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The linear sampling method for cracks

Fioralba Cakoni and David Colton[1]

Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

E-mail: colton@math.udel.edu

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Abstract

We consider the inverse scattering problem of determining the shape of an infinite cylinder having an open arc as cross section from a knowledge of the TM-polarized scattered electromagnetic field corresponding to time-harmonic incident plane waves propagating from arbitrary directions. We assume that the arc is a (possibly) partially coated perfect conductor and develop the linear sampling method, which was originally developed for solving the inverse scattering problem for obstacles with nonempty interior, to include the above case of obstacles with empty interior.

1. Introduction

In this paper we consider the scattering of an electromagnetic time-harmonic plane wave by an infinite cylinder having an open arc in $\mathbb{R}^2$ as cross section. We assume that the cylinder is a perfect conductor that is (possibly) coated on one side by a material with surface impedance λ. This corresponds to the situation when a thin cylindrical object is (possibly) coated on one side to avoid detection using probes facing this coating. Assuming that the electric field is polarized in the TM mode, this leads to a (possibly) mixed boundary value problem for the Helmholtz equation defined in the exterior of an open arc in $\mathbb{R}^2$. Our aim is to establish the existence and uniqueness of a solution to this scattering problem and to then use this knowledge to study the inverse scattering problem of determining the shape of the open arc (or 'crack') from a knowledge of the far-field pattern of the scattered field. In particular, we propose to use the *linear sampling method* [5] to arrive at the solution of this inverse scattering problem and difficulties arise from the fact that the crack has an empty interior. The aim of this paper is to overcome this difficulty by using the ideas of Kirch and Ritter [12] who used the *factorization method* to study the case of a perfectly conducting crack.

The inverse scattering problem for cracks was initiated by Kress [14]. In particular, Kress considered the inverse scattering problem for a perfectly conducting crack and used Newton's method to reconstruct the shape of the crack from a knowledge of the far-field

[1] Author to whom any correspondence should be addressed.

pattern corresponding to a single incident wave. In order to do this it is necessary to know the type of singularity the scattered field has at the tip of the crack. The case of a sound-hard crack was considered by Mönch [17]. The investigations initiated by Kress were continued by Kirsch and Ritter in [12] who, as mentioned above, used the factorization method to reconstruct the shape of the open arc from a knowledge of the far-field pattern. The advantage of this approach over Newton's method is that it is no longer necessary to solve a forward problem at each step of an iterative process to reconstruct the arc but instead only involves the solution of an integral equation of the first kind with right-hand side dependent on a 'sampling point' z. However, its implementation requires a knowledge of the far-field pattern for a set of incident directions that is dense on the unit sphere and the method is not applicable to cracks with mixed boundary conditions.

In this paper we shall adapt the linear sampling method, which was originally developed for obstacles with nonempty interior, to the case of cracks (which have empty interior). The advantages of this method over Newton's method and the factorization method are that iterative methods are avoided while at the same time it is possible to consider limited-aperture scattering data and mixed boundary conditions (without knowing *a priori* what these conditions are). However, in contrast to Newton's method, it is still necessary to have data corresponding to incident waves from many directions. The plan of our paper is the following. In the next section we shall use integral equations of the first kind to study the direct scattering problem for both perfectly conducting and partially coated cracks. Then, in section 3, we shall examine the approximation properties of Herglotz wavefunctions in Sobolev spaces defined on open arcs. These results will then be used in section 4 to develop the basis of the linear sampling method for scattering of plane waves by cracks. Central to this analysis is the factorization of the far-field operator into a product of injective operators with dense range and a Herglotz integral operator (cf [8, p 147]). Finally, in section 5, we will present some numerical examples that establish the viability of our approach.

Although the analysis of this paper is done in $\mathbb{R}^2$ all the results remain valid in $\mathbb{R}^3$, i.e. (scalar) screens can be handled in the same way as cracks.

2. The direct scattering problems for cracks

Let $\Gamma \subset \mathbb{R}^2$ be an oriented piecewise smooth nonintersecting arc without cusps, i.e. $\Gamma = \{\varrho(s) : s \in [s_0, s_1]\}$ where $\varrho : [s_0, s_1] \to \mathbb{R}^2$ is an injective piecewise C^1 function. The normal vector pointing to the right side of Γ is denoted by ν and is defined everywhere except a finite number of points on Γ. We denote the right-hand side of Γ with respect to the chosen orientation by Γ^+ and the left-hand side by Γ^-. The scattering of time-harmonic electromagnetic plane waves from a thin infinitely long cylindrical perfect conductor with the electromagnetic field E-polarized leads to the following problem:

$$\Delta U + k^2 U = 0 \qquad \text{in } \mathbb{R}^2\backslash\Gamma \tag{1a}$$

$$U^{\pm} = 0 \qquad \text{on } \Gamma^{\pm}, \tag{1b}$$

where $U^{\pm}(x) = \lim_{h\to 0^+} U(x \pm h\nu)$ for $x \in \Gamma$. The total field U is decomposed $U = u + u^i$ into the given incident field $u^i(x) = \mathrm{e}^{\mathrm{i}kx\cdot d}$, $|d| = 1$, and the unknown scattered field u which is required to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0 \tag{2}$$

uniformly in $\hat{x} = x/|x|$ with $r = |x|$. In the case where the positive side Γ^+ of the thin cylindrical obstacle Γ is coated by a material with surface impedance $\lambda > 0$ we obtain the following mixed crack problem for the total field $U(x) = u(x) + e^{ikx\cdot d}$:

$$\Delta U + k^2 U = 0 \qquad \text{in } \mathbb{R}^2\backslash\Gamma \tag{3a}$$

$$U^- = 0 \qquad \text{on } \Gamma^-, \tag{3b}$$

$$\frac{\partial U^+}{\partial \nu} + ik\lambda U^+ = 0 \qquad \text{on } \Gamma^+, \tag{3c}$$

where again $\frac{\partial U^\pm}{\partial \nu}(x) = \lim_{h\to 0^+} \nu \cdot \nabla U(x \pm h\nu)$ for $x \in \Gamma$, and u satisfies the Sommerfeld radiation condition (2).

We remark that the scattering problem (1*a*), (1*b*) can be seen as a particular case of the scattering problem by a one-side-coated crack (3*a*)–(3*c*) if the surface impedance λ is very large. In this case as $\lambda \to \infty$ the condition (3*c*) says that $U^+ = 0$ as well.

For further considerations we extend the arc Γ to an arbitrary piecewise smooth, simply connected, closed curve ∂D enclosing a bounded domain D such that the normal vector ν on Γ coincides with the outward normal vector on ∂D which we again denote by ν. In order to formulate the above scattering problems more precisely we need to properly define the trace spaces on Γ. The classical reference for such trace spaces is Lions and Magenes [15]. The notation there is different from that in [16] and [19]. However, we use the notation of McLean [16] because this is our main reference for the potential theory needed here. If $H^1_{loc}(\mathbb{R}^2)$, $L^2(\partial D)$, $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$ denote the usual Sobolev spaces we define the following spaces (see [16, p 99, 91]):

$$L^2(\Gamma) := \{u|_\Gamma : u \in L^2(\partial D)\}$$

$$H^{\frac{1}{2}}(\Gamma) := \{u|_\Gamma : u \in H^{\frac{1}{2}}(\partial D)\}$$

$$\tilde{H}^{\frac{1}{2}}(\Gamma) := \{u \in H^{\frac{1}{2}}(\Gamma) : \ \operatorname{supp} u \subseteq \overline{\Gamma}\}.$$

In other words, $\tilde{H}^{\frac{1}{2}}(\Gamma)$ contains functions $u \in H^{\frac{1}{2}}(\Gamma)$ such that their extension by zero to the whole boundary ∂D is in $H^{\frac{1}{2}}(\partial D)$ (theorem 3.33 in [16]). As noted in [4, p 43], and [1] (among many others) $\tilde{H}^{\frac{1}{2}}(\Gamma)$ coincides with the space $H^{\frac{1}{2}}_{00}(\Gamma)$ introduced by Lions and Magenes in [15]:

$$H^{\frac{1}{2}}_{00}(\Gamma) := \{u \in H^{\frac{1}{2}}(\Gamma) : \rho^{-\frac{1}{2}} u \in L^2(\Gamma)\}.$$

Now we denote by $H^{-\frac{1}{2}}(\Gamma)$ the dual space of $\tilde{H}^{\frac{1}{2}}(\Gamma)$ and by $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ the dual space of $H^{\frac{1}{2}}(\Gamma)$. Hence we have the chain

$$\mathcal{D}(\Gamma) \subset \tilde{H}^{\frac{1}{2}}(\Gamma) \subset H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma) \subset \tilde{H}^{-\frac{1}{2}}(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma) \subset \mathcal{D}'(\Gamma)$$

where $\mathcal{D}(\Gamma) := C_0^\infty(\Gamma)$. We note that $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ can also be identified with $H^{-\frac{1}{2}}_{\overline{\Gamma}}(\partial D) := \{u \in H^{-\frac{1}{2}}(\partial D) : \ \operatorname{supp} u \in \overline{\Gamma}\}$ (theorems 3.29 in [16]).

To be able to compare our results with previous work done for the Dirichlet crack problem [12] and [14] we will consider both Dirichlet and mixed crack problems.

Dirichlet crack problem (DCP)

Given $f \in H^{\frac{1}{2}}(\Gamma)$ find $u \in H^1_{loc}(\mathbb{R}^2\backslash\overline{\Gamma})$ such that

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2\backslash\Gamma \tag{4a}$$

$$u^\pm = f \qquad \text{on } \Gamma^\pm \tag{4b}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0. \tag{4c}$$

Mixed crack problem (MCP)

Given $f \in H^{\frac{1}{2}}(\Gamma)$ and $h \in H^{-\frac{1}{2}}(\Gamma)$ find $u \in H^1_{loc}(\mathbb{R}^2\backslash\overline{\Gamma})$ such that

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2\backslash\Gamma \tag{5a}$$

$$u^- = f \qquad \text{on } \Gamma^- \tag{5b}$$

$$\frac{\partial u^+}{\partial \nu} + \mathrm{i}k\lambda u^+ = h \qquad \text{on } \Gamma^+ \tag{5c}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0. \tag{5d}$$

First we establish uniqueness for the DCP and MCP.

Theorem 2.1. *The DCP and MCP have at most one solution.*

Proof. Denote by B_R a sufficiently large ball with radius R containing $\overline{D}$ and by ∂B_R its boundary. Let u be a solution to the homogeneous Dirichlet or MCP, i.e. u satisfies DCP with $f = 0$ or MCP with $f = h = 0$. Obviously, this solution $u \in H^1(B_R\backslash\overline{D}) \cup H^1(D)$ satisfies the Helmholtz equation in $B_R\backslash\overline{D}$ and D and the following transmission conditions on the complementary part $\partial D\backslash\overline{\Gamma}$ of ∂D:

$$u^+ = u^- \qquad \text{and} \qquad \frac{\partial u^+}{\partial \nu} = \frac{\partial u^-}{\partial \nu} \qquad \text{on } \partial D\backslash\overline{\Gamma} \tag{6}$$

where the '+' denotes the limit approaching ∂D from inside D and '−' the limit approaching ∂D from outside of D. An application of the Green formula for u and $\overline{u}$ in D and $B_R\backslash\overline{D}$ and using the transmission conditions (6) then yields

$$\int_{\partial B_R} u\frac{\partial \overline{u}}{\partial \nu}\,\mathrm{d}s = \int_{B_R\backslash\overline{D}} |\nabla u|^2\,\mathrm{d}x + \int_D |\nabla u|^2\,\mathrm{d}x - k^2\int_{B_R\backslash\overline{D}} |u|^2\,\mathrm{d}x$$
$$- k^2\int_D |u|^2\,\mathrm{d}x + \int_\Gamma u^+\frac{\partial \overline{u}^+}{\partial \nu}\,\mathrm{d}s - \int_\Gamma u^-\frac{\partial \overline{u}^-}{\partial \nu}\,\mathrm{d}s. \tag{7}$$

Now, for the DCP the boundary condition ($4b$) implies

$$\int_\Gamma u^+\frac{\partial \overline{u}^+}{\partial \nu}\,\mathrm{d}s - \int_\Gamma u^-\frac{\partial \overline{u}^-}{\partial \nu}\,\mathrm{d}s = 0,$$

while for the MCP, since $k > 0$ and $\lambda > 0$, the boundary conditions ($5c$) and ($5b$) imply

$$\int_\Gamma u^+\frac{\partial \overline{u}^+}{\partial \nu}\,\mathrm{d}s - \int_\Gamma u^-\frac{\partial \overline{u}^-}{\partial \nu}\,\mathrm{d}s = \mathrm{i}k\lambda\int_\Gamma |u^+|^2\,\mathrm{d}s.$$

Hence, for both problems we can conclude that

$$\mathrm{Im}\int_{\partial B_R} u\frac{\partial \overline{u}}{\partial \nu}\,\mathrm{d}s \geqslant 0,$$

whence from [8, theorem 2.12] and a unique continuation argument we obtain that $u = 0$ in $\mathbb{R}^2\backslash\overline{\Gamma}$. □

We define by $[u] := u^+ - u^-$ and $\left[\frac{\partial u}{\partial \nu}\right] := \frac{\partial u^+}{\partial \nu} - \frac{\partial u^-}{\partial \nu}$, the jump of u and $\frac{\partial u}{\partial \nu}$ respectively, across the crack Γ.

Lemma 2.2. *If u is a solution of the DCP and MCP then $[u] \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ and $\left[\frac{\partial u}{\partial \nu}\right] \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$.*

Proof. The proof for the Dirichlet case can be found in [18] and [19]. Following [18] and [19] we give the proof for the MCP. To this end let $u \in H^1_{loc}(\mathbb{R}^2\backslash\overline{\Gamma})$ be a solution to MCP. Then $[u] \in H^{\frac{1}{2}}(\partial D)$ and $\left[\frac{\partial u}{\partial \nu}\right] \in H^{-\frac{1}{2}}(\partial D)$. Now by local regularity for solutions of Helmholtz equation we have that $u \in C^\infty$ away from Γ, whence $[u] = \left[\frac{\partial u}{\partial \nu}\right] = 0$ on $\partial D\backslash\overline{\Gamma}$. The assertion of the lemma now follows from the definition of $\tilde{H}^{\frac{1}{2}}(\Gamma)$ and $\tilde{H}^{-\frac{1}{2}}(\Gamma)$. □

We are now ready to prove the existence of a solution for the above crack problems by using an integral equation approach. We start with the Green representation formula

$$u(x) = \begin{cases} \displaystyle\int_{\partial D} \frac{\partial u(y)}{\partial \nu_y}\Phi(x,y)\,\mathrm{d}s_y - \int_{\partial D} u(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)\,\mathrm{d}s_y, & x \in D \\ \displaystyle -\int_{\partial D} \frac{\partial u(y)}{\partial \nu_y}\Phi(x,y)\,\mathrm{d}s_y + \int_{\partial D} u(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)\,\mathrm{d}s_y, & x \in \mathbb{R}^2\backslash\overline{D} \end{cases} \tag{8}$$

where Φ is the fundamental solution to the Helmholtz equation defined by

$$\Phi(x,y) := \frac{\mathrm{i}}{4}H_0^{(1)}(k|x-y|) \tag{9}$$

with $H_0^{(1)}$ being a Hankel function of the first kind of order zero. Next by making use of the known jump relations of the single- and double-layer potentials across the boundary ∂D [16], and by eliminating the integrals over $\partial D\backslash\overline{\Gamma}$ from (6) we obtain

$$u^- + u^+ = -S_\Gamma\left[\frac{\partial u}{\partial \nu}\right] + K_\Gamma[u] \qquad \text{on} \quad \Gamma \tag{10}$$

$$\frac{\partial u^-}{\partial \nu} + \frac{\partial u^+}{\partial \nu} = -K'_\Gamma\left[\frac{\partial u}{\partial \nu}\right] + T_\Gamma[u] \qquad \text{on} \quad \Gamma, \tag{11}$$

where S, K, K', T are boundary integral operators

$$\begin{aligned} &S : H^{-\frac{1}{2}}(\partial D) \longrightarrow H^{\frac{1}{2}}(\partial D) && K : H^{\frac{1}{2}}(\partial D) \longrightarrow H^{\frac{1}{2}}(\partial D) \\ &K' : H^{-\frac{1}{2}}(\partial D) \longrightarrow H^{-\frac{1}{2}}(\partial D) && T : H^{\frac{1}{2}}(\partial D) \longrightarrow H^{-\frac{1}{2}}(\partial D), \end{aligned}$$

defined by

$$\begin{aligned} &S\psi(x) := 2\int_{\partial D}\psi(y)\Phi(x,y)\,\mathrm{d}s_y && K\psi(x) := 2\int_{\partial D}\psi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)\,\mathrm{d}s_y \\ &K'\psi(x) := 2\int_{\partial D}\psi(y)\frac{\partial}{\partial \nu_x}\Phi(x,y)\,\mathrm{d}s_y && T\psi(x) := 2\frac{\partial}{\partial \nu_x}\int_{\partial D}\psi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)\,\mathrm{d}s_y \end{aligned}$$

and $S_\Gamma, K_\Gamma, K'_\Gamma, T_\Gamma$ are the corresponding operators restricted to Γ. These restricted operators have the mapping properties [16]

$$\begin{aligned} &S_\Gamma : \tilde{H}^{-\frac{1}{2}}(\Gamma) \longrightarrow H^{\frac{1}{2}}(\Gamma) && K_\Gamma : \tilde{H}^{\frac{1}{2}}(\Gamma) \longrightarrow H^{\frac{1}{2}}(\Gamma) \\ &K'_\Gamma : \tilde{H}^{-\frac{1}{2}}(\Gamma) \longrightarrow H^{-\frac{1}{2}}(\Gamma) && T_\Gamma : \tilde{H}^{\frac{1}{2}}(\Gamma) \longrightarrow H^{-\frac{1}{2}}(\Gamma). \end{aligned}$$

In the case of the DCP, since $[u] = 0$ and $u^+ = u^- = f$, the relation (10) gives the following first-kind integral equation for the unknown jump of the normal derivative of the solution across Γ:

$$2f = -S_\Gamma\left[\frac{\partial u}{\partial \nu}\right]. \tag{12}$$

In the case of the MCP the unknowns are both $[u] \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ and $\left[\frac{\partial u}{\partial \nu}\right] \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$. Using the boundary conditions (5*b*) and (5*c*), together with the relations (10) and (11), we obtain the following integral equation of the first kind for the unknowns $[u]$ and $\left[\frac{\partial u}{\partial \nu}\right]$:

$$\begin{pmatrix} S_\Gamma & -K_\Gamma + I \\ K'_\Gamma - I & -T_\Gamma - 2\mathrm{i}k\lambda I \end{pmatrix}\begin{pmatrix} \left[\frac{\partial u}{\partial \nu}\right] \\ [u] \end{pmatrix} = \begin{pmatrix} -2f \\ 2\mathrm{i}k\lambda f - 2h \end{pmatrix}. \tag{13}$$

We define $H := \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$, $H^* := H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ (which is the dual space of H), $\zeta := \left(\left[\frac{\partial u}{\partial \nu}\right], [u]\right) \in H$ and denote by A the matrix operator on the left-hand side of (13). In particular, A maps H continuously into H^*.

Note again that, in the case of the scattering problem, (13) becomes (12) as $\lambda \to \infty$ because $f - \frac{1}{\mathrm{i}k\lambda}h := -\frac{1}{\mathrm{i}k\lambda}\frac{\partial u^{inc}}{\partial \nu} \to 0$ and, from the second equation of (13), $[u] = 0$.

Lemma 2.3. *The operator A is Fredholm with index zero. In addition A has a trivial kernel.*

Proof. Let $\phi \in H^{-\frac{1}{2}}(\partial D)$ and $\psi \in H^{\frac{1}{2}}(\partial D)$ be the extension by zero to ∂D of $\left[\frac{\partial u}{\partial \nu}\right] \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$ and $[u] \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ respectively. It is known [16] that the operators S and $-T$ are positive and bounded below up to a compact perturbation. We denote by L_S and L_T the compact operators

$$L_S : H^{-\frac{1}{2}}(\partial D) \longrightarrow H^{\frac{1}{2}}(\partial D), \qquad L_T : H^{\frac{1}{2}}(\partial D) \longrightarrow H^{-\frac{1}{2}}(\partial D)$$

such that

$$\mathrm{Re}\langle (S + L_S)\phi, \bar{\phi}\rangle \geqslant C\|\phi\|^2_{H^{-\frac{1}{2}}(\partial D)} \qquad \text{for } \phi \in H^{-\frac{1}{2}}(\partial D) \tag{14}$$

$$\mathrm{Re}\langle -(T + L_T)\psi, \bar{\psi}\rangle \geqslant C\|\psi\|^2_{H^{\frac{1}{2}}(\partial D)} \qquad \text{for } \psi \in H^{\frac{1}{2}}(\partial D) \tag{15}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$. We define $S_0 := S + L_S$ and $T_0 := -(T + L_T)$. Then S_0 and T_0 are bounded below and positive. Furthermore, let K_0 and K_0' be the operators corresponding to the Laplace operator, i.e. defined as K and K' with kernel $\Phi(x, y)$ replaced by $\Phi_0(x, y) = -\frac{1}{2\pi}\ln|x - y|$. Then $L_K = K_0 - K$ and $L_{K'} = K_0' - K'$ are compact since they have continuous kernels and K_0 and K_0' are adjoint since their kernels are real. Collecting together all the compact terms we can write $A = (A_0 + L_A)$ where for $\zeta := \left(\left[\frac{\partial u}{\partial \nu}\right], [u]\right) \in H$

$$A_0\zeta = \begin{pmatrix} S_0\phi|_\Gamma + (-K_0 + I)\psi|_\Gamma \\ (K_0' - I)\phi|_\Gamma + (T_0 - 2\mathrm{i}k\lambda I)\psi|_\Gamma \end{pmatrix} \qquad \text{and} \qquad L_A\zeta = \begin{pmatrix} -L_S\phi|_\Gamma + L_K\psi|_\Gamma \\ -L_{K'}\phi|_\Gamma + L_T\psi|_\Gamma \end{pmatrix}.$$

In this decomposition $L_A : H \to H^*$ is compact and $A_0 : H \to H^*$ defines the sesquilinear form

$$\begin{aligned}\langle A_0\zeta, \overline{\zeta}\rangle_{H,H^*} = {} & (S_0\phi, \phi) + (-K_0\psi, \phi) + (\psi, \phi) + (K_0'\phi, \psi) \\ & - (\phi, \psi) + (T_0\psi, \psi) - 2\mathrm{i}k\lambda(\psi, \psi)\end{aligned} \tag{16}$$

where (u, v) denotes the scalar product on $L^2(\Gamma)$ defined by $\int_\Gamma u\bar{v}\,\mathrm{d}s$. We now take the real part of (16). From (14) and (15) and the fact that $\operatorname{supp}\phi \subseteq \Gamma$ and $\operatorname{supp}\psi \subseteq \Gamma$, we obtain

$$\mathrm{Re}[(S_0\phi, \phi) + (T_0\psi, \psi)] \geqslant C\left(\|\phi\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \|\psi\|^2_{H^{\frac{1}{2}}(\Gamma)}\right) = C\|\zeta\|^2_H. \tag{17}$$

Furthermore, since K_0 and K_0' are adjoint, we have

$$\begin{aligned}\mathrm{Re}[(-K_0\psi, \phi) + (K_0'\phi, \psi)] &= \mathrm{Re}[(-K_0\psi, \phi) + (\phi, K_0\psi)] \\ &= \mathrm{Re}[(-K_0\psi, \phi) + \overline{(K_0\psi, \phi)}] = 0\end{aligned} \tag{18}$$

and furthermore

$$\mathrm{Re}[(\psi, \phi) - (\phi, \psi) - 2\mathrm{i}k\lambda(\psi, \psi)] = \mathrm{Re}[(\psi, \phi) - \overline{(\psi, \phi)} - 2\mathrm{i}k\lambda\|\psi\|^2] = 0. \tag{19}$$

Combining (17), (18) and (19) we obtain that the sesquilinear form defined by the operator A_0 is coercive, i.e.

$$\mathrm{Re}\langle (A - L_A)\zeta, \overline{\zeta}\rangle_{H,H^*} \geqslant C\|\zeta\|^2_H \qquad \text{for } \zeta \in \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma), \tag{20}$$

whence the operator A is Fredholm with index zero (theorem 2.33 in [16]).

We now want to show that the kern $A = \{0\}$. To this end let $\zeta = (\alpha, \beta) \in H$ be such that $A\zeta = 0$. Define the potential

$$u(x) = -\int_\Gamma \alpha(y)\Phi(x, y)\,\mathrm{d}s_y + \int_\Gamma \beta(y)\frac{\partial}{\partial \nu_y}\Phi(x, y)\,\mathrm{d}s_y \qquad x \in \mathbb{R}^3\backslash\overline{\Gamma}. \tag{21}$$

This potential is well defined in $\mathbb{R}^2\backslash\overline{\Gamma}$ since the densities α and β can be extended by zero to functions in $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$ respectively. Moreover, $u \in H^1_{loc}(\mathbb{R}^2\backslash\overline{\Gamma})$ satisfies the Helmholtz equation in $\mathbb{R}^2\backslash\overline{\Gamma}$ and the Sommerfeld radiation condition. One can in fact show by computing the jump of u and $\frac{\partial u}{\partial \nu}$ across Γ (cf [16, p 203]) that $\alpha = \left[\frac{\partial u}{\partial \nu}\right]$ and $\beta = [u]$. In particular, the jump relations of the single- and double-layer potentials and the first equation of $A\zeta = 0$ imply

$$2u^-|_\Gamma = -S\left[\frac{\partial u}{\partial \nu}\right] + K[u] - [u] = 0. \tag{22}$$

Moreover, we also have

$$2\frac{\partial u^+}{\partial \nu}\bigg|_\Gamma = -K'\left[\frac{\partial u}{\partial \nu}\right] + T[u] + \left[\frac{\partial u}{\partial \nu}\right]$$

and from the fact that $u^+ = [u]$ on Γ (22) and the second equation of $A\zeta = 0$ we have

$$2\frac{\partial u^+}{\partial \nu} + 2\mathrm{i}k\lambda u^+\bigg|_\Gamma = -K'\left[\frac{\partial u}{\partial \nu}\right] + \left[\frac{\partial u}{\partial \nu}\right] + T[u] + 2\mathrm{i}k\lambda[u] = 0. \tag{23}$$

Hence u defined by (21) is a solution of MCP with zero boundary data and from the uniqueness theorem 2.1 $u \equiv 0$ in $\mathbb{R}^2\backslash\overline{\Gamma}$ and so $\zeta := \left(\left[\frac{\partial u}{\partial \nu}\right], [u]\right) \equiv 0.$ □

It follows from this lemma that the operator A has a bounded inverse $A^{-1} : H^* \to H$. We also note that from (14) it follows that the operator S is a Fredholm operator with index zero and therefore $S^{-1} : H^{\frac{1}{2}}(\Gamma) \to \tilde{H}^{-\frac{1}{2}}(\Gamma)$ exists and it is bounded.

Theorem 2.4. *The DCP has a unique solution. Moreover, this solution satisfies the estimate*

$$\|u\|_{H^1(B_R\backslash\overline{\Gamma})} \leqslant C\left(\|f\|_{H^{\frac{1}{2}}(\Gamma)}\right) \tag{24}$$

where the positive constant C depends on R but not on f.

Proof. Uniqueness follows from theorem 2.1. The solution of the DCP is given by

$$u(x) = -\int_\Gamma \left[\frac{\partial u(y)}{\partial \nu}\right]\Phi(x, y)\,\mathrm{d}s_y, \qquad x \in \mathbb{R}^3\backslash\overline{\Gamma}$$

where $\left[\frac{\partial u(y)}{\partial \nu_y}\right]$ is the unique solution of (12). The estimate (24) is a consequence of the continuity of S^{-1} from $H^{\frac{1}{2}}(\Gamma)$ to $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ and the continuity of the single-layer potential between $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ and $H^1_{loc}(\mathbb{R}^2\backslash\overline{\Gamma})$. □

Theorem 2.5. *The MCP has a unique solution. Moreover, this solution satisfies the estimate*

$$\|u\|_{H^1(B_R\backslash\overline{\Gamma})} \leqslant C\left(\|f\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)} + \|h\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma)}\right) \qquad x \in \mathbb{R}^3\backslash\overline{\Gamma} \tag{25}$$

where the positive constant C depends on R but not on f and h.

Proof. Uniqueness follows from theorem 2.1. The solution of the MCP is given by

$$u(x) = -\int_\Gamma \left[\frac{\partial u(y)}{\partial \nu_y}\right]\Phi(x, y)\,\mathrm{d}s_y + \int_\Gamma [u(y)]\frac{\partial}{\partial \nu_y}\Phi(x, y)\,\mathrm{d}s_y \qquad x \in \mathbb{R}^3\backslash\overline{\Gamma},$$

where $\left(\left[\frac{\partial u}{\partial \nu}\right], [u]\right)$ is the unique solution of (13). The estimate (25) is a consequence of the continuity of A^{-1} from $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ to $\tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$, the continuity of the single-layer potential from $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ to $H^1_{loc}(\mathbb{R}^2\backslash\overline{\Gamma})$ and the continuity of the double-layer potential from $\tilde{H}^{\frac{1}{2}}(\Gamma)$ to $H^1_{loc}(\mathbb{R}^2\backslash\overline{\Gamma})$. □

We end this section with a remark on the regularity of solutions to crack problems. It is in fact known that the solution of the crack problem with Dirichlet boundary conditions has a singularity near a crack tip no matter how smooth the boundary data. In particular, the solution does not belong to $H^{\frac{3}{2}}(\mathbb{R}^2\backslash\overline{\Gamma})$ due to the fact that this solution has a singularity of the form $r^{\frac{1}{2}}\phi(\theta)$, where (r, θ) are the polar coordinates centred at the crack tip [11, 13]. In the case of the crack problem with mixed boundary conditions one would expect a stronger singular behaviour of the solution near the tips. Indeed, for this case the solution of the MCP with smooth boundary data belongs to $H^{\frac{5}{4}-\epsilon}(\mathbb{R}^2\backslash\overline{\Gamma})$ for all $\epsilon > 0$ but not to $H^{\frac{5}{4}}(\mathbb{R}^2\backslash\overline{\Gamma})$ due to the presence of a term of the form $r^{\frac{1}{4}+\mathrm{i}\eta}\phi(\theta)$ in the asymptotic expansion of the solution in a neighbourhood of the crack tip where η is a real number. A complete investigation of crack singularities can be found in the recent paper by Costabel and Dauge [11].

3. Approximation properties

Approximation properties of *Herglotz wavefunctions* are a fundamental ingredient of the *linear sampling method* for solving the inverse problem. A Herglotz wavefunction is a solution of the Helmholtz equation in $\mathbb{R}^2$ of the form

$$v_g(x) := \int_\Omega g(d)\mathrm{e}^{\mathrm{i}kx\cdot d}\,\mathrm{d}s(d) \tag{26}$$

where $\Omega := \{x \in \mathbb{R}^2 : |x| = 1\}$ is the unit sphere and $g \in L^2(\Omega)$ is called the kernel of v_g. In [9] and [10] it is shown that a solution of Helmholtz equation in a bounded domain D with connected boundary can be approximated by a Herglotz wavefunction with respect to the $H^1(D)$ norm. For crack problems we cannot make use of this result. However, we can show that the traces corresponding to DCP or MCP of the solution on the both sides of Γ can be approximated by the appropriate traces of v_g. To this end we define the corresponding trace operator $\mathcal{H} : L^2(\Omega) \to H^* := H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ by

$$\mathcal{H}g(x) := \begin{cases} \displaystyle\int_\Omega g(d)\mathrm{e}^{\mathrm{i}kx\cdot d}\,\mathrm{d}s(d) & x \in \Gamma^- \\ \displaystyle\frac{\partial}{\partial \nu_x}\int_\Omega g(d)\mathrm{e}^{\mathrm{i}kx\cdot d}\,\mathrm{d}s(d) + \mathrm{i}k\lambda\int_\Omega g(d)\mathrm{e}^{\mathrm{i}kx\cdot d}\,\mathrm{d}s(d) & x \in \Gamma^+. \end{cases} \tag{27}$$

Theorem 3.1. *The range of the operator $\mathcal{H} : L^2(\Omega) \to H^*$ is dense.*

Proof. By the change of variables $d \to -d$ it suffices to show that the operator $\tilde{\mathcal{H}} : L^2(\Omega) \to H^*$ defined by

$$\tilde{\mathcal{H}}g(x) := \begin{cases} \displaystyle\int_\Omega g(d)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(d) & x \in \Gamma^- \\ \displaystyle\frac{\partial}{\partial \nu_x}\int_\Omega g(d)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(d) + \mathrm{i}k\lambda\int_\Omega g(d)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(d) & x \in \Gamma^+ \end{cases}$$

has dense range. To this end we need to show that the dual operator $\tilde{\mathcal{H}}^\top : H := \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma) \to L^2(\Omega)$ satisfying

$$\langle \tilde{\mathcal{H}}g, (\alpha, \beta)\rangle_{H^*,H} = \langle g, \tilde{\mathcal{H}}^\top(\alpha, \beta)\rangle_{L^2(\Omega),L^2(\Omega)} \tag{28}$$

for $g \in L^2(\Omega)$ and $(\alpha, \beta) \in H$, is injective. Then the assertion of the theorem follows from the fact that the range of $\tilde{\mathcal{H}}$ can be characterized as [16, p 23]

$$\overline{(\text{range } \tilde{\mathcal{H}})} = {}^a\text{kern } \tilde{\mathcal{H}}^\top$$

where

$${}^a\text{kern } \tilde{\mathcal{H}}^\top := \{(f, h) \in H^* : \langle (f, h), (\alpha, \beta)\rangle_{H^*,H} = 0 \; \forall (\alpha, \beta) \in \text{kern } \tilde{\mathcal{H}}^\top\}.$$

One can easily see from (28) by changing the order of integration that

$$\begin{aligned}\tilde{\mathcal{H}}^\top(\alpha, \beta)(d) := & \int_\Gamma \alpha(x)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s_x + \mathrm{i}k\lambda \int_\Gamma \beta(x)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s_x \\ & + \int_\Gamma \beta(x)\frac{\partial}{\partial \nu_x}\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s_x, \qquad d \in \Omega.\end{aligned}$$

Hence $\tilde{\mathcal{H}}^\top(\alpha, \beta)$ coincides with the far-field pattern of the potential

$$\begin{aligned}\gamma^{-1}V(z) := & \int_\Gamma \alpha(x)\Phi(z, x)\,\mathrm{d}s_x + \mathrm{i}k\lambda \int_\Gamma \beta(x)\Phi(z, x)\,\mathrm{d}s_x \\ & + \int_\Gamma \beta(x)\frac{\partial}{\partial \nu_x}\Phi(z, x)\,\mathrm{d}s_x, \qquad z \in \mathbb{R}^2\backslash\overline{\Gamma}\end{aligned}$$

where $\gamma = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}}$. Note that V is well defined in $\mathbb{R}^2\backslash\overline{\Gamma}$ since the layers α and β can be extended by zero to functions in $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$ respectively. Moreover, $V \in H^1_{loc}(\mathbb{R}^2\backslash\overline{\Gamma})$ satisfies the Helmholtz equation in $\mathbb{R}^2\backslash\overline{\Gamma}$ and the Sommerfeld radiation condition. Now assume that $\tilde{\mathcal{H}}^\top(\alpha, \beta) = 0$. This means that the far-field pattern of V is zero and from Rellich's lemma and the unique continuation principle we conclude that $V = 0$ in $\mathbb{R}^2\backslash\overline{\Gamma}$. By now using the jump relations across ∂D for the single- and double-layer potentials with α and β defined to be zero on $\partial D\backslash\overline{\Gamma}$ we obtain that

$$\beta = [V]_\Gamma \qquad \alpha + \mathrm{i}k\lambda\beta = -\left[\frac{\partial V}{\partial \nu}\right]_\Gamma$$

and hence $\alpha = \beta = 0$. Thus $\tilde{\mathcal{H}}^\top$ is injective and the theorem is proven. □

As a special case of the above theorem we obtain the following theorem.

Theorem 3.2. *Every function in $H^{\frac{1}{2}}(\Gamma)$ can be approximated by the trace of a Herglotz wavefunction $v_g|_\Gamma$ on Γ with respect to the $H^{\frac{1}{2}}(\Gamma)$ norm.*

4. Inverse scattering problem

We now consider the scattering of an electromagnetic time-harmonic wave by a perfectly conducting infinite cylindrical surface that is possibly coated on one side by a material with surface impedance λ. Assuming the electric field is polarized in the TM mode and the plane wave is propagating in the direction d, the scattered field u satisfies MCP with $f := -\mathrm{e}^{\mathrm{i}kx\cdot d}|_{\Gamma^-}$ and $h := -\left(\frac{\partial}{\partial\nu} + \mathrm{i}k\lambda\right)\mathrm{e}^{\mathrm{i}kx\cdot d}\big|_{\Gamma^+}$ (or DCP with $f := -\mathrm{e}^{\mathrm{i}kx\cdot d}|_{\Gamma^\pm}$ if there is no coating). It is easy to show [8] that the scattered field has the asymptotic behaviour

$$u(x) = \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}}u_\infty(\hat{x}, d) + \mathrm{O}(r^{-3/2}) \tag{29}$$

where u_∞ is the *far-field pattern* of the scattered wave $\hat{x} = x/|x|$ and $r = |x|$.

The inverse scattering problem we will consider in this section of our paper is to determine Γ from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}$ and d on the unit circle Ω. By using the ideas of [3] we could also easily consider the limited aperture case where $\hat{x}, d \in \Omega_0 \subset \Omega$. We will adapt the *linear sampling method* developed for scattering from objects with nonempty interior (see [2, 6, 7]) to solve the inverse problem for cracks. Let us define the far-field operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, \mathrm{d}s(d) \tag{30}$$

and consider the far-field equation

$$(Fg) = \Phi^e_\infty, \tag{31}$$

where Φ^e_∞ is the far-field pattern of a suitable (to be defined later) solution to the scattering problem. We want to characterize the crack Γ by the behaviour of an approximate solution g of the far-field equation (31). To understand the far-field equation better we define an operator $\mathcal{B} : H^* \to L^2(\Omega)$ which maps the boundary data $(f, h) \in H^*$ (recall $H^* := H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$) to the far-field pattern of the solution to the corresponding MCP. By superposition we have the following relation:

$$(Fg) = -\mathcal{B}(\mathcal{H}g)$$

where $\mathcal{H}g$ is defined by (27). We now define the compact operator $\mathcal{F} : H \longrightarrow L^2(\Omega)$ by

$$\mathcal{F}(\alpha, \beta)(\hat{x}) = \int_\Gamma \alpha(y) \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\, \mathrm{d}s_y + \int_\Gamma \beta(y) \frac{\partial}{\partial \nu_y} \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\, \mathrm{d}s_y \tag{32}$$

with $H := \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$ and observe that for a given pair $(\alpha, \beta) \in H$, the function $\mathcal{F}(\alpha, \beta)(\hat{x})$ is the far-field pattern of the radiating solution $\gamma^{-1} P(\alpha, \beta)(x)$ of the Helmholtz equation in $\mathbb{R}^2 \backslash \overline{\Gamma}$ where $\gamma = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}}$ and the potential P is defined by

$$P(\alpha, \beta)(x) := \int_\Gamma \alpha(y) \Phi(x, y)\, \mathrm{d}s_y + \int_\Gamma \beta(y) \frac{\partial}{\partial \nu_y} \Phi(x, y)\, \mathrm{d}s_y. \tag{33}$$

Proceeding as in the proof of lemma 2.3 by using the jump relations across ∂D for the single- and double-layer potential with densities extended by zero to ∂D we obtain that $\alpha := -\left[\frac{\partial P}{\partial \nu}\right]_\Gamma$ and $\beta := [P]_\Gamma$. Moreover, P satisfies

$$\begin{pmatrix} P^-(\alpha, \beta)|_{\Gamma^-} \\ \left(\frac{\partial}{\partial \nu} + \mathrm{i}k\lambda\right) P^+(\alpha, \beta)|_{\Gamma^+} \end{pmatrix} = M \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{34}$$

where the operator $M : H \to H^*$ is given by

$$\frac{1}{2} \begin{pmatrix} S_\Gamma & K_\Gamma - I \\ K'_\Gamma - I + \mathrm{i}k\lambda S_\Gamma & T_\Gamma + \mathrm{i}k\lambda(I + K_\Gamma) \end{pmatrix}.$$

The operator M is related to the operator A of section 2 by $M = \frac{1}{2} \begin{pmatrix} I & 0 \\ \mathrm{i}\lambda k I & I \end{pmatrix} A \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, whence $M^{-1} : H^* \to H$ exists and is bounded. In particular, we have that

$$\mathcal{F}(\alpha, \beta) = \gamma^{-1} \mathcal{B} M(\alpha, \beta). \tag{35}$$

In the special case of DCP we have $\mathcal{F}_D(\alpha) = \gamma^{-1} \mathcal{B} S_\Gamma(\alpha)$ where $\alpha \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$, $\mathcal{B} : H^{\frac{1}{2}}(\Gamma) \to L^2(\Omega)$ and $\mathcal{F}_D : \tilde{H}^{-\frac{1}{2}}(\Gamma) \to L^2(\Omega)$ is defined by

$$\mathcal{F}_D(\alpha)(\hat{x}) := \int_\Gamma \alpha(y) \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\, \mathrm{d}s_y. \tag{36}$$

Lemma 4.1. *The operator $\mathcal{F} : H \to L^2(\Omega)$ defined by (32) is injective and has dense range.*

Proof. Injectivity follows from the fact that $\mathcal{F}(\alpha, \beta)$ is the far-field pattern of $P(\alpha, \beta)$ for $(\alpha, \beta) \in H$. Hence $\mathcal{F}(\alpha, \beta) = 0$ implies $P(\alpha, \beta) = 0$ and so $\alpha := -\left[\frac{\partial P}{\partial \nu}\right]_\Gamma = 0$ and $\beta := [P]_\Gamma = 0$. Next the dual operator $\mathcal{F}^\top : L^2(\Omega) \to H^*$ is given by

$$\mathcal{F}^\top g(y) := \begin{cases} \displaystyle\int_\Omega g(\hat{x}) \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} \, \mathrm{d}s(\hat{x}) & y \in \Gamma^- \\ \displaystyle\frac{\partial}{\partial \nu_y} \int_\Omega g(\hat{x}) \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} \, \mathrm{d}s(\hat{x}) & y \in \Gamma^+. \end{cases} \tag{37}$$

As in the proof of theorem 3.1, it is enough to show that $\mathcal{F}^\top$ is injective. In particular, $\mathcal{F}^\top g = 0$ implies that there exists a Herglotz wavefunction v_g (with kernel $g(-\hat{x})$) such that $v_g|_\Gamma = 0$ and $\frac{\partial v_g}{\partial \nu}\big|_\Gamma = 0$ (note that the limit of v_g and its normal derivative from both sides of the crack is the same). From the Green representation formula and the analyticity of v_g the latter implies that $v_g \equiv 0$ in $\mathbb{R}^2$ and therefore $g = 0$. This proves the lemma. □

We obtain a similar result for the operator $\mathcal{F}_D$ corresponding to the DCP. But in this case it has dense range under some restriction. More precisely, the following result holds.

Lemma 4.2. *The operator $\mathcal{F}_D : \tilde{H}^{-\frac{1}{2}}(\Gamma) \to L^2(\Omega)$ defined by (36) is injective. The range of $\mathcal{F}_D$ is dense in $L^2(\Omega)$ if and only if there does not exist a Herglotz wavefunction which vanishes on Γ.*

Proof. The injectivity can be proved in the same way as in lemma 4.1 if one replaces the potential V by the single-layer potential.

The dual operator $\mathcal{F}_D^\top : L^2(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ in this case coincides with $v_g|_\Gamma$. Hence $\mathcal{F}_D^\top$ is injective if and only if there does not exist a Herglotz wavefunction which vanishes on Γ. □

A similar result for the Dirichlet case is obtained by Kress in [14] (theorem 3.2).

From the above analysis we can factorize the far-field operator (30) corresponding to the MCP as

$$(Fg) = -\gamma \mathcal{F} M^{-1} \mathcal{H} g, \qquad g \in L^2(\Omega). \tag{38}$$

In the case of the DCP we have

$$(Fg) = -\gamma \mathcal{F}_D S_\Gamma^{-1} (v_g|_\Gamma), \qquad g \in L^2(\Omega). \tag{39}$$

The following lemma will help us to choose the right-hand side of the far-field equation (31).

Lemma 4.3. *For any piecewise smooth nonintersecting arc L without cusps and two functions $\alpha_L \in \tilde{H}^{-\frac{1}{2}}(L)$, $\beta_L \in \tilde{H}^{\frac{1}{2}}(L)$ we define $\Phi_\infty^L \in L^2(\Omega)$ by*

$$\Phi_\infty^L(\hat{x}) := \int_L \alpha_L(y) \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} \, \mathrm{d}s_y + \int_L \beta_L(y) \frac{\partial}{\partial \nu_y} \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} \, \mathrm{d}s_y. \tag{40}$$

Then, $\Phi_\infty^L(\hat{x}) \in \mathrm{range}(\mathcal{F})$ if and only if $L \subset \Gamma$.

Proof. First assume that $L \subset \Gamma$. Then since $\tilde{H}^{\pm\frac{1}{2}}(L) \subset \tilde{H}^{\pm\frac{1}{2}}(\Gamma)$ it follows directly from the definition of $\mathcal{F}$ that $\Phi_\infty^L(\hat{x}) \in \mathrm{range}(\mathcal{F})$.

Now let $L \not\subset \Gamma$ and assume, on the contrary, that $\Phi_\infty^L(\hat{x}) \in \mathrm{range}(\mathcal{F})$, i.e. there exists $\alpha \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$ and $\beta \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ such that

$$\Phi_\infty^L(\hat{x}) = \int_\Gamma \alpha(y) \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} \, \mathrm{d}s_y + \int_\Gamma \beta(y) \frac{\partial}{\partial \nu_y} \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} \, \mathrm{d}s_y.$$

Hence by Rellich's lemma and the unique continuation principle we have that the potentials

$$\Phi^L(x) = \int_L \alpha_L(y)\Phi(x, y)\,\mathrm{d}s_y + \int_L \beta_L(y)\frac{\partial}{\partial \nu_y}\Phi(x, y)\,\mathrm{d}s_y \qquad x \in \mathbb{R}^2\backslash\overline{L}$$

$$P(x) = \int_\Gamma \alpha(y)\Phi(x, y)\,\mathrm{d}s_y + \int_\Gamma \beta(y)\frac{\partial}{\partial \nu_y}\Phi(x, y)\,\mathrm{d}s_y \qquad x \in \mathbb{R}^2\backslash\overline{\Gamma}$$

coincide in $\mathbb{R}^2\backslash(\overline{\Gamma} \cup \overline{L})$. Now let $x_0 \in L$, $x_0 \notin \Gamma$, and let $B_\epsilon(x_0)$ be a small ball with centre at x_0 such that $B_\epsilon(x_0) \cap \Gamma = \emptyset$. Hence P is analytic in $B_\epsilon(x_0)$ while Φ^L has a singularity at x_0 which is a contradiction. This proves the lemma. □

We note that the statement and proof of lemma 4.3 remain valid for the DCP if we set $\beta_L = 0$ and $\mathcal{F} = \mathcal{F}_D$.

Now let us denote by $\mathcal{L}$ the set of open nonintersecting piecewise smooth arcs without cusps and look for a solution $g \in L^2(\Omega)$ of the far-field equation

$$-\gamma^{-1}Fg = \mathcal{F}M^{-1}\mathcal{H}g = \Phi^L_\infty \qquad \text{for } L \in \mathcal{L} \tag{41}$$

where Φ^L_∞ is the far-field pattern of Φ^L. If $L \subset \Gamma$ then the corresponding (α_L, β_L) is in $H := \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$. Since $M(\alpha_L, \beta_L) \in H^* := H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ then, from theorem 3.1, for every $\epsilon > 0$ there exists a $g^L_\epsilon \in L^2(\Omega)$ such that

$$\|M(\alpha_L, \beta_L) - \mathcal{H}g^L_\epsilon\|_{H^*} \leqslant \epsilon$$

whence from the continuity of M^{-1}

$$\|(\alpha_L, \beta_L) - M^{-1}\mathcal{H}g^L_\epsilon\|_{H^*} \leqslant C\epsilon \tag{42}$$

with a positive constant C. Finally, (38), the continuity of $\mathcal{F}$ and the fact that $\mathcal{F}(\alpha_L, \beta_L) = \Phi^L_\infty$ imply that

$$\|\gamma^{-1}Fg^L_\epsilon + \Phi^L_\infty\|_{L^2(\Omega)} \leqslant \tilde{C}\epsilon. \tag{43}$$

Next, we assume that $L \not\subset \Gamma$. In this case Φ^L_∞ does not belong to the range of $\mathcal{F}$. But, from theorem 4.1 and the fact that $\mathcal{F}$ is compact, by using Tikhonov regularization we can construct a regularized solution of

$$\mathcal{F}(\alpha, \beta) = \Phi^L_\infty. \tag{44}$$

In particular, if $(\alpha^\rho_L, \beta^\rho_L) \in H$ is the regularized solution of (44) corresponding to the regularization parameter ρ (chosen by a regular regularization strategy, e.g. the Morozov discrepancy principle), we have for a given $\delta > 0$

$$\|\mathcal{F}(\alpha^\rho_L, \beta^\rho_L) - \Phi^L_\infty\|_{L^2(\Omega)} < \delta, \tag{45}$$

and

$$\lim_{\rho\to 0} \|(\alpha^\rho_L, \beta^\rho_L)\|_H = \infty. \tag{46}$$

Now the above considerations for (α_L, β_L) can be applied to $(\alpha^\rho_L, \beta^\rho_L)$. In particular, let $g^L_{\epsilon,\rho} \in L^2(\Omega)$ be such that

$$\|M(\alpha^\rho_L, \beta^\rho_L) - \mathcal{H}g^L_{\epsilon,\rho}\|_{H^*} \leqslant \epsilon'$$

and

$$\|(\alpha^\rho_L, \beta^\rho_L) - M^{-1}\mathcal{H}g^L_{\epsilon,\rho}\|_{H^*} \leqslant \epsilon''. \tag{47}$$

Combining (45) and (47) we obtain that for every $\epsilon > 0$ and $\delta > 0$ there exists a $g^L_{\epsilon,\rho} \in L^2(\Omega)$ such that

$$\|\gamma^{-1}Fg^L_{\epsilon,\rho} + \Phi^L_\infty\|_{L^2(\Omega)} \leqslant \epsilon + \delta. \tag{48}$$

Furthermore, from (46) and the boundness of M and M^{-1} we have that

$$\lim_{\rho\to 0}\|\mathcal{H}g^L_{\epsilon,\rho}\|_{H^*}=\infty \qquad \text{and} \qquad \lim_{\rho\to 0}\|v_{g^L_{\epsilon,\rho}}\|_{H^1(B_R)}=\infty$$

where $v_{g^L_{\epsilon,\rho}}$ is the Herglotz wavefunction with kernel $g^L_{\epsilon,\rho}$,

$$\lim_{\rho\to 0}\|g^L_{\epsilon,\rho}\|_{L^2(\Omega)}=\infty.$$

We summarize these results in the following theorem, noting that for $L\in\mathcal{L}$ we have that $\rho\to 0$ as $\delta\to 0$.

Theorem 4.4. *Assume that Γ is an oriented nonintersecting piecewise smooth arc without cusps. Then if F is the far-field operator corresponding to the scattering problem for mixed boundary conditions, i.e. (3a)–(3c) and (2), the following are true:*

(1) If $L\subset\Gamma$ then for every $\epsilon>0$ there exists a solution $g^L_\epsilon\in L^2(\Omega)$ of the inequality

$$\|\gamma^{-1}Fg^L_\epsilon+\Phi^L_\infty\|_{L^2(\Omega)}\leqslant\epsilon.$$

(2) If $L\not\subset\Gamma$ then for every $\epsilon>0$ and $\delta>0$ there exists a solution $g^L_{\epsilon,\delta}\in L^2(\Omega)$ of the inequality

$$\|\gamma^{-1}Fg^L_{\epsilon,\delta}+\Phi^L_\infty\|_{L^2(\Omega)}\leqslant\epsilon+\delta$$

such that

$$\lim_{\delta\to 0}\|g^L_{\epsilon,\delta}\|_{L^2(\Omega)}=\infty \qquad \textit{and} \qquad \lim_{\delta\to 0}\|v_{g^L_{\epsilon,\delta}}\|_{H^1(B_R)}=\infty$$

where $v_{g^L_{\epsilon,\delta}}$ is the Herglotz wavefunction with kernel $g^L_{\epsilon,\delta}$.

The statement and proof of theorem 4.4 remain valid for the DCP if we set $\beta_L=0$ in the definition of Φ^L_∞ and assume that there does not exist a Herglotz wavefunction which vanishes on Γ.

In particular, if $L\subset\Gamma$ we can find a bounded solution to the far-field equation (41) with discrepancy ϵ whereas if $L\not\subset\Gamma$ then there exist solutions of the far-field equation with discrepancy $\epsilon+\delta$ with arbitrary large norm in the limit as $\delta\to 0$. For numerical purposes we need to replace Φ^L_∞ in the far-field equation (41) by an expression independent of L. To this end, assuming that there does not exist a Herglotz wavefunction which vanishes on L, we can conclude from lemma 4.2 that the class of potentials of the form

$$\int_L \alpha(y)\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\,\mathrm{d}s_y, \qquad \alpha\in\tilde{H}^{-\frac{1}{2}}(L) \tag{49}$$

is dense in $L^2(\Omega)$ and hence for numerical purposes we can replace Φ^L_∞ in (41) by an expression of the form (49). Finally, we note that as L degenerates to a point z with α_L an appropriate delta sequence we have that the integral in (49) approaches $-\gamma\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}$. Hence, it is reasonable to replace Φ^L_∞ by $-\Phi_\infty$ where $\Phi_\infty:=-\gamma\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}$ when numerically solving the far-field equation (41). (This is perhaps not too surprising since from [2] we can reconstruct the shape of an arbitrary thin domain surrounding the crack Γ by solving the far-field equation $Fg=\gamma\Phi_\infty$.)

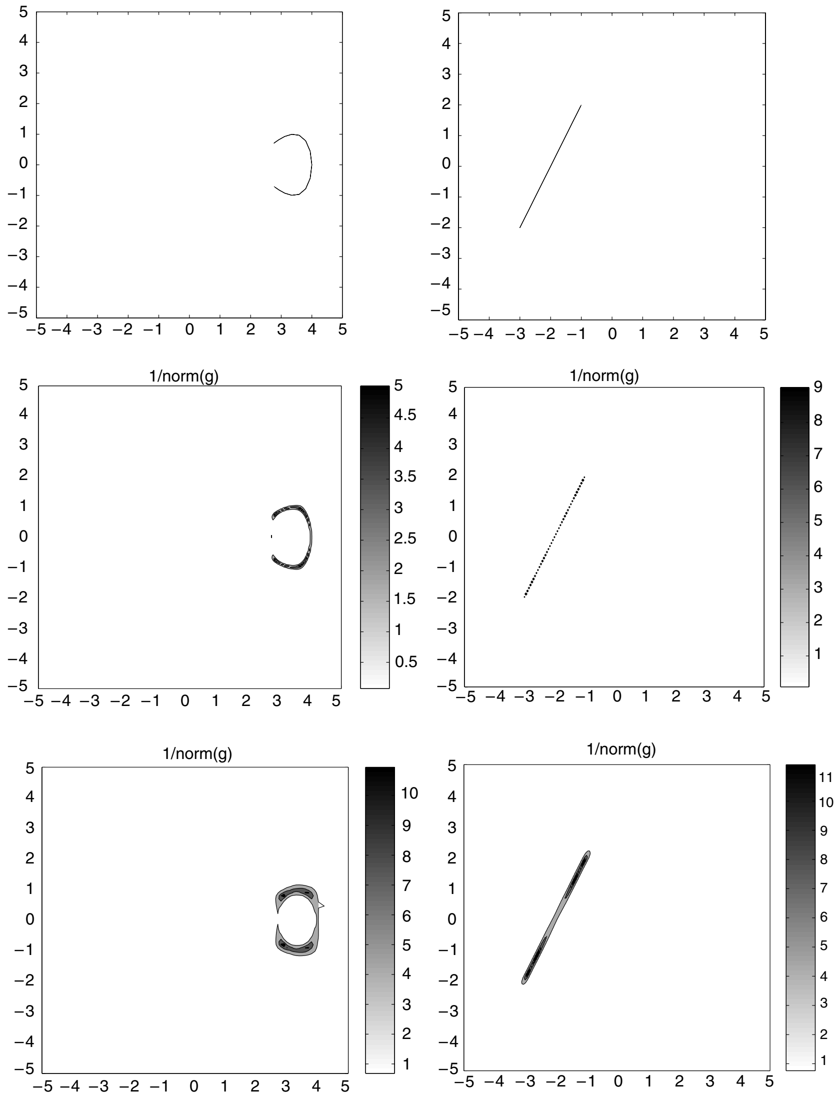

Figure 1. The true object (top), reconstruction with 0.5% noise (middle) and with 5% noise (bottom). The wavenumber is $k = 3$.

5. Numerical examples

In this section we will give some results of numerical experiments to reconstruct an open arc. The far-field data are synthetic, but corrupted by random noise added pointwise to the measurements. We test our method only for the case of the Dirichlet boundary condition since a forward solver for the mixed boundary value problem for cracks needed to produce the corresponding far-field data is not available to us.

The forward data for the Dirichlet boundary conditions have been computed by the Nyström method presented in [14].

For the inverse problem we first select an open curve Γ and compute the far-field pattern.

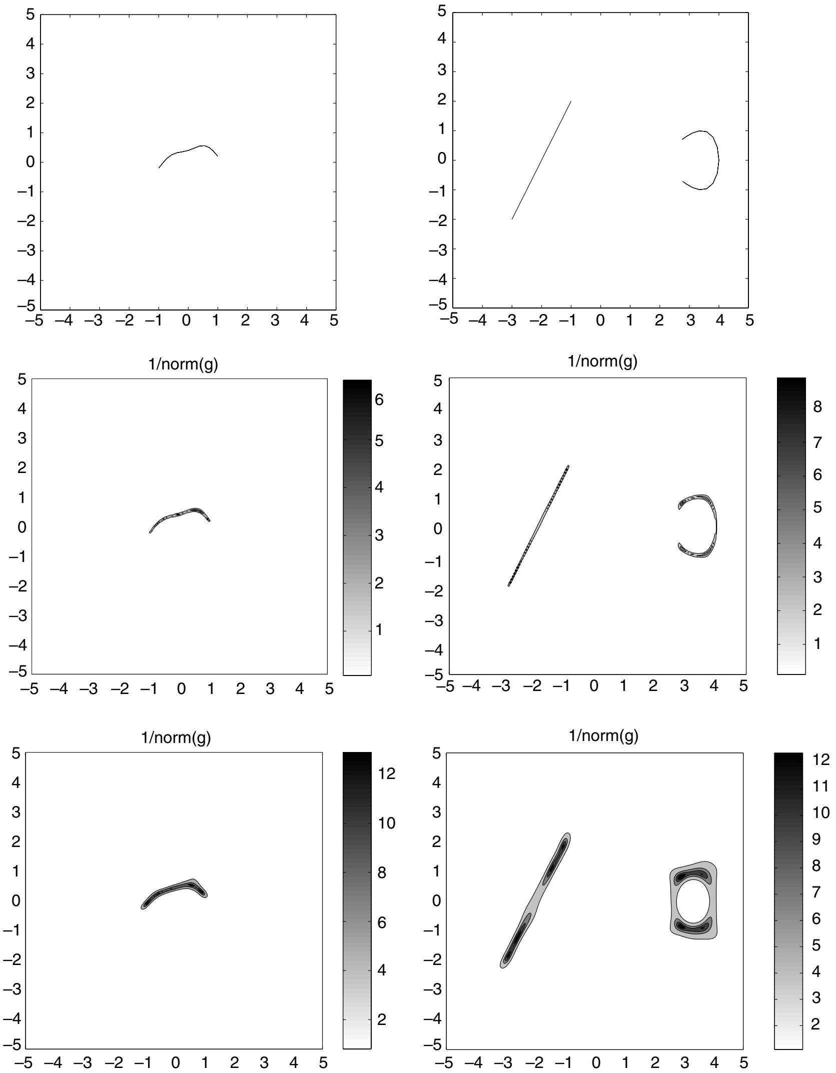

Figure 2. The true object (top), reconstruction with 0.5% noise (middle) and with 5% noise (bottom). The wavenumber is $k = 3$.

This is obtained as a trigonometric series

$$u_\infty = \sum_{n=-N}^{N} u_{\infty,n} \exp(\mathrm{i}n\theta).$$

We then add random noise to the Fourier coefficients of u_∞ to obtain the approximate far-field pattern

$$u_{\infty,a} = \sum_{n=-N}^{N} u_{\infty,a,n} \exp(\mathrm{i}n\theta)$$

where $u_{\infty,a,n} = u_{\infty,n}(1+\epsilon\chi_n)$ with χ_n a random variable in $[-1\ 1]$ and $\epsilon = 0.005$ and 0.05 in our examples. The inverse problem is solved using Tikhonov regularization and the Morozov discrepancy principle as in [6]. In particular, using the above expression for $u_{\infty,a}$ the far-field equation

$$\int_\Omega u_{\infty,a}(\hat{x}, d)g(d)\,\mathrm{d}s(d) = \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}$$

is rewritten as an ill-conditioned matrix equation for the Fourier coefficients of g which we write in the form

$$Ag_z = f_z. \tag{50}$$

As already noted, this equation needs to be regularized. To do this, we begin by computing the singular value decomposition of A, i.e. $A = U\Lambda V^*$ where U and V are unitary and Λ is real diagonal with $\Lambda_{l,l} = \sigma_l$, $1 \leqslant l \leqslant n$, where σ_l are the singular values of A. The solution of (50) is then equivalent to solving

$$\Lambda V^* g_z = U^* f_z. \tag{51}$$

Now let $\rho = (\rho_{z,1}, \rho_{z,2}, \ldots, \rho_{z,n})^\top = U^* f_z$. Then the Tikhonov regularization of (51) leads to the problem of solving

$$\min_{g_z\in\mathbb{R}^n} \|\Lambda V^* g_z - \rho\|_{l^2}^2 + \alpha\|g_z\|_{l^2}^2$$

where $\alpha > 0$ is the Tikhonov regularization parameter. Defining $u_z = V^* g_z$, we see that the solution to this problem is

$$u_{z,l} = \frac{\sigma_l}{\sigma_l^2 + \alpha}\rho_{z,l}, \qquad 1 \leqslant l \leqslant n,$$

and hence

$$\|g_z\|_{l^2} = \|u\|_{l^2} = \left(\sum_{l=1}^{n} \frac{\sigma_l^2}{(\sigma_l^2 + \alpha)^2}\|\rho_{z,l}\|^2\right)^{\frac{1}{2}}.$$

Note that we use the discrete l^2 norm of g_z rather than the $L^2(\Omega)$ norm of g. The regularization parameter α depends on both z and the error in the data $\{u_{\infty,a}\}$. In order to choose α, we use the Morozov discrepancy principle. In particular, suppose that we know an estimate for the error in the far-field operator so that $\|F - F_h^a\|_{L^2(\Omega)} \leqslant \delta$ for some $\delta > 0$ (for example in the left-hand side of figure 1, $\delta = 0.008$ when $\epsilon = 0.5\%$ and $\delta = 0.09$ when $\epsilon = 5\%$). Then, ignoring the error in the right-hand side of the far-field equation, the Morozov procedure picks $\alpha = \alpha(z)$ to be the zero of

$$\mu_z(\alpha) = \sum_{j=1}^{n} \frac{\delta^2\sigma_j^2 - \alpha^2}{(\alpha + \sigma_j^2)^2}\|\rho_{z,l}\|^2, \qquad \alpha > 0.$$

In order to compare the performance of the linear sampling method with the factorization method [12] and the Newton method [14], in the following numerical examples we consider the reconstruction of the open arc (figure 1, top left)

$$\Gamma := \left\{\varrho(s) = \left(2\sin\frac{s}{2}, \sin s\right) : \frac{\pi}{4} \leqslant s \leqslant \frac{7\pi}{4}\right\},$$

of the line (figure 1, top right)

$$\Gamma := \{\varrho(s) = (-2 + s, 2s) : -1 \leqslant s \leqslant 1\},$$

both used by Kirsch in [12] and of the curve (figure 2, top left)

$$\Gamma := \left\{\varrho(s) = \left(s, 0.5\cos\frac{\pi s}{2} + 0.2\sin\frac{\pi s}{2} - 0.1\cos\frac{3\pi s}{2}\right) : -1 \leqslant s \leqslant 1\right\}$$

chosen by Kress in [14].

In our examples we fix the wavenumber $k = 3$. The sampling domain is the square $[-5, 5]^2$ and we use a uniform grid for the sampling points z with 201×201 points. The far-field data are given for 32 incident directions and 32 observation directions equally distributed on the unit circle.

An attractive feature of the linear sampling method is that neither the boundary conditions nor number of components need to be known *a priori*. In particular, even though we have only given examples of reconstruction for Dirichlet boundary conditions, for mixed boundary conditions one would still solve the same far-field equation. In figure 2 we give an example of the reconstruction of two disconnected open arcs. As in [12] the forward data are computed separately for each arc. Then the far-field data inserted in the inverse code are the superposition of the far-field patterns of the two arcs.

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