THE INTERIOR TRANSMISSION EIGENVALUE PROBLEM*

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Abstract. We consider the inverse problem of determining the spherically symmetric index of refraction $n(r)$ from a knowledge of the corresponding transmission eigenvalues (which can be determined from field pattern of the scattered wave). We also show that for constant index of refraction $n(r) = n$, the smallest transmission eigenvalue suffices to determine n, complex eigenvalues exist for n sufficiently small and, for homogeneous media of general shape, determine a region in the complex plane where complex eigenvalues must lie.

Key words. Interior transmission problem, transmission eigenvalues, inhomogeneous medium, inverse scattering.

AMS subject classifications. 35R30, 35Q60, 35J40, 78A25.

1. Introduction.

The transmission eigenvalue problem has come to play an important role in inverse scattering theory for non-absorbing media. This is due to the fact that these eigenvalues can be determined from the far field pattern of the scattered wave and used to determine lower bounds for the index of refraction [1], [2], [9]. In particular, if $k > 0$ is the wave number and $n(x)$ the index of refraction such that $n(x) = 1$ for $x \in \mathbb{R}^d \setminus \overline{D}$, $d = 2, 3$ where D is simply connected bounded domain with piecewise smooth boundary ∂D, then the *transmission eigenvalue problem* corresponding to the scattering problem (c.f. [8])

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^3 \tag{1.1}$$

$$u(x) = \exp(ikx \cdot d) + u^s(x) \tag{1.2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{1.3}$$

where $r = |x|$ and d is a unit vector, is to determine values of $k > 0$ such that there exists a nontrivial solution to

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in } D \tag{1.4}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{1.5}$$

$$w = v \qquad \text{in } \partial D \tag{1.6}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D \tag{1.7}$$

where ν is the unit outward normal to ∂D. Such values of k are called transmission eigenvalues. It is known that transmission eigenvalues exist and form a discrete set whose only accumulation point is plus infinity [4], [11], [14]. Note that in our definition transmission eigenvalues are positive. Until now it was not known whether or not

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complex eigenvalue can exist. We will provide an example in Section 3 of this paper showing that complex eigenvalues can exist.

In this paper we will consider the inverse spectral problem corresponding to the transmission eigenvalue problem for the special case when the index of refraction is spherically stratified. In particular, we ask the question does a knowledge of all transmission eigenvalues uniquely determine the spherically stratified index of refraction $n(r)$? Under the assumption that the eigenfunction w and v depend only on $r = |x|$, this problem was previously considered by McLaughlin and Polyakov [12] and McLaughlin, Polyakov and Sachs [13] who showed that in general $n(r)$ is uniquely determined only if certain restrictions are made on the magnitude of $n(r)$ and the distribution of the transmission eigenvalues in the complex plane. Here we will show, under the assumption that $n(0)$ is known but without assuming that w and v are spherically stratified, that the transmission eigenvalues (including the possible complex eigenvalues) uniquely determine $n(r)$. We will also show that if $n(r) = n$ is a constant than the smallest real transmission eigenvalue uniquely determines n. Finally, in this case we will determine a region in the complex plane where complex transmission eigenvalues must lie.

2. The Transmission Eigenvalue Problem for Spherically Stratified Media.

2.1. The inverse spectral problem. We are interested in the inverse spectral problem for the interior transmission problem

$$\Delta w + k^2 n(r) w = 0 \qquad \text{in } B \tag{2.1}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } B \tag{2.2}$$

$$w = v \qquad \text{in } \partial B \tag{2.3}$$

$$\frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} \qquad \text{on } \partial B \tag{2.4}$$

where $B := \left\{x \in \mathbb{R}^3 : |x| < a\right\}$ and $n(r) > 1$ or $n(r) < 1$ for $r < a$, $0 < n(r) = 1$ for $r > a$ and $n \in C^2[0, \infty)$. Introducing spherical coordinates (r, θ, φ) we look for solutions of (2.1)-(2.4) in the form

$$v(r, \theta) = a_\ell j_\ell(kr) P_\ell(\cos\theta)$$

$$w(r, \theta) = b_\ell y_\ell(r) P_\ell(\cos\theta)$$

where P_ℓ is Legendre's polynomial, j_ℓ is a spherical Bessel function, a_ℓ and b_ℓ are constants and y_ℓ is a solution of

$$y'' + \frac{2}{r} y' + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) y_\ell = 0$$

for $r > 0$ such that $y_\ell(r)$ behaves like $j_\ell(kr)$ as $r \to 0$, i.e.

$$\lim_{r \to 0} r^{-\ell} y_\ell(r) = \frac{\sqrt{\pi} k^\ell}{2^{\ell+1}\Gamma(\ell + 3/2)}.$$

From [7], pp. 261-264, in particular Theorem 9.9, we can deduce (note that in equation (9.35) of Theorem 9.9, λ should be $\lambda = \ell + \frac{1}{2}$) that k is a (possibly complex) transmission eigenvalue if and only if

$$d_\ell(k) = \det \begin{pmatrix} y_\ell(a) & -j_\ell(ka) \\ y'_\ell(a) & -kj'_\ell(ka) \end{pmatrix} = 0 \tag{2.5}$$

and that $d_\ell(k)$ has the asymptotic behavior

$$d_\ell(k) = \frac{1}{a^2 k\,[n(0)]^{\ell/2+1/4}} \sin k \left(a - \int_0^a [n(r)]^{1/2} dr \right) + O\left(\frac{\ln k}{k^2}\right) \tag{2.6}$$

From [6] (see also [5], pp. 46-50) we can also represent $y_\ell(r)$ in the form

$$y_\ell(r) = j_\ell(kr) + \int_0^r G(r,s,k) j_\ell(ks) ds \tag{2.7}$$

where $G(r,s,k)$ satisfies the Goursat problem

$$r^2 \left[\frac{\partial^2 G}{\partial r^2} + \frac{2}{r}\frac{\partial G}{\partial r} + k^2 n(r) G \right] = s^2 \left[\frac{\partial^2 G}{\partial s^2} + \frac{2}{s}\frac{\partial G}{\partial s} + k^2 G \right] \tag{2.8}$$

$$G(r,r,k) = \frac{k^2}{2r} \int_0^r \rho\, m(\rho) d\rho \tag{2.9}$$

$$G(r,s,k) = O\left((rs)^{1/2} \right) \tag{2.10}$$

and $m := 1 - n$ (see Fig. 2.1). It is shown in [5] and [6] that G can be solved

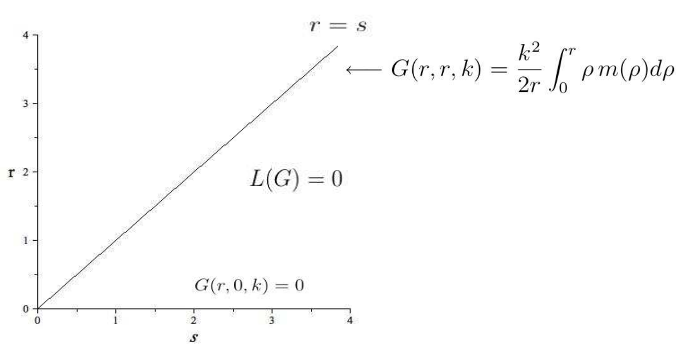


FIG. 2.1. *Configuration of the Goursat problem. Here $L(G) = 0$ denotes (2.8).*

by iteration, is an even function of k and is an entire function of exponential type satisfying

$$G(r,s,k) = \frac{k^2}{2\sqrt{rs}} \int_0^{\sqrt{rs}} \rho\, m(\rho)\, d\rho \left(1 + O(k^2)\right). \tag{2.11}$$

We now return to the determinant (2.5) and compute the coefficient $c_{2\ell+2}$ of the term $k^{2\ell+2}$. A short computation using using (2.5), (2.7), (2.11) and the order estimate

$$j_\ell(kr) = \frac{\sqrt{\pi}(kr)^\ell}{2^{\ell+1}\Gamma(\ell + 3/2)} \left(1 + O(k^2 r^2)\right) \tag{2.12}$$

shows that

$$c_{2\ell+2}\left[\frac{2^{\ell+1}\Gamma(\ell+3/2)}{\sqrt{\pi}a^{(\ell-1)/2}}\right]^2 = a\int_0^a \frac{d}{dr}\left(\frac{1}{2\sqrt{rs}}\int_0^{\sqrt{rs}}\rho\, m(\rho)\,d\rho\right)_{r=a} s^\ell\,ds \qquad (2.13)$$
$$-\,\ell\int_0^a \frac{1}{2\sqrt{as}}\int_0^{\sqrt{as}}\rho\, m(\rho)\,d\rho\, s^\ell\,ds + \frac{a^\ell}{2}\int_0^a \rho\, m(\rho)\,d\rho.$$

After a rather tedious calculation involving a change of variables and interchange of orders of integration, the identity (2.13) remarkably simplifies to

$$c_{2\ell+2} = \frac{\pi a^2}{2^{\ell+1}\Gamma(\ell+3/2)}\int_0^a \rho^{2\ell+2}\, m(\rho)\,d\rho. \qquad (2.14)$$

We now note that $j_\ell(r)$ is odd if ℓ is odd and even if ℓ is even. Hence, since G is an even function of k, we have that $d_\ell(k)$ is an even function of k. Furthermore, since both G and j_ℓ are entire function of k of exponential type, so is $d_\ell(k)$. From the asymptotic behavior of $d_\ell(k)$ for $k\to\infty$, i.e. (2.6), we see that the rank of $d_\ell(k)$ is one and hence by Hadamard's factorization theorem [15],

$$d_\ell(k) = k^{2\ell+2}e^{a_\ell k + b_\ell}\prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(1-\frac{k}{k_{n\ell}}\right)e^{k/k_{n\ell}}$$

where a_ℓ, b_ℓ are constants or, since d_ℓ is even,

$$d_\ell(k) = k^{2\ell+2}c_{2\ell+2}\prod_{n=1}^{\infty}\left(1-\frac{k^2}{k_{n\ell}^2}\right) \qquad (2.15)$$

where $c_{2\ell+2}$ is a constant given by (2.14) and $k_{n\ell}$ are zeros in the right half plane (possibly complex). In particular, $k_{n\ell}$ are the (possibly complex) *transmission eigenvalues* in the right half plane. Thus if the transmission eigenvalues are known so is

$$\frac{d_\ell(k)}{c_{2\ell+2}} = k^{2\ell+2}\prod_{n=1}^{\infty}\left(1-\frac{k^2}{k_{n\ell}^2}\right)$$

as well as (from (2.6)) a nonzero constant γ_ℓ independent of k such that

$$\frac{d_\ell(k)}{c_{2\ell+2}} = \frac{\gamma_\ell}{a^2 k}\sin k\left(a-\int_0^a [n(r)]^{1/2}\,dr\right) + O\left(\frac{\ln k}{k^2}\right),$$

i.e.

$$\frac{1}{c_{2\ell+2}\,[n(0)]^{\ell/2+1/4}} = \gamma_\ell.$$

From (2.14) we now have

$$\int_0^a \rho^{2\ell+2}\, m(\rho)\,d\rho = \frac{\left(2^{\ell+1}\Gamma(\ell+3/2)\right)^2}{[n(0)]^{\ell/2+1/4}\,\gamma_\ell \pi a^2}.$$

If $n(0)$ is given then $m(\rho)$ is uniquely determined by Müntz's theorem [10].

THEOREM 2.1. *If $n(0)$ is given then $n(r)$ is uniquely determined from a knowledge of the transmission eigenvalues.*

2.2. Complex transmission eigenvalues. In the previous section of this paper we showed that for a spherically symmetric index of refraction the real and complex transmission eigenvalues uniquely determine the index of refraction up to a normalizing constant. As noted in the Introduction, the existence of real transmission eigenvalues is well known. This raises the question as to whether or not complex transmission eigenvalues can exist. The following simple example shows that in general complex transmission eigenvalues exist.

Consider the interior transmission problem in a disk Ω of radius one in $\mathbb{R}^2$ and constant index of refraction $n \neq 1$, i.e.

$$\Delta w + k^2 n w = 0 \qquad \text{in} \quad \Omega \tag{2.16}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \quad \Omega \tag{2.17}$$

$$w = v \qquad \text{in} \quad \partial\Omega \tag{2.18}$$

$$\frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} \qquad \text{on} \quad \partial\Omega \tag{2.19}$$

We will show that if n is sufficiently small there exist complex transmission eigenvalues to (2.16)-(2.19). To this end we note that k is a transmission eigenvalue to (2.16)-(2.19) provided

$$d_0(k) = k\left(J_1(k)J_0(k\sqrt{n}) - \sqrt{n}J_0(k)J_1(k\sqrt{n})\right) = 0.$$

Viewing d_0 as a function of $\sqrt{n}$ we compute

$$d_0'(k) = k\left(kJ_1(k)J_0'(k\sqrt{n}) - J_0(k)J_1(k\sqrt{n}) - k\sqrt{n}J_0(k)J_1'(k\sqrt{n})\right)$$

where differentiation is with respect to $\sqrt{n}$. Hence

$$d_0'(k)|_{\sqrt{n}=1} = k\left(kJ_1(k)J_0'(k) - J_0(k)J_1(k) - kJ_0(k)J_1'(k)\right).$$

But $J_0'(t) = -J_1(t)$ and $\frac{d}{dt}\left(tJ_1(t)\right) = tJ_0(t)$ and hence

$$d_0'(k)|_{\sqrt{n}=1} = -k^2\left(J_1^2(k) + J_0^2(k)\right) \tag{2.20}$$

i.e.

$$f(k) = \lim_{\sqrt{n}\to 1^+} \frac{d_0(k)}{\sqrt{n}-1} = -k^2\left(J_1^2(k) + J_0^2(k)\right) \tag{2.21}$$

Since $J_1(k)$ and $J_0(k)$ do not have any common zeros, $f(k)$ is strictly negative for $k \neq 0$ real, i.e. the only zeros of $f(k)$, $k \neq 0$, are complex. Furthermore, $f(k)$ is an even entire function of exponential type that is bounded on the real axis and hence by Hadamard's factorization theorem [15] $f(k)$ has an infinite number of complex zeros. By Hurwitz's theorem in analytic function theory (c.f. [7], p. 213) we can now conclude that for n close enough to one $d_0(k) = 0$ has complex roots, thus establishing the existence of complex transmission eigenvalues for (2.16)-(2.19) for $n > 1$ sufficiently small (Note that by Montel's theorem ([7], p. 213) the convergence in (2.21) is uniform on compact subsets of the complex plane).

3. The Interior Transmission Problem for Homogeneous Media of General Shape.

3.1. A uniqueness result. We now consider the interior transmission problem corresponding to the scattering problem for a homogeneous medium with support $D \in \mathbb{R}^d$, $d = 2, 3$, which satisfies the assumptions in the Introduction, i.e (1.4)-(1.7) with $n(x) := n$ being a constant such that $0 < n \neq 1$. As shown in [4], the homogenous eigenvalue problem (1.4)-(1.7) can be written as the fourth order equation

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0 \tag{3.1}$$

for $u = w - v \in H_0^2(D)$ where

$$H_0^2(D) = \left\{ u \in H^2(D) : \ u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial D \right\}.$$

In the variational form (3.1) is formulated as the problem of finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1} (\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v})\, dx = 0 \qquad \text{for all } v \in H_0^2(D). \tag{3.2}$$

Setting $k^2 := \tau$, it is shown in [3] that $k_{1,n} > 0$ such that $k_{1,n}^2 = \tau_{1,n}$ is the first transmission eigenvalue corresponding to n if and only if $\tau_{1,n}$ is the smallest zero of

$$\lambda(\tau, n) - \tau = 0 \tag{3.3}$$

where

$$\lambda(\tau, n) = \inf_{\substack{u \in H_0^2(D) \\ \|\nabla u\|_D = 1}} \left(\frac{1}{n-1} \|\Delta u + \tau u\|_D^2 + \tau^2 \|u\|_D^2 \right), \qquad \text{if } n > 1 \tag{3.4}$$

$$\lambda(\tau, n) = \inf_{\substack{u \in H_0^2(D) \\ \|\nabla u\|_D = 1}} \left(\frac{n}{1-n} \|\Delta u + \tau u\|_D^2 + \|\Delta u\|_D^2 \right), \quad \text{if } 0 < n < 1. \tag{3.5}$$

Here $\| \cdot \|_D$ denotes the $L^2(D)$-norm. Obviously $\lambda(\tau, n)$ is a continuous function of $\tau \in (0, +\infty)$.

THEOREM 3.1. *The constant index of refraction n is uniquely determined from a knowledge of the corresponding smallest transmission eigenvalue $k_{1,n} > 0$ provided that it is known a priori that either $n > 1$ or $0 < n < 1$.*

Proof. We first consider the case of $n > 1$ and assume that we have two homogeneous media with constant index of refraction n_1 and n_2 such that $1 < n_1 < n_2$. It is obvious that $\lambda(\tau, n_2) \leq \lambda(\tau, n_1)$ for all $\tau > 0$. Now let k_{1,n_1}, be the first transmission eigenvalue for (1.4)-(1.7) with $n(x) := n_1$ and let $u_1 := w_1 - v_1$ where w_1 and v_1 are the corresponding nonzero solution of (1.4)-(1.7). We normalize u_1 such that $\|\nabla u_1\|_D = 1$. Setting $\tau_1 = k_1^2$, we first notice that, by definition (see [3], [4])

$$\frac{1}{n-1} \|\Delta u_1 + \tau_1 u_1\|_D^2 + \tau_1^2 \|u_1\|_D^2 - \tau_1 = 0$$

(u_1 is the minimizer of (3.4)). Furthermore, we have

$$\frac{1}{n_2 - 1} \|\Delta u + \tau u\|_D^2 + \tau^2 \|u\|_D^2 < \frac{1}{n_1 - 1} \|\Delta u + \tau u\|_D^2 + \tau^2 \|u\|_D^2$$

for all $u \in H_0^2(D)$ such that $\|\nabla u\|_D = 1$ and all $\tau > 0$. Now, for $u = u_1$ and $\tau = \tau_1$

$$\frac{1}{n_2 - 1}\|\Delta u_1 + \tau_1 u_1\|_D^2 + \tau_1^2\|u_1\|_D^2 < \frac{1}{n_1 - 1}\|\Delta u_1 + \tau_1 u_1\|_D^2 + \tau_1^2\|u_1\|_D^2 = \lambda(\tau_1, n_1).$$

But

$$\lambda(\tau_1, n_2) \leq \frac{1}{n_2 - 1}\|\Delta u_1 + \tau_1 u_1\|_D^2 + \tau_1^2\|u_1\|_D^2 < \lambda(\tau_1, n_1)$$

and hence for this τ_1 we have a strict inequality, i.e.

$$\lambda(\tau_1, n_2) < \lambda(\tau_1, n_1). \tag{3.6}$$

Next we look for zeros of the equation $\lambda(\tau, n_2) - \tau = 0$. As shown in [3] and [4], for all $\tilde{\tau} > 0$ small enough such that $\tilde{\tau} \in (0,\ \lambda(D)/n_2)$ where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D, we have that $\lambda(\tilde{\tau}, n_2) - \tilde{\tau} > 0$. On the other hand from the above $\lambda(\tau_1, n_2) - \tau_1 < \lambda(\tau_1, n_1) - \tau_1 = 0$. Hence by continuity there is a $\tau_2 \leq \tau_1$ such that $\lambda(\tau_2, n_2) - \tau_2 = 0$, and $k_2 > 0$ such that $k_2^2 = \tau_2$ is a transmission eigenvalue for (1.4)-(1.7) with $n(x) := n_2$. We can choose τ_2 such that $\tau_2 < \tau_1$. Indeed if $\tau_2 = \tau_1$ then $0 = \lambda(\tau_2, n_2) - \tau_2 = \lambda(\tau_1, n_2) - \tau_1$ and since also $\lambda(\tau_1, n_1) - \tau_1 = 0$, we conclude that $\lambda(\tau_1, n_2) = \lambda(\tau_1, n_1)$ which contradicts (3.6). Hence we proved that if $n_1 > 1$ and $n_2 > 1$ are such $n_1 \neq n_2$ then $k_1 \neq k_2$, which proves the uniqueness. The case of $0 < n < 1$ can be treated exactly in the same way using (3.5) for the definition of $\lambda(\tau, n)$. $\square$

3.2. Eigenvalue free zones in the complex plane. Now that we know that complex transmission eigenvalues can exist it is natural to investigate where in the complex plane they may lie. To this end we again consider the case where $n(x) := n$ is a constant and $D \subset \mathbb{R}^d$, $d = 2.3$ is a bounded simply connected region. Here a transmission eigenvalue $k \in \mathbb{C}$ may be a complex number $k := x + iy$. We set $k^2 := \tau + i\mu$, i.e. $\tau := x^2 - y^2$ and $\mu = 2xy$. As mentioned in Section 3.1, the interior transmission eigenvalue problem can be written in the following equivalent variational form

$$\int_D \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v})\, dx = 0 \qquad \text{for all } v \in H_0^2(D). \tag{3.7}$$

We ask the question that under what conditions can we guarantee the uniqueness of the variational equation (3.7). To this let we assume that $n > 1$ (note that similar estimates using the same techniques can be obtained if $0 < n < 1$). Now taking (3.7) for $v = u \in H_0^2(D)$, regrouping terms and integrating by parts we obtain

$$\begin{aligned}
0 = &\int_D \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{u} + k^2 n \overline{u})\, dx = \int_D \frac{1}{n-1}|\Delta u + k^2 n u|^2\, dx \\
&+ \int_D \frac{1}{n-1}(\Delta u + k^2 n u)(k^2 - \overline{k^2}) n \overline{u})\, dx - \int_D k^2 u(\Delta \overline{u} + k^2 n \overline{u})\, dx \\
= &\int_D \frac{1}{n-1}|\Delta u + k^2 n u|^2\, dx + k^2 \int_D |\nabla u|^2\, dx - k^4 \int_D n|u|^2\, dx \\
&- (k^2 - \overline{k^2}) \int_D \frac{n}{n-1}|\nabla u|^2\, dx + k^2(k^2 - \overline{k^2}) \int_D \frac{n^2}{n-1}|u|^2\, dx.
\end{aligned} \tag{3.8}$$

Setting $k^2 := \tau + i\mu$ we now have that

$$0 = \int_D \frac{1}{n-1}|\Delta u + (\tau + i\mu)nu|^2\,dx + \int_D \left(\tau + i\mu - 2\mu\frac{n}{n-1}i\right)|\nabla u|^2\,dx$$
$$- \int_D n\left(\tau^2 - \mu^2 + 2\tau\mu i + 2\mu^2\frac{n}{n-1} - 2\tau\mu\frac{n}{n-1}i\right)|u|^2\,dx. \tag{3.9}$$

Taking the imaginary part of (3.9), and dividing by $\mu \neq 0$ yields

$$0 = -\int_D \frac{n+1}{n-1}|\nabla u|^2\,dx + 2\tau\int_D \frac{n}{n-1}|u|^2\,dx \tag{3.10}$$

whence we obtain that $u = 0$ in D as long as $\tau \le 0$ which in terms of the real and imaginary part of k, x and y respectively, means that $x^2 \le y^2$. Thus a complex number $k = x + iy$ can be a transmission eigenvalue only if $x^2 > y^2$. Now taking the real part of (3.9) we obtain

$$0 = \int_D \frac{1}{n-1}|\Delta u + (\tau + i\mu)nu|^2\,dx + \tau\int_D |\nabla u|^2\,dx$$
$$- \int_D \left[(\tau^2 - \mu^2)n + 2\mu^2\frac{n^2}{n-1}\right]|u|^2\,dx.$$

But

$$\tau\int_D |\nabla u|^2\,dx - \int_D \left[(\tau^2 - \mu^2)n + 2\mu^2\frac{n^2}{n-1}\right]|u|^2\,dx$$
$$\ge \left(\tau\lambda(D) - (\tau^2 - \mu^2)n - 2\mu^2\frac{n^2}{n-1}\right)\|u\|^2_{L^2(D)}. \tag{3.11}$$

Hence we have the uniqueness of the homogeneous interior transmission problem provided that

$$\tau\lambda(D) - (\tau^2 - \mu^2)n - 2\mu^2\frac{n^2}{n-1} \ge 0$$

where $\lambda(D) = \inf_{u\in H_0^1(D)} \frac{\|\nabla u\|^2_{L^2(D)}}{\|u\|^2_{L^2(D)}}$ is the first Dirichlet eigenvalue for $-\Delta$ in D. Thus the real and imaginary part of a complex eigenvalue k must satisfy

$$\tau\lambda(D) - (\tau^2 - \mu^2)n - 2\mu^2\frac{n^2}{n-1} < 0, \tag{3.12}$$

where again $k^2 = \tau + i\mu$. We can rewritten (3.12) as

$$\tau^2 - \frac{\lambda(D)}{n}\tau + \frac{n+1}{n-1}\mu^2 > 0. \tag{3.13}$$

For the case of real transmission eigenvalues (i.e. $\mu = 0$) (3.13) recovers the known Faber-Krahn estimate $k^2 > \lambda(D)/n$ [9]. The relation (3.12) in terms of the real and imaginary part of k, x and y respectively, can be written as

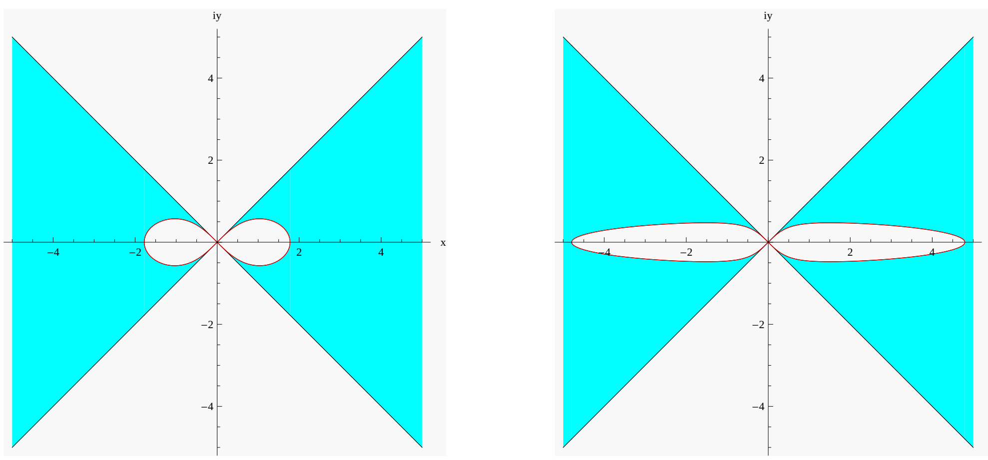
$$x^4 + y^4 + \frac{2n+6}{n-1}x^2y^2 - \frac{\lambda(D)}{n}(x^2 - y^2) > 0. \tag{3.14}$$

Combining both conditions we can conclude that complex transmission eigenvalues $k = x + iy$ (if they exist) lie in the region Σ of the complex plane (x, y) defined by

$$\Sigma := \begin{cases} x^4 + y^4 + \dfrac{2n+6}{n-1}x^2y^2 - \dfrac{\lambda(D)}{n}(x^2 - y^2) > 0 \\ x^2 > y^2. \end{cases} \tag{3.15}$$

The first inequality defines a region in the complex plane in the exterior of a lemniscate centered at the origin, whereas the second inequality defines a symmetric sector about the x-axis the intersection of these two region defines the possible location of transmission eigenvalues (see Fig. 3.1). The lemniscate intersect the real axis at $\pm\sqrt{\lambda(D)/n}$, where $\lambda(D)$ is the smallest Dirichlet eigenvalue of $-\Delta$ in D which states the known fact that the real positive transmission eigenvalues (which are known to exist [4]) are bigger than $\pm\sqrt{\lambda(D)/n}$.

Similar results can be obtained for the case of negative contrast, i.e if $0 < n < 1$. We do not present here the calculations to avoid repetition.



(a) (b)

FIG. 3.1. *A plot of the region in the complex plane $\mathbb{C}$ where transmission eigenvalues lie for the rectangle $D := [-0.5, 0.5] \times [0.4, 0.4]$, and for $n = 8$ in panel (a) and $n = 1.1$ in panel (b). The unshaded region is an eigenvalue free zone; transmission eiegnvalues can lie only in the shaded region. The lemniscate intersects the x-axis at $\pm\sqrt{\lambda(D)/n}$ which corresponds to $\pm\sqrt{25.3/8} = \pm 1.8$ for the case of panel (a) and $\pm\sqrt{25.3/1.1} = \pm 4.8$ for the case of panel (b). This is in agreement with the known location of real positive eigenvalues.*

REMARK 3.1. Similar calculations can be done for smooth $n(x)$ not equal to a constant in which case the lemniscate in Figure 3.1 remains the same but the lines $x = \pm y$ now become hyperbolas with $x = \pm y$ as asymptotes and the shape of the hyperbolas depend on n and ∇n.

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