JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 21, Number 2, Summer 2009

THE COMPUTATION OF LOWER BOUNDS FOR THE NORM OF THE INDEX OF REFRACTION IN AN ANISOTROPIC MEDIA FROM FAR FIELD DATA

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Communicated by Charles Groetsch

Dedicated to Professor Dr. Rainer Kress on the occasion of his 65^{th} birthday and the pleasure that knowing him has given to our lives!

ABSTRACT. We consider the scattering of time harmonic electromagnetic plane waves by a bounded, inhomogeneous, anisotropic dielectric medium and show that under certain assumptions a lower bound on the norm of the (matrix) index of refraction can be obtained from a knowledge of the smallest transmission eigenvalue corresponding to the medium. Numerical examples are given showing the efficaciousness of our estimates.

1. Introduction. Anisotropic material play a special role in electromagnetic inverse scattering theory. This is due to the fact that from far field data only the support D of the scatterer is uniquely determined [**3**], [**15**] and little can be said about the material properties of the scatterer [**13**]. This remains true even if multifrequency data is used. Although specific information about the material properties may be unavailable, there remains the possibility of obtaining upper or lower bounds on certain norms of the (matrix) index of refraction and it is to this task that this paper is directed. In particular, are there certain inequalities that the index of refraction must satisfy for a given measured far field pattern? For the case of a dielectric isotropic

2000 AMS *Mathematics subject classification.* Primary 35R30, 35Q60, 35J40, 78A25.
Keywords and phrases. Anisotropic media, direct and inverse scattering, far field data, trans- mission eigenvalues, Maxwell's equations.
The research of F. Cakoni and D. Colton was supported in part by the U.S. Air Force Office of Scientific Research under Grant FA9550-05-1-0127
Received by the editors on April 2, 2007, and in revised form on December 18, 2007.
DOI:10.1216/JIE-2009-21-2-203

scatterer, this question was considered in [**5**] and [**11**] where it was shown that if the (scalar) index of refraction is greater than one then it is bounded below by $\lambda(D)/k_1^2$ where $\lambda(D)$ is the first Dirichlet eigenvalue for the Laplacian in D of the scattering obstacle and k_1 is the first transmission eigenvalue [**4**]. Since D and k_1 can be determined from the far field data [**5**], this then provides a lower bound for the index of refraction. In this paper we will show that a similar inequality is valid in the case of a dielectric anisotropic media where the supremum of the (scalar) index of refraction is replaced by the Euclidean norm of the (matrix) index of refraction.

The plan of our paper is as follows. In the following section we formulate the inverse scattering problem for Maxwell's equations in the frequency domain for an anisotropic media. In Section 3 we consider the special case when the scattering object is an infinite cylinder and the medium is orthotropic. In this case the three dimensional vector problem reduces to the simpler case of a two dimensional scalar problem. Following the ideas of [**7**], we will then show in this case that we can obtain bounds on the Euclidian norm of the index of refraction both in the case when the norm is greater than one as well as when the norm is less than one. In Section 4 we extend these results for the scalar case to the case of Maxwell"s equations in three dimensions. Finally, in Section 5, we give numerical examples showing the practicality of our estimates. Although the estimates that are obtained for the refractive index are crude, we believe that these estimates can be significantly improved and this is a topic of current research.

It gives each of the authors particular pleasure in dedicating this paper to Professor Rainer Kress on the occasion of his sixty fifth birthday. Each of us has written papers with Rainer and had the pleasure of visiting him in Göttingen. In particular, the second author has been a close friend of Rainer's for over thirty years, has written two books with him [**8**], [**9**] and has visited Göttingen more times than he can remember! This combination of friendship and mathematical collaboration has played a unique role in the second author's life and so on behalf of all of us, but particularly from David Colton, happy birthday Rainer!

2. Formulation of the problem. Let $D \subset \mathbb{R}^3$ be a bounded open set having a picewise smooth boundary ∂D such that the exterior domain $D_e := \mathbb{R}^3 \setminus \overline{D}$ is connected. The unit normal vector to ∂D directed into the exterior of D is denoted by ν. We assume that the domain D is the support of an anisotropic (possibly disconnected) object and the incident field is a time-harmonic electromagnetic plane wave with frequency ω. The exterior electric and magnetic fields $\tilde{E}^{ext}$, $\tilde{H}^{ext}$ and the interior electric and magnetic fields $\tilde{E}^{int}$, $\tilde{H}^{int}$, satisfy

$$\begin{aligned} \nabla \times \tilde{E}^{ext} - i\omega\mu_0 \tilde{H}^{ext} &= 0 \\ \nabla \times \tilde{H}^{ext} + i\omega\epsilon_0 \tilde{E}^{ext} &= 0 \qquad \text{in } D_e \end{aligned} \tag{1}$$

$$\begin{aligned} \nabla \times \tilde{E}^{int} - i\omega\mu_0 \tilde{H}^{int} &= 0 \\ \nabla \times \tilde{H}^{int} + (i\omega\epsilon(x) - \sigma(x))\tilde{E}^{int} &= 0 \qquad \text{in } D \end{aligned} \tag{2}$$

and on the boundary ∂D we assume the continuity of the tangential component of both fields, i.e.

$$\tilde{E}^{ext} \times \nu - \tilde{E}^{int} \times \nu = 0 \quad \text{on} \quad \partial D \tag{3}$$

$$\tilde{H}^{ext} \times \nu - \tilde{H}^{int} \times \nu = 0 \quad \text{on} \quad \partial D \tag{4}$$

The electric permittivity ϵ_0 and magnetic permeability μ_0 of the exterior dielectric medium are positive constants whereas the dielectric scatterer has the same magnetic permeability μ_0 as the exterior medium but the electric permittivity ϵ is a real 3×3 symmetric matrix valued function. If we define $\tilde{E}^{(ext,int)} = \frac{1}{\sqrt{\epsilon_0}} E^{(ext,int)}$, $\tilde{H}^{(ext,int)} = \frac{1}{\sqrt{\mu_0}} H^{(ext,int)}$, $k^2 = \epsilon_0\mu_0\omega^2$ and $N(x) = \epsilon(x)/\epsilon_0$, the direct scattering problem for an anisotropic medium reads

$$\begin{aligned} \nabla \times E^{ext} - ikH^{ext} &= 0 \qquad \text{and} \\ \nabla \times H^{ext} + ikE^{ext} &= 0 \qquad \text{in} \quad D_e \end{aligned} \tag{5}$$

$$\begin{aligned} \nabla \times E^{int} - ikH^{int} &= 0 \qquad \text{and} \\ \nabla \times H^{int} + ikN(x)E^{int} &= 0 \qquad \text{in} \quad D \end{aligned} \tag{6}$$

$$E^{ext} \times \nu - E^{int} \times \nu = 0 \qquad \text{on} \quad \partial D \tag{7}$$

$$H^{ext} \times \nu - H^{int} \times \nu = 0 \qquad \text{on} \quad \partial D \tag{8}$$

where

$$E^{ext} = E^s + E^i \quad \text{and} \quad H^{ext} = H^s + H^i,$$

the scattered electric and magnetic fields E^s and H^s satisfy the Silver-Müller radiation condition

$$\lim_{r\to\infty}(H^s \times x - rE^s) = 0 \tag{9}$$

uniformly in $\hat{x} = x/|x|$, $r = |x|$, and the incident electric field E^i and incident magnetic field H^i are time harmonic plane waves given by

$$E^i(x) := \frac{i}{k}\nabla \times \nabla \times p\, e^{ikx\cdot d} \quad \text{and} \quad H^i(x) := \frac{1}{k^2}\nabla \times p\, e^{ikx\cdot d} \tag{10}$$

where d is a unit vector giving the direction of propagation and p is the polarization vector.

3. The scalar case. Now we assume the scatterer is an infinitely long dielectric cylinder with axis in the z-direction and assume that the incident electromagnetic field is a plane wave propagating in the direction perpendicular to the cylinder. Let the bounded domain $D \subset \mathbb{R}^2$ with piecewise smooth boundary ∂D be the cross section of the cylinder such that the exterior domain $D_e := \mathbb{R}^2 \setminus \overline{D}$ is connected. We denote by ν the outward unit normal to ∂D defined almost everywhere and assume that the dielectric cylinder is orthotropic, i.e. the matrix N is of the form

$$N = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix}$$

If we consider incident waves such that the electric field is polarized perpendicular to the z axis, then the magnetic fields have only a component in the z direction, i.e.

$$H^i = (0,\ 0,\ u^i), \qquad H_0 = (0,\ 0,\ w), \qquad H^s = (0,\ 0,\ u^s).$$

Assuming that N^{-1} exists and expressing the electric fields in terms of magnetic fields the equations $(5-8)$ now lead to the following transmission problem for v and u:

$$\nabla \cdot A\nabla w + k^2\, w = 0 \qquad \text{in} \quad D \tag{11}$$

$$\Delta u + k^2\, u = 0 \qquad \text{in} \quad D_e \tag{12}$$

$$w - u = 0 \qquad \text{on} \quad \partial D \tag{13}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \quad \partial D \tag{14}$$

$$u = u^s + u^i \tag{15}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{16}$$

where u^s is the scattered field and u^i is the given incident field. In the case of plane waves the incident field is given by $u^i := e^{ikx\cdot d}$, $d \in \Omega := \{x : |x| = 1\}$. Moreover

$$\frac{\partial w}{\partial \nu_A}(x) := \nu(x) \cdot A(x)\nabla w(x), \qquad x \in \partial D,$$

$$A = \frac{1}{n_{11}n_{22} - n_{12}n_{21}} \begin{pmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \end{pmatrix}$$

and the radiation condition (16) holds uniformly with respect to $\hat{x} = x/|x|$. In the following we assume that A is a real valued 2×2 matrix-valued function whose entries are piecewise continuously differentiable functions in $\overline{D}$ with (possible) jumps along piecewise smooth curves such that A is symmetric and $\bar{\xi} \cdot A\,\xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and $x \in \overline{D}$ where γ is a positive constant. The existence of a unique solution to (11 − 16) can be established by variational methods [**4, 15**]. Note that (11 − 16) is also well posed for complex matrices A provided that $\mathcal{I}m\left(\bar{\xi} \cdot A\,\xi\right) \leq 0$ (Theorem 5.24 in [**4**]).

It can then be shown [**4, 9**] that the scattered field u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right) \tag{17}$$

as $r \to \infty$ uniformly in $\hat{x}$ where u_∞ is the *far field pattern.*

In this paper we are concerned with the *inverse scattering problem* of determining D and A from a knowledge of $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \Omega$

(for the case of limited aperture data see [**2**]). In [**15**] (see also [**4**]) it is shown that D is uniquely determined by $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \Omega$ and fixed k. However it is known [**13**] that $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \Omega$ does not uniquely determine the matrix A even if it is known for an interval of values of k. Our aim is to first determine D from the above data and then provide inequalities that are satisfied by all dielectric anisotropic media that give raise to the same far field data. To this end, we note that D can be determined by using the *linear sampling method* to solve the *far field equation*

$$\int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d) = \Phi_\infty(\hat{x}, z) \tag{18}$$

where Φ_∞ is the far field pattern of the radiating fundamental solution

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}\,(k|x - z|) \tag{19}$$

and $H_0^{(1)}$ denotes a Hankel function of the first kind of order zero. In particular, from [**4**] we have that the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Fg)\,(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d) ds(d) \tag{20}$$

is injective with dense range provided k is not a *transmission eigenvalue*, i.e. a value of k for which the (homogeneous) *interior transmission problem*

$$\nabla \cdot A \nabla w + k^2\, w = 0 \qquad \text{in} \quad D \tag{21}$$

$$\Delta v + k^2\, v = 0 \qquad \text{in} \quad D \tag{22}$$

$$w - v = 0 \qquad \text{on} \quad \partial D \tag{23}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = 0 \qquad \text{on} \quad \partial D \tag{24}$$

has a nontrivial solution $w, v \in H^1(D)$. (It follows from Theorem 6.4 of [**4**] that transmission eigenvalues can only exist if A is real, i.e. the scatterer is a dielectric). In [**1**] and [**6**] it is shown that provided that k is not a transmission eigenvalue and under the assumptions that $\bar{\xi} \cdot A\,\xi \ge \delta|\xi|^2$ or $\bar{\xi} \cdot A^{-1}\,\xi \ge \delta|\xi|^2$ for some constant $\delta > 1$, then ∂D can be characterized from the behavior of $\|g\|_{L^2(\Omega)}$, where g is an approximate solution of (18). Having determined D, we now want to recover information about A. Following [**5**] we will make use of transmission eigenvalues (which we avoided when determining D) to obtain a lower bound for the Euclidian norm of A. Due to the lack of injectivity and the denseness of the range of the far field operator F, when k is a transmission eigenvalue the L^2-norm of the (regularized) solution to

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z_0), \quad \text{for a fixed} \quad z_0 \in D$$

can be expected to be large for such values of k. (This expectation will be numerically verified for several examples in Section 5 of this paper). This provides a method for determining the smallest transmission eigenvalue. In the following subsection we establish a relationship between the smallest transmission eigenvalue and the Euclidian norm of A.

3.1. *A lower bound for* $\|A\|_2$. The interior transmission problem for an anisotropic inhomogeneous scattering problem has been studied in [**1, 6, 10 and 12**] (see also [**19**]). However the approach used in these papers does not include the eigenvalue problem (21 − 24). The main idea to study (21 − 24) is to observe that by making an appropriate substitution one can rewriting (21 − 24) as an eigenvalue problem for a fourth order differential equation.

Let $w \in H^1(D)$ and $v \in H^1(D)$ satisfy (21 − 24) and make the substitution

$$\mathbf{w} = A\nabla w \in L^2(D)^2, \quad \text{and} \quad \mathbf{v} = \nabla v \in L^2(D)^2.$$

Assuming that A^{-1} exists and is bounded, we have that

$$\nabla w = A^{-1}\mathbf{w}.$$

Taking the gradient of (21) and (22), we obtain that $\mathbf{w}$ and $\mathbf{v}$ satisfy

$$\nabla(\nabla \cdot \mathbf{w}) + k^2 A^{-1}\mathbf{w} = 0 \tag{25}$$

and

$$\nabla(\nabla\cdot\mathbf{v})+k^2\mathbf{v}=0, \tag{26}$$

respectively, in D. Obviously (24) implies that

$$\nu\cdot\mathbf{w}=\nu\cdot\mathbf{v}\qquad\text{on }\ \partial D. \tag{27}$$

Furthermore, from (21) and (22) we have that

$$-k^2w=\nabla\cdot\mathbf{w}\qquad\text{and}\qquad -k^2v=\nabla\cdot\mathbf{v}$$

and the transmission condition (23) yields

$$\nabla\cdot\mathbf{w}=\nabla\cdot\mathbf{v}\qquad\text{on }\ \partial D. \tag{28}$$

We now formulate the interior transmission eigenvalue problem in terms of $\mathbf{w}$ and $\mathbf{v}$. In addition to the usual energy spaces

$$\begin{aligned}H^1(D)&:=\left\{u\in L^2(D):\ \nabla u\in L^2(D)^2\right\}\\ H^1_0(D)&:=\left\{u\in H^1(D):\ u=0\text{ on }\partial D\right\}\end{aligned}$$

we introduce the Sobolev spaces

$$\begin{aligned}H(\mathrm{div}\,,D)&:=\left\{\mathbf{u}\in L^2(D)^2:\ \nabla\cdot\mathbf{u}\in L^2(D)\right\}\\ H_0(\mathrm{div}\,,D)&:=\{\mathbf{u}\in H(\mathrm{div}\,,D):\ \nu\cdot\mathbf{u}=0\text{ on }\partial D\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}(D)&:=\left\{\mathbf{u}\in H(\mathrm{div}\,,D):\ \nabla\cdot\mathbf{u}\in H^1(D)\right\}\\ \mathcal{H}_0(D)&:=\left\{\mathbf{u}\in H_0(\mathrm{div}\,,D):\ \nabla\cdot\mathbf{u}\in H^1_0(D)\right\}\end{aligned}$$

equipped with the scalar product

$$(\mathbf{u},\mathbf{v})_{\mathcal{H}(D)}:=(\mathbf{u},\mathbf{v})_{L^2(D)}+(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_{H^1(D)}\,.$$

Note that

$$\int_D\varphi\,\mathrm{div}\,\boldsymbol{\psi}\,dx+\int_D\nabla\varphi\cdot\boldsymbol{\psi}\,dx=\int_{\partial D}\varphi\,\boldsymbol{\psi}\cdot\nu\,ds \tag{29}$$

for $(\varphi, \boldsymbol{\psi}) \in H^1(D) \times H(\operatorname{div}, D)$.
The interior transmission eigenvalue problem in terms of $\mathbf{w}$ and $\mathbf{v}$ now becomes the following: Find $\mathbf{w} \in L^2(D)$ and $\mathbf{v} \in L^2(D)$ such that $\mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$ satisfies

$$\nabla(\nabla \cdot \mathbf{w}) + k^2 A^{-1}\mathbf{w} = 0 \qquad \text{in} \quad D \tag{30}$$

$$\nabla(\nabla \cdot \mathbf{v}) + k^2 \mathbf{v} = 0 \qquad \text{in} \quad D. \tag{31}$$

Note that the boundary conditions (27) and (28) are incorporated in the fact that $\mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$.
From the above analysis we have the following result:

Lemma 3.1. *If k is a transmission eigenvalue, i.e. if $w \in H^1(D)$ and $v \in H^1(D)$ satisfy* (21 – 24)*, then $\mathbf{w} = A\nabla w \in L^2(D)^2$ and $\mathbf{v} = \nabla v \in L^2(D)^2$ satisfy $\mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$ and* (30 – 31)*.*

We now formulate (30 – 31) as an eigenvalue problem for a fourth order differential equation. To this end, we have that $\mathbf{u} = \mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$ satisfies

$$\nabla(\nabla \cdot \mathbf{u}) + k^2 \mathbf{u} = k^2 \left(I - A^{-1}\right) \mathbf{w} \qquad \text{in} \quad D. \tag{32}$$

Assuming that $(A^{-1} - I)^{-1}$ exists and is bounded, from (32) using (30) we obtain the fourth order differential equation

$$\left(\nabla\nabla \cdot + k^2 A^{-1}\right) (A^{-1} - I)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + k^2 \mathbf{u}\right) = 0 \qquad \text{in} \quad D. \tag{33}$$

Note that in addition $\mathbf{u} \in \mathcal{H}_0(D)$ implies that

$$\nu \cdot \mathbf{u} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \qquad \text{on} \quad \partial D \tag{34}$$

and if k is a transmission eigenvalue than k is an eigenvalue of (33 – 34).

To study the eigenvalue problem for (33 – 34) we write it in variational form. To this end multiplying (33) by a function $\overline{\boldsymbol{\psi}} \in \mathcal{H}_0(D)$, using (29), the zero boundary values $\nu \cdot \overline{\boldsymbol{\psi}} = 0$ and $\nabla \cdot \overline{\boldsymbol{\psi}}|_{\partial D} = 0$ and the symmetry of A we obtain

$$\int_D (A^{-1} - I)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + k^2 \mathbf{u}\right) \cdot \left(\nabla\nabla \cdot \overline{\boldsymbol{\psi}} + k^2 A^{-1} \overline{\boldsymbol{\psi}}\right) dx = 0. \tag{35}$$

Using the denseness in $\mathcal{H}_0(D)$ of C^∞ functions with compact support on D one can easily see that $\mathbf{u} \in \mathcal{H}_0(D)$ satisfies (33) if and only if $\mathbf{u} \in \mathcal{H}_0(D)$ satisfies (35) for every $\boldsymbol{\psi} \in \mathcal{H}_0(D)$. Since $A^{-1}(A^{-1}-I)^{-1} = I + (A^{-1} - I)^{-1}$, we can write (35) in the following equivalent forms

$$\mathcal{B}_k(\mathbf{u}, \boldsymbol{\psi}) - k^2 \mathcal{C}(\mathbf{u}, \boldsymbol{\psi}) = 0 \qquad \text{for all } \boldsymbol{\psi} \in \mathcal{H}_0(D) \tag{36}$$

or

$$\tilde{\mathcal{B}}_k(\mathbf{u}, \boldsymbol{\psi}) - k^2 \mathcal{C}(\mathbf{u}, \boldsymbol{\psi}) = 0 \qquad \text{for all } \boldsymbol{\psi} \in \mathcal{H}_0(D) \tag{37}$$

where

$$\begin{aligned} \mathcal{B}_k(\mathbf{u}, \boldsymbol{\psi}) :=& \left((A^{-1} - I)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + k^2 \mathbf{u}\right), \left(\nabla\nabla \cdot \boldsymbol{\psi} + k^2 \boldsymbol{\psi}\right)\right)_{L^2(D)} \\ &+ k^4 \left(\mathbf{u}, \boldsymbol{\psi}\right)_{L^2(D)} \\ \tilde{\mathcal{B}}_k(\mathbf{u}, \boldsymbol{\psi}) :=& \left((I - A^{-1})^{-1} \left(\nabla\nabla \cdot \mathbf{u} + k^2 A^{-1} \mathbf{u}\right), \left(\nabla\nabla \cdot \boldsymbol{\psi} + k^2 A^{-1} \boldsymbol{\psi}\right)\right)_{L^2(D)} \\ &+ k^4 \left(A^{-1}\mathbf{u}, \boldsymbol{\psi}\right)_{L^2(D)} \end{aligned}$$

and

$$\mathcal{C}(\mathbf{u}, \boldsymbol{\psi}) := \left(\nabla \cdot \mathbf{u}, \nabla \cdot \boldsymbol{\psi}\right)_{L^2(D)}.$$

Obviously, $\mathcal{B}_k(\cdot\,,\,\cdot)$, $\tilde{\mathcal{B}}_k(\cdot\,,\,\cdot)$ and $\mathcal{C}(\cdot\,,\,\cdot)$ are continuous sesquilinear forms in $\mathcal{H}_0(D) \times \mathcal{H}_0(D)$. Let us denote by B_k, $\tilde{B}_k$ and C the bounded linear operators from $\mathcal{H}_0(D)$ to $\mathcal{H}_0(D)$ defined using the Riesz representation theorem by

$$\begin{aligned} (B_k \mathbf{u},\, \boldsymbol{\psi})_{\mathcal{H}_0(D)} &= \mathcal{B}_k(\mathbf{u}, \boldsymbol{\psi}), \\ \left(\tilde{B}_k \mathbf{u},\, \boldsymbol{\psi}\right)_{\mathcal{H}_0(D)} &= \tilde{\mathcal{B}}_k(\mathbf{u}, \boldsymbol{\psi}), \text{ and} \\ (C \mathbf{u},\, \boldsymbol{\psi})_{\mathcal{H}_0(D)} &= \mathcal{C}(\mathbf{u}, \boldsymbol{\psi}) \end{aligned}$$

for all $\boldsymbol{\psi} \in \mathcal{H}_0(D)$.

Lemma 3.2. *$C : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is a compact operator.*

Proof. Let $\mathbf{u}_n$ be a bounded sequence in $\mathcal{H}_0(D)$. Hence there exists a subsequence, denoted again by $\mathbf{u}_n$, which converges weakly to $\mathbf{u}^0$

in $\mathcal{H}_0(D)$. Since $\nabla \cdot \mathbf{u}_n$ is also bounded in $H^1(D)$, from the Rellich compactness theorem we have that $\nabla \cdot \mathbf{u}_n$ converges strongly to $\nabla \cdot \mathbf{u}^0$ in $L^2(D)$. But

$$\|C(\mathbf{u}_n - \mathbf{u}^0)\|_{\mathcal{H}_0(D)} \leq \|\nabla \cdot (\mathbf{u}_n - \mathbf{u}^0)\|_{L^2(D)}$$

which proves that $C\mathbf{u}_n$ converges strongly to $C\mathbf{u}^0$. ◻

Theorem 3.1. *Assume that $\bar{\xi} \cdot (A^{-1} - I)^{-1}\, \xi \geq \alpha|\xi|^2$ in D and for all $\xi \in \mathbb{C}^2$ where $\alpha > 0$ is a constant. Then*

1. *The set of transmission eigenvalues is discrete and does not accumulate at* 0.

2. *All transmission eigenvalues (if they exist) are such that $k^2 \geq \dfrac{\alpha}{1+\alpha}\lambda(D)$ where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.*

Proof. In order to prove the first part of the theorem we consider the formulation (36). Using the assumption on $(A^{-1} - I)^{-1}$ we have that

$$\mathcal{B}_k(\mathbf{u}, \mathbf{u}) \geq \alpha \|\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}\|^2_{L^2(D)} + k^4\|\mathbf{u}\|^2_{L^2(D)}.$$

Setting $X = \|\nabla\nabla \cdot \mathbf{u}\|_{L^2(D)}$ and $Y = k^2\|\mathbf{u}\|_{L^2(D)}$ we have that

$$\|\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}\|^2_{L^2(D)} \geq X^2 - 2XY + Y^2$$

and therefore

$$\mathcal{B}_k(\mathbf{u}, \mathbf{u}) \geq \alpha X^2 - 2\alpha XY + (\alpha + 1)Y^2 \tag{38}$$

From the identity,

$$\alpha X^2 - 2\alpha XY + (\alpha+1)Y^2 = \epsilon\left(Y - \frac{\alpha}{\epsilon}X\right)^2 + \left(\alpha - \frac{\alpha^2}{\epsilon}\right)X^2 + (1 + \alpha - \epsilon)Y^2 \tag{39}$$

for $\alpha < \epsilon < \alpha + 1$, setting $\epsilon = \alpha + 1/2$ we now obtain that

$$\mathcal{B}_k(\mathbf{u}, \mathbf{u}) \geq \frac{\alpha}{1 + 2\alpha}(X^2 + Y^2). \tag{40}$$

From (29) we have

$$\|\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}\|^2_{L^2(D)} = \|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)} - 2k^2\|\nabla\cdot\mathbf{u}\|^2_{L^2(D)} + k^4\|\mathbf{u}\|^2_{L^2(D)}$$

which implies that

$$2k^2\|\nabla\cdot\mathbf{u}\|^2_{L^2(D)} \le X^2+Y^2.$$

Finally combining the above estimates yields the existence of a constant $c_k>0$ (independent of $\mathbf{u}$ and α) such that

$$\mathcal{B}_k(\mathbf{u},\mathbf{u}) \ge c_k\frac{\alpha}{1+2\alpha}\|\mathbf{u}\|^2_{\mathcal{H}(\mathcal{D})}. \tag{41}$$

Hence the sesquilinear form $\mathcal{B}_k(\cdot\,,\,\cdot)$ is coercive in $\mathcal{H}_0(D)\times\mathcal{H}_0(D)$ and consequently the operator $B_k:\mathcal{H}_0(D)\to\mathcal{H}_0(D)$ is a bijection for fixed k. To use the analytic Fredholm theory, note that, since the sesquilinear form $\mathcal{B}_k(\cdot,\,\cdot)$ is obviously analytic in k, $k\to B_k$ is weakly analytic and hence strongly analytic. By the Lax-Milgram theorem we can conclude that in a neighborhood of the positive real axis, B_k^{-1} exists and $k\to B_k^{-1}$ is strongly analytic.

Next we need to show that the operator $B_k - C:\mathcal{H}_0(D)\to\mathcal{H}_0(D)$ is an isomorphism for $k>0$ small enough. To this end, for $\nabla\cdot u\in H_0^1(D)$, using the Poincaré inequality [**16**] we have that

$$\|\nabla\cdot\mathbf{u}\|^2_{L^2(D)} \le \frac{1}{\lambda(D)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)} \tag{42}$$

where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D. Hence, from (39) and (42) for $\alpha<\epsilon<\alpha+1$ we have that

$$\mathcal{B}_k(\mathbf{u},\mathbf{u})-k^2\mathcal{C}(\mathbf{u},\mathbf{u}) \ge \left(\alpha-\frac{\alpha^2}{\epsilon}\right)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)}+(1+\alpha-\epsilon)k^2\|\mathbf{u}\|^2_{L^2(D)}$$
$$-\,k^2\frac{1}{\lambda(D)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)}$$

Therefore, if $k^2 < \left(\alpha-\alpha^2/\epsilon\right)\lambda(D)$ for every $\alpha<\epsilon<\alpha+1$, then $B_k - k^2C$ is invertible, whence the analytic Fredholm theory implies that the set of transmission eigenvalue is at most discrete [**9**]. In particular taking ϵ arbitrarily close to $\alpha+1$ we have that if $k^2 <$

$\frac{\alpha}{1+\alpha}\lambda(D)$ then k is not a transmission eigenvalue.
The second part of the theorem is merely a consequence of part 1.
$\square$

Theorem 3.2. *Assume that $\bar{\xi}\cdot A^{-1}(I-A^{-1})^{-1}\,\xi \geq \alpha|\xi|^2$ in D and for all $\xi \in \mathbb{C}^2$, where $\alpha > 0$ is a constant. Then*

1. *The set of transmission eigenvalues is discrete and does not accumulate at* 0.

2. *All transmission eigenvalues (if they exist) are such that $k^2 \geq \lambda(D)$ where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.*

Proof. The proof is similar to the proof of Theorem 3.1. Here we need to use the sesquilinear form $\tilde{\mathcal{B}}_k(\mathbf{u},\mathbf{u})$ which, since A is symmetric, can be re-written as

$$\tilde{\mathcal{B}}_k(\mathbf{u},\boldsymbol{\psi}) := \left(A^{-1}(I-A^{-1})^{-1}\left(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}\right),\left(\nabla\nabla\cdot\boldsymbol{\psi}+k^2\boldsymbol{\psi}\right)\right)_{L^2(D)} + \left(\nabla\nabla\cdot\mathbf{u},\nabla\nabla\cdot\boldsymbol{\psi}\right)_{L^2(D)}.$$

From the assumption that $\bar{\xi}\cdot A^{-1}(I-A^{-1})^{-1}\,\xi \geq \alpha|\xi|^2$ we have that

$$\tilde{\mathcal{B}}_k(\mathbf{u},\mathbf{u}) \geq \alpha\|\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}\|^2_{L^2(D)} + \|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)}$$

which implies that

$$\tilde{\mathcal{B}}_k(\mathbf{u},\mathbf{u}) \geq (\alpha+1)X^2 - 2\alpha XY + \alpha Y \tag{43}$$

where again $X = \|\nabla\nabla\cdot\mathbf{u}\|_{L^2(D)}$ and $Y = k^2\|\mathbf{u}\|_{L^2(D)}$. Proceeding in the same way as in the first part of Theorem 3.2 we conclude that

$$\tilde{\mathcal{B}}_k(\mathbf{u},\mathbf{u}) \geq c_k\frac{\alpha}{1+2\alpha}\|\mathbf{u}\|^2_{\mathcal{H}(\mathcal{D})}. \tag{44}$$

where $c_k > 0$ is a constant independent of $\mathbf{u}$ and α, whence $\tilde{\mathcal{B}}_k(\cdot\,,\,\cdot)$ is a coercive sesquilinear form in $\mathcal{H}_0(D)\times\mathcal{H}_0(D)$. Arguing exactly in the same way as in part 1 of Theorem 3.2 we conclude that $\tilde{B}_k$ and $\tilde{B}_k^{-1}$ depend analytically on k. Finally, to show that $\tilde{B}_k - C$ is invertible for small enough k, using (39, 42 and 43) for $\alpha < \epsilon < \alpha+1$ we have that

$$\tilde{\mathcal{B}}_k(\mathbf{u},\mathbf{u}) - k^2\mathcal{C}(\mathbf{u},\mathbf{u}) \geq (1+\alpha-\epsilon)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)} + \left(\alpha - \frac{\alpha^2}{\epsilon}\right)k^2\|\mathbf{u}\|^2_{L^2(D)} - k^2\frac{1}{\lambda(D)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)}$$

where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D. In particular, $\mathcal{B}_k(\mathbf{u}, \mathbf{u}) - k^2\mathcal{C}(\mathbf{u}, \mathbf{u})$ is coercive as long as $k^2 < (\alpha + 1 - \epsilon)\lambda(D)$ for every $\alpha < \epsilon < \alpha + 1$. In particular, by taking $\epsilon > 0$ arbitrarily close to α we have that k such that $k^2 < \lambda(D)$ are not transmission eigenvalues.

The second part of the theorem is a straight forward consequence of part 1. $\square$

We now are ready to formulate the main result of this paper which provide estimates on the matrix index of refraction A under the assumption that the anisotropic material is a dielectric. Note that in this case the symmetric matrices A and A^{-1} are bounded bellow, i.e. $\bar{\xi} \cdot A\,\xi \geq \gamma|\xi|^2$ and $\bar{\xi} \cdot A^{-1}\,\xi \geq \beta|\xi|^2$, for all $\xi \in \mathbb{C}^2 \setminus \{0\}$ and all $x \in D$ for some constant $\gamma > 0$ and $\beta > 0$.

We denote by $\|A^{-1}\|_2$ the Euclidian norm of A^{-1} which is the largest eigenvalue of A^{-1} since the matrix is positive definite. We denote by $\lambda_1(x) \leq \lambda_2(x)$ the two eigenvalues of A^{-1} for $x \in D$. The above assumptions guaranty that $\beta < \lambda_1(x)$ and $\gamma < 1/\lambda_2(x)$ for $x \in D$, since λ_2 is the reciprocal of the smallest eigenvalue of A which by assumption is bigger then γ. We recall that

$$\|A^{-1}\|_2 = \lambda_2 = \sup_{\|\xi\|=1} (\bar{\xi} \cdot A^{-1}\,\xi)$$

and

$$\lambda_1 = \inf_{\|\xi\|=1} (\bar{\xi} \cdot A^{-1}\,\xi).$$

Theorem 3.3. 1. *Assume that* $\|A^{-1}(x)\|_2 \geq \delta > 1$ *for all* $x \in D$ *and some constant* δ. *Then,*

$$\sup_D \|A^{-1}\|_2 \geq \frac{\lambda(D)}{k^2} \tag{45}$$

where k *is a transmission eigenvalue and* $\lambda(D)$ *is the first eigenvalue of* $-\Delta$ *on* D.

2. *Assume that* $0 < \beta \leq \|A^{-1}(x)\|_2 \leq \delta < 1$ *for all* $x \in D$ *and some constants* β *and* δ. *Then, if* k *is a transmission eigenvalue,*

$$k^2 \geq \lambda(D)$$

where $\lambda(D)$ is the first eigenvalue of $-\Delta$ on D.

Proof. To prove the first part of the theorem, we let k be a transmission eigenvalue. The assumptions on A^{-1} imply that there exists a constant $\alpha > 0$ such that $\bar{\xi} \cdot (A^{-1} - I)^{-1}\, \xi \geq \alpha|\xi|^2$. Indeed

$$\inf_{\xi\in\mathbb{C}^2} \bar{\xi} \cdot (A^{-1} - I)^{-1}\, \xi = \frac{1}{\lambda_2 - 1}|\xi|^2 \geq \frac{1}{1/\gamma - 1}|\xi|^2 = \alpha|\xi|^2, \quad x \in D$$

since $1 < \lambda_2 \leq 1/\gamma$. Now, without loss of generality, we take

$$\alpha = \inf_{|\xi|=1} \bar{\xi} \cdot (A^{-1}(x_0) - I)^{-1}\, \xi, \qquad \text{for an appropriate } x_0 \in D.$$

From Theorem 3.1 we have that $\dfrac{\alpha}{\alpha + 1} < k^2/\lambda(D)$. Using the fact that α is the reciprocal of the largest eigenvalue of $A^{-1}(x_0) - I$ we have

$$\alpha = \frac{1}{\lambda_2(x_0) - 1} = \frac{1}{\|A^{-1}(x_0)\|_2 - 1} \geq \frac{1}{\sup_D \|A^{-1}(x)\|_2 - 1}.$$

Now since $\sup_D \|A^{-1}\|_2 > 1$ by assumption we conclude that

$$\sup_D \|A^{-1}\|_2 \geq \frac{1}{\alpha} + 1 > \frac{\lambda(D)}{k^2}$$

which proves the first part of the theorem.

In order to show the second part of the theorem, it suffices to show that the assumptions on A^{-1} imply that there exists a constant $\alpha > 0$ such that $\bar{\xi} \cdot A^{-1}(I - A^{-1})^{-1}\, \xi \geq \alpha|\xi|^2$. Then the result follows from the second part of Theorem 3.2. To this end, we have that $A^{-1}(I - A^{-1})^{-1} = (I - A^{-1})^{-1} - I$.

$$\begin{aligned}\inf_{\xi\in\mathbb{C}^2} \left(\bar{\xi} \cdot A^{-1}(I - A^{-1})^{-1}\, \xi\right) &= \inf_{\xi\in\mathbb{C}^2} \left(\bar{\xi} \cdot (I - A^{-1})^{-1}\, \xi - |\xi|^2\right) \\ &= \left(\frac{1}{1 - \lambda_2} - 1\right)|\xi|^2 \\ &\geq \left(\frac{1}{1 - \beta} - 1\right)|\xi|^2 = \alpha|\xi|^2, \quad x \in D.\end{aligned}$$

This ends the proof. □

4. The vector case. We now return to the three dimensional scattering problem for Maxwell's equations in an anisotropic medium which in terms of electric fields become

$$\begin{aligned}
\nabla \times \nabla \times E^{ext} - k^2 E^{ext} &= 0 \qquad \text{in} \quad D_e \\
\nabla \times \nabla \times E^{int} - k^2 N(x) E^{int} &= 0 \qquad \text{in} \quad D \\
E^{ext} \times \nu - E^{int} \times \nu &= 0 \qquad \text{on} \quad \partial D \\
\nabla \times E^{ext} \times \nu - \nabla \times E^{int} \times \nu &= 0 \qquad \text{on} \quad \partial D \\
E^{ext} &= E^s + E^i \\
\lim_{r \to \infty} (\nabla \times E^s \times x - ikrE^s) &= 0
\end{aligned}$$

where E^i is given by (10). In [**18**] it is shown that the above problem has a unique solution in $H(\mathrm{curl}, B_R)$ for any ball B_R of radius R. Moreover, the scattered electric field E^s has the asymptotic behavior [**9**]

$$E^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right) \right\}$$

as $|x| \to \infty$, where E_∞ is a tangential vector field defined on Ω and is known as the *electric far field pattern.* Note that $E_\infty(\hat{x}, d, p)$ depends linearly on the polarization p. As in the previous section, the *inverse scattering problem* we are interested in is to determine D and $N(x)$ from a knowledge of $E_\infty(\hat{x}, d, p)$ for all $\hat{x}, d \in \Omega$ and three linearly independent polarizations $p_1, p_2, p_3 \in \mathbb{R}^3$. In [**3**], it is shown that D is uniquely determined by $E_\infty(\hat{x}, d, p)$ for all $\hat{x}, d \in \Omega$ and three linearly independent polarizations $p_1, p_2, p_3 \in \mathbb{R}^3$. However, as in the scalar case, it is expected that $E_\infty(\hat{x}, d, p)$ for all $\hat{x}, d \in \Omega$ and three linearly independent polarizations $p_1, p_2, p_3 \in \mathbb{R}^3$ does not uniquely determine the matrix N even if it is known for an interval of values of k. Our goal is to extend the ideas presented in the previous section for the scalar case to the vector case. We define *the electric far field operator* $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(46) \qquad (Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, ds(d), \qquad \hat{x} \in \Omega,$$

for $g \in L^2_t(\Omega)$, where $L^2_t(\Omega)$ is the space of square integrable tangential vector-valued functions defined on the unit sphere Ω. Note that F depends linearly on g. As in the scalar case, D can be determined by the linear sampling method [**14**] which is based on the behavior of the (regularized) solution to the far field equation

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q) \qquad \hat{x} \in \Omega, \text{ and } z, q \in \mathbb{R}^3 \tag{47}$$

where

$$E_{e,\infty}(\hat{x}, z, q) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x}\, e^{-ik\hat{x}\cdot z} \tag{48}$$

is the far field pattern of the electric field E_e of an electric dipole located at z with polarization q. In particular, E_e is defined by

$$E_e(x, z, q) := \frac{i}{k}\mathrm{curl}_x\, \mathrm{curl}_x\, q\, \Phi(x, z) \tag{49}$$

where Φ is the fundamental solution of Helmholtz equation in $\mathbb{R}^3$ defined by

$$\Phi(x, z) := \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x-z|}, \quad x \neq z \quad \text{and} \quad x, z \in \mathbb{R}^3.$$

As in the scalar case, in order to apply the linear sampling method it is necessary that the operator F is injective with dense range which holds provided that k is not a *transmission eigenvalue* [**7, 14**], i.e. a value of k for which the interior transmission problem

$$\mathrm{curl}\,\mathrm{curl}\, E - k^2 N E = 0 \qquad \text{in } D \tag{50}$$

$$\mathrm{curl}\,\mathrm{curl}\, E_0 - k^2 E_0 = 0 \qquad \text{in } D \tag{51}$$

$$\nu \times E = \nu \times E_0 \qquad \text{on } \partial D \tag{52}$$

$$\nu \times \mathrm{curl}\, E = \nu \times \mathrm{curl}\, E_0 \qquad \text{on } \partial D \tag{53}$$

has a nontrivial solution, E, E_0, where $E, E_0 \in L^2(D)$ and $E - E_0 \in \mathcal{U}_0(D)$ where

$$\mathcal{U}_0(D) := \{u \in H_0(\mathrm{curl}\,, D) :\ \mathrm{curl}\ u \in H_0(\mathrm{curl}\,, D)\}$$

equipped with the scalar product

$$(u,v)_{\mathcal{U}(D)} = (u,v)_{L^2(D)} + (\text{curl } u, \text{curl } v)_{L^2(D)} \\ + (\text{curl curl } u, \text{curl curl } v)_{L^2(D)}.$$

The interior transmission problem (50 – 53) is studied in [**7**] (for the isotropic case see [**17**]) where it is proven that the transmission eigenvalues form at most a discrete set provided that either $\bar{\xi} \cdot (N - I)^{-1}\,\xi \geq \alpha|\xi|^2$ or $\bar{\xi} \cdot N(I-N)^{-1}\,\xi \geq \alpha|\xi|^2$ in D and for all $\xi \in \mathbb{C}^2$, where $\alpha > 0$ is a constant. In the following we will assume that the above assumption on N holds in addition to the fact that both N and N^{-1} are positive definite in D.

As in the scalar case, we expect that the norm of the (regularized) solution to

$$(Fg)\,(\hat{x}) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x} e^{-ik\hat{x}\cdot z_0}, \qquad z_0 \in D \tag{54}$$

should be large if k is a transmission eigenvalue, thus providing us with a method for determining transmission eigenvalues from far field data. In [**7**] it is also shown that, if $\bar{\xi} \cdot \mathcal{I}m(N)\xi > 0$ in D and for all $\xi \in \mathbb{C}^3 \setminus \{0\}$, there are no transmission eigenvalues. Hence, in the following we assume that $\bar{\xi} \cdot \mathcal{I}m(N)\xi = 0$ in D and for all $\xi \in \mathbb{C}^3$, i.e. the scatterer is a dielectric. The following results are a consequence of Theorem 3.3 of [**7**].

Theorem 4.1. *Assume that $\bar{\xi} \cdot (N-I)^{-1}\,\xi \geq \alpha|\xi|^2$ in D, for all $\xi \in \mathbb{C}^3$ and some $\alpha > 0$. Than all transmission eigenvalues (if they exist) satisfy $k^2 \geq \dfrac{\alpha}{1+\alpha}\lambda(D)$ where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.*

Theorem 4.2. *Assume that $\bar{\xi} \cdot N(I-N)^{-1}\,\xi \geq \alpha|\xi|^2$ in D, for all $\xi \in \mathbb{C}^3$ and some $\alpha > 0$. Than all transmission eigenvalues (if they exist) satisfy $k^2 \geq \lambda(D)$ where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.*

Finally, exactly in the same way as in the proofs of Theorem 3.3 where A^{-1} is replaced by N, we can conclude from the above theorems the following estimates for the index of refraction.

Theorem 4.3. 1. *If* $\|N(x)\|_2 \geq \delta > 1$ *for all* $x \in D$ *and some* δ *constant, then*

$$\sup_D \|N\|_2 \geq \frac{\lambda(D)}{k^2} \tag{55}$$

where k *is a transmission eigenvalue and* $\lambda(D)$ *is the first eigenvalue of* Δ *on* D.

2. *If* $0 < \beta \leq \|N(x)\|_2 \leq \delta < 1$ *for all* $x \in D$ *and some* β *and* δ *constants, then all the transmission eigenvalue satisfy* $k^2 > \lambda(D)$ *where* $\lambda(D)$ *is the first Dirichlet eigenvalue of* $-\Delta$ *on* D.

5. Numerical examples. This section will present some numerical investigations on the behavior of transmission eigenvalues with respect to both the shape of the medium and the index of refraction. We shall restrict ourselves to a simplified 2-D orthotropic media where

$$A = \frac{1}{n(x)} I$$

(i.e. $n_{11} = n_{22} = n$ and $n_{12} = n_{21} = 0$) with $n(x) > 0$ being a piecewise constant function. This numercial study is complementary to the one presented in [**5**] where the isotropic case is considered. As indicated in the third section, the transmission eigenvalues are numerically computed using the behavior of the solution g to the far field equation (18) in terms of the wave number k for a fixed point z lying inside the orthotropic medium. We expect $\|g\|$ to have peaks at the transmission eigenvalues.

The set of data (i.e. $\{u_\infty(\hat{x}, d; k)$, for $\hat{x} \in \Omega$, $d \in \Omega$ and $k \in [k_{min}, k_{max}]\}$) is generated using a forward solver based on reformulating the scattering problem as an integral equation on the interface between media with constant n. The integral equation is then solved using a variational approach and P^1 finite elements. The discretization step has been adapted to the frequency in order to maintain a comparable precision for different wave numbers and index. We have also adapted the number of direction and observation points $(\hat{x}, d)$ for the discretization of the far field operator (20) to the frequency.

The computed data is then corrupted with 1% random relative noise and equation (18) is solved using Tikhonov regularization combined with the generalized Morozov principle to evaluate the regularization parameter [**4**].

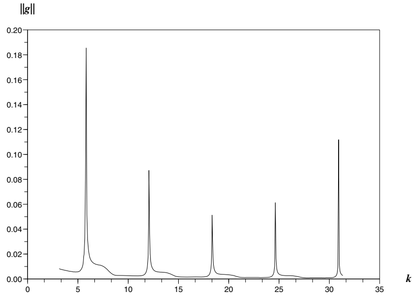
A validating example: We shall first consider the simple case of a circle with a constant index of refraction where some transmission eigenvalues can be computed explicitly. Let r_0 be the radius of this circle and assume that n is constant. Then, seeking solutions to the interior transmission problem in the form $v(x) = \alpha J_0(k|x|)$ and $w(x) = \beta J_0(k\sqrt{n}|x|)$, one easily checks that the existence of a solution is equivalent to k being a zero of the equation

$$\frac{1}{\sqrt{n}} J_0(kr_0) J_1(k\sqrt{n}r_0) - J_1(kr_0) J_0(k\sqrt{n}r_0) = 0. \tag{56}$$

Notice that this equation is different from the one obtained for isotropic inclusions. Figure 1 shows the numerical evaluation of $k \longmapsto \|g\|$ for $r_0 = 0.5$, $n = 4$ and $k \leq 35$ and compares the abscissa of the observed peaks with the first zeros of equation (56). One clearly sees a very good agreement between the two sets of computed values.

The example of Figure 1 can be considered as a validating example for both the forward solver that generates the synthetic data and the proposed numerical procedure to compute transmission eigenvalues.

We also compared the values of the first transmission eigenvalues for different refractive indices as shown in Table 1. Again, one observes a very good agreement between the two sets of computed values.



Peaks' abscissa	Zeros of (56)
5.8	5.81
12.0	12.08
18.3	18.36
24.6	24.64
31	30.92

Figure 1: $\|g\|$ in terms of the wave number k for the circle with $r_0 = 0.5$ and $n = 4$.. The peaks coincide with the computed transmission eigenvalues using (56) as shown in the table.

n	2.	3.	4.	6.	9.	12.
$k_{0,\text{ex}}$	15.97	9.38	5.81	4.45	3.54	3.04
k_0	16.0	9.4	5.8	4.4	3.5	3.0

Table 1: First transmission eigenvalues for a circle of radius 0.5 and a varying index of refraction. $k_{0,\text{ex}}$: first zero of (56). k_0: abscissa of the first peak of $k \mapsto \|g\|$.

Transmission eigenvalues for other geometries: The nice structure of the transmission eigenvalues observed in the case of the circle is less pronounced for other geometries. Figures 2-3 respectively show $k \mapsto \|g\|$ for a domain D respectively being a square $= [-r_0, r_0] \times [-r_0, r_0]$ and an L-shape $= \{[-r_0, r_0] \times [-r_0, r_0]\} \setminus \{]0, r_0] \times]0, r_0]\}$, with $r_0 = 0.5$ and $n = 4.$. However, we observe in both cases a very good localization of the first transmission eigenvalue since the first peak is well separated from the other ones.

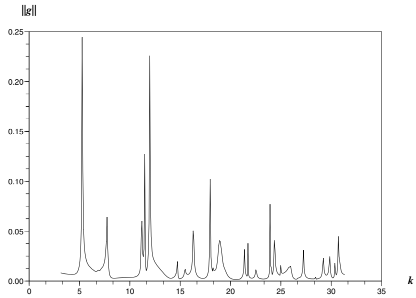


Figure 2: $\|g\|$ in terms of the wave number k for the square with $r_0 = 0.5$ and $n = 4.$.

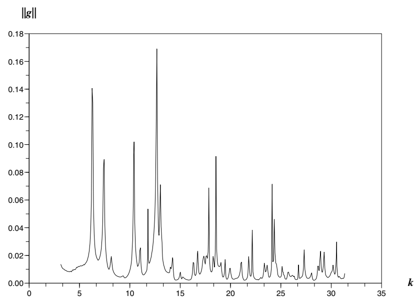


Figure 3: $\|g\|$ in terms of the wave number k for the L-shape with $r_0 = 0.5$ and $n = 4.$.

Numerical estimates on the index: The following tables (Tables 2-4) illustrate the dependence of the first eigenvalues k_0 in terms of the index of refraction n and give the value of the resulting lower bound

$$n_{\min} = \lambda(D)/k_0^2$$

where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D. We recall that $\lambda(D) = (t_0/r_0)^2$ in the case of the circle, with $t_0 \simeq 2.40\ldots$ denoting the first zero of J_0, $\lambda(D) = 2(\pi/(2r_0))^2$ in the case of the square and $\lambda(D) = \lambda_0/r_0^2$ in the case of the L-shape with $\lambda_0 \simeq 9.65\ldots$ denoting the Dirichlet eigenvalue for $r_0 = 1$.

In all examples, the value of k_0 is evaluated by computing the function $k \mapsto \|g\|$ at equidistant points with a step $\Delta k = 0.1$ for $k \in [k_{min}, k_{min} + \tilde{k}_0]$ where $k_{min} = \sqrt{\lambda(D)/n}$ and $\tilde{k}_0$ denotes the first eigenvalue computed with the aid of (56) for a circle having roughly the same Dirichlet eigenvalue as the considered shape.

According to Theorem 3.3, one has some extra information on n only in the cases where the index of refraction is greater than 1. Therefore, if no information is available on n one can only state that $\sup_D n \geq n_{\min}$ if $n_{\min} > 1$. In our numerical examples, one then observes that no information is obtained for $n \leq 4$. For a larger index of refraction the accuracy of obtained lower bounds increases with n. One also observes a monotonically decreasing dependence between k_0 and n which indicates that the value of k_0 could also be used as a comparative test between the index of refraction of two unknown media (at least if we already know that in both cases the index of refraction is greater than 1).

n	2.	3.	4.	6.	9.	12.	16.
k_0	16.0	9.4	5.8	4.4	3.5	3.0	2.6
$n_{\min}$	0.1	0.3	0.7	1.2	2.0	2.6	3.4

Table 2: First transmission eigenvalues (k_0) and lower bounds of the index of refraction ($n_{\min}$) for the **circle** with $r_0 = 0.5$.

n	2.	3.	4.	6.	9.	12.	16.
k_0	13.0	8.0	5.3	4.4	3.14	2.7	2.3
$n_{\min}$	0.1	0.3	0.7	1.0	2.0	2.7	3.7

Table 3: First transmission eigenvalues (k_0) and lower bounds of the index of refraction ($n_{\min}$) for the **square** with $r_0 = 0.5$.

n	2.	3.	4.	6.	9.	12.	16.
k_0	15.5	8.1	6.3	4.5	3.3	2.8	2.3
$n_{\min}$	0.2	0.6	1.	1.9	3.5	4.9	7.2

Table 4: First transmission eigenvalues (k_0) and lower bounds of the index of refraction ($n_{\min}$) for the **L-shape** with $r_0 = 0.5$.

We end this section with a numerical example for a nonhomogeneous media. More precisely we consider the case of two embedded squares $[-r_0, r_0] \times [-r_0, r_0]$ and $[-r_1, r_1] \times [-r_1, r_1]$ with $r_0 = 0.5$ and $r_1 \leq r_0$ where $n = 2$ inside the smaller square and $n = 4$ in the region between the two squares. We then vary the value of r_1 between 0 and r_0. The values of the first transmission eigenvalue detected by our algorithm are reported in the following table (Table 5). One clearly observe how k_0 increases with r_1, which indeed causes the lower bound on $\sup_D n$ to become worse. However, the observed dependence of k_0 in terms of r_1 suggests that a sharper estimate on n in terms of L^p norms with $p < \infty$ would be possible, but this issue definitely needs deeper numerical and theoretical investigations.

r_1	0	0.2	0.4	0.5 (=r_0)
k_0	5.3	7.4	10.9	13.

Table 5: First transmission eigenvalue (k_0) for two concentric boxes with different index of refraction.

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