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INTEGRAL EQUATIONS FOR INVERSE PROBLEMS IN CORROSION DETECTION FROM PARTIAL CAUCHY DATA

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Abstract. We consider the inverse problem to recover a part Γ_c of the boundary of a simply connected planar domain D from a pair of Cauchy data of a harmonic function u in D on the remaining part $\partial D \setminus \Gamma_c$ when u satisfies a homogeneous impedance boundary condition on Γ_c. Our approach extends a method that has been suggested by Kress and Rundell [17] for recovering the interior boundary curve of a doubly connected planar domain from a pair of Cauchy data on the exterior boundary curve and is based on a system of nonlinear integral equations. As a byproduct, these integral equations can also be used for the problem to extend incomplete Cauchy data and to solve the inverse problem to recover an impedance profile on a known boundary curve. We present the mathematical foundation of the method and illustrate its feasibility by numerical examples.

1. Introduction

We consider a simply connected bounded domain $D \subset \mathbb{R}^2$ with piece-wise smooth boundary ∂D. By ν we denote the outward unit normal to ∂D. We assume that the boundary ∂D is written as $\partial D = \overline{\Gamma}_m \cup \overline{\Gamma}_c$ where Γ_m and Γ_c are two open and connected disjoint portions of ∂D and consider the following boundary value problem

$$\Delta u = 0 \quad \text{in } D, \tag{1.1}$$

$$u = f \quad \text{on } \Gamma_m, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} + \lambda u = 0 \quad \text{on} \quad \Gamma_c, \tag{1.3}$$

where λ is a nonnegative L^∞ function on Γ_c. The inverse problem we are concerned with is: *given the Dirichlet data f on Γ_m and the (measured) Neumann data*

$$g := \frac{\partial u}{\partial \nu} \quad \textit{on } \Gamma_m$$

determine the shape of the portion Γ_c of the boundary or the impedance function λ.

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This problem arises in electrostatic or thermal imagining methods in nondestructive testing and evaluations. For instance, one can think of u as representing the electrostatic potential in a conducting body D of which only the portion Γ_m of the boundary is accessible to measurements. Hence, in this application, the above inverse problem can be interpreted as to determine the shape of the inaccessible portion Γ_c of the boundary from a knowledge of the imposed voltage $u|_{\Gamma_m}$ and the measured resulting current $\partial u/\partial\nu|_{\Gamma_m}$ on Γ_m. Various applications of this problem (or slightly modified versions) are discussed in [1, 2, 5] (see also the references therein) where, in general, the authors consider only the reconstruction of the boundary impedance λ as a function of space on the inaccessible portion of the boundary.

Remark 1.1. In particular, in the above formulation we can consider the cases $\lambda = 0$ and $\lambda = \infty$ which correspond to a homogeneous Neumann boundary condition and a homogeneous Dirichlet boundary condition, respectively, on the unknown part Γ_c of the boundary.

In order to formulate (1.1)–(1.3) and the corresponding inverse problem more precisely we need to define the trace $u|_\Gamma$ for $u \in H^1(D)$ where Γ is a generic open subset of ∂D. To this end, let $H^{\frac{1}{2}}(\partial D)$ be the trace space of $H^1(D)$ and $H^{-\frac{1}{2}}(\partial D)$ the dual of $H^{\frac{1}{2}}(\partial D)$ with $L^2(\partial D)$ as a pivot space. We define

$$H^{\frac{1}{2}}(\Gamma) := \{u|_\Gamma : u \in H^{\frac{1}{2}}(\partial D)\}$$

with the norm

$$\|u\|_{H^{\frac{1}{2}}(\Gamma)} = \inf_{\substack{U \in H^{\frac{1}{2}}(\partial D)\\ U|_\Gamma = u}} \{\|U\|_{H^{\frac{1}{2}}(\partial D)}\}$$

and

$$\widetilde{H}^{\frac{1}{2}}(\Gamma) := \{u \in H^{\frac{1}{2}}(\Gamma) : \operatorname{supp} u \subseteq \overline{\Gamma}\}.$$

In other words, $\widetilde{H}^{\frac{1}{2}}(\Gamma)$ contains functions $u \in H^{\frac{1}{2}}(\Gamma)$ such that their extension by zero to the whole boundary ∂D is in $H^{\frac{1}{2}}(\partial D)$ (Theorem 3.33 in [18]). Now we denote by $H^{-\frac{1}{2}}(\Gamma)$ the dual space of $\widetilde{H}^{\frac{1}{2}}(\Gamma)$ and by $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ the dual space of $H^{\frac{1}{2}}(\Gamma)$. The following chain of inclusions holds

$$\widetilde{H}^{\frac{1}{2}}(\Gamma) \subset H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma) \subset \widetilde{H}^{-\frac{1}{2}}(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma).$$

We note that $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ can also be identified with $H^{-\frac{1}{2}}_{\overline{\Gamma}}(\partial D) := \{u \in H^{-\frac{1}{2}}(\partial D) : \operatorname{supp} u \in \overline{\Gamma}\}$ (Theorem 3.29 in [18]).

It is known [4, 21] that for $f \in H^{\frac{1}{2}}(\Gamma_m)$ there exists a unique solution $u \in H^1(D)$ of (1.1)–(1.3). Hence, our *inverse problem* can be formulated as: given $u|_{\Gamma_m} = f \in H^{\frac{1}{2}}(\Gamma_m)$ and $\partial u/\partial\nu|_{\Gamma_m} = g \in H^{-\frac{1}{2}}(\Gamma_m)$ determine Γ_c. Our approach for solving it is based on a system of nonlinear and ill-posed integral equations for the unknown boundary and the density of a single-layer potential on ∂D that is solved using regularized iterations. This method has been recently suggested by Kress and Rundell [17] to determine the shape of a perfectly conducting obstacle in a homogeneous background from overdetermined Cauchy data. Ivanyshyn and Kress [8] have extended it to the Neumann boundary condition and to cracks. Furthermore, this method has also been employed for inverse obstacle scattering problems [9, 11].

Our presentation is organized as follows. In Section 2 we briefly discuss the open issue of uniqueness. Then, although a general uniqueness result is lacking, in Section 3, we proceed with deriving our system of nonlinear integral equations. Then we pause with the inverse problem by an intermezzo on the completion of Cauchy data in Section 4. Then in Section 5, this is followed by details on an iterative solution procedure. The paper is concluded with some numerical examples in Section 6.

2. UNIQUENESS FOR THE INVERSE PROBLEM

In this section, we begin by discussing the question of whether a single pair of Cauchy data on Γ_m uniquely determines the missing part Γ_c of the boundary ∂D. In general, for $\lambda \in (0,\infty)$ this is not the case as can be seen from the following simple example for non-uniqueness. More precisely we show that, for a fixed constant impedance λ, a single measurement of f, g on Γ_m can give rise to infinitely many different domains D.

Example 2.1. Let D be a rectangle defined by

$$D = \big\{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 < \pi/2,\ -a < x_2 < 1\big\}$$

for some $a > 0$ and set

$$\Gamma_m := \{(0,x_2) : 0 < x_2 < 1\} \cup \{(\pi/2,x_2) : 0 < x_2 < 1\} \cup \{(x_1,1) : 0 < x_1 < \pi/2\}.$$

We consider the entire harmonic function u given by

$$u(x_1,x_2) = (\cos x_1 + \sin x_1)\, e^{x_2}$$

and choose $\lambda = 1$. Then, elementary calculations show that on $\Gamma_c := \partial D \setminus \overline{\Gamma}_m$ we have that

$$\frac{\partial u}{\partial \nu} + u = 0$$

with the outward normal ν to ∂D. Hence, if we choose as Cauchy data on Γ_m the restrictions $f = u|_{\Gamma_m}$ and $g = \partial u/\partial\nu|_{\Gamma_m}$ we have infinitely many solutions to the inverse problem, since $a > 0$ can be chosen arbitrarily.

The following example indicates that it is impossible to simultaneously recover the shape and the impedance.

Example 2.2. Let D be the rectangle with corners $(0,0), (\pi,0), (0,a)$, and (π,a) for some positive a. Then the entire harmonic function u given by

$$u(x_1,x_2) = \cos x_1 \left(\cosh x_2 - \frac{\lambda\cosh a + \sinh a}{\cosh a + \lambda \sinh a}\ \sinh x_2\right)$$

has Dirichlet values $u(x_1,0) = \cos x_1$ on the lower horizontal part of ∂D, and satisfies a homogeneous Neumann condition on the two vertical parts of ∂D and a homogeneous impedance boundary condition with constant impedance λ on the upper horizontal part of ∂D. From

$$\frac{\partial u}{\partial x_2}\,(x_1,0) = -\frac{\lambda\cosh a + \sinh a}{\cosh a + \lambda \sinh a}\ \cos x_1$$

we observe that we cannot recover simultaneously both a and λ from the normal derivative of u on the lower horizontal boundary since this provides only one equation for two unknowns.

However, in the particular case when homogeneous Dirichlet or Neumann boundary conditions are assumed on the unknown part of the boundary, which corresponds to $\lambda = \infty$ and $\lambda = 0$, respectively, it is easy to show that one pair of Cauchy data uniquely determines the missing part of the boundary.

Theorem 2.3. *Assume that in* (1.1)–(1.3) *we have* $u = 0$ *on* Γ_c, *then* $f = u|_{\Gamma_m}$ *and* $g = \partial u/\partial\nu|_{\Gamma_m}$ *uniquely determine* Γ_c *provided that* $f \neq 0$.

Proof: We suppose that there are two bounded domains D_1 and D_2 having Γ_m as part of their boundary such that the corresponding solutions u_i for $i = 1, 2$ satisfy $\Delta u_i = 0$ in D_i, $u_i = 0$ on $\partial D_i \setminus \overline{\Gamma}_m$ and $u_1 = u_2 = f$ and $\partial u_1/\partial\nu = \partial u_2/\partial\nu = g$ on Γ_m. Then Holmgren's theorem implies that $u_1 = u_2$ in $D_1 \cap D_2$.

Without loss of generality we may assume that there exists a nonempty connected component Ω of $D_2 \setminus \overline{D}_1$. Then from $u_1 = u_2$ in $D_1 \cap D_2$ and the boundary conditions for u_1 and u_2 we can conclude that $u_2 = 0$ on the boundary of Ω. Now the maximum-minimum principle for harmonic functions implies that $u_2 = 0$ in Ω and consequently, by analyticity, $u_2 = 0$ in D_2. However, this contradicts the fact that f is not identically zero. □

Remark 2.4. In the case of Neumann boundary data on Γ_c, i.e. for $\lambda = 0$ by using the boundary value problem for the harmonic conjugate of u one can prove that $f = u|_{\Gamma_m}$ and $g = \partial u/\partial\nu|_{\Gamma_m}$ uniquely determine Γ_c provided that f is not a constant.

We note that as a consequence of Holmgren's theorem it is easy to show [5] that for a fixed D the impedance coefficient λ as a function of space is uniquely determined from one pair of Cauchy data on an open subset of the boundary ∂D.

3. Nonlinear integral equations

To derive nonlinear integral equations that are equivalent to the inverse problem we represent the solution u of (1.1)–(1.3) as surface superposition of point sources given by the fundamental solution

$$\Phi(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|}, \quad x \neq y,$$

with an unknown density φ defined on the boundary ∂D. In particular, we write

$$u(x) = \int_{\partial D} \Phi(x, y)\varphi(y)\, ds(y), \quad x \in D, \tag{3.1}$$

with a density $\varphi \in H^{-\frac{1}{2}}(\partial D)$. From now on, without loss of generality, we assume that there exists a point $x_0 \in D$ such that $|x - x_0| \neq 1$ for all $x \in \partial D$. Then Theorem 3.16 in [12] guarantees that the corresponding single-layer boundary integral operator is injective. (An alternative approach to guarantee the injectivity via boundedness of u at infinity is to modify the above definition (3.1) by adding an appropriate term as in Theorem 7.30 in [16].)

In (3.1) the portion $\Gamma_c = \partial D \setminus \overline{\Gamma}_m$ of the boundary and the density φ are the unknowns. To set up a system of equations to solve for these two unknowns we take the traces of (3.1) on the boundary ∂D requiring that $u|_{\Gamma_m} = f$ and $\partial u/\partial\nu|_{\Gamma_m} = g$.

Hence we obtain

$$S\varphi = f \quad \text{on } \Gamma_m, \tag{3.2}$$
$$K'\varphi + \frac{\varphi}{2} = g \quad \text{on } \Gamma_m$$

where $S : H^{-\frac{1}{2}+s}(\partial D) \to H^{\frac{1}{2}+s}(\partial D)$ and $K' : H^{-\frac{1}{2}+s}(\partial D) \to H^{-\frac{1}{2}+s}(\partial D)$, $-1/2 \le s \le 1/2$, are continuous boundary integral operators defined by

$$(S\varphi)(x) := \int_{\partial D} \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \partial D,$$

and

$$(K'\varphi)(x) := \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\, \varphi(y)\, ds(y), \quad x \in \partial D.$$

In addition, on the unknown part $\Gamma_c = \partial D \setminus \overline{\Gamma}_m$ of the boundary ∂D the equation

$$K'\varphi + \frac{\varphi}{2} + \lambda\, S\varphi = 0 \quad \text{on } \Gamma_c \tag{3.3}$$

is satisfied.

Conversely, if ∂D and $\varphi \in H^{-\frac{1}{2}}(\partial D)$ satisfy the system (3.2) and (3.3) then $\Gamma_c = \partial D \setminus \overline{\Gamma}_m$ solves the inverse problem. Indeed, if we define u by (3.1) then we obtain an $H^1(D)$ solution of the Laplace equation [18]. Furthermore, approaching the boundary ∂D from inside D from (3.2) and (3.3) we also have that u satisfies the mixed boundary value problem (1.1)–(1.3) and $\partial u/\partial \nu|_{\Gamma_m} = g$. Hence, $\Gamma_c = \partial D \setminus \overline{\Gamma}_m$ provides a solution of the inverse problem and we can state the following result.

Theorem 3.1. *The inverse problem and the system of integral equations (3.2) and (3.3) are equivalent.*

The system of integral equations (3.2) and (3.3) equivalent to our inverse problem is not unique. For instance, representing the solution u as a combination of a single- and double-layer potential one can derive a different system of integral equations equivalent to our inverse problem [17]. The benefit of the approach presented here is that it avoids hypersingular integral equations.

For the further investigation of the nonlinear integral equations and, in particular, for the numerical solution a parameterization is required. In this preliminary study, for the sake of simplicity we confine ourselves to smooth boundaries ∂D of class C^2, that is, we represent

$$\partial D := \{z(t) : t \in [0,\, 2\pi]\} \tag{3.4}$$

with a 2π periodic C^2–smooth function $z : \mathbb{R} \to \mathbb{R}^2$ such that z is injective on $[0,\, 2\pi)$ and satisfies $z'(t) \neq 0$ for all t. Without loss of generality we may assume that the known part of the boundary Γ_m and the unknown part of the boundary Γ_c are given by

$$\Gamma_m := \{z(t) : t \in (0,\pi)\} \quad \text{and} \quad \Gamma_c := \{z(t) : t \in (\pi, 2\pi)\}\,.$$

In order to incorporate possible singularities of the solutions at the end points of Γ_c and Γ_m, in principle, we could use suitable transformations such as the cosine transform introduced in [22] and used in [14, 19] or sigmoidal transformations as investigated in [6] and used in [7, 13] for domains with corners. However, for the present paper we have chosen not to pursue this idea any further.

In view of (3.4), setting

$$\psi(t) := |z'(t)|\,\varphi(z(t)) \tag{3.5}$$

we obtain the parameterized integral operators

$$(\widetilde{S}\psi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \ln\frac{1}{|z(t)-z(\tau)|}\;\psi(\tau)\,d\tau \tag{3.6}$$

and

$$(\widetilde{K}'\psi)(t) = \frac{1}{2\pi\,|z'(t)|}\int_0^{2\pi} \frac{[z'(t)]^\perp\cdot[z(\tau)-z(t)]}{|z(t)-z(\tau)|^2}\;\psi(\tau)\,d\tau + \frac{\psi(t)}{2\,|z'(t)|} \tag{3.7}$$

for $t \in [0,2\pi]$. Here, we used the notation $a^\perp = (a_2,-a_1)$ for any vector $a = (a_1,a_2)$, that is, $a^\perp$ is obtained by rotating a counter clockwise by 90 degrees. The parameterized form of the equations (3.2) and (3.3) now reads

$$\begin{aligned} \widetilde{S}\psi &= f\circ z \quad \text{on } [0,\pi],\\ \widetilde{K}'\psi &= g\circ z \quad \text{on } [0,\pi], \end{aligned} \tag{3.8}$$

and

$$\widetilde{K}'\psi + \lambda\,\widetilde{S}\psi = 0 \quad \text{on } [\pi,2\pi]. \tag{3.9}$$

For the discretization of the integral operators we note that the 2π periodic kernel of $\widetilde{S}$ can be decomposed in the form

$$\ln\frac{1}{|z(t)-z(\tau)|} = -\ln\left|\sin\frac{t-\tau}{2}\right| + \ln\frac{\left|\sin\dfrac{t-\tau}{2}\right|}{|z(t)-z(\tau)|}$$

where the second term is smooth with diagonal values

$$\lim_{\tau\to t}\ln\frac{\left|\sin\dfrac{t-\tau}{2}\right|}{|z(t)-z(\tau)|} = -\ln 2|z'(t)|.$$

Hence, the well established trigonometric interpolation quadrature rules on equidistant meshes for logarithmic singularities as described in [16] are available. The 2π periodic kernel of $\widetilde{K}'$ is smooth with diagonal term

$$\frac{[z'(t)]^\perp\cdot z''(t)}{4\pi\,|z'(t)|^3}$$

and therefore the trapezoidal rule (which is also a trigonometric interpolation quadrature) can be employed.

4. Completion of Cauchy data

A particular case of our setting is the completion of Cauchy data. This problem can be formulated as follows: Given $f \in H^{\frac{1}{2}}(\Gamma_m)$ and $g \in H^{-\frac{1}{2}}(\Gamma_m)$ find $\alpha \in H^{\frac{1}{2}}(\Gamma_c)$ and $\beta \in H^{-\frac{1}{2}}(\Gamma_c)$ such that there exists a harmonic function $u \in H^1(D)$ satisfying

$$u = f \quad\text{and}\quad \frac{\partial u}{\partial\nu} = g \quad \text{on } \Gamma_m$$

and $u = \alpha$ and $\dfrac{\partial u}{\partial \nu} = \beta$ on Γ_c. Note that this Cauchy problem admits at most one solution and is known to be ill-posed. In the literature many approaches have been developed to address the completion of Cauchy data (see e.g. [3, 10] and the references therein). Our integral equation method provides an alternative solution. In particular, if $\varphi \in H^{-\frac{1}{2}}(\partial D)$ solves the system of linear integral equations

$$S\varphi = f \qquad \text{on } \Gamma_m, \tag{4.1}$$
$$K'\varphi + \frac{\varphi}{2} = g \qquad \text{on } \Gamma_m,$$

then $\alpha = u|_{\Gamma_c}$ and $\beta = \partial u/\partial\nu|_{\Gamma_c}$ where $u \in H^1(D)$ is given by

$$u(x) = \int_{\partial D} \Phi(x,y)\varphi(y)\, dy, \quad x \in D, \tag{4.2}$$

provides the solution of the Cauchy problem.

In practice, given the measured data f and g we solve the ill-posed equation (4.1) by using regularization methods such as Tikhonov regularization. Since the L^2-norm is the appropriate norm to measure the data error, it is natural to apply the regularization scheme in the space of square integrable functions. For this reason, we consider the operator $A : L^2(\partial D) \to L^2(\Gamma_m) \times L^2(\Gamma_m)$ defined by

$$A\varphi = \begin{pmatrix} S\varphi \\ \\ K'\varphi + \dfrac{\varphi}{2} \end{pmatrix}.$$

Note that $S : L^2(\partial D) \to H^1(\partial D)$ whereas $K' : L^2(\partial D) \to L^2(\partial D)$. In order to apply the Tikhonov regularization scheme to (4.1) we need the following result.

Theorem 4.1. *The operator A is injective with dense range.*

Proof: Injectivity follows from Holmgren's theorem. Indeed, if $A\varphi = 0$ for some $\varphi \in L^2(\partial D)$ then u defined by (4.2) satisfies $u|_{\Gamma_m} = 0$ and $\partial u/\partial\nu|_{\Gamma_m} = 0$ from inside D whence $u = 0$ in D follows. The trace theorem now implies that $S\varphi = 0$. Since our geometric assumptions on D guarantee injectivity of S we conclude that $\varphi = 0$.

Next we prove that A has dense range. To this end, we consider the adjoint operator $A^* : L^2(\Gamma_m) \times L^2(\Gamma_m) \to L^2(\partial D)$ defined by

$$(A\varphi, [\alpha,\beta])_{L^2(\Gamma_m)\times L^2(\Gamma_m), L^2(\Gamma_m)\times L^2(\Gamma_m)} = (\varphi, A^*[\alpha,\beta])_{L^2(\partial D), L^2(\partial D)}$$

where $(\cdot,\cdot)$ denotes the respective L^2 inner product. We want to show that A^* is injective which implies that A has dense range. Let $\tilde{\alpha} \in L^2(\partial D)$ and $\tilde{\beta} \in L^2(\partial D)$ be the extensions of $\alpha \in L^2(\Gamma_m)$ and $\beta \in L^2(\Gamma_m)$ by zero to the whole boundary ∂D. Then, for every $\varphi \in L^2(\partial D)$ we have that

$$(A\varphi, [\alpha,\beta]) = (\varphi, S\tilde{\alpha}) + \left(\varphi, K\tilde{\beta}\right) + \left(\varphi, \frac{\tilde{\beta}}{2}\right) = \left(\varphi, S_m\alpha + K_m\beta + \frac{\beta}{2}\right)$$

where S_m and K_m are defined by

$$(S_m\alpha)(x) := \int_{\Gamma_m} \Phi(x,y)\alpha(y)\, ds(y), \quad x \in \partial D,$$

and

$$(K_m\beta)(x) := \int_{\Gamma_m} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\, \beta(y)\, ds(y), \quad x \in \partial D.$$

Hence, we conclude that

$$A^*(\alpha,\beta) = S_m\alpha + K_m\beta + \frac{\beta}{2}\,.$$

Now, let $A^*(\alpha,\beta) = 0$ for some $\alpha \in L^2(\Gamma_m)$ and $\beta \in L^2(\Gamma_m)$. We define

$$u(x) = \int_{\Gamma_m} \Phi(x,y)\alpha(y)\,ds(y) + \int_{\Gamma_m} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\beta(y)\,ds(y)$$

which is a solution of the Laplace equation in $\mathbb{R}^2 \setminus \overline{\Gamma}_m$. Letting $x \to \partial D$ from outside D and using the jump relations for single- and double-layer potentials with L^2 densities we obtain that

$$u|_{\partial D} = S_m\alpha + K_m\beta + \frac{\beta}{2} = 0. \tag{4.3}$$

Following the proof of Theorem 3.16 in [12] to deal with the logarithmic behavior of the single-layer potential at infinity, from (4.3) we obtain that $u = 0$ in $\mathbb{R}^2 \setminus \overline{D}$ and consequently, by analyticity, $u = 0$ in D as well. From this, finally, the jump relations across ∂D imply that $\alpha = \beta = 0$ which proves that A^* is injective. □

In order to indicate the feasibility of this approach to completing Cauchy data we illustrate its application to the inverse problem of determining the impedance for a fixed domain D, that is, we want to recover the impedance function λ on Γ_c from a Cauchy pair (f,g) on Γ_m. To this end, using the notations introduced at the beginning of this section, we just observe that after completing the Cauchy data, that is, after determining α and β on Γ_c we obtain the impedance function from the equation

$$\beta + \lambda\alpha = 0 \quad \text{on } \Gamma_c. \tag{4.4}$$

Therefore, we have to carry out three steps: first we need to solve the ill-posed equation (4.1), for example, by Tikhonov regularization for the density φ on ∂D. For this, of course, we use the parameterized version (3.8) of (4.1) and the trigonometric quadratures based on a equidistant mesh $t_j = j\pi/n$, $j = 1,\dots,2n$, for the parameter t in (3.4) as mentioned at the end of Section 3. Then we obtain α and β as the traces of the potential (4.2) on Γ_c, that is, $\alpha = S\varphi|_{\Gamma_c}$ and $\beta = (K'\varphi + \varphi/2)|_{\Gamma_c}$. Finally, we compute the impedance function λ at the collocation points $x_i = z(t_{n+i})$, $i = 1,\dots,n$, on Γ_c by solving

$$\beta(x_i) + \lambda(x_i)\alpha(x_i) = 0, \quad i = 1,\dots,n. \tag{4.5}$$

In order to avoid instabilities arising from dividing by small values of $\alpha(x_i)$ we represent the unknown λ as a linear combination

$$\lambda = \sum_{k=1}^{K} a_k w_k \tag{4.6}$$

of appropriate basis functions w_k and solve the equation that is obtained by inserting (4.6) into (4.5) in the least squares sense for the coefficients a_k.

In numerical examples we used cubic B-splines on an equidistant subdivision (with respect to the parameter in the parameterization (3.4)). The example is for an ellipse with parameterization

$$z(t) = (a\cos t, b\sin t), \quad t \in [0,2\pi],$$

and the impedance profile

$$\lambda(t) = \begin{cases} 0, & t \in [0, \pi], \\ \sin^4 t, & t \in [\pi, 2\pi]. \end{cases}$$

The synthetic Cauchy data (f, g) on Γ_m were obtained by solving the impedance problem in D with boundary condition

$$\frac{\partial u}{\partial \nu} + \lambda u = h$$

with

$$h(t) = \begin{cases} \sin^4 t, & t \in [0, \pi], \\ 0, & t \in [\pi, 2\pi], \end{cases}$$

by the double-layer boundary integral equation approach (avoiding an inverse crime!). The numerical solution of the resulting hypersingular integral equation was obtained via trigonometric collocation and quadratures as described in [15]. The reconstructions were performed by using 64 grid points for discretizing the single-layer potential. The Figures 1 and 2 show the reconstructed profile both for exact data and for 3% random noise added to the Neumann data g (with respect to the L^2 norm). The Tikhonov regularization parameter was chosen by trial and error as 10^{-9} for exact data and 10^{-6} for noisy data. For the B-spline approximation of the impedance profile the dimension $K = 10$ was used. The reconstructions are for $a = 0.3$ and $b = 0.2$ in Figure 1 and for $a = 0.3$ and $b = 0.4$ in Figure 2. As to be expected, the reconstructions are slightly better for the smaller ellipse since here the Cauchy problem has to be solved over a smaller distance.

5. THE ITERATION SCHEME

We now return to the inverse problem to determine the part Γ_c of the boundary curve ∂D assuming that the impedance is known. For the sake of simplicity, from now on we only consider the case where λ is constant. Because of the linearity of the integral operators with respect to ψ, the linearization of (3.8) and (3.9) leads to

$$\begin{aligned} \widetilde{S}(\psi, z) + \widetilde{S}(\chi, z) + d\widetilde{S}(\psi, z; \zeta) &= f \circ z \quad \text{on } [0, \pi], \\ \widetilde{K}'(\psi, z) + \widetilde{K}'(\chi, z) + d\widetilde{K}'(\psi, z; \zeta) &= g \circ z \quad \text{on } [0, \pi] \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &\widetilde{K}'(\psi, z) + \widetilde{K}'(\chi, z) + d\widetilde{K}'(\psi, z; \zeta) \\ &\quad + \lambda[\widetilde{S}(\psi, z) + \widetilde{S}(\chi, z) + d\widetilde{S}(\psi, z; \zeta)] = 0 \quad \text{on } [\pi, 2\pi]. \end{aligned} \tag{5.2}$$

Given a current approximation for z and ψ, the linear system (5.1) and (5.2) needs to be solved for ζ and χ to obtain the update $z + \zeta$ for the parameterization of Γ_c and $\psi + \chi$ for the single-layer density. Of course, the perturbation ζ is assumed different from zero only on the unknown part Γ_c of ∂D. Then, in an obvious way, this procedure is iterated.

The following alternative approach is more in the spirit of Section 3 on the Cauchy problem and it resembles a decomposition method in the sense that it decomposes the inverse problem into a severely ill-posed linear problem and an at

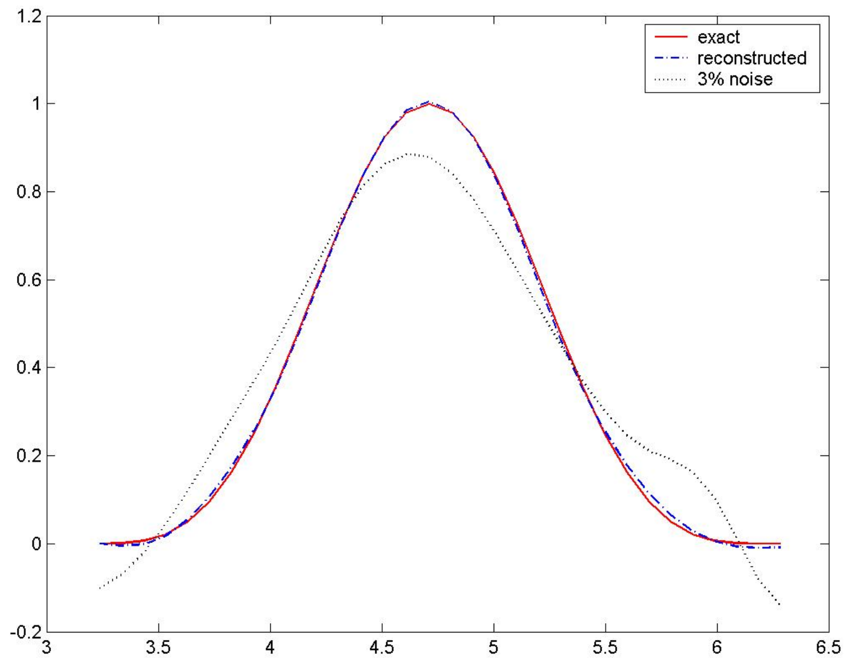

FIGURE 1. Reconstruction of an impedance profile for an ellipse with semi-axis $a = 0.3$ and $b = 0.2$.

most mildly ill-posed nonlinear problem. For this, given a current approximation z for the parameterization, we first solve the ill-posed linear equation

$$\begin{aligned} \widetilde{S}\psi &= f \circ z \quad \text{on } [0, \pi], \\ \widetilde{K}'\psi &= g \circ z \quad \text{on } [0, \pi] \end{aligned} \tag{5.3}$$

for ψ and then, keeping ψ fixed, we solve the linearized equation

$$\widetilde{K}(\psi, z) + d\widetilde{K}'(\psi, z; \zeta) + \lambda[\widetilde{S}(\psi, z) + d\widetilde{S}(\psi, z; \zeta)] = 0 \quad \text{on } [\pi, 2\pi] \tag{5.4}$$

to obtain the update $z + \zeta$ for the boundary parameterization. In principle, this second method has the advantage that the computational cost of one iteration step is smaller as compared to the above full linearization. In our numerical examples described at the end of the paper we only used this second approach.

Clearly, in both approaches the ill-posedness requires that a stabilization is incorporated. Since we are only performing an initial analysis, we restrict ourselves to the well-established Tikhonov regularization.

The Fréchet derivatives of the integral operators $\widetilde{S}$ and $\widetilde{K}'$ with respect to z can be obtained by differentiating their kernels with respect to z (see Potthast [20]). In

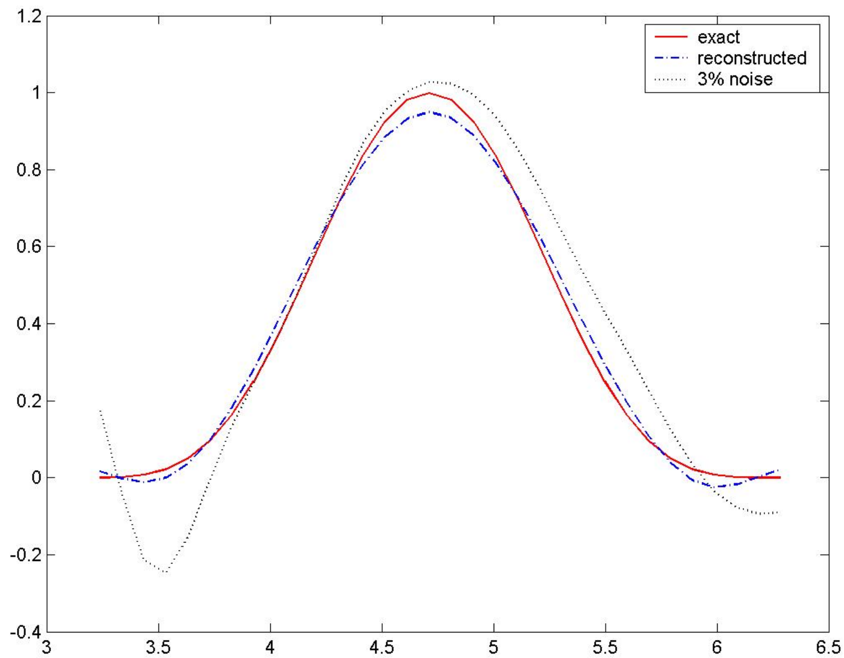

FIGURE 2. Reconstruction of an impedance profile for an ellipse with semi-axis $a = 0.3$ and $b = 0.4$.

particular we have

$$d\widetilde{S}(\psi, z; \zeta)(t) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{[z(t) - z(\tau)] \cdot [\zeta(t) - \zeta(\tau)]}{|z(t) - z(\tau)|^2} \, \psi(\tau)\, d\tau, \quad t \in [0, 2\pi],$$

and

$$\begin{aligned} d\widetilde{K}'(\psi, z; \zeta)(t) \;=\; & \frac{1}{2\pi |z'(t)|} \int_0^{2\pi} \left\{ \frac{2[z'(t)]^\perp \cdot [z(t) - z(\tau)][z(t) - z(\tau)] \cdot [\zeta(t) - \zeta(\tau)]}{|z(t) - z(\tau)|^4} \right. \\ & \left. - \, \frac{[z'(t)]^\perp \cdot [\zeta(t) - \zeta(\tau)] + [\zeta'(t)]^\perp \cdot [z(t) - z(\tau)]}{|z(t) - z(\tau)|^2} \right\} \psi(\tau)\, d\tau \\ & - \frac{z'(t) \cdot \zeta'(t)}{|z'(t)|^2} \, \widetilde{K}'(\psi, z)(t), \quad t \in [0, 2\pi]. \end{aligned}$$

We note that in both expressions we need to keep in mind that both for the integration and for the evaluation, the perturbation ζ is different from zero only in the interval $[\pi, 2\pi]$. We also note that the kernels of $d\widetilde{S}$ and $d\widetilde{K}'$ are smooth with their diagonal values given by

$$-\frac{z'(t)\cdot\zeta'(t)}{2\pi\,|z'(t)|^2} \quad \text{and} \quad -\frac{[z'(t)]^\perp\cdot z''(t)\, z'(t)\cdot\zeta'(t)}{2\pi\,|z'(t)|^5} + \frac{[z'(t)]^\perp\cdot\zeta''(t) + [\zeta'(t)]^\perp\cdot z''(t)}{4\pi\,|z'(t)|^3}\,,$$

respectively.

We complete our analysis with proving the injectivity of the full linearization at the exact solution for the limiting case $\lambda = \infty$. Note that injectivity for the linearization in the general case $0 < \lambda < \infty$ is open.

Theorem 5.1. *Let z be the parameterization of the exact boundary ∂D and let $\psi = |z'|\,\varphi\circ z$ where φ satisfies (3.2)–(3.3) for $\lambda = \infty$. Assume that $\chi \in H^{-\frac{1}{2}}[0, 2\pi]$ and $\zeta \in C^2[0, 2\pi]$ with $\zeta = 0$ on $[0, \pi]$ and $\zeta(t)\cdot\nu(z(t)) \neq 0$ for $t \in (\pi, 2\pi)$ satisfy the homogeneous system*

$$\begin{aligned} \widetilde{S}(\chi, z) + d\widetilde{S}(\psi, z; \zeta) \;&=\; 0 \quad \text{on } [0,\pi], \\ \widetilde{K}(\chi, z) + d\widetilde{K}'(\psi, z; \zeta) \;&=\; 0 \quad \text{on } [0,\pi] \end{aligned} \tag{5.5}$$

and

$$\widetilde{S}(\chi, z) + d\widetilde{S}(\psi, z; \zeta) = 0 \quad \text{on } [\pi, 2\pi]. \tag{5.6}$$

Then $\chi = 0$ and $\zeta = 0$.

Proof: Let us define

$$W(x) = \int_0^{2\pi} \Phi(x, z(\tau))\chi(\tau)\,d\tau + \int_0^{2\pi} \psi(\tau)\,\mathrm{grad}_x\,\Phi(x, z(\tau))\cdot\zeta(\tau)\,d\tau, \quad x \in D.$$

Taking the boundary values of W and the normal derivative of W when approaching the boundary ∂D from inside D, from (5.5) we obtain that $W|_{\Gamma_m} = 0$ and $\partial W/\partial\nu|_{\Gamma_m} = 0$. Therefore Holmgren's theorem implies that $W = 0$ in D. Now subtracting $W = 0$ on Γ_c from (5.6) we observe that $\zeta\cdot\mathrm{grad}\,u = 0$ on Γ_c for the solution u of (1.1)–(1.3) (for $\lambda = \infty$). In view of the boundary condition $u = 0$ on Γ_c and $\zeta(t)\cdot\nu(z(t)) \neq 0$ for $t \in (\pi, 2\pi)$ a second application of Holmgren's theorem yields $\zeta = 0$. Finally, due to our geometric assumption on D, from the injectivity of the single-layer operator for ∂D we obtain $\chi = 0$. □

6. Numerical examples

In this final section we present some numerical results to illustrate the feasibility of the reconstruction method as described in the previous section. We confine ourselves to the second method, that is, each iteration step consists of first solving (5.3) for the density ψ and then solving (5.4) to update the boundary.

For further simplicity we assume that the unknown part Γ_c can be represented in polar coordinates, that is, we express

$$z(t) = r(t)(\cos t, \sin t), \quad t \in [\pi, 2\pi],$$

with a C^2 function $r : [\pi, 2\pi] \to (0, \infty)$. For the numerical examples we approximate r by a cubic B-splines on an equidistant subdivision of $[\pi, 2\pi]$.

We begin with presenting examples for the homogeneous Dirichlet boundary condition on the unknown boundary part Γ_c, that is, for the limiting case $\lambda = \infty$.

The synthetic Cauchy data (f, g) on Γ_m were obtained by solving the Dirichlet problem in D with boundary condition $u = f$ with

$$f(t) = \begin{cases} \sin^4 t, & t \in [0, \pi], \\ 0, & t \in [\pi, 2\pi]. \end{cases}$$

by the double-layer boundary integral equation approach (avoiding again an inverse crime!). For the solution of the ill-posed integral equation (5.3) via Tikhonov regularization the single-layer potential was discretized as indicated at the end of Section 3 using 64 equidistant grid points. The corresponding regularization parameter γ_1 was chosen by trial and error. For the solution of (5.4) we also used a regularization with an H^1 penalty term on r with regularization parameter γ_2. Furthermore, we observed the need of an additional regularization by updating the density ψ by $\psi_{\text{new}} = \gamma\psi + (1-\gamma)\psi_{\text{old}}$ where ψ denotes the solution of (5.3) and γ had to be chosen from an interval close to $[0.4, 0.7]$.

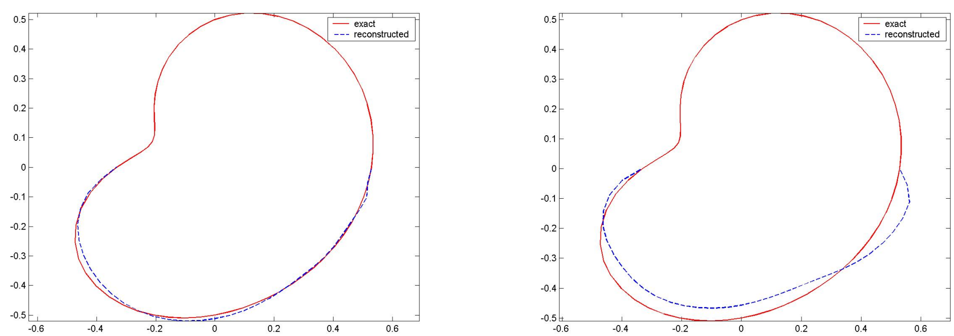

FIGURE 3. Reconstruction of (6.1) for exact (left) and noisy data (right).

We started the iterations by choosing as an initial approximation for Γ_c the half circle in the lower half plane with end points coinciding with the end points $z(0)$ and $z(\pi)$ of Γ_m. We started the iteration by performing L iteration steps for the cubic spline approximation of r on a subdivision of $[\pi, 2\pi]$ in five equidistant intervals. Then we successively increased the number of equidistant subintervals of $[\pi, 2\pi]$ for the spline approximation of r, using the result for a subdivision into m subintervals as an initial guess for Γ_c and performed again L iterations on $m + 1$ subintervals. This process was repeated until a final number M of subintervals was reached. We note that the iteration number L and the disretization level M can be considered as additional regularization parameters. In the following three examples, by trial and error, we chose $L = 8$ and $M = 10$. The figures give reconstructions for exact data and for 1% random noise added to the Neumann data.

Figure 3 shows reconstructions for an apple shaped contour with the parameterization

$$z(t) = \frac{0.5 + 0.4\cos t + 0.07\sin 2t}{1 + 0.7\cos t}\,(\cos t, \sin t), \quad t \in [0, 2\pi]. \tag{6.1}$$

The full lines represent the exact boundary and the broken lines the reconstructions. The reconstructions with exact data are for $\gamma_1 = 10^{-9}$ and $\gamma_2 = 10^{-6}$ and the reconstructions with random noise for $\gamma_1 = 10^{-7}$ and $\gamma_2 = 10^{-4}$.

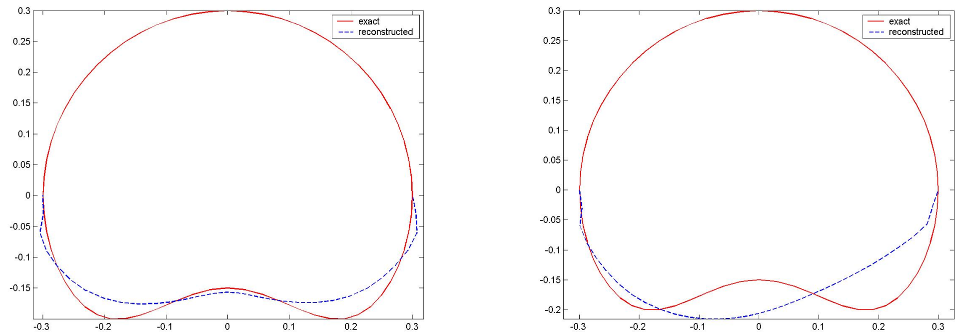

FIGURE 4. Reconstruction of (6.2) for exact (left) and noisy data (right).

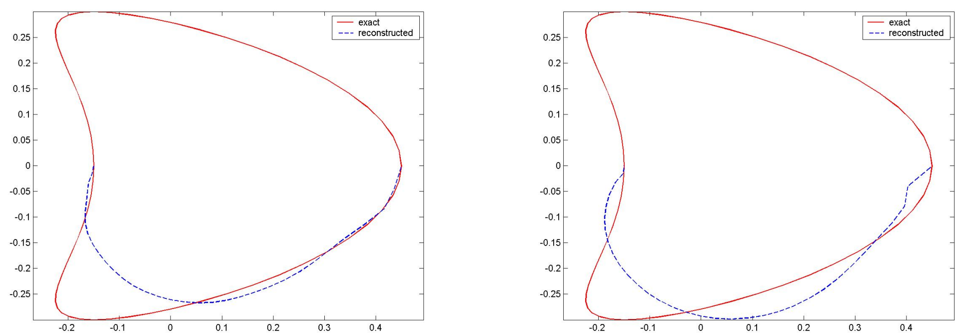

FIGURE 5. Reconstruction of (6.3) for exact (left) and noisy data (right).

Reconstructions for a perturbed circle with the parameterization

$$z(t) = \begin{cases} (0.3\cos t, 0.3\sin t), & t \in [0,\pi], \\ (0.3\cos t, 0.3\sin t + 0.15\sin^6 t), & t \in [\pi, 2\pi]. \end{cases} \tag{6.2}$$

are shown in Figure 4. The regularization parameters are $\gamma_1 = 10^{-7}$ and $\gamma_2 = 10^{-5}$ for exact data and $\gamma_1 = 10^{-4}$ and $\gamma_2 = 10^{-3}$ for noisy data.

Finally, Figure 5 illustrates reconstructions for a kite shaped contour with the parameterization

$$z(t) = (0.3\cos t + 0.15\cos 2t, 0.3\sin t), \quad t \in [0, 2\pi]. \tag{6.3}$$

The reconstructions are obtained with $\gamma_1 = 10^{-7}$ and $\gamma_2 = 10^{-5}$ for exact data and with $\gamma_1 = 10^{-4}$ and $\gamma_2 = 10^{-3}$ for noisy data.

Summarizing, the numerical experiments show rather satisfying reconstructions for the Dirichlet boundary condition with reasonable stability against noisy data. Further research is required on a more sophisticated choice of the regularization parameters including the dimension of the space for the boundary approximation and a stopping rule for terminating the iterations. In addition, we expect better reconstructions by incorporating graded meshes in the neighborhood of the end

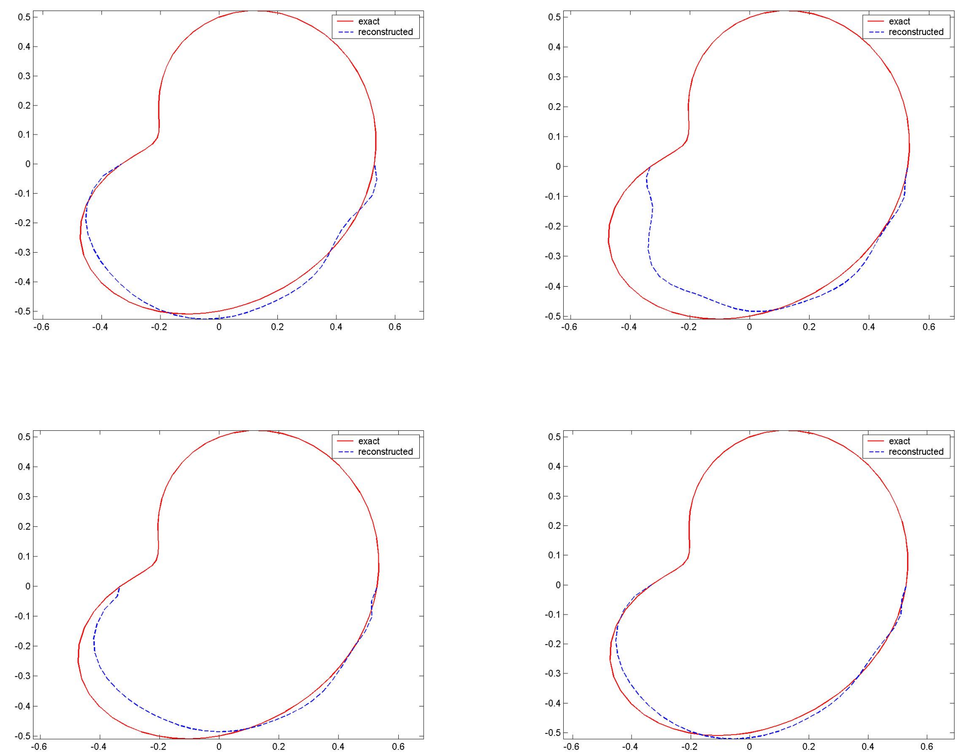

FIGURE 6. Reconstruction of (6.1) for $\lambda = 5$ (upper left), $\lambda = 10$ (upper right), $\lambda = 50$ (lower left) and $\lambda = 100$ (lower right).

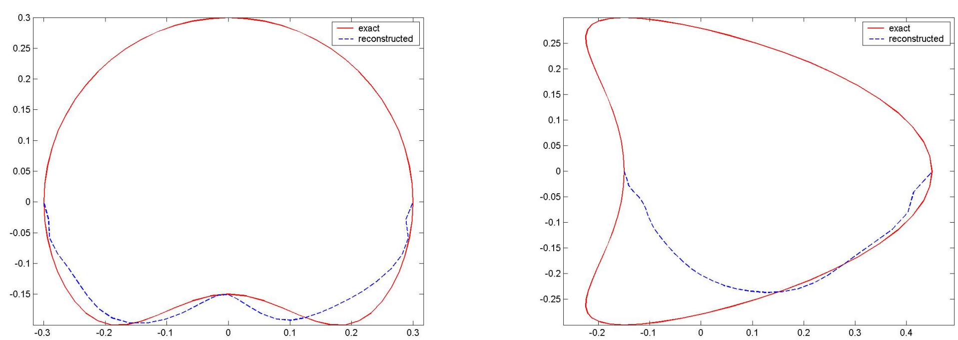

FIGURE 7. Reconstruction of (6.2) (left) and (6.3) (right) for $\lambda = 10$.

points of Γ_m and Γ_c do deal with the singularities of the solutions as mentioned in Section 3. This also would open up the possibility of using different Cauchy data sets.

We finish the paper with a few examples for the homogeneous impedance boundary condition on Γ_c. In principle, we proceeded as in the case of the Dirichlet boundary condition. The synthetic Cauchy data (f, g) on Γ_m were obtained as in the examples for recovering the impedance at the end of Section 4, but with constant impedance λ. The reconstructions were obtained with $M = 8$ and $L = 6$. Figure 6 shows the reconstructions for the apple shaped contour for different values of the impedance constant λ with $\gamma_1 = 10^{-6}$ and $\gamma_2 = 10^{-4}$. Figure 7 shows the reconstructions for the perturbed circle and the kite for $\lambda = 10$ with $\gamma_1 = 10^{-6}$ and $\gamma_2 = 10^{-8}$.

For smaller values of λ the reconstructions start to deteriorate which could be due to the way the synthetic data were created since for small λ we get close to the not uniquely solvable Neumann problem. In general, as mentioned above for the Dirichlet problem, even more research is required in order to improve on the performance of the algorithm for the impedance boundary condition.

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