitable domain D_2 and matrix A, this problem is well posed and we can define the far field pattern of the scattered field defined as usual by

$$u^s(x) = \frac{\exp ik|x|}{\sqrt{|x|}} \left(u_\infty(\hat{x}, d) + O(1/|x|) \right).$$

The analogue of the electric far field operator is then $F : L^2(\Omega_2) \to L^2(\Omega_2)$

$$(Fg)(\hat{x}) = \int_{\Omega_2} u_\infty(\hat{x}, d) g(d)\, ds(d)$$

where Ω_2 is the unit circle in two dimensions.

The far field operator is injective with dense range provided k is not an interior transmission eigenvalue for the two dimensional domain. From (15), (16), k is a transmission eigenvalue if the problem

$$\begin{aligned} \nabla \cdot A\nabla u + k^2 u &= 0 \text{ in } D_2 \\ \Delta u_0 + k^2 u_0 &= 0 \text{ in } D_2, \\ \frac{\partial}{\partial \nu} u_0 &= \frac{\partial}{\partial \nu_A} u \text{ on } \Gamma_2, \\ u &= u_0 \text{ on } \Gamma \end{aligned} \tag{34}$$

has a non-trivial solution (u, u_0). For a precise statement of these problems and a derivation of the properties claimed here in appropriate Sobolev spaces see [3].

In the following numerical study we approximate u^s and u^{int} using a cubic finite element method with perfectly matched layer, and hence can evaluate an approximation to u_∞ which is then modified by adding random noise as in [9]. We thus have an approximate far field patter u_∞^a. Then choosing N_a uniformly spaced measurement points on the unit circle given by

$$\hat{x}_j = (\cos(2\pi(j-1)/N_a), \sin(2\pi(j-1)/N_a)), \quad 1 \leq j \leq N_a,$$

we define the $N_a \times N_a$ far field matrix $\mathcal{F}$ by $\mathcal{F}_{r,s} = hu_\infty^a(\hat{x}_r, \hat{x}_s)$ for $1 \leq r, s \leq N_a$ where $h = 2\pi/N_a$. We set $N_a = 61$ in this study. The matrix $\mathcal{F}$ discretizes the far field operator.

Then, given any z, we can solve the discrete far field equation $\mathcal{F}g_z = b$ where $b_r = \exp(-ik\hat{x}_r \cdot z)$, $1 \leq r \leq N_a$ (corresponding to the standard continuous far field equation $(Fg_z)(\hat{x}) = \exp(-ikz \cdot \hat{x})$). Since the exact far field operator is compact, the discrete equation is increasingly ill-conditioned as N_a increases and so we use the Tikhonov-Morozov scheme outlined in [9] to regularize the problem. Having computed an approximate solution $\vec{g}_z$ we can then use $|h^{1/2}\vec{g}_z|_{l^2}$ to approximate the norm of the Herglotz kernel $g_z \in L^2(\Omega)$ resulting from solving $Fg_z = \exp(-ik\hat{x} \cdot z)$ approximately. A graph of $|h^{1/2}\vec{g}_z|_{l^2}$ against wave number k should then reveal the transmission eigenvalues according to [4].

In order to help verify these transmission eigenvalues, and in order to test the algorithm following the proof of Theorem 3.3, we need to be able to calculate transmission eigenvalues directly. This is done by an extension of one of the methods in [11]: the continuous finite element method. We choose this method because, on a given grid, it results in a smaller linear system than any of the other methods in that paper. The method is based on decomposing $H^1(D_2) = H_0^1(D_2) \oplus S$ where S is the H^1 orthogonal complement of $H_0^1(D_2)$. Then we can write the transmission eigenvalue problem (34) as the problem of finding $u^I \in H^1(D_2)$, $u^B \in S$ and $u_0^I \in H_0^1$ such that $u = u^I + u^B$ and $u_0 = u_0^I + u^B$. The relevant weak formulation is to find $(u^I, u_0^I, u^B) \in H_0^1(D_2)^2 \times S$ such that

$$\int_{D_2} A\nabla(u^I + u^B) \cdot \xi - k^2(u^I + u^B)\xi \, dA = 0 \quad \text{for all } \xi \in H_0^1(D_2) \tag{35}$$

$$\int_{D_2} \nabla(u_0^I + u^B) \cdot \chi - k^2(u_0^I + u^B)\chi \, d = 0 \qquad \text{for all } \chi \in H_0^1(D_2) \tag{36}$$

$$\int_{D_2} A\nabla(u^I + u^B) \cdot \mu - k^2(u^I + u^B)\mu, dA =$$

$$\int_{D_2} A\nabla(u_0^I + u^B) \cdot \mu - k^2(u_0^I + u^B)\mu \, dA \text{ for all } \mu \in S. \tag{37}$$

The last equation represents the equality of the normal derivatives on Γ given in (34). Obviously, if k is a transmission eigenvalue then the above system is satisfied provided the corresponding non-zero eigenfunction $(u, u_0) \in H^1(D_2)^2$. This is not known to be generally true for transmission eigenfunctions, except in the case we are considering here [3]. Contrariwise, if $(u^I, u^B, u_0^I) \neq 0$ and k satisfy the above weak formulation, then k is a transmission eigenvalue.

A numerical scheme for computing transmission eigenvalues can be obtained by using finite elements. We mesh the domain D_2 using triangular elements of maximum diameter h and use piecewise linear elements on the resulting triangles to construct a subspace $V^h \subset H^1(D_2)$. The subspace $V_0^h \subset H_0^1(D_2)$ is then constructed in the usual way by setting the degrees of freeedom of the finite element functions on Γ to zero. Then we have $V^h = V_0^h \oplus S_h$ where S_h discretizes S. We seek nontrivial $(u_h^I, u_h^B, u_{0,h}) \in V_0^h \times S_h \times V_0^h$ and k_h which satisfy (35)-(37) (k_h replaces k). We expect that $k_h \to k$ as $h \to 0$, but the convergence theory of the method is not complete. In any case, the resulting discrete generalized eigenproblem does not have symmetric matrices and we have found it to be necessary to solve for all the eigenvalues and eigenvectors of the discrete problem [11]. This full solver limits the number of elements in the mesh because of Matlab memory constraints.

In this study we shall consider three domain used previously in [5]: a circle of unit diameter, the unit square and an L-shaped domain $D_2 = (-0.5, 0.5)^2 \setminus [0, 0.5] \times [0, -0.5]$.

4.1. Isotropic media

We start by choosing A to be an isotropic matrix as in [5]:

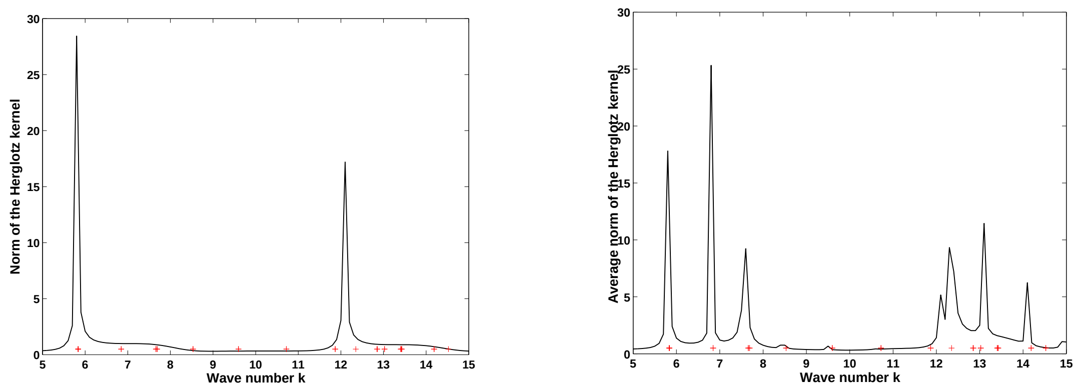
$$A_{iso} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}. \tag{38}$$

We can then solve the forward problem for N_k wave numbers k_j, $1 \leq j \leq N_k$, between k_{min} and k_{max}, then compute the discrete Herglotz kernel as above for randomly chosen z in D. Finally we can read off the lowest positive real eigenvalue. Results for the circle are shown in Fig. 1a) when we choose the single point $z = (0, 0)$ as was used in [5]. We also superimpose crosses to show the first few non=zero and real transmission eigenvalues computed by the finite element method outlined earlier in this section. Clearly many eigenvalues are missing from the graph of the Herglotz kernel (unexpected in view of the theoretical results in [5]). However, by choosing a single source at the origin, only eigenmodes that are proportional to the Bessel function J_0 are probed. In order to detect other eigenmodes, other choices of z are needed, and we use 25 randomly chosen points z in the interior of the scatterer. With this choice, Fig 1b) shows that more eigenvalues are detected.

Turning now to the inverse permittivity problem, once the first eigenvalue $k_{1,D,N(x)}$ is known, we can then find the isotropic $A = aI$ that gives the same first eigenvalue. To do this we solve the problem of finding a such that

$$k_{1,D,a} = k_{1,D,A(x)}^{meas} \tag{39}$$

where $k_{1,D,A(x)}^{meas}$ is the first eigenvalue measured from the Herglotz data, and $k_{1,D,a}$ is the first eigenvalue of the isotropic medium. This is solved by the secant method using starting guesses $a_1 = 1/8$ and $a_2 = 1/2$ together with the method for computing transmission eigenvalues mentioned above. One problem is that this method is currently quite inefficient because of the need to compute all transmission eigenvalues just to obtain the smallest non-zero real transmission eigenvalue. For the case of an isotropic medium we expect to compute a good approximation to A, and this is seen in Table 1.



(a) (b)

Figure 1. Results for the circle using the isotropic coefficient in (38). a) the norm of the Herglotz kernel using the single auxiliary source point $z = (0, 0)$. Crosses show the real transmission eigenvalues predicted by the finite element code. Only the eigenmodes associated with J_0 are visible from the kernel calculation. b) the average norm of the Herglotz kernel using 25 randomly chosen points inside the circle. Now other eigenvalues are detected.

4.2. Anisotropic media

Turning now to the anisotropic case, we choose A to be one of three anisotropies as follows:

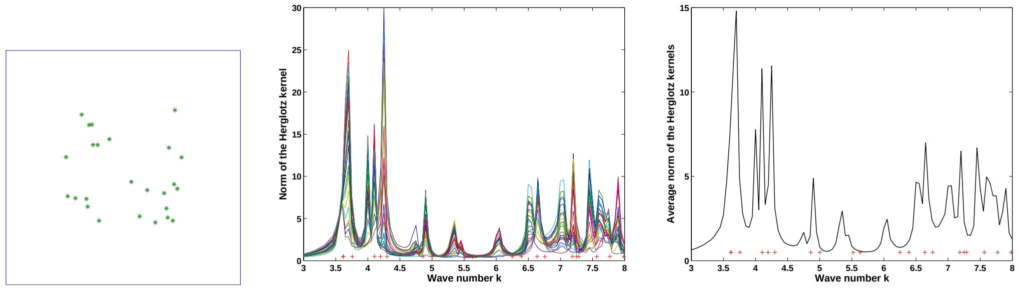
$$A_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/8 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1/6 & 0 \\ 0 & 1/8 \end{pmatrix}, \tag{40}$$

$$A_{2r} = \begin{pmatrix} 0.1372 & 0.0189 \\ 0.0189 & 0.1545 \end{pmatrix}. \tag{41}$$

The first two are diagonal and the third is obtained by rotating matrix A_2 by 1 radian. Thus A_2 and A_{2r} have the same eigenvalues. As in the previous sub-section we can compute the forward problem for a range of wave numbers and predict the first non-zero real eigenvalue from peaks in the graph of the average norm of g against k. Fig. 2b) shows the a composite plot of the norm of the Herglotz kernel $\|g_z\|_{L^2(\Omega_2)}$ for the 25 random points in the square shown in Fig. 2a) using anisotropy A_{2r}. The average norm is plotted in Fig. 2c). It is the graph of the average norm of the Herglotz kernel that we use to determine the first transmission eigenvalue.

Once we have determined $k_{1,D,A(x)}$ we can then compute a by (39) as before. From Theorem 3.3 and the continuity of the eigenvalues, we expect that a should lie between the upper and lower eigenvalues of A. This can be seen from the results in Table 1.

The poorest results are obtained for the L-shape domain. We have noted that, for this shape, the agreement between our estimate of $k_{1,D,A(x)} \approx 6.45$ and the corresponding result in [5] ($k_{1,D,A(x)} \approx 6.3$) is the worst across the domains, and in addition the finite element estimates of this quantity produce a lowest eigenvalue $k_{1,D,A(x)} \approx 6.77$ (see Fig. 3). It appears that this is a challenging numerical problem and will require improved



(a) (b) (c)

Figure 2. Results for the square using anisotropy A_{2r} from (41). a) the position of the 25 random z points. b) a composite plot of all $\|g_z\|_{L^2(D_2)}$ against k for each point. We also mark the computed eigenvalues from the finite element code (shown as + along the bottom of the graph). Good agreement is seen with the lowest computed eigenvalue and the first peak of the norms of g_z. c) The average norm of g_z over all choics of z (and the computed eigenvalues). We use this graph to determine $k_{1,d,A(x)}$ in each case.

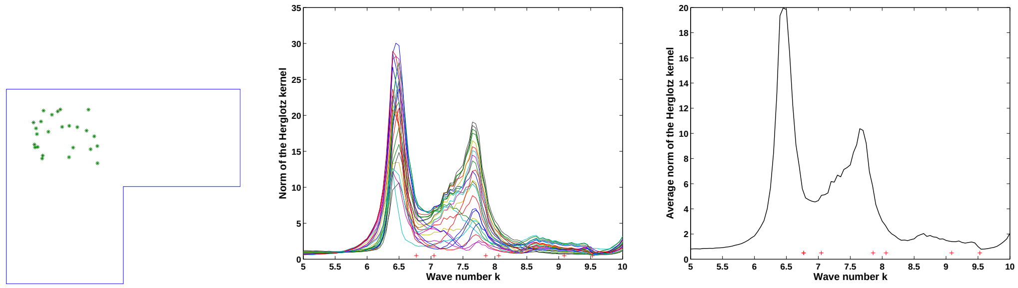
Domain	Matrix	Eigenvalues	Predicted $k_{1,D,A(x)}$	Predicted a
Circle	A_{iso}	1/4,1/4	5.8	0.248
	A_1	1/2,1/8	4.81	0.188
	A_2	1/6,1/8	3.95	0.134
	A_{2r}	1/6,1/8	3.95	0.134
Square	A_{iso}	1/4,1/4	5.3	0.248
	A_1	1/2,1/8	4.1	0.172
	A_2	1/6,1/8	3.55	0.135
	A_{2r}	1/6,1/8	3.7	0.145
L shape	A_{iso}	1/4,1/4	6.45	0.228
	A_1	1/2,1/8	5.2	0.182
	A_2	1/6,1/8	4	0.125
	A_{2r}	1/6,1/8	4.1	0.130

Table 1. Table of results. Our theory implies that the scalar a reconstructed from the first non-zero real transmission eigenvalue should lie between the eigenvalues of the matrix A. In the case of an isotropic A, the predicted a should reconstruct the diagonal of A. The table supports both these claims.

efficiency of the eigenvalue solver (so that we can move to a higher precision) to better characterize the anisotropy.

5. Conclusion

Transmission eigenvalues can be reliably identified from the far field pattern of the scattered wave provided sufficiently many source points are used for the far field equation. Using these eigenvalues we can obtain reconstructions of the electric



(a) (b) (c)

Figure 3. Results for the L-shape using anisotropy A_{iso} from (38). a) the position of the 25 random z points. b) a composite plot of all $\|g_z\|_{L^2(D_2)}$ against k for each point. We also mark the computed eigenvalues from the finite element code (shown as $+$ along the bottom of the graph). Poor agreement is seen with the lowest computed eigenvalue and the first peak of the norms of g_z. c)The average norm of g_z over all choics of z (and the computed eigenvalues).

permittivity (if it is a scalar constant) or an estimate of the eigenvalues of the matrix in the case of anisotropic permittivity (inhomogeneous media can also be considered).

This technique rests on having an efficient and robust method for computing transmission eigenvalues for a scalar permittivity. Unfortunately the current method is restricted because of the need to compute all numerical transmission eigenvalues. Improving the calculation of the transmission eigenvalues is currently an important part of our future research program.

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