Integral equations for shape and impedance reconstruction in corrosion detection

Fioralba Cakoni*, Rainer Kress† and Christian Schuft†

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Abstract

In a simply connected planar domain D a pair of Cauchy data of a harmonic function u is given on an accessible part of the boundary curve, and on the non-accessible part u is supposed to satisfy a homogeneous impedance boundary condition. We consider the inverse problems to recover the non-accessible part of the boundary or the impedance function. Our approach extends the method proposed by Kress and Rundell [21] for the corresponding problem to recover the interior boundary curve of a doubly connected planar domain and can be considered complementary to the potential approach developed by Cakoni and Kress [7]. It is based on a system of nonlinear and ill-posed integral equations which is solved iteratively by linearization. We will present the mathematical foundation of the method and, in particular, establish injectivity for the linearized system at the exact solution when the impedance function is known. Numerical reconstructions will show the feasibility of the method.

1 Introduction

We consider an inverse problem originating from corrosion detection. Let $D \subset \mathbb{R}^2$ be a simply connected bounded domain with piece-wise smooth boundary ∂D. By ν we denote the outward unit normal to ∂D. We assume that the boundary is composed of $\partial D = \overline{\Gamma}_m \cup \overline{\Gamma}_c$ where Γ_m and Γ_c are two open disjoint portions of ∂D. The electrostatic potential u in a conductor D with the non-accessible boundary

*Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716, USA
†Institut für Numerische und Angewandte Mathematik, Universität Göttingen, 37083 Göttingen, Germany

part Γ_c affected by corrosion is modeled by the following boundary value problem

$$\Delta u = 0 \quad \text{in } D, \tag{1.1}$$

$$u = f \quad \text{on } \Gamma_m, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} + \lambda u = 0 \quad \text{on } \Gamma_c, \tag{1.3}$$

where λ is a nonnegative L^∞ function on Γ_c which can be interpreted as the corrosion coefficient and f is the imposed voltage on the accessible boundary part Γ_m.

The inverse problem we are concerned with is to determine the shape of Γ_c or the impedance function λ from an imposed voltage f on Γ_m and the measured current

$$g = \frac{\partial u}{\partial \nu} \quad \text{on } \Gamma_m,$$

i.e. the resulting Neumann data. Various applications of this problem and modified versions are discussed in the literature (see e.g. [1, 3, 8]). In general, only the reconstruction of the impedance function λ as a function in space on the inaccessible portion of the boundary is considered whereas in [7] both inverse problems were investigated.

To formulate the boundary value problem (1.1)–(1.3) and the inverse problems more precisely we recall the definitions of some Sobolev spaces. Let $\Gamma \subset \partial D$ be a generic open subset of the boundary. If $H^1(D)$ denotes the usual Sobolev space and $H^{1/2}(\partial D)$ its usual trace space, then we define

$$\begin{aligned}
&H^{1/2}(\Gamma) := \{u|_\Gamma : u \in H^{1/2}(\partial D)\},\\
&\widetilde{H}^{1/2}(\Gamma) := \{u \in H^{1/2}(\Gamma) : \operatorname{supp} u \subseteq \overline{\Gamma}\},\\
&H^{-1/2}(\Gamma) := (\widetilde{H}^{1/2}(\Gamma))' \text{ the dual space of } \widetilde{H}^{1/2}(\Gamma),\\
&\widetilde{H}^{-1/2}(\Gamma) := (H^{1/2}(\Gamma))' \text{ the dual space of } H^{1/2}(\Gamma).
\end{aligned}$$

The norm on $H^{1/2}(\Gamma)$ is given by

$$\|u\|_{H^{1/2}(\Gamma)} = \inf\{\|v\|_{H^{1/2}(\partial D)} : v \in H^{1/2}(\partial D),\, v|_\Gamma = u\}$$

and the following chain of inclusion holds

$$\widetilde{H}^{1/2}(\Gamma) \subset H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset \widetilde{H}^{-1/2}(\Gamma) \subset H^{-1/2}(\Gamma)$$

(see [22] for further discussion on these Sobolev spaces). It is known [6, 14] that for $f \in H^{1/2}(\Gamma_m)$ there exists a unique solution $u \in H^1(D)$ of (1.1)–(1.3). Hence our inverse problems can be formulated as: given λ as a function in space, $f \in H^{1/2}(\Gamma_m)$

and $g \in H^{-1/2}(\Gamma_m)$ determine Γ_c such that the unique solution $u \in H^1(D)$ of (1.1)–(1.3) satisfies $\partial u/\partial\nu|_{\Gamma_m} = g$. We call this inverse problem the *inverse shape problem*. By the *inverse impedance problem* we understand the determination of the impedance function from a given pair of Cauchy data (f, g) on Γ_m assuming the whole boundary ∂D is known. That means, given Γ_c, $f \in H^{1/2}(\Gamma_m)$ and $g \in H^{-1/2}(\Gamma_m)$, we want to determine λ such that the unique solution $u \in H^1(D)$ of (1.1)–(1.3) again satisfies $\partial u/\partial\nu|_{\Gamma_m} = g$. In [7], by two of us, it was suggested to solve these problems using an approach based on a single-layer potential with a density on ∂D leading to a system of nonlinear and ill-posed integral equations that is solved using regularized iterations. In general, for direct and inverse boundary value problems in potential theory one has the choice between two complementary solution methods via boundary integral equations: the potential approach and the direct approach via Green's representation theorem. To establish this principle for the inverse problems under consideration, in this paper we follow a method suggested by Kress and Rundell [21] based on Green's theorem to determine the shape of a perfectly conducting inclusion in a homogeneous background from overdetermined Cauchy data. Extensions of that method to inverse problems with other boundary conditions were given among others in [11, 15]. The inverse problem to simultaneously recover the shape *and* impedance was recently considered by Rundell in [24] where, in particular, an algorithm was proposed which is also based on [21].

Addressing the ill-posedness of the inverse problems different stability estimates for the impedance λ were proved in [1, 8, 9] and recently reviewed by Alessandrini et al. [2]. The question of uniqueness for the inverse shape problem was considered in [7] where by a counterexample it was shown that a single pair of Cauchy data on Γ_m does not uniquely determine the missing part Γ_c. However, Bacchelli [5] recently established that two pairs of Cauchy data on Γ_m, that is, $(f_1, g_1), (f_2, g_2)$ uniquely determine both the impedance function λ and the shape of the domain D provided that f_1, f_2 are linearly independent and one of them, say f_1, is positive. Given one pair of Cauchy data the inverse impedance problem is known to be uniquely solvable (see [8]).

The plan of the paper is as follows. In section 2 we will derive our systems of integral equations and prove equivalence to the inverse shape and the inverse impedance problem in a Sobolev space setting. However, the ill-posedness of the inverse problems suggests to treat these systems in an L^2 setting appropriate for the discussion of their regularization. We then proceed considering the regularization of the inverse impedance problem in section 3 including numerical examples. After describing the linearization and the iteration scheme for the inverse shape problem in section 4 the paper is concluded with some numerical examples for shape reconstructions in section 5.

2 Integral equations

In this section we will develop the systems of integral equations that we are going to employ for the solution of the two inverse problems. We begin by noting that the inverse problems are related to the following problem of completion of Cauchy data: Given $f \in H^{1/2}(\Gamma_m)$ and $g \in H^{-1/2}(\Gamma_m)$ find $\alpha \in H^{1/2}(\Gamma_c)$ and $\beta \in H^{-1/2}(\Gamma_c)$ such that there exists a harmonic function $u \in H^1(D)$ satisfying

$$u = f \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma_m$$

and $u = \alpha$ and $\partial u/\partial \nu = \beta$ on Γ_c. Note that this Cauchy problem admits at most one solution and is known to be ill-posed. In the literature many approaches have been developed for its solution (see e.g. [4], [7] and the references therein). Our solution method is based on Green's theorem and provides an alternative approach.

To this end, in terms of the fundamental solution

$$\Phi(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|}\,, \quad x \neq y, \tag{2.1}$$

we introduce the single- and double-layer potential operators

$$S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D) \quad \text{and} \quad K : H^{1/2}(\partial D) \to H^{1/2}(\partial D)$$

defined by

$$(S\varphi)(x) := \int_{\partial D} \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \partial D, \tag{2.2}$$

and

$$(K\varphi)(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \partial D, \tag{2.3}$$

as well as their restrictions to the boundary portions given by

$$(S_{kj}\varphi)(x) := \int_{\Gamma_k} \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \Gamma_j, \tag{2.4}$$

and

$$(K_{kj}\varphi)(x) := \int_{\Gamma_k} \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \Gamma_j, \tag{2.5}$$

for $k, j = m, c$.

From now on, without loss of generality because of the possibility of scaling, we assume that there exists a point $x_0 \in D$ such that $|x - x_0| \neq 1$ for all $x \in \partial D$. Then Theorem 3.16 in [17] guarantees that the operator S defined by (2.2) is injective. Now we can state the following theorem.

Theorem 2.1 *Let* $\alpha \in H^{1/2}(\Gamma_c)$ *and* $\beta \in H^{-1/2}(\Gamma_c)$ *be a solution to the Cauchy problem. Then there exist* $\varphi \in H^{1/2}(\partial D)$ *and* $\psi \in H^{-1/2}(\partial D)$ *such that*

$$\frac{\varphi}{2} + K\varphi - S\psi = 0 \tag{2.6}$$

and φ *and* ψ *have restrictions* $\varphi|_{\Gamma_m} = f$, $\varphi|_{\Gamma_c} = \alpha$ *and* $\psi|_{\Gamma_m} = g$, $\psi|_{\Gamma_c} = \beta$, *respectively. Conversely, for any solution* $\varphi \in H^{1/2}(\partial D)$ *and* $\psi \in H^{-1/2}(\partial D)$ *of* (2.6) *satisfying* $\varphi|_{\Gamma_m} = f$ *and* $\psi|_{\Gamma_m} = g$ *we have that* $\alpha := \varphi|_{\Gamma_c}$ *and* $\beta := \psi|_{\Gamma_c}$ *is a solution of the Cauchy problem.*

Proof. Let $u \in H^1(D)$ correspond to a solution to the Cauchy problem. Then for $\varphi := u|_{\partial D} \in H^{1/2}(\partial D)$ and $\psi := \partial u/\partial\nu|_{\partial D} \in H^{-1/2}(\partial D)$ clearly we have that $\varphi|_{\Gamma_m} = f$, $\varphi|_{\Gamma_c} = \alpha$ and $\psi|_{\Gamma_m} = g$, $\psi|_{\Gamma_c} = \beta$. From Green's representation formula for $u \in H^1(D)$ it follows that

$$u(x) = \int_{\partial D} \left\{ \psi(y)\Phi(x,y) - \varphi(y)\,\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right\} ds(y), \quad x \in D,$$

and (2.6) is obtained by restricting this to ∂D with the aid of the jump relations.

Conversely, if $\varphi \in H^{1/2}(\partial D)$ and $\psi \in H^{-1/2}(\partial D)$ solve (2.6) then we see that $\widetilde{u}$ defined by

$$\widetilde{u}(x) = \int_{\partial D} \left\{ \psi(y)\Phi(x,y) - \varphi(y)\,\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right\} ds(y), \quad x \in \mathbb{R}^2 \setminus \partial D,$$

belongs to $H^1(D)$ and $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$ and satisfies $\Delta\widetilde{u} = 0$ in D and $\mathbb{R}^2 \setminus \overline{D}$. From (2.6) we conclude $\widetilde{u}^+|_{\partial D} = 0$ where by $+$ we indicate the limit obtained by approaching ∂D from outside D. Using our assumption that there exists $x_0 \in D$ such that $|x - x_0| \neq 1$ for all $x \in \partial D$, following and modifying the proof of Theorem 3.16 in [17] to deal with the logarithmic behavior of the single-layer potential at infinity from $\widetilde{u}^+ = 0$ on ∂D we obtain that $\widetilde{u}$ vanishes in $\mathbb{R}^2 \setminus \overline{D}$. Approaching the boundary ∂D from inside D by the jump relations from $\widetilde{u} = 0$ in $\mathbb{R}^2 \setminus \overline{D}$ we conclude that $\widetilde{u}^-|_{\partial D} = \varphi$ and $\partial\widetilde{u}^-/\partial\nu|_{\partial D} = \psi$ on ∂D. Therefore, in view of the condition $\varphi|_{\Gamma_m} = f$ and $\psi|_{\Gamma_m} = g$ it follows that $\alpha := \varphi|_{\Gamma_c}$ and $\beta := \psi|_{\Gamma_c}$ provide a solution of the Cauchy problem. □

Corollary 2.2 *The inverse shape problem is equivalent to solving* (2.6) *under the constraints* $\varphi|_{\Gamma_m} = f$ *and* $\psi|_{\Gamma_m} = g$ *and*

$$\psi + \lambda\varphi = 0 \quad \text{on } \Gamma_c$$

for Γ_c *and* $\varphi|_{\Gamma_c}$.

Corollary 2.3 *The inverse impedance problem is equivalent to solving* (2.6) *for* $\varphi|_{\Gamma_c}$ *and* $\psi|_{\Gamma_c}$. *The unknown impedance follows from*

$$\psi|_{\Gamma_c} + \lambda\varphi|_{\Gamma_c} = 0.$$

The question of existence of a solution to the ill-posed integral equation (2.6) stated in the Corollaries 2.2 and 2.3, that is, a characterization of Cauchy data (f, g) for which a solution to the inverse shape and the inverse impedance problem, respectively, exists, is the wrong question to ask. Instead of this, assuming that we have correct data or small perturbations thereof, for a stable numerical solution regularization schemes need to be applied. Since the L^2-norm is the appropriate norm to measure the data error, it is natural to consider the equation in L^2 spaces rather than in the trace spaces that are appropriate only for the corresponding forward problems. Hence, for the remainder of the paper we will assume that the data f and g are in $L^2(\Gamma_m)$ and look for solutions of (2.6) with $\varphi|_{\Gamma_c}$ and $\psi|_{\Gamma_c}$ in $L^2(\Gamma_c)$.

To simplify notations, in terms of the given functions f and g we define the combined single- and double-layer potential

$$w(x) := \int_{\Gamma_m} \left\{ f(y)\,\frac{\partial\Phi(x,y)}{\partial\nu(y)} - g(y)\Phi(x,y) \right\} ds(y), \quad x \in \mathbb{R}^2 \setminus \overline{\Gamma}_m. \tag{2.7}$$

Then, after separating and renaming the unknowns, in view of Corollary 2.2 we solve the inverse shape problem by a regularized solution Γ_c and $\varphi \in L^2(\Gamma_c)$ of the system of integral equations

$$\frac{\varphi}{2} + K_{cc}\varphi + S_{cc}(\lambda\varphi) = -w|_{\Gamma_c} \tag{2.8}$$

and

$$K_{cm}\varphi + S_{cm}(\lambda\varphi) = -w|_{\Gamma_m}, \tag{2.9}$$

where $w|_{\Gamma_m}$ in (2.9) represents the limit obtained by approaching Γ_m from outside D. Clearly, these equations are nonlinear with respect to Γ_c. For convenience we note that

$$w|_{\Gamma_c} = K_{mc}f - S_{mc}g \quad \text{and} \quad w|_{\Gamma_m} = \frac{f}{2} + K_{mm}f - S_{mm}g. \tag{2.10}$$

Analogously, in view of Corollary 2.3 the inverse inverse impedance problem is solved by a regularized solution $\varphi, \psi \in L^2(\Gamma_c)$ of the system of integral equations

$$\begin{aligned} \frac{\varphi}{2} + K_{cc}\varphi - S_{cc}\psi &= -w|_{\Gamma_c}, \\ K_{cm}\varphi - S_{cm}\psi &= -w|_{\Gamma_m}. \end{aligned} \tag{2.11}$$

For the further investigation of the integral equations and, in particular, for the numerical solution a parameterization is required. For the sake of simplicity we confine ourselves to smooth boundaries ∂D of class C^2, that is, we represent

$$\partial D = \{z(t) : t \in [0, 2\pi]\}$$

with a 2π periodic C^2-smooth function $z : \mathbb{R} \to \mathbb{R}^2$ such that z is injective on $[0, 2\pi)$ and satisfies $z'(t) \neq 0$ for all t. Without loss of generality we may assume that Γ_c and Γ_m are given by

$$\Gamma_c = \{z(t) : t \in (0, \pi)\}, \qquad \Gamma_m = \{z(t) : t \in (\pi, 2\pi)\}.$$

From now on we denote by z_c the parameterization function z for $t \in (0, \pi)$ and by z_m for $t \in (\pi, 2\pi)$. In applications it is natural that at the connection points of Γ_c and Γ_m corners can develop. In order to incorporate the corresponding singularities of the solution u one can employ sigmoidal transformations as investigated in [12] and used in [13] and [19]. For the following analysis we did not pursue this idea. However, in sections 3 and 5 we will show some numerical examples for corner domains where we have incorporated sigmoidal transformations in order to improve the accuracy. Setting $\psi = \varphi \circ z_c$ we obtain from (2.4) and (2.5) the parameterized integral operators

$$(\widetilde{S}_{cj}\psi)(t) = \frac{1}{2\pi} \int_0^\pi \ln \frac{1}{|z_j(t) - z_c(\tau)|} \, |z_c'(\tau)| \psi(\tau) \, d\tau$$

and

$$(\widetilde{K}_{cj}\psi)(t) = \frac{1}{2\pi} \int_0^\pi \frac{[z_c'(\tau)]^\perp \cdot [z_j(t) - z_c(\tau)]}{|z_j(t) - z_c(\tau)|^2} \, \psi(\tau) \, d\tau + \frac{\delta_{cj}}{2} \psi(t)$$

for $t \in [0, 2\pi]$. Here we used the convention

$$\delta_{cj} = \begin{cases} 1 & \text{if } j = c, \\ 0 & \text{if } j = m, \end{cases}$$

and the notation $a^\perp = (a_2, -a_1)$ for any vector $a = (a_1, a_2)$, that is, $a^\perp$ is obtained by rotating a clockwise by 90 degrees. For the discretization of the integral operators we note that the kernel of $\widetilde{S}_{cc}$ can be decomposed in the form

$$\ln \frac{1}{|z_c(t) - z_c(\tau)|} = -\ln \left| \sin \frac{t - \tau}{2} \right| + \ln \frac{\left| \sin \frac{t-\tau}{2} \right|}{|z_c(t) - z_c(\tau)|} \,,$$

where the second term is smooth with diagonal values

$$\lim_{t \to \tau} \ln \frac{\left| \sin \frac{t-\tau}{2} \right|}{|z_c(t) - z_c(\tau)|} = -\ln 2 |z_c'(t)|.$$

Hence, the well established trigonometric interpolation quadrature rules on equidistant meshes for logarithmic singularities as described in [20] are available. The kernels of $\widetilde{K}_{cj}$ are smooth with the diagonal values for $\widetilde{K}_{cc}$ given through the limit

$$\lim_{\tau\to t} \frac{[z_j'(\tau)]^\perp \cdot [z_j(t) - z_j(\tau)]}{|z_j(t) - z_j(\tau)|^2} = \frac{[z_j'(t)]^\perp \cdot z_j''(t)}{2|z_j'(t)|^2}\ , \quad j = c, m, \tag{2.12}$$

for $j = c$. For the parameterized form of the combined single- and double-layer potentials $w_j = w \circ z_j$ evaluated on Γ_j, $j = c, m$, due to the jump relations, we have

$$\begin{aligned} w_m(t) =& \frac{1}{2\pi} \int_\pi^{2\pi} f(z_m(\tau))\ \frac{[z_m'(\tau)]^\perp \cdot [z_m(t) - z_m(\tau)]}{|z_m(t) - z_m(\tau)|^2}\ d\tau + \frac{1}{2} f(z_m(t)) \\ & - \int_\pi^{2\pi} g(z_m(\tau))\Phi(z_m(t), z_m(\tau))|z_m'(\tau)|\, d\tau, \quad t \in [\pi, 2\pi] \end{aligned}$$

and

$$\begin{aligned} w_c(t) =& \frac{1}{2\pi} \int_\pi^{2\pi} f(z_m(\tau))\ \frac{[z_m'(\tau)]^\perp \cdot [z_c(t) - z_m(\tau)]}{|z_c(t) - z_m(\tau)|^2}\ d\tau \\ & - \int_\pi^{2\pi} g(z_m(\tau))\Phi(z_c(t), z_m(\tau))|z_m'(\tau)|\, d\tau, \quad t \in [0, \pi]. \end{aligned}$$

The kernel of w_c is smooth and in the kernel of w_m, the term arising from the single-layer potential has again a logarithmic singularity which can be treated as the one for the operator $\widetilde{S}_{cc}$. The term stemming from the double-layer potential is smooth with diagonal values given by (2.12) for $j = m$. For the smooth kernels in all the operators, of course, the trapezoidal rule can be employed for the numerical approximation.

With the identification of $\lambda = \lambda \circ z_c$ the parameterized form of the equations (2.8) and (2.9) now reads

$$\widetilde{K}_{cc}\psi + \widetilde{S}_{cc}(\lambda\psi) = -w_c \tag{2.13}$$

and

$$\widetilde{K}_{cm}\psi + \widetilde{S}_{cm}(\lambda\psi) = -w_m. \tag{2.14}$$

Analogously, the system (2.11) has a similar transformed version.

3 Solution of the inverse impedance problem

We continue with the discussion of the inverse impedance problem, i.e. the completion of Cauchy data. For this we recall the ill-posed linear system (2.11) and consider the corresponding operator $A : L^2(\Gamma_c) \times L^2(\Gamma_c) \to L^2(\Gamma_c) \times L^2(\Gamma_m)$ defined by

$$A(\varphi, \psi) = \begin{pmatrix} \frac{1}{2}\,\mathrm{I} + K_{cc} & -S_{cc} \\ K_{cm} & -S_{cm} \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

Theorem 3.1 *The operator A is injective with dense range.*

Proof. Let $A(\varphi, \psi) = 0$ for some $\varphi \in L^2(\Gamma_c)$ and $\psi \in L^2(\Gamma_c)$. We define

$$v(x) = \int_{\Gamma_c} \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, \varphi(y)\, ds(y) - \int_{\Gamma_c} \Phi(x, y) \psi(y)\, ds(y) \tag{3.1}$$

which is a solution of the Laplace equation in $\mathbb{R}^2 \setminus \overline{\Gamma}_c$. For $x \to \partial D$ from outside D by the jump relations for single- and double-layer potentials with L^2 densities it follows from $A(\varphi, \psi) = 0$ that $v^+ = 0$ on ∂D. By our geometric assumption as in the proof of theorem 2.1 we obtain that $v = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Now, by analyticity, it follows that $v = 0$ in D and from this the jump relations across ∂D imply that $\varphi = \psi = 0$.

To prove that A has dense range we consider the adjoint operator $A^* : L^2(\Gamma_c) \times L^2(\Gamma_m) \to L^2(\Gamma_c) \times L^2(\Gamma_c)$ which is given by

$$A^*(\alpha, \beta) = \begin{pmatrix} \frac{1}{2}\,\mathrm{I} + K'_{cc} & K'_{mc} \\ -S_{cc} & -S_{mc} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Here K'_{cc} and K'_{mc} denote obvious restrictions of the normal derivative of the single-layer potential

$$K' : L^2(\partial D) \to L^2(\partial D)$$

defined by

$$(K'\varphi)(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(x)}\, \varphi(y)\, ds(y), \quad x \in \partial D. \tag{3.2}$$

Now we define $\chi \in L^2(\partial D)$ by

$$\chi := \begin{cases} \alpha & \text{on } \Gamma_c, \\ \beta & \text{on } \Gamma_m, \end{cases}$$

and obtain that $A^*(\alpha, \beta) = \widetilde{A}^*(\chi)$ where

$$\widetilde{A}^* \chi := \begin{pmatrix} K'\chi + \frac{\chi}{2} \\ -S\chi \end{pmatrix}\Bigg|_{\Gamma_c}.$$

If $\widetilde{A}^*\chi = 0$ for some $\chi \in L^2(\partial D)$ then u defined by

$$u(x) = \int_{\partial D} \Phi(x,y)\chi(y)\,ds(y), \qquad x \in D,$$

satisfies $u|_{\Gamma_c} = 0$ and $\partial u/\partial\nu|_{\Gamma_c} = 0$ from inside D whence $u = 0$ in D follows by Holmgren's theorem. The trace theorem now implies that $S\chi = 0$. By our geometric assumption on D we have injectivity of S and therefore we conclude that $\chi = 0$. Hence, $\alpha = \beta = 0$ follows which proves that A^* is injective. □

To show the feasibility of this approach to completing Cauchy data we want to use it for the inverse problem of determining the impedance for a fixed domain D, i.e. we want to recover the impedance function λ on Γ_c from a pair of Cauchy data (f, g) on Γ_m. To this end, we just recall that after completing the Cauchy data, i.e. after determining φ and ψ on Γ_c, we obtain the impedance function from the equation

$$\psi + \lambda\varphi = 0 \quad \text{on } \Gamma_c. \tag{3.3}$$

Therefore, we have to carry out two steps: first we need to solve the ill-posed equation (2.11), for example, by Tikhonov regularization for the densities φ and ψ on Γ_c. For this, of course, we use the parameterized version of (2.11) and the trigonometric quadratures based on a graded mesh with a sigmoidal transformation

$$\omega : [0, \pi] \to [0, \pi],$$

which is bijective, strictly monotonically increasing and sufficiently smooth (see [19]). We will use two different transformations in the examples. The first one is a modified version of a transformation introduced by Korobov [18]

$$\omega_p(t) = \frac{2p-1}{2(2\pi)^{2p-2}} \begin{pmatrix} 2p-2 \\ p-1 \end{pmatrix} \int_0^{2t} [s(2\pi - s)]^{p-1}\,ds, \quad t \in [0, \pi], \tag{3.4}$$

and the second one is a modification of a rational transformation introduced by Kress [19]

$$\omega_p(t) = \pi \frac{[v(t)]^p}{[v(t)]^p + [v(2\pi - 2t)]^p}, \quad t \in [0, \pi], \tag{3.5}$$

with the cubic polynomial v given by

$$v(t) = \left(\frac{2}{p} - 1\right) \frac{(\pi - 2t)^3}{\pi^2} + \frac{2}{p}(2t - \pi) + \pi, \quad t \in [0, \pi].$$

The parameter p in the substitution functions is the so-called grading parameter. For larger values of p the grid points are more densely accumulated at the end points of the integration interval.

The final step in the computation is to obtain the impedance function λ at the collocation points $x_i = z_c(\omega(t_i)), i = 1, \dots, n$, on Γ_c by solving

$$\psi(x_i) + \lambda(x_i)\varphi(x_i) = 0, \quad i = 1, \dots, n. \tag{3.6}$$

In order to avoid instabilities arising from dividing by small values of $\alpha(x_i)$ we represent the unknown λ as a linear combination

$$\lambda = \sum_{k=1}^{K} a_k w_k \tag{3.7}$$

of appropriate basis functions w_k and solve the equation that is obtained by inserting (3.7) into (3.6) in the least squares sense for the coefficients a_k. In numerical examples we used cubic B-splines on an equidistant subdivision with respect to the parameter t in the parameterization

$$\Gamma_c = \{z(\omega(t)) : t \in (0, \pi)\}.$$

In the first example we consider an ellipse with parameterization

$$z_c(t) = -\left(0.3\cos t, 0.2\sin t\right), \quad t \in [0, \pi],$$

$$z_m(t) = -\left(0.3\cos t, 0.2\sin t\right), \quad t \in [\pi, 2\pi],$$

whereas in the second example the boundary is parameterized by half of an ellipse

$$z_c(t) = -\left(0.3\cos t, 0.2\sin t\right), \quad t \in [0, \pi],$$

and half of a bowl shaped contour

$$z_m(t) = -(1 + \sin t)(0.3\cos t, 0.2\sin t), \quad t \in [\pi, 2\pi].$$

We note that in the second example there are corners at the connection of the two boundary parts. The impedance profile in both examples is

$$\lambda(t) = \begin{cases} \sin^4 t, & t \in [0, \pi], \\ 0, & t \in [\pi, 2\pi], \end{cases}$$

and the synthetic Cauchy data (f, g) on Γ_m were obtained by solving the impedance problem in D with boundary condition

$$\frac{\partial u}{\partial \nu} + \lambda u = h,$$

with

$$h(t) = \begin{cases} 0, & t \in [0, \pi], \\ \sin^4 t, & t \in [\pi, 2\pi], \end{cases}$$

by a method based on Green's formula with double the number of discretization points than in the inverse solver and the sigmoidal transformation (3.4) with grading parameter $p = 4$ (to avoid an inverse crime). The reconstructions were performed by using $2n = 128$ grid points for discretizing the integral operators on the boundary ∂D and the transformation function (3.5) with $p = 6$. The figures 3.1 and 3.2 show the reconstructed profile both for exact data and for 3% random noise added to the Neumann data g (with respect to the L^2-norm). The exact impedance profile is represented by the full (blue) lines and the reconstructions by the dash-dotted (green) lines for exact data and the dotted (red) lines for perturbed data. The Tikhonov regularization parameter was chosen by trial and error as 10^{-9} for exact data and 10^{-6} for noisy data. For the B-spline approximation of the impedance profile the dimension $K = 11$ was used. As to be expected, the quality of the reconstructions in the vicinity of the corner points in the case of the bowl-ellipse shaped contour (figure 3.2) is not as accurate as the one for the smooth boundary in the example of the ellipse (figure 3.1).

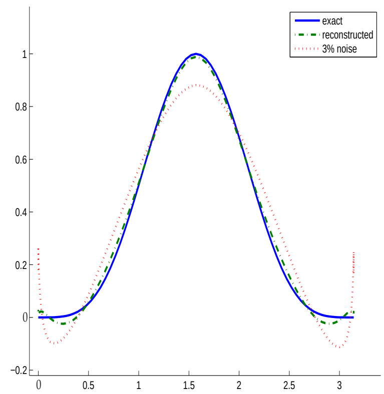


Figure 3.1: Reconstruction of an impedance profile for an ellipse with semi-axis $a = 0.3$ and $b = 0.2$.

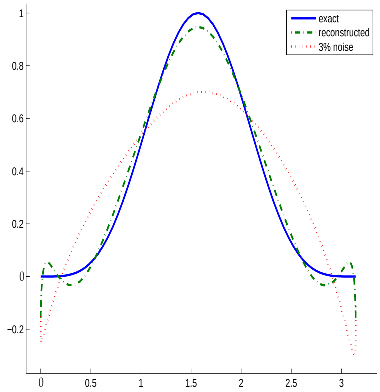


Figure 3.2: Reconstruction of an impedance profile for a bowl-ellipse shaped contour.

4 The iteration scheme for the inverse shape problem

We now return to the inverse shape problem, i.e. to determine the non-accessible part Γ_c of the boundary curve ∂D assuming that the impedance as a function of space is known. Because of the linearity of the integral operators with respect to ψ, the linearization of (2.13) and (2.14) with respect to ψ and z_c leads to

$$\begin{aligned} &\widetilde{K}_{cc}(\psi, z_c) + \widetilde{K}_{cc}(\chi, z_c) + d\widetilde{K}_{cc}(\psi, z_c; \zeta) + \widetilde{S}_{cc}(\lambda\psi, z_c) \\ &\quad + \widetilde{S}_{cc}(\lambda\chi, z_c) + d\widetilde{S}_{cc}(\lambda\psi, z_c; \zeta) = -w_c - dw_c(z_c, \zeta) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\widetilde{K}_{cm}(\psi, z_c) + \widetilde{K}_{cm}(\chi, z_c) + d\widetilde{K}_{cm}(\psi, z_c; \zeta) + \widetilde{S}_{cm}(\lambda\psi, z_c) \\ &\quad + \widetilde{S}_{cm}(\lambda\chi, z_c) + d\widetilde{S}_{cm}(\lambda\psi, z_c; \zeta) = -w_m. \end{aligned} \tag{4.2}$$

Given an approximation for z_c and ψ, the linear system (4.1) and (4.2) needs to be solved for ζ and χ to obtain the update $z_c + \zeta$ for the parameterization of Γ_c and $\psi + \chi$ for the boundary values. Then, in an obvious way, this procedure is iterated. Clearly, the ill-posedness requires to incorporate a regularization in order to achieve stability. For this, we employed Tikhonov regularization with a Sobolev penalty term on the parameterization and an L^2 penalty term on the boundary values.

The Fréchet derivatives of the operators $\widetilde{S}_{cj}$, $\widetilde{K}_{cj}$ and the potential w_c with respect to z_c can be obtained by formally differentiating their kernels with respect to z_c (see

[23]). In particular, we have

$$d\widetilde{S}_{cc}[\psi, z_c; \zeta](t) = -\frac{1}{2\pi}\int_0^\pi \frac{[z_c(t)-z_c(\tau)]\cdot[\zeta(t)-\zeta(\tau)]}{|z_c(t)-z_c(\tau)|^2}\,|z_c'(\tau)|\psi(\tau)\,d\tau$$
$$+\frac{1}{2\pi}\int_0^\pi \ln\frac{1}{|z_c(t)-z_c(\tau)|}\,\psi(\tau)\,\frac{z_c'(\tau)\cdot\zeta'(\tau)}{|z_c'(\tau)|}\,d\tau, \quad t\in[0,\pi].$$

We note that the perturbation ζ is different from zero only on Γ_c and that $\zeta(0) = \zeta(\pi) = 0$. The kernel of the first term of $d\widetilde{S}_{cc}$ is smooth with diagonal values

$$\lim_{\tau\to t}\frac{[z_c(t)-z_c(\tau)]\cdot[\zeta(t)-\zeta(\tau)]}{|z_c(t)-z_c(\tau)|^2} = \frac{z_c'(t)\cdot\zeta'(t)}{|z_c'(t)|^2}\,.$$

The second term can be treated as in the case of the operator $\widetilde{S}_{cc}$. The derivative of $\widetilde{K}_{cc}$ in direction ζ is given by

$$d\widetilde{K}_{cc}[\psi, z_c; \zeta] = -\frac{1}{\pi}\int_0^\pi \frac{[z_c'(\tau)]^\perp\cdot[z_c(t)-z_c(\tau)][z_c(t)-z_c(\tau)]\cdot[\zeta(t)-\zeta(\tau)]}{|z_c(t)-z_c(\tau)|^4}\,\psi(\tau)\,d\tau$$
$$+\frac{1}{2\pi}\int_0^\pi \frac{[z_c'(\tau)]^\perp\cdot[\zeta(t)-\zeta(\tau)]+[\zeta'(\tau)]^\perp\cdot[z_c(t)-z_c(\tau)]}{|z_c(t)-z_c(\tau)|^2}\,\psi(\tau)\,d\tau$$

for $t\in[0,\pi]$. The kernel $H(t,\tau)$ of the operator $d\widetilde{K}_{cc}$ is smooth with the diagonal values

$$\widetilde{H}(t) := \lim_{\tau\to t} H(t,\tau)$$

given by

$$\widetilde{H}(t) = \frac{[z_c'(\tau)]^\perp\cdot\zeta''(t)+[\zeta'(\tau)]^\perp\cdot z_c''(t)}{2|z_c'(t)|^2} - \frac{[z_c'(\tau)]^\perp\cdot z_c''(t) z_c'(t)\cdot\zeta'(t)}{|z_c'(t)|^4}\,.$$

Analogously, the Fréchet derivatives of the operators $\widetilde{S}_{cm}$, $\widetilde{K}_{cm}$ and the potential w_c are given by

$$d\widetilde{S}_{cm}[\psi, z_c; \zeta](t) = \frac{1}{2\pi}\int_0^\pi \frac{[z_m(t)-z_c(\tau)]\cdot\zeta(\tau)}{|z_m(t)-z_c(\tau)|^2}\,|z_c'(\tau)|\psi(\tau)\,d\tau$$
$$+\frac{1}{2\pi}\int_0^\pi \ln\frac{1}{|z_m(t)-z_c(\tau)|}\,\psi(\tau)\,\frac{z_c'(\tau)\cdot\zeta'(\tau)}{|z_c'(\tau)|}\,d\tau,$$

for $t\in[\pi, 2\pi]$,

$$d\widetilde{K}_{cm}[\psi, z_c; \zeta] = \frac{1}{\pi}\int_0^\pi \frac{[z_c'(\tau)]^\perp\cdot[z_m(t)-z_c(\tau)][z_m(t)-z_c(\tau)]\cdot\zeta(\tau)}{|z_m(t)-z_c(\tau)|^4}\,\psi(\tau)\,d\tau$$
$$+\frac{1}{2\pi}\int_0^\pi \frac{[\zeta'(\tau)]^\perp\cdot[z_m(t)-z_c(\tau)]-[z_c'(\tau)]^\perp\cdot\zeta(\tau)}{|z_m(t)-z_c(\tau)|^2}\,\psi(\tau)\,d\tau,$$

for $t \in [\pi, 2\pi]$ and

$$
\begin{aligned}
dw_c[z_c;\zeta](t) =& \frac{1}{2\pi}\int_\pi^{2\pi} f(z_m(\tau))\,\frac{[z_m'(\tau)]^\perp\cdot\zeta(t)}{|z_c(t)-z_m(\tau)|^2}\,d\tau\\
&-\frac{1}{\pi}\int_\pi^{2\pi} f(z_m(\tau))\frac{[z_m'(\tau)]^\perp\cdot[z_c(t)-z_m(\tau)][z_c(t)-z_m(\tau)]\cdot\zeta(t)}{|z_c(t)-z_m(\tau)|^4}\,d\tau\\
&+\frac{1}{2\pi}\int_\pi^{2\pi} g(z_m(\tau))\,\frac{[z_c(t)-z_m(\tau)]\cdot\zeta(t)}{|z_c(t)-z_m(\tau)|^2}\,|z_m'(\tau)|\,d\tau,\quad t\in[0,\pi].
\end{aligned}
$$

The operators $d\widetilde{S}_{cm}$, $d\widetilde{K}_{cm}$ and dw_c all have smooth kernels and, of course, $dw_c[z_c;\zeta] = \zeta\cdot(\operatorname{grad} w)\circ z_c$.

For the following theorem on injectivity of the linearization (4.1)–(4.2) at the exact solution we need some restricting assumptions on the geometry of the domain and the regularity of the solution u on the boundary. We assume that a subset of Γ_m is part of a closed analytic curve such that $\overline{\Gamma}_c$ does not intersect with the closed interior of that curve. Then, by the uniqueness for the interior Dirichlet problem and analyticity, any harmonic function defined in $\mathbb{R}^2\setminus\overline{\Gamma}_c$ that vanishes on Γ_m is identically zero. Further we assume that the exact solution u is continuous on $\overline{\Gamma}_c$. In view of the regularity results for the direct problem (see [10]) the latter regularity assumption is not too restrictive.

Theorem 4.1 *Let z_c be the parameterization of Γ_c, let $\psi\in C([0,\pi])$ satisfy* (2.13)–(2.14) *for a nonnegative $\lambda\in C([0,\pi])$ and let Γ_m satisfy the above geometric assumption. Then for any solution $\zeta\in C^2([0,\pi])$ and $\chi\in L^2([0,\pi])$ to the homogeneous system*

$$\widetilde{K}_{cc}(\chi,z_c)+d\widetilde{K}_{cc}(\psi,z_c;\zeta)+\widetilde{S}_{cc}(\lambda\chi,z_c)+d\widetilde{S}_{cc}(\lambda\psi,z_c;\zeta)+dw_c(z_c,\zeta)=0 \tag{4.3}$$

and

$$\widetilde{K}_{cm}(\chi,z_c)+d\widetilde{K}_{cm}(\psi,z_c;\zeta)+\widetilde{S}_{cm}(\lambda\chi,z_c)+d\widetilde{S}_{cm}(\lambda\psi,z_c;\zeta)=0 \tag{4.4}$$

we have that $\zeta=0$ and $\chi=0$.

Proof. We begin by noting that, for sufficiently small ζ, the perturbed boundary part Γ_c as given by

$$\Gamma_{z_c+\zeta}=\{z(t)+\zeta(t): t\in[0,\pi]\}$$

can be represented in the form

$$\Gamma_{z_c+\zeta}=\{z(t)+q(t)[z'(t)]^\perp: t\in[0,\pi]\}$$

in terms of the normal vector $[z'(t)]^\perp$ to the unperturbed boundary and a function q with $q(0) = q(\pi) = 0$ (for a proof in the case of closed curves see [16]). Therefore in the Fréchet derivatives $d\widetilde{S}_{cc}, d\widetilde{S}_{cm}, d\widetilde{K}_{cc}, d\widetilde{K}_{cm}$ and dw_c we can replace the perturbation vector ζ by $\zeta = q[z_c']^\perp$.

We define a harmonic function V in $\mathbb{R}^2 \setminus \overline{\Gamma}_c$ by

$$
\begin{aligned}
V(x) = &- \int_0^\pi \chi(\tau)\nabla_x\Phi(x, z(\tau)) \cdot [z'(\tau)]^\perp \, d\tau \\
&+ \int_0^\pi \psi(\tau)\nabla_x \left(\nabla_x\Phi(x, z(\tau)) \cdot [z'(\tau)]^\perp\right) \cdot \zeta(\tau) \, d\tau \\
&- \int_0^\pi \psi(\tau)\nabla_x\Phi(x, z(\tau)) \cdot [\zeta'(\tau)]^\perp \, d\tau \\
&+ \int_0^\pi \lambda(\tau)\chi(\tau)\Phi(x, z(\tau))|z'(\tau)| \, d\tau \\
&- \int_0^\pi \lambda(\tau)\psi(\tau)\nabla_x\Phi(x, z(\tau)) \cdot \zeta(\tau) \, |z'(\tau)| \, d\tau \\
&+ \int_0^\pi \lambda(\tau)\psi(\tau)\Phi(x, z(\tau)) \, \frac{z'(\tau) \cdot \zeta'(\tau)}{|z'(\tau)|} \, d\tau, \quad x \in \mathbb{R}^2 \setminus \overline{\Gamma}_c.
\end{aligned}
\tag{4.5}
$$

Then from the representation of the involved operators it can be seen that equation (4.4) implies that $V = 0$ on Γ_m. Hence, $V = 0$ in $\mathbb{R}^2 \setminus \overline{\Gamma}_c$ as consequence of our geometric assumption. Inserting $\zeta = q[z_c']^\perp$ in (4.5) we observe that in addition to single-layer potentials the definition of V contains double-layer potentials in line one and five, the normal derivative of a double-layer potential in line two and a derivative of a single-layer potential in line three. Therefore, the jump relations imply the relation

$$
\chi + \frac{\zeta' \cdot z'}{|z'|^2} \, \psi + \lambda q \psi \, |z'| = 0 \quad \text{on } \Gamma_c
$$

and from this, by the assumptions on ψ and λ, we can conclude that χ is continuous on $\overline{\Gamma}_c$.

Since the single- and double-layer potentials with continuous density in (4.5) represent continuous bounded functions in D, the only parts of V that can be become unbounded are the second and the third term. We rewrite $V = 0$ in D into the form

$$
V_1 + V_2 + V_3 + V_r = 0 \quad \text{in } D \tag{4.6}
$$

where V_r is the sum of the bounded terms of V corresponding to single- and double-layer potentials and

$$V_1(x) := 2\int_0^\pi \frac{[(x - z(\tau)) \cdot \nu(z(\tau))]^2}{|x - z(\tau)|^4}\, q(\tau)\psi(\tau)\, d\tau,$$

$$V_2(x) := -\int_0^\pi \frac{1}{|x - z(\tau)|^2}\, q(\tau)\psi(\tau)\, d\tau,$$

$$V_3(x) := \int_0^\pi \frac{[x - z(\tau)] \cdot [\zeta'(\tau)]^\perp}{|x - z(\tau)|^2}\, \psi(\tau)\, d\tau$$

for $x \in D$. The kernel in the integral for V_1 coincides with the square of the kernel of the double-layer potential. Hence, one can proceed as in the proof of Theorem 6.17 in [20] to see that the function V_1 is bounded in D. Therefore, in view of (4.6), the sum $V_2 + V_3$ also must be bounded.

Now observing that the singularity in the integral for V_2 is stronger than the singularity for V_3, following the proof of theorem 4.1 in [16], the assumption that $q\psi \neq 0$ can be brought to a contradiction to the boundedness of $V_2 + V_3$ in D. Hence, $q\psi = 0$. An application of Holmgren's theorem and the homogeneous impedance boundary condition for u on Γ_c leads to the conclusion that u cannot vanish on an open subset of Γ_c. Therefore, in view of $\psi = u \circ z_c$, we see that $q = 0$ which also gives $\zeta = 0$.

So V reduces to

$$V(x) = -\int_0^\pi \chi(\tau)\nabla_x\Phi(x, z(\tau)) \cdot [z'(\tau)]^\perp\, d\tau + \int_0^\pi \lambda(\tau)\chi(\tau)\Phi(x, z(\tau))|z'(\tau)|\, d\tau$$

for $x \in \mathbb{R}^2 \setminus \overline{\Gamma}_c$. Since $V = 0$ in $\mathbb{R}^2 \setminus \overline{\Gamma}_c$ by approaching Γ_c from inside D in view of the jump relations for single- and double-layer potentials we conclude that

$$-\chi + \widetilde{K}_{cc}(\chi, z_c) + \widetilde{S}_{cc}(\lambda\chi, z_c) = 0.$$

This together with (4.3) finally yields $\chi = 0$ and this concludes the proof. □

5 Numerical examples

In this final section we present some numerical results to illustrate the feasibility of the reconstruction method. The direct data were obtained using a solver based on Green's formula using the sigmoidal transformation (3.4) in the parameterization of the boundary parts with parameter $p = 6$ and twice the number of discretization points than in the inverse solver. Furthermore, in the inverse solver we used the substitution function (3.5) with parameter $p = 4$. This clearly avoids an inverse crime. In all the examples the synthetic Cauchy data (f, g) on Γ_m were obtained by solving the impedance boundary value problem with given Neumann data

$$g(t) = \sin^4 t, \quad t \in [\pi, 2\pi],$$

on Γ_m and with different impedance functions λ on Γ_c, namely

$$\lambda_1(t) = 0.5,$$

$$\lambda_2(t) = 2.5,$$

$$\lambda_3(t) = \sin^4 t + 1$$

for $t \in [0, \pi]$. The system of integral equations (4.1)–(4.2) was solved using Tikhonov regularization with an H^2 penalty term on ζ with regularization parameter β and an L^2 penalty term on the density ψ with parameter α. The parameters were found by trial and error and are indicated in the figures below. The potentials were discretized using $2n = 64$ grid points on each boundary part. The update ζ of the boundary part Γ_c is given by

$$\zeta = \sum_{j=1}^{N} a_j q_j \in Q_N,$$

with basis elements of the approximation space $Q_N, N \geq 3$, chosen as

$$q_j(t) = -r_j(t)(\cos t, \sin t), \quad j = 1, \ldots, N, \quad 0 \leq t \leq \pi,$$

with radial parts

$$r_1(t) = t(\pi - t)^2, \quad r_2(t) = t^2(\pi - t)$$

and

$$r_j(t) = \sin(j-2)t, \quad j = 3, \ldots, N,$$

(see [25]). We started the iterations with an initial approximation for Γ_c given by the half circle in the lower half plane with end points coinciding with the end points $z(\pi)$ and $z(2\pi)$ of Γ_m. The iteration was started by performing L iteration steps on a subdivision of $[0, \pi]$ in five intervals, graded by the sigmoidal transformation, for the approximation of q_j. Then we successively increased the number of subintervals of $[0, \pi]$ for the approximation of q_j using the result for a subdivision into m subintervals as initial guess for Γ_c and performed again L iterations on $m + 1$ subintervals. This process was repeated until a final number M of subintervals was reached. We fixed the parameters M, L to be $M = 10$ and $L = 8$ in all the examples. The figures give reconstructions for exact data and for 3% random noise added to the Neumann data (with respect to the L^2-norm).

We start with two examples where the boundary is smooth. Figures 5.1–5.3 show reconstructions for an apple shaped contour with parameterization

$$z(t) = 0.5\,\frac{0.5 + 0.4\cos t + 0.1\sin 2t}{1 + 0.8\cos t}\,(\cos t, \sin t), \quad t \in [0, 2\pi]. \tag{5.1}$$

In all figures the dash-dotted (blue) lines represent the exact Γ_c and the full (red) lines the reconstructions. The initial guess is given by the dotted (blue) lines. Reconstructions for a kite with parameterization

$$z(t) = (0.3\cos t + 0.15\cos 2t, 0.3\sin t)\,, \quad t \in [0, 2\pi], \tag{5.2}$$

are shown in figures 5.4-5.6.

Finally, in the last example the boundary is only piecewise smooth with corners at the connections of Γ_m and Γ_c. The boundary part Γ_m has the form of half a peanut

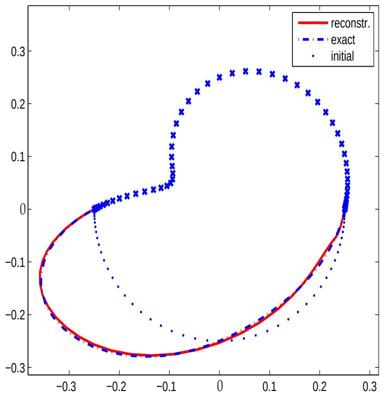
$$z_m(t) = \left(1 - \frac{1}{3}\sin t + \frac{1}{6}\sin 3t\right)\,(a\cos t, b\sin t), \quad t \in [\pi, 2\pi]$$

and Γ_c has the form of a sink

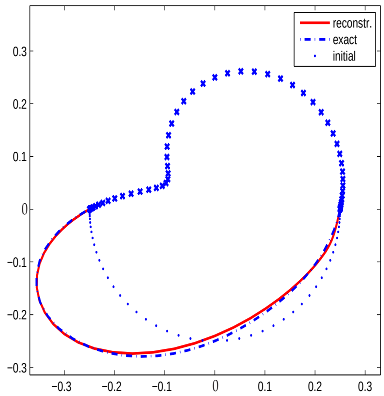
$$z_c(t) = \left(a\frac{2t - \pi}{\pi}, -b\sin t\right), \quad t \in [0, \pi], \tag{5.3}$$

with $a = 0.3$ and $b = 0.2$. The reconstructions are shown in figures 5.7–5.9.

We can summarize that the numerical experiments show satisfactory reconstructions for the case of a constant impedance on Γ_c with also reasonable stability against noisy data. For a non-constant impedance the reconstructions are slightly worse but still reasonable. We expect that by a more sophisticated choice of the regularization parameters the reconstructions can be improved. Compared to [7] the numerical results were improved using our method in combination with graded meshes. Furthermore we were able to justify our numerical experiments by showing a local uniqueness result in theorem 4.1 for a general impedance function whereas in [7] a similar result was obtained only for the limiting case $\lambda = \infty$.

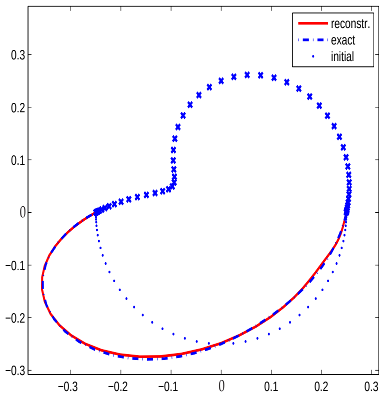


(a) exact data: $\alpha = 10^{-6}, \beta = 10^{-3}$

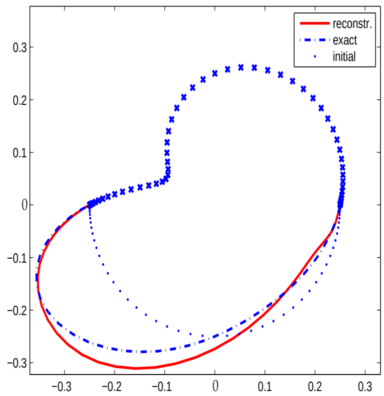


(b) noisy data: $\alpha = 10^{-5}, \beta = 10^{-2}$

Figure 5.1: Reconstruction of (5.1) for $\lambda = 0.5$

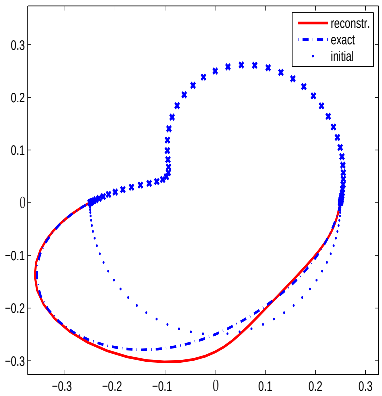


(a) exact data: $\alpha = 10^{-6}, \beta = 10^{-3}$

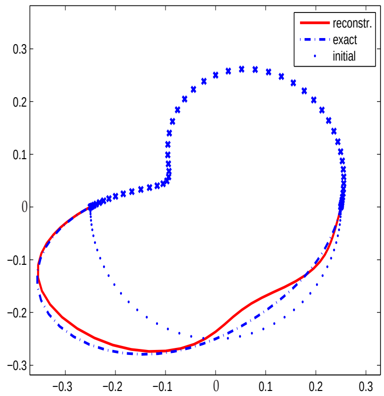


(b) noisy data: $\alpha = 10^{-5}, \beta = 10^{-3}$

Figure 5.2: Reconstruction of (5.1) for $\lambda = 2.5$

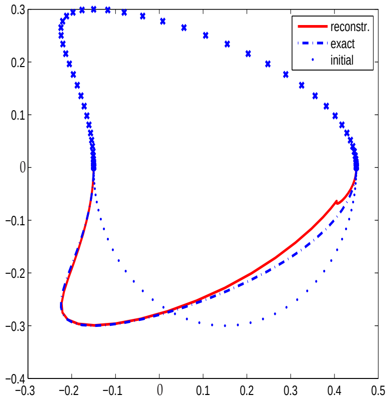


(a) exact data: $\alpha = 10^{-7}, \beta = 10^{-4}$

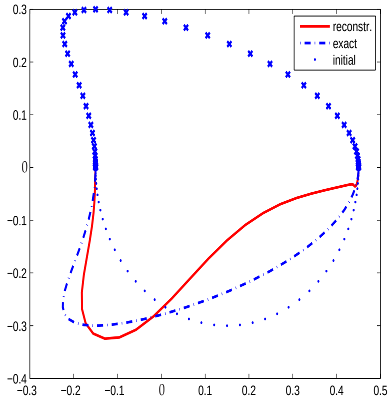


(b) noisy data: $\alpha = 10^{-6}, \beta = 10^{-4}$

Figure 5.3: Reconstruction of (5.1) for $\lambda(t) = \sin^4 t + 1, t \in [0, \pi]$

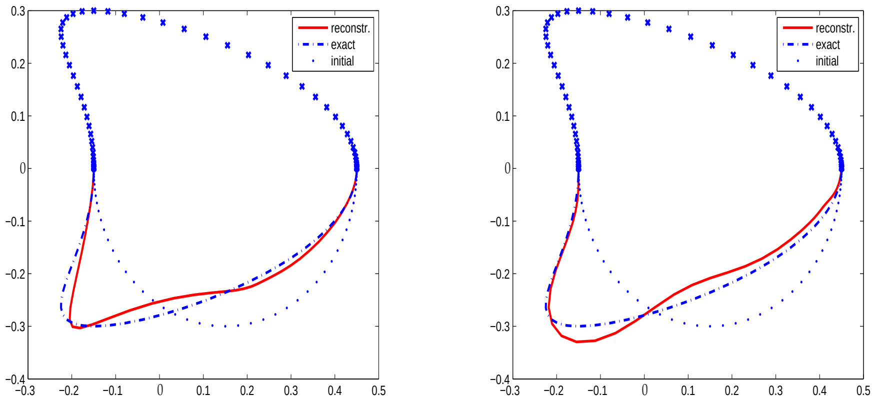


(a) exact data: $\alpha = 10^{-8}, \beta = 10^{-5}$



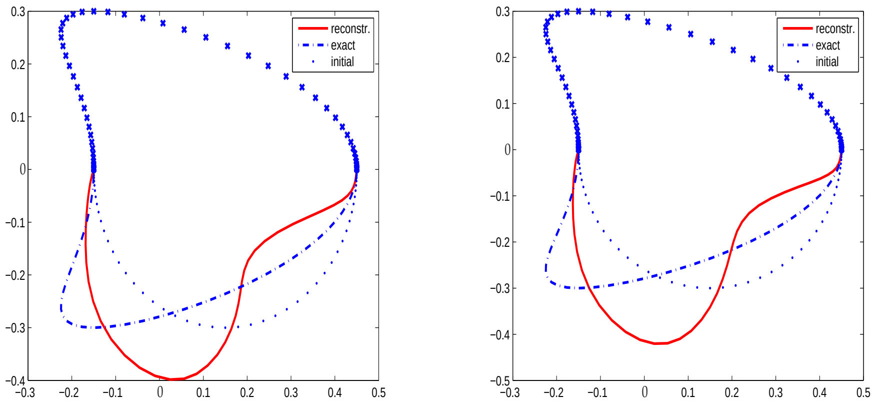
(b) noisy data: $\alpha = 10^{-5}, \beta = 10^{-2}$

Figure 5.4: Reconstruction of (5.2) for $\lambda = 0.5$



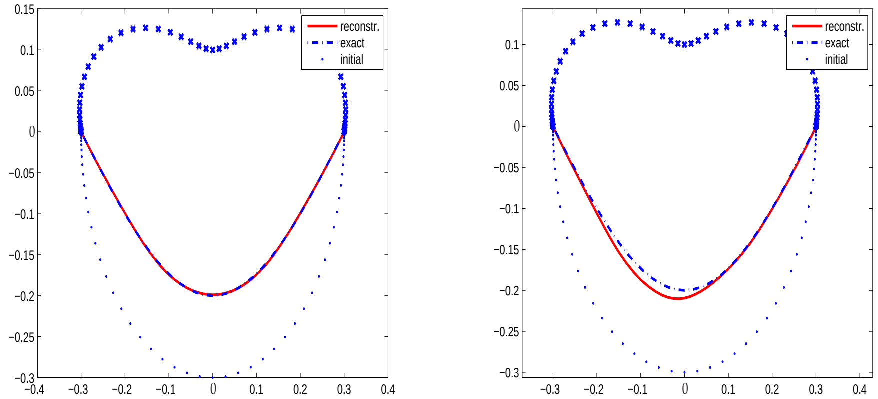
(a) exact data: $\alpha = 10^{-7}, \beta = 10^{-5}$ (b) noisy data: $\alpha = 10^{-5}, \beta = 10^{-3}$

Figure 5.5: Reconstruction of (5.2) for $\lambda = 2.5$



(a) exact data: $\alpha = 10^{-6}, \beta = 10^{-2}$ (b) noisy data: $\alpha = 10^{-5}, \beta = 10^{-2}$

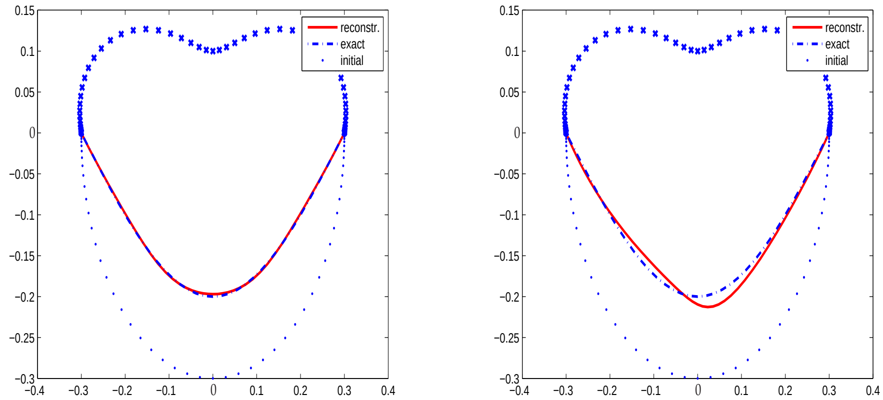
Figure 5.6: Reconstruction of (5.2) for $\lambda(t) = \sin^4 t + 1, t \in [0, \pi]$



(a) exact data: $\alpha = 10^{-5}, \beta = 10^{-3}$

(b) noisy data: $\alpha = 10^{-3}, \beta = 10^{-2}$

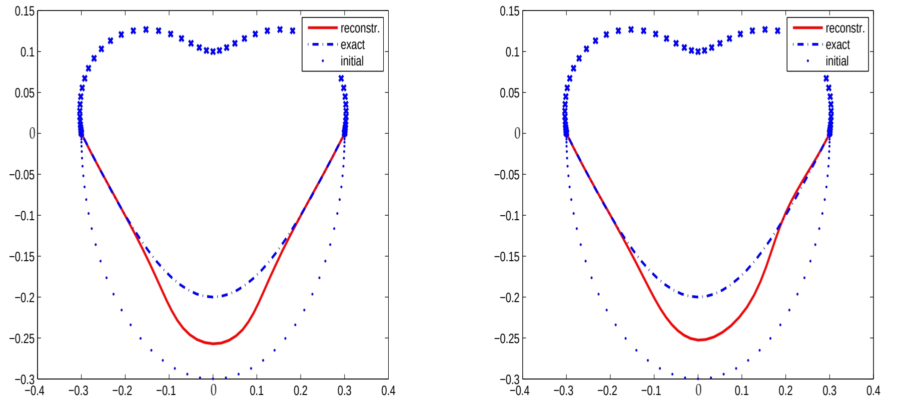
Figure 5.7: Reconstruction of (5.3) for $\lambda = 0.5$



(a) exact data: $\alpha = 10^{-7}, \beta = 10^{-4}$

(b) noisy data: $\alpha = 10^{-3}, \beta = 10^{-2}$

Figure 5.8: Reconstruction of (5.3) for $\lambda = 2.5$



(a) exact data: $\alpha = 10^{-9}, \beta = 10^{-5}$ (b) noisy data: $\alpha = 10^{-7}, \beta = 10^{-4}$

Figure 5.9: Reconstruction of (5.3) for $\lambda(t) = \sin^4 t + 1, t \in [0, \pi]$

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