Integral Equation Methods for the Inverse Obstacle Problem with Generalized Impedance Boundary Condition

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Abstract.

Determining the shape of an inclusion within a conducting medium from voltage and current measurements on the accessible boundary of the medium can be modeled as an inverse boundary value problem for the Laplace equation. We present a solution method for such an inverse boundary value problem with a generalized impedance boundary condition on the inclusion via boundary integral equations. Both the determination of the unknown boundary and the determination of the unknown impedance functions are considered. In addition to describing the reconstruction algorithms and illustrating their feasibility by numerical examples we also obtain a uniqueness result on determining the impedance coefficients.

Keywords: Inverse boundary value problem, integral equations, partial boundary measurements, generalized impedance boundary condition, uniqueness.

1. Introduction

Problems in electrostatic or thermal imaging are modeled in terms of the Laplace equation for the potential or the temperature in a bounded domain with some appropriate boundary condition. In this work we will consider the so-called generalized impedance boundary condition. More specifically, we have in mind the following imaging configuration. Let Ω be a doubly connected bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ that consists of two disjoint smooth closed Jordan curves Γ_m and Γ_c such that $\partial\Omega := \Gamma_m \cup \Gamma_c$ and Γ_c is contained in the interior of the domain bounded by Γ_m. We denote by ν the unit normal vector to these curves oriented towards the exterior of Ω. We assume that the electrostatic potential u satisfies

$$\Delta u = 0 \quad \text{in } \Omega \tag{1.1}$$

subject to the generalized impedance boundary condition

$$\frac{\partial u}{\partial \nu} - \operatorname{div}_{\Gamma_c}(\mu \operatorname{grad}_{\Gamma_c} u) + \lambda u = 0 \quad \text{on } \Gamma_c \tag{1.2}$$

where $\operatorname{div}_{\Gamma_c}$ and $\operatorname{grad}_{\Gamma_c}$ are the surface divergence and surface gradient, respectively, and $\mu \in C^1(\Gamma_c)$ is positive and $\lambda \in C^1(\Gamma_c)$ is nonnegative and not identically zero. Furthermore we assume that Dirichlet data (i.e., voltage) is prescribed on Γ_m, namely

$$u = f \quad \text{on } \Gamma_m. \tag{1.3}$$

The inverse problem is to determine Γ_c and/or μ and λ from a knowledge of the measured current density on Γ_m corresponding to the prescribed voltage f, i.e., from a knowledge of (the measured) Neumann data

$$g := \frac{\partial u}{\partial \nu} \quad \text{on } \Gamma_m \tag{1.4}$$

where u satisfies (1.1), (1.2) and (1.3). In the two-dimensional case, the inhomogeneous Laplace–Beltrami differential operator $\operatorname{div}_{\Gamma_c}(\mu \operatorname{grad}_{\Gamma_c} u)$ becomes

$$\operatorname{div}_{\Gamma_c}(\mu \operatorname{grad}_{\Gamma_c} u) = \frac{d}{ds}\, \mu\, \frac{du}{ds}$$

where d/ds is the tangential derivative and s is the arc length. The latter is the form we use from now on.

Our approach to the inverse problems is based on a boundary integral equation method in the spirit of the method proposed by Kress and Rundell [17] for an inverse boundary value problem for the Laplace equation with a Dirichlet boundary condition. For the application of these ideas for impedance boundary conditions we refer to [5–7] and the references therein. To this end we first need to develop the method of integral equations for boundary value problems with a generalized impedance boundary condition. In this case, the boundary condition is given in terms of a second order differential operator on the boundary which is of the same order as the differential equation inside the domain, and hence the existing theory of boundary integral equations does not cover it.

Section 2 is devoted to the solution of the direct boundary value problem with the generalized impedance boundary condition via a potential approach including a short discussion on the numerical solution. In Section 3 on the inverse problems we start with a uniqueness theorem stating that, knowing Γ_c, three Cauchy pairs uniquely determine both impedance functions λ and μ. Counter examples show that, in general, two Cauchy pairs are not sufficient to determine both λ and μ, knowing Γ_c, or Γ_c, knowing λ and μ. We then proceed with reconstruction schemes both for the inverse shape problem to determine Γ_c when λ and μ are known and for the inverse impedance problem where the roles are interchanged. Both algorithms are based on boundary integral equations derived from Green's representation theorem. We present the mathematical foundation of both methods and provide numerical examples illustrating their feasibility.

2. The direct problem

In this section we develop an integral equation method for solving the boundary value problem for the Laplace equation with generalized impedance boundary condition. Such problems in the context of scattering theory are investigated in [2–4] where the forward problem is analyzed by means of a variational approach. In order to present the main idea of our integral equation approach and to study the properties of the involved boundary integral operators, we first consider the boundary value problem where the generalized impedance boundary condition is prescribed on the entire boundary. This problem is the main building block for the analysis of the direct problem (1.1)-(1.3). To this end, let D be a bounded domain in $\mathbb{R}^2$ with C^2 boundary ∂D and let ν denote the unit normal vector on ∂D directed towards the exterior of D. The boundary value problem we want to solve is: Given $h \in H^{-\frac{1}{2}}(\partial D)$ find $u \in H^2(D)$ such that

$$\Delta u = 0 \qquad \text{in } D, \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} - \frac{d}{ds}\,\mu\,\frac{du}{ds} + \lambda u = h \quad \text{on } \partial D, \tag{2.2}$$

where $\mu \in C^1(\partial D)$ is positive and $\lambda \in C^1(\partial D)$ nonnegative and not identically zero and s is the arc length variable on ∂D. The derivative for $u|_{\partial D} \in H^{\frac{3}{2}}(\partial D)$ with respect to arc length in (2.2) has to be understood in the weak sense, that is,

$$\int_{\partial D} \left[\mu\, \frac{d\eta}{ds}\, \frac{du}{ds} + \lambda \eta u + \eta\, \frac{\partial u}{\partial \nu} \right] ds = \int_{\partial D} \eta h\, ds \tag{2.3}$$

for all $\eta \in H^{\frac{3}{2}}(\partial D)$.

Theorem 2.1 *The boundary value problem (2.1)–(2.2) has at most one solution.*

Proof. Let u denote the difference of two solutions to (2.1)–(2.2). Applying Green's first integral theorem and using the homogeneous form of the boundary condition (2.3) for $\eta = u|_{\partial D}$ we have that

$$\int_D |\operatorname{grad} u|^2\, dx = \int_{\partial D} u\, \frac{\partial u}{\partial \nu}\, ds = -\int_{\partial D} \mu \left| \frac{du}{ds} \right|^2 ds - \int_{\partial D} \lambda |u|^2 ds.$$

Hence in view of our assumptions on the positivity of μ and λ we can conclude that $u = 0$ in D. □

We try to find the solution of (2.1)–(2.2) in the form of a single-layer potential

$$u(x) = \int_{\partial D} \varphi(y) \Phi(x, y)\, ds(y), \quad x \in D, \tag{2.4}$$

where $\varphi \in H^{\frac{1}{2}}(\partial D)$ and

$$\Phi(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|}$$

is the fundamental solution of Laplace's equation in $\mathbb{R}^2$. It is known that $\varphi \in H^{\frac{1}{2}}(\partial D)$ implies that $u \in H^2(D)$. From now on, without loss of generality, we assume that there exists a point $x_0 \in D$ such that $|x - x_0| \neq 1$ for all $x \in \partial D$ and refer to this as **Assumption G**. Then Theorem 3.16 in [14] guarantees that the corresponding single-layer boundary integral operator is injective. (An alternative approach to guaranty the injectivity via boundedness of u at infinity is to modify the above definition (2.4) by adding an appropriate term as in Theorem 7.30 in [16].) Letting x approach the boundary ∂D from inside D, we observe that the boundary condition (2.2) is satisfied provided φ solves the boundary integral equation

$$-\frac{d}{ds}\,\mu\,\frac{d}{ds}\,S\varphi + \lambda S\varphi + \varphi + K'\varphi = 2h \tag{2.5}$$

where $S : H^{-\frac{1}{2}+s}(\partial D) \to H^{\frac{1}{2}+s}(\partial D)$ and $K' : H^{-\frac{1}{2}+s}(\partial D) \to H^{-\frac{1}{2}+s}(\partial D)$, $-1 \leq s \leq 1$, are the bounded integral operators defined by

$$(S\varphi)(x) := 2\int_{\partial D} \Phi(x,y)\varphi(y)\,ds(y), \quad x \in \partial D, \tag{2.6}$$

and

$$(K'\varphi)(x) := 2\int_{\partial D} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\,\varphi(y)\,ds(y), \quad x \in \partial D. \tag{2.7}$$

For later use we also introduce the bounded integral operator $K : H^{\frac{1}{2}+s}(\partial D) \to H^{\frac{1}{2}+s}(\partial D)$, $-1 \leq s \leq 1$, by

$$(K\varphi)(x) := 2\int_{\partial D} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\,\varphi(y)\,ds(y), \quad x \in \partial D. \tag{2.8}$$

Next let us define bounded linear operators $A, B : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ by

$$A\varphi := \frac{d^2}{ds^2}\,S\varphi + \int_{\partial D} S\varphi\,ds \tag{2.9}$$

and

$$B\varphi := \frac{1}{\mu}\,\frac{d\mu}{ds}\,\frac{d}{ds}\,S\varphi - \frac{\lambda}{\mu}\,S\varphi - \frac{1}{\mu}\,(K'\varphi + \varphi) - \int_{\partial D} S\varphi\,ds. \tag{2.10}$$

Then we can summarize the above analysis into the following theorem.

Theorem 2.2 *The single-layer potential (2.4) solves the boundary value problem (2.1)–(2.2) provided the density φ satisfies the integral equation*

$$(A + B)\varphi = -\frac{2}{\mu}\,h. \tag{2.11}$$

Lemma 2.1 *The operator $A : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is invertible with a bounded inverse $A^{-1} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$.*

Proof. We parametrize the boundary ∂D with the arc length s as parameter and for $-\frac{1}{2} \le r \le \frac{3}{2}$ we can identify $H^r(\partial D)$ with $H^r_{\text{per}}[0,L]$ where L is the length of ∂D and $H^r_{\text{per}}[0,L] \subset H^r[0,L]$ is the subspace of L periodic functions. Using the Fourier series representation of $H^r_{\text{per}}[0,L]$ it can be seen that the operator $M : H^{\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ given by

$$M\varphi := \frac{d^2\varphi}{ds^2} + \int_{\partial D} \varphi\, ds \tag{2.12}$$

is bounded and has a bounded inverse $M^{-1} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$. Since by our Assumption G on ∂D, the operator $S : H^{\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ has a bounded inverse $S^{-1} : H^{\frac{3}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ clearly $A = MS$ is bounded and has the bounded inverse $A^{-1} = S^{-1}M^{-1}$. □

Lemma 2.2 *The operator $B : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact.*

Proof. Since S, $\frac{d}{ds}\, S$ and K are bounded from $H^{\frac{1}{2}}(\partial D)$ into $H^{\frac{1}{2}}(\partial D)$ and both $1/\mu$ and λ/μ are in $C^1(\partial D)$ by our assumptions on μ and λ the operator $B : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is bounded. Thus the statement follows from the compact embedding of $H^{\frac{1}{2}}(\partial D)$ into $H^{-\frac{1}{2}}(\partial D)$. □

Theorem 2.3 *The generalized impedance boundary value problem (2.1)–(2.2) has a unique solution which depends continuously on h.*

Proof. In view of Theorem 2.1, Lemma 2.1 and Lemma 2.2 it suffices to show that the operator $A + B$ is injective. Assume that $\varphi \in H^{\frac{1}{2}}(\partial D)$ satisfies $(A + B)\varphi = 0$. Then, by Theorem 2.2 the single-layer potential u defined by (2.4) solves the boundary value problem (2.1)–(2.2) for $h = 0$. Hence, by Theorem 2.1 we have $u = 0$ in D. Taking the boundary trace of u we conclude $S\varphi = 0$. From Assumption G on ∂D we have injectivity of S and therefore $\varphi = 0$ which ends the proof. □

Now we turn our attention to the direct problem formulated in the layer Ω bounded by the closed smooth curves Γ_m and Γ_c. Here we would like to solve the following mixed boundary value problem: Given $f \in H^{\frac{3}{2}}(\Gamma_m)$ find $u \in H^2(\Omega)$ such that

$$\Delta u = 0 \qquad \text{in } \Omega, \tag{2.13}$$

$$u = f \qquad \text{on } \Gamma_m, \tag{2.14}$$

$$\frac{\partial u}{\partial \nu} - \frac{d}{ds}\,\mu\,\frac{du}{ds} + \lambda u = 0 \quad \text{on } \Gamma_c, \tag{2.15}$$

where we remind the reader that ν is oriented into the exterior of Ω. For the functions μ and λ we make the same assumptions as above in the problem (2.1)–(2.2). Analogous to Theorem 2.1 we can prove the following uniqueness theorem.

Theorem 2.4 *The boundary value problem (2.13)–(2.15) has at most one solution.*

We look for a solution to (2.13)–(2.15) in the form of a single layer potential

$$u(x) = \int_{\Gamma_m} \varphi_m(y)\Phi(x,y)ds(y) + \int_{\Gamma_c} \varphi_c(y)\Phi(x,y)ds(y), \quad x \in \Omega, \tag{2.16}$$

with densities $\varphi_m \in H^{\frac{1}{2}}(\Gamma_m)$ and $\varphi_c \in H^{\frac{1}{2}}(\Gamma_c)$. We assume that the Assumption G stated on page 4 is satisfied for the interior domains of both Γ_m and Γ_c. Analogously to (2.6)–(2.8) for $j,k = m,c$ we define operators $S_{jk} : H^{-\frac{1}{2}+s}(\Gamma_j) \to H^{\frac{1}{2}+s}(\Gamma_k)$, $K'_{jk} : H^{-\frac{1}{2}+s}(\Gamma_j) \to H^{-\frac{1}{2}+s}(\Gamma_k)$, and $K_{jk} : H^{\frac{1}{2}+s}(\Gamma_j) \to H^{-\frac{1}{2}+s}(\Gamma_k)$, $-1 \le s \le 1$, by integrating in (2.6)–(2.8), respectively, over $y \in \Gamma_j$ and evaluating for $x \in \Gamma_k$. Further we define bounded linear operators $A_{cc}, B_{cc} : H^{\frac{1}{2}}(\Gamma_c) \to H^{-\frac{1}{2}}(\Gamma_c)$ by replacing ∂D by Γ_c in (2.9) and (2.10), respectively, and a bounded linear operator $B_{mc} : H^{\frac{1}{2}}(\Gamma_m) \to H^{\frac{3}{2}}(\Gamma_c)$ by setting

$$B_{mc}\varphi := \frac{1}{\mu}\frac{d}{ds}\,\mu\,\frac{d}{ds}\,S_{mc}\varphi - \frac{\lambda}{\mu}\,S_{mc}\varphi - \frac{1}{\mu}\,K'_{mc}\varphi. \tag{2.17}$$

Then analogously to Theorem 2.3 we can state the following result.

Theorem 2.5 *The single-layer potential (2.16) solves the boundary value problem (2.13)–(2.15) provided the densities φ_m and φ_c satisfy the system of integral equations*

$$S_{mm}\varphi_m + S_{cm}\varphi_c = f, \tag{2.18}$$

$$B_{mc}\varphi_m + (A_{cc} + B_{cc})\varphi_c = 0. \tag{2.19}$$

Let us introduce the bounded operators $\mathbb{A}, \mathbb{B} : H^{\frac{1}{2}}(\Gamma_m) \times H^{\frac{1}{2}}(\Gamma_c) \to H^{\frac{3}{2}}(\Gamma_m) \times H^{-\frac{1}{2}}(\Gamma_c)$ by

$$\mathbb{A}\begin{pmatrix}\varphi_m\\ \varphi_c\end{pmatrix} := \begin{pmatrix} S_{mm} & 0\\ 0 & A_{cc}\end{pmatrix}\begin{pmatrix}\varphi_m\\ \varphi_c\end{pmatrix}$$

and

$$\mathbb{B}\begin{pmatrix}\varphi_m\\ \varphi_c\end{pmatrix} := \begin{pmatrix} 0 & S_{cm}\\ B_{mc} & B_{cc}\end{pmatrix}\begin{pmatrix}\varphi_m\\ \varphi_c\end{pmatrix}.$$

Because of our Assumption G on the interior of Γ_m the single-layer operator $S_{mm} : H^{\frac{1}{2}}(\Gamma_m) \to H^{\frac{3}{2}}(\Gamma_m)$ has a bounded inverse and by Lemma 2.1, applied to the interior of Γ_c, also $A_{cc} : H^{\frac{1}{2}}(\Gamma_c) \to H^{-\frac{1}{2}}(\Gamma_c)$ has a bounded inverse. Consequently $\mathbb{A}$ has a bounded inverse. On the other hand the operator $\mathbb{B}$ is compact. In the same way as in the proof of Theorem 2.3 one can show that $\mathbb{A}+\mathbb{B}$ is injective. Indeed, let $\varphi_m \in H^{\frac{1}{2}}(\Gamma_m)$ and $\varphi_c \in H^{\frac{1}{2}}(\Gamma_c)$ be in the null space of $\mathbb{A}+\mathbb{B}$. Then the single-layer potential u defined by (2.16) in all of $\mathbb{R}^2 \setminus \partial\Omega$ solves the boundary value (2.13)–(2.15) with $f = 0$. Hence by Theorem 2.4 we have $u = 0$ in Ω. Continuity of the single-layer potential implies that v is harmonic in the interior of Γ_c and vanishes on Γ_c, whence $v = 0$ in the interior of Γ_c follows. Now the jump relation for the normal derivative of the single layer potential across Γ_c yields $\varphi_c = 0$. Next we have that $S_{mm}\varphi_m = 0$ and injectivity of S_{mm} implies that $\varphi_m = 0$. Thus have proven the following theorem.

Theorem 2.6 *Let $f \in H^{\frac{3}{2}}(\Gamma_m)$. Then the generalized impedance boundary value problem (2.13)–(2.15) has a unique solution in $H^2(\Omega)$ which depends continuously on f.*

2.1. Numerical solution

For the numerical solution we employed a collocation method based on numerical quadratures using trigonometric polynomial approximations as the most efficient method for solving potential theoretic boundary integral equations in planar domains with smooth boundaries (see [16]). Here we only need to be concerned with the approximation of the combination of the Laplace–Beltrami operator and the single-layer operator as the new feature in the integral equations for the generalized impedance boundary condition. Again we confine our presentation to the boundary value problem (2.1)–(2.2), i.e., the integral equation (2.5).

We assume that the boundary curve ∂D is given by a regular 2π–periodic parameterization

$$\partial D = \{z(t) : 0 \le t \le 2\pi\} \tag{2.20}$$

with counter-clockwise orientation. Then, via $\psi = \varphi \circ z$ we introduce the parameterized operators $\widetilde{S} : H^{-\frac{1}{2}+s}_{\mathrm{per}}[0, 2\pi] \to H^{\frac{1}{2}+s}_{\mathrm{per}}[0, 2\pi]$ and $\widetilde{K}' : H^{-\frac{1}{2}+s}_{\mathrm{per}}[0, 2\pi] \to H^{-\frac{1}{2}+s}_{\mathrm{per}}[0, 2\pi]$, $-1 \le s \le 1$, by

$$\widetilde{S}(\psi)(t) := \frac{1}{2\pi} \int_0^{2\pi} \ln \frac{1}{|z(t) - z(\tau)|}\, |z'(\tau)|\, \psi(\tau)\, d\tau, \quad t \in [0, 2\pi],$$

and

$$\widetilde{K}'(\psi)(t) := \frac{1}{\pi\, |z'(t)|} \int_0^{2\pi} \frac{[z'(t)]^\perp \cdot [z(\tau) - z(t)]}{|z(t) - z(\tau)|^2}\, |z'(\tau)|\, \psi(\tau)\, d\tau, \quad t \in [0, 2\pi],$$

where we write $a^\perp = (a_2, -a_1)$ for any vector $a = (a_1, a_2)$, that is, $a^\perp$ is obtained by rotating a clockwise by 90 degrees. Then the parameterized form of (2.5) is given by

$$-\frac{1}{|z'|} \frac{d}{dt} \frac{\mu}{|z'|} \frac{d}{dt} \widetilde{S}\psi + \lambda \widetilde{S}\psi + \psi + \widetilde{K}'\psi = 2h \tag{2.21}$$

where for convenience we identify μ, λ and h with $\mu \circ z, \lambda \circ z$ and $h \circ z$.

We note that, in principle, we also could incorporate arc length into the parameterized densities, i.e., parametrize via $\psi = |z'|\, \varphi \circ z$ instead of $\psi = \varphi \circ z$. However, in our numerical experiments using these discretizations later on in our inverse algorithm we observed that the latter leads to a poorer accuracy in the numerical reconstructions.

Since the operator $\widetilde{K}'$ has a smooth kernel with the diagonal values given by

$$\lim_{\tau \to t} \frac{[z'(t)]^\perp \cdot [z(\tau) - z_j(t)]}{|z(t) - z(\tau)|^2} = \frac{[z'(t)]^\perp \cdot z''(t)}{2|z'(t)|^2} \tag{2.22}$$

the most efficient approximation is obtained via numerical quadrature by the composite rectangular rule with the nodal points $t_j = j\pi/n$, $j = 1, \dots, 2n$. Do deal with the logarithmic singularity of the operator $\widetilde{S}$ we split

$$\ln \frac{1}{|z(t) - z(\tau)|} = -\ln 4 \sin^2 \frac{t - \tau}{2} + \ln \frac{4 \sin^2 \frac{t-\tau}{2}}{|z(t) - z(\tau)||}$$

and apply the rectangular rule for the smooth second term on the right hand side. For the first term we use a weighted trigonometric interpolation quadrature with the above quadrature points and refer to [16] for the quadrature weights and further details. This way we obtain approximations $\widetilde{S}_n$ and $\widetilde{K}'_n$ for the operators $\widetilde{S}$ and $\widetilde{K}'$.

To approximate the Laplace–Beltrami operator we simply use numerical differentiation via trigonometric interpolation differentiation, i.e., we approximate the derivative ψ' of a 2π periodic function ψ by the derivative $(T_n\psi)'$ of the unique trigonometric polynomial $T_n\psi$ of degree n (without the term $\sin nt$) that interpolates $(T_n\psi)(t_j) = \psi(t_j)$, $j = 1, \dots, 2n$. Since the explicit weights for trigonometric differentiation are hard to find in the literature, for the reader's convenience we note that

$$(T_n\psi)'(t_j) = \sum_{k=1}^{2n} d_{|k-j|}^{(n)} \psi(t_k), \quad j = 1, \dots, 2n,$$

where

$$d_j = \begin{cases} \dfrac{(-1)^j}{2} \cot \dfrac{j\pi}{2n}\,, & j = 1, \dots, 2n-1, \\[2ex] 0, & j = 0. \end{cases}$$

We set $T'_n\psi := (T_n\psi)'$ and approximate

$$\frac{1}{|z'|} \frac{d}{dt} \frac{\mu}{|z'|} \frac{d}{dt} \widetilde{S}\psi \approx \frac{1}{|z'|} T'_n \frac{\mu}{|z'|} T'_n \widetilde{S}_n\psi.$$

Summarizing, our numerical solution method approximates the integral equation (2.21) by

$$-\frac{1}{|z'|} T'_n \frac{\mu}{|z'|} T'_n \widetilde{S}_n\psi_n + \lambda\widetilde{S}_n\psi_n + \psi_n + \widetilde{K}'\psi_n = 2h \tag{2.23}$$

which is solved for the trigonometric polynomial ψ_n by collocation at the nodal points $t_j = j\pi/n$ for $j = 1, \dots, 2n$.

Using partial integration it can be seen that

$$\frac{d^2}{dt^2} \int_0^{2\pi} \ln 4 \sin^2 \frac{t-\tau}{2}\, \psi(\tau)\, d\tau = \int_0^{2\pi} \cot \frac{t-\tau}{2}\, \psi'(\tau)\, d\tau.$$

From this it can be concluded that the principal part, i.e., the leading singularity of $\frac{d^2}{ds^2} S$ coincides with that of the normal derivative of the double-layer potential. Hence, it is to expected that the error analysis for hypersingular operator equations involving the normal derivative of the double-layer potential such as presented, for example, in [15] can be carried over to an error analysis for (2.23). In particular, such an analysis would predict exponential convergence in the case of analytic h, μ, λ and z. However, since our main emphasis is on the inverse problem we refrain from carrying out these ideas here. Instead of this we will conclude with a numerical example exhibiting the conjectured exponential convergence.

In the spirit of the inverse problem our numerical example will be for the boundary value problem (2.13)–(2.15) in the annulus Ω, that is, for the solution of the system (2.18)–(2.19). For the interior boundary Γ_c we choose the non-convex kite-shaped curve with the parametric representation

$$z_c(t) = 0.3\,(\cos t + 0.65\cos 2t - 0.65, 1.5\sin t), \quad 0 \le t \le 2\pi, \tag{2.24}$$

and for Γ_m the circle with the representation $z_m(t) = 0.9\,(\cos t, \sin t)$ for $t \in [0, 2\pi]$. The impedance functions are

$$\lambda(z_c(t)) = \frac{1}{1 - 0.1\sin 2t} \quad \text{and} \quad \mu(z_c(t)) = \frac{1}{1 + 0.3\cos t} \tag{2.25}$$

for $t \in [0, 2\pi]$ and the Dirichlet values $f(z_m(t)) = 1$ for $t \in [0, 2\pi]$. Table 1 gives some approximate values for $\tilde{g} = g \circ z_m$ with the normal derivative $g = \partial u/\partial\nu$ on Γ_m evaluated by

$$2g = \varphi_m + K'_{mm}\varphi_m + K'_{cm}\varphi_c.$$

The exponential convergence is clearly exhibited.

n	$\tilde{g}(0)$	$\tilde{g}(\pi/2)$	$\tilde{g}(\pi)$	$\tilde{g}(3\pi/2)$
8	0.2589149999	0.3462201537	0.3821973923	0.3353863415
16	0.2585276953	0.3456400586	0.3811830317	0.3348042890
32	0.2585217163	0.3456288996	0.3811614158	0.3347928309
64	0.2585217166	0.3456289007	0.3811614281	0.3347928324

Table 1. Numerical solution for direct problem

3. The Inverse Problems

We now turn our attention to inverse problems corresponding to the generalized impedance boundary value problem (1.1)–(1.3). In the following, the pair of function $(f, g) \in H^{\frac{3}{2}}(\Gamma_m) \times H^{\frac{1}{2}}(\Gamma_m)$ such that $g := \frac{\partial u}{\partial \nu}$ on Γ_m where $u \in H^2(\Omega)$ satisfies (2.13)–(2.15) is referred to as a Cauchy pair. The most general inverse problem is the *inverse shape and impedance problem* to determine Γ_c, μ and λ from a knowledge of one (or finitely many) Cauchy pairs. In this paper we will be only concerned with two less general cases, namely the *inverse shape problem* and the *inverse impedance problem* as a preparation for a subsequent investigation of the full inverse shape and impedance problem. The inverse shape problem consists in determining Γ_c from one (or finitely many) Cauchy pairs knowing the impedance coefficients μ and λ. With the roles reversed, the inverse impedance problem requires to determine the impedance functions μ and λ from one (or finitely many) Cauchy pairs for a known shape Γ_c.

3.1. Uniqueness of the inverse problems

The first question to ask is what is the minimum amount of data, i.e., the minimal number of Cauchy pairs, to guaranty the uniqueness of the solution for the inverse impedance problem or the inverse shape problem. In this subsection we provide partial answers to this uniqueness question.

The following counter example, inspired by the example in [10] which also appears in [19], illustrates the non-uniqueness issues for the above inverse problems using two Cauchy pairs. Let Ω be the annulus bounded by $\Gamma_m := \{x : |x| = R\}$ and $\Gamma_c := \{x : |x| = \rho < R\}$. We consider the complex valued function

$$u(r,\theta) = \left(\frac{r^n}{\rho^n} + b\,\frac{\rho^n}{r^n}\right) e^{in\theta} \tag{3.1}$$

for $n \in \mathbb{N}$ and $b \in \mathbb{R}$ in polar coordinates (r,θ), whose real and imaginary parts are harmonic functions in Ω. The generalized impedance boundary condition on the circle $\rho < R$ with constant $\mu > 0$ and $\lambda > 0$ reads

$$-\frac{\partial u}{\partial r} - \frac{\mu}{\rho^2}\,\frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0 \quad \text{on } r = \rho.$$

Hence, (3.1) satisfies the generalized impedance boundary condition provided the constant b is chosen such that

$$b = \frac{n\rho - \mu n^2 - \lambda\rho^2}{n\rho + \mu n^2 + \lambda\rho^2}\,. \tag{3.2}$$

We see that b in fact depends on the combined expression $\mu n^2 + \lambda\rho^2$ and thus for fixed $\rho < R$ (i.e., for given boundary Γ_c) and fixed $n \in \mathbb{N}$ two Cauchy pairs corresponding to the real and imaginary part of u given by (3.1) and (3.2) on the circle Γ_m of radius R, provide more than one solution for μ and λ. In fact we obtain infinitely many solutions for μ and λ.

Regarding the unique determination of the boundary, in view of the above discussion, it is reasonable to ask whether two pairs of Cauchy data can uniquely determine the boundary Γ_c for known boundary coefficients μ and λ. To this end we take a second domain bounded by $\tilde{\Gamma}_c := \{x : |x| = \tilde{\rho}\}$, $\tilde{\rho} \neq \rho$ and $\tilde{\rho} < R$. Then (3.1) satisfies the generalized impedance boundary condition with the same μ and λ on $\tilde{\Gamma}_c$ provided the positive ratio $t := \tilde{\rho}/\rho$ satisfies

$$\frac{t^{2n} - b}{t^{2n} + b} = \mu\,\frac{n}{\rho}\,\frac{1}{t} + \lambda\,\frac{\rho}{n}\,t. \tag{3.3}$$

In the particular case of $b = 0$ which occurs if

$$\mu\,\frac{n}{\rho} + \lambda\,\frac{\rho}{n} = 1 \tag{3.4}$$

the equation (3.3) becomes

$$t^2 - \frac{n}{\lambda\rho}\,t + \frac{n^2\mu}{\rho^2\lambda} = 0,$$

and its zeros are $t_1 = 1$ and $t_2 = n^2\mu/\rho^2\lambda$. Hence, if we choose $n \in \mathbb{N}$, $\rho < R$, μ and λ such that (3.4) and the two conditions $n^2\mu < \lambda\rho R$ and $n^2\mu \neq \rho^2\lambda$ are satisfied (which is possible) then the Cauchy data on Γ_c corresponding to the real and imaginary part of (3.1) allow two different solutions Γ_c and $\tilde{\Gamma}_c$ of the inverse shape problem. An example for a possible parameter choice is $R = 2$, $n = 1$, $\mu = 1/4$ and $\lambda = 3/4$ which gives rise to two solutions of the inverse shape problem, namely $\rho = 1$ and $\tilde{\rho} = 1/3$.

The following theorem shows that indeed three Cauchy pairs uniquely determine both impedance functions λ and μ provided that Γ_c is known.

Theorem 3.1 *Let f_1, f_2 and f_3 be linearly independent. Then the three Cauchy pairs (f_1, g_1), (f_2, g_2) and (f_3, g_3) for the solutions u_1, u_2 and u_3 of the generalized impedance problem (2.13)–(2.15) uniquely determine the coefficient functions μ and λ.*

Proof. By Holmgren's theorem and the impedance boundary condition (2.15), the linear independence of f_1, f_2 and f_3 implies linear independence of $u_1|_{\Gamma_c}$, $u_2|_{\Gamma_c}$ and $u_3|_{\Gamma_c}$. From this it follows that for $j, k = 1, 2, 3$ with $j \neq k$ the Wronskians

$$W(u_j, u_k) := u_j \frac{du_k}{ds} - u_k \frac{du_j}{ds}$$

do not vanish on open subsets of Γ_c. Multiplying the impedance condition for u_1 by u_2 and the impedance condition of u_2 by u_1 and subtract we obtain

$$\frac{d}{ds}\,\mu\left(u_1 \frac{du_2}{ds} - u_2 \frac{du_1}{ds}\right) = u_1 \frac{\partial u_2}{\partial \nu} - u_2 \frac{\partial u_1}{\partial \nu} \quad \text{on } \Gamma_c. \tag{3.5}$$

From this we observe that due to the constant occurring in the integration of (3.5) the coefficient μ, in general, cannot be recovered from only two pairs of Cauchy data. In particular, the difference of two coefficients μ and $\tilde{\mu}$ that are compatible with the two Cauchy pairs (f_1, g_1) and (f_2, g_2) is given by

$$\mu - \tilde{\mu} = \frac{\alpha}{W(u_1, u_2)} \tag{3.6}$$

for some constant α.

Now assume that μ and $\tilde{\mu}$ are two different impedance functions that are compatible with all three Cauchy pairs (f_1, g_1), (f_2, g_2) and (f_3, g_3). Then from (3.6), applied to the three possible combinations of Cauchy pairs we find that there exist constants α_1, α_2 and α_3 with $\alpha_1\alpha_2\alpha_3 \neq 0$ such that

$$\frac{\alpha_3}{W(u_1, u_2)} = \frac{\alpha_1}{W(u_2, u_3)} = \frac{\alpha_2}{W(u_3, u_1)}\,,$$

that is,

$$\alpha_3 W(u_2, u_3) = \alpha_1 W(u_1, u_2) \quad \text{and} \quad \alpha_3 W(u_3, u_1) = \alpha_2 W(u_1, u_2) \quad \text{on } \Gamma_c. \tag{3.7}$$

Multiplying the first equation in (3.7) by u_1 and the second equation by u_2 and adding we obtain

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0 \quad \text{on } \Gamma_c$$

in contradiction to the linear independence of $u_1|_{\Gamma_c}$, $u_2|_{\Gamma_c}$ and $u_3|_{\Gamma_c}$.

Hence, the impedance function μ is uniquely determined by three Cauchy pairs. Once we know μ, the remaining coefficient λ can be obtained from the impedance condition (2.15) for any of the three functions u_1, u_2, and u_3 since by Holmgren's theorem neither of them can vanish on open subsets of Γ_c. □

Remark 3.1 Note that the regularity assumption on λ in Theorem 3.1 can be relaxed, for example, to piecewise continuity. Further note that the proof uses only the differential equation on the boundary Γ_c and hence, the same proof can be carried over to the inverse scattering problem for acoustic waves with generalized impedance boundary condition in $\mathbb{R}^2$ [2–4]. More specifically for this scattering problem, our proof shows that the C^1 function μ and the piecewise C^0 function λ are uniquely determined from the far field pattern corresponding to three incident plane waves with different incident directions, provided that the shape of the scattering object is known.

The uniqueness of the boundary Γ_c with finitely many Cauchy pairs is an open problem, even if it is assumed that μ and λ are known. For the case of the Robin condition, i.e., $\mu = 0$, it was shown in [1,19] that two Cauchy pairs (f_1, g_1) and (f_2, g_2) such that f_1 and f_2 are linearly independent and $f_1 > 0$ uniquely determine both Γ_c and λ. Similar result can be also stated for the unique determination of μ and Γ_c if $\lambda = 0$, since in this case the problem for the conjugate harmonic of u becomes a Robin problem with impedance $1/\mu$ (see [10]).

3.2. Solution of the inverse shape problem

Our solution method for the inverse problems is based on an equivalent system of nonlinear integral equations. This method was first introduced by Kress and Rundell in [17] and then further developed for various inverse problems in [5–7, 11, 12] (see also references therein). To simplify our notations, for a solution $u \in H^2(\Omega)$ to the boundary value problem (2.13)–(2.15), we set

$$\eta := u|_{\Gamma_c}$$

and abbreviate

$$\chi := \frac{d}{ds}\,\mu\frac{d\eta}{ds} - \lambda\eta. \tag{3.8}$$

Then from Green's formula we have

$$\begin{aligned} u(x) = &\int_{\Gamma_m} \left\{ \Phi(x,y)g(y) - \frac{\partial\Phi(x,y)}{\partial\nu(y)}\, f(y) \right\} ds(y) \\ &+ \int_{\Gamma_c} \left\{ \Phi(x,y)\chi(y) - \frac{\partial\Phi(x,y)}{\partial\nu(y)}\, \eta(y) \right\} ds(y), \quad x \in \Omega, \end{aligned} \tag{3.9}$$

in terms of the Cauchy data f and g for the inverse problem. Letting x tend to Γ_m and Γ_c from inside Ω, we obtain the integral equations

$$S_{mm}g + S_{cm}\left(\frac{d}{ds}\,\mu\frac{d\eta}{ds} - \lambda\eta\right) - K_{cm}\eta = f + K_{mm}f \tag{3.10}$$

and

$$S_{mc}g + S_{cc}\left(\frac{d}{ds}\,\mu\frac{d\eta}{ds} - \lambda\eta\right) - \eta - K_{cc}\eta = K_{mc}f \tag{3.11}$$

for $g := \frac{\partial u}{\partial \nu}|_{\Gamma_m} \in H^{\frac{1}{2}}(\Gamma_m)$ and $\eta \in H^{\frac{3}{2}}(\Gamma_c)$, with the notation using the subscripts m, c for the operators as introduced in Section 2.

Since we settled existence of a solution to the boundary value problem in Theorem 2.6, existence of a solution to the system (3.10)–(3.11) is evident and we only need to be concerned with uniqueness. For this we observe that the integral operator on the left-hand side of (3.10)–(3.11) is the adjoint of the integral operator $\mu(\mathbb{A}+\mathbb{B})$ in Theorem 2.6 with respect to the L^2-dual system. Hence, by the Fredholm theory injectivity of the latter implies injectivity for the system (3.10)–(3.11) (note we assume that $\mu > 0$).

Thus by rearranging the unknowns we have established the following main theorem for our approach to solving the inverse shape problem.

Theorem 3.2 *The inverse shape problem is equivalent to solving the system of integral equations*

$$S_{cm}\left(\frac{d}{ds}\,\mu\frac{d\eta}{ds} - \lambda\eta\right) - K_{cm}\eta = f + K_{mm}f - S_{mm}g \tag{3.12}$$

and

$$S_{cc}\left(\frac{d}{ds}\,\mu\frac{d\eta}{ds} - \lambda\eta\right) - \eta - K_{cc}\eta = K_{mc}f - S_{mc}g \tag{3.13}$$

for Γ_c and η.

In the sequel we will refer to (3.12) as data equation and to (3.13) as field equation. Although, in principle, we are not interested in the Dirichlet values $\eta = u|_{\Gamma_c}$ in methods based on the simultaneous solution of (3.12)–(3.13) we cannot avoid solving also for η. Both equations are linear with respect to η and nonlinear with respect to Γ_c. We note that the data equation (3.12) is severely ill-posed reflecting the ill-posedness of the inverse shape problem.

Obviously we have three options for an iterative solution of (3.12)–(3.13). In a first method, given an approximation for the boundary curve Γ_c we can solve (3.13) as linear equation for η and compute χ via (3.8). Then, keeping η and χ, that is, $u|_{\Gamma_c}$ and $\partial_\nu u|_{\Gamma_c}$, fixed we linearize equation (3.12) with respect to Γ_c to update the boundary approximation. This approach has been suggested by Johansson and Sleeman [13] in inverse obstacle scattering. In the second method, following ideas first developed by Kress and Rundell [17], one also can solve the system (3.12)–(3.13) simultaneously for Γ_c and η by Newton iterations, i.e., by linearizing both equations with respect to both unknowns. Finally, in a third method given an approximation for the boundary curve

Γ_c we could solve the severely ill-posed equation (3.12) for η and then and linearize (3.13) for gaining the boundary update.

Here, we will pursue only the first approach, i.e., solving the field equation (3.13) for η and linearizing the data equation (3.12) to update the boundary curve Γ_c. The solvability of (3.13) is ensured by the following theorem.

Theorem 3.3 *The integral equation (3.13) has a unique solution* $\eta \in H^{\frac{3}{2}}(\Gamma_c)$.

Proof. In view of Lemma 2.1 and 2.2, it suffices to prove injectivity of the adjoint operator $\mu(A_{cc}+B_{cc}) : H^{\frac{1}{2}}(\Gamma_c) \to H^{\frac{3}{2}}(\Gamma_c)$ given by

$$\mu(A_{cc}+B_{cc}) = \frac{d}{ds}\,\mu\,\frac{d}{ds}\,S_{cc} - \lambda S_{cc} - K'_{cc} - I$$

with respect to the L^2-dual system, where A_{cc}, B_{cc} are introduced in Theorem 2.5. Assume that $\varphi \in H^{\frac{1}{2}}(\Gamma_c)$ is a solution to $(A_{cc}+B_{cc})\varphi = 0$ and define the single–layer potential

$$u(x) := \int_{\Gamma_c} \Phi(x,y)\varphi(y)\,ds(y), \quad x \in \Omega_c,$$

where Ω_c is the unbounded exterior of Γ_c. Then the integral equation implies that u satisfies the impedance condition

$$\frac{\partial u}{\partial \nu} - \frac{d}{ds}\,\mu\frac{du}{ds} + \lambda u = 0 \quad \text{on } \Gamma_c. \tag{3.14}$$

In order to conclude from this that u vanishes identically in Ω_c we are faced with the difficulty arising from the behavior of the logarithmic fundamental solution at infinity. To this end we will show that $\int_{\Gamma_c}\varphi\,ds = 0$. Then u is bounded and $u(x) \to 0$ as $x \to \infty$ and consequently $u(x) = O(1/|x|)$ and $\operatorname{grad} u(x) = O(1/|x|)$ as $x \to \infty$. This decay at infinity is fast enough to allow the application of Green's integral theorem and proceed analogous to the proof of Theorem 2.1 to conclude that $u = 0$ in Ω_c. Then the boundary trace implies $S_{cc}\varphi = 0$ and consequently $\varphi = 0$ which establishes the required injectivity.

With the boundedness of $S_{cc} : H^{-\frac{1}{2}+s}(\Gamma_c) \to H^{\frac{1}{2}+s}(\Gamma_c)$ and $K'_{cc} : H^{-\frac{1}{2}+s}(\Gamma_c) \to H^{-\frac{1}{2}+s}(\Gamma_s)$, $-3 \le s \le 3$, if Γ_{cc} is C^4 smooth (see [18]) as in Lemma 2.1 and 2.2 it can be seen that $A_{cc} : H^{\frac{5}{2}}(\Gamma_{cc}) \to H^{\frac{3}{2}}(\Gamma_{cc})$ is bounded and has a bounded inverse and that $B_{cc} : H^{\frac{5}{2}}(\Gamma_{cc}) \to H^{\frac{3}{2}}(\Gamma_{cc})$ is compact. Then using the Fredholm alternative in the two dual systems $(H^{\frac{5}{2}}(\Gamma_{cc}), H^{\frac{1}{2}}(\Gamma_{cc})$ and $(H^{\frac{1}{2}}(\Gamma_{cc}), H^{\frac{1}{2}}(\Gamma_{cc})$, both with respect to the L^2 bilinear form, proceeding as in [8, Theorem 3.27] it can be proven that the null spaces of $A_{cc}+B_{cc}$ in $H^{\frac{1}{2}}(\Gamma_{cc})$ and $H^{\frac{5}{2}}(\Gamma_{cc})$ coincide. Hence, we have $\varphi \in H^{\frac{5}{2}}(\Gamma_{cc})$ which implies that $u \in H^4_{\text{loc}}(\Omega_c)$ and by the Sobolev imbedding theorem we have that $u \in C^2(\overline{\Omega}_c)$. Hence, we can apply Hopf's lemma (see e.g. [9]).

In view of the geometric Assumption G we first consider the case where x_0 exists in the interior of Γ_c such that $|x - x_0| < 1$ for all $x \in \Gamma_c$. By continuity we can choose a disk U contained in the interior of Γ_c such that $|x - z| < 1$ for all $x \in \Gamma_c$ and all $z \in U$. Without loss of generality we may assume that $\int_{\Gamma_c}\varphi\,ds \le 0$. Then $u(x) \to \infty$

as $|x| \to \infty$ and consequently u assumes its minimum in some $x_{\min} \in \Gamma_c$. Assume that $u(x_{\min})$ is negative. For the minimum on the boundary we have that

$$\frac{du}{ds}\,(x_{\min}) = 0 \quad \text{and} \quad \frac{d^2u}{ds^2}\,(x_{\min}) \geq 0$$

and therefore

$$\frac{\partial u}{\partial \nu}\,(x_{\min}) \geq 0$$

as consequence of (3.14). However, this is a contradiction to Hopf's lemma. Consequently we have $u \geq 0$ on Γ_c. Now we consider the harmonic function

$$w(x) := \frac{1}{2\pi} \ln|x - z| \int_{\Gamma_c} \varphi\, ds$$

for $x \in \overline{\Omega}_c$ and $z \in U$ which is non-negative for all $x \in \Gamma_c$. Therefore the function $v := u + w$ is also non-negative on Γ_c. Since $v(x) \to 0$ as $x \to \infty$ by the maximum-minimum principle we have that v is non-negative in Ω_c for all $z \in U$. From this, proceeding as in the proof of Theorem 3.16 in [14], it can be concluded that $\int_{\Gamma_c} \varphi\, ds = 0$. The case where x_0 exists in the interior of Γ_c such that $|x - x_0| > 1$ for all $x \in \Gamma_c$ is treated analogously by assuming that $\int_{\Gamma_c} \varphi\, ds \geq 0$ and considering the maximum of u. □

Based on Theorem 3.3 we can describe our inversion method as a two step technique for which the first step is to solve the well-posed linear integral equation (3.12) for η and the second step to solve the non-linear ill-posed equation (3.13) for Γ_c. For the latter we need to linearize with respect to the boundary Γ_c. To this end without loss of generality we assume that the C^2-boundaries Γ_c and Γ_m have a parametric representation

$$\Gamma_c = \{z_c(t) : t \in [0, 2\pi]\} \qquad \text{and} \qquad \Gamma_m = \{z_m(t) : t \in [0, 2\pi]\}, \tag{3.15}$$

respectively, with 2π periodic C^2-smooth functions $z_c : \mathbb{R} \to \mathbb{R}^2$ and $z_m : \mathbb{R} \to \mathbb{R}^2$ such that z_c and z_m are injective on $[0, 2\pi)$. In order to have the relation $|z_j'|\, \nu \circ z_j = [z_j']^\perp$ between the normal vector ν to Γ_j directed into the exterior of Ω both for $j = m, c$ for the unified representation (3.17) of the parameterized double-layer integral operators we need to assume z_m with counter-clockwise and z_c with clockwise orientation. Setting $\psi = \phi \circ z_j$ for $j, k = m, c$ we obtain from (2.6) and (2.8) the parameterized versions $\widetilde{S}_{jk} : H_{\rm per}^{-\frac{1}{2}+s}[0, 2\pi] \to H_{\rm per}^{\frac{1}{2}+s}[0, 2\pi]$ and $\widetilde{K}_{jk} : H_{\rm per}^{\frac{1}{2}+s}[0, 2\pi] \to H_{\rm per}^{\frac{1}{2}+s}[0, 2\pi]$ $-1 \leq s \leq 1$, of the operators S_{jk} and K_{jk} given by

$$\widetilde{S}_{jk}(\psi)(t) := \frac{1}{2\pi} \int_0^{2\pi} \ln \frac{1}{|z_k(t) - z_j(\tau)|}\, |z_j'(\tau)|\, \psi(\tau)\, d\tau, \quad t \in [0, 2\pi], \tag{3.16}$$

and

$$\widetilde{K}_{jk}(\psi)(t) := \frac{1}{\pi} \int_0^{2\pi} \frac{[z_j'(\tau)]^\perp \cdot [z_k(t) - z_j(\tau)]}{|z_k(t) - z_j(\tau)|^2}\, \psi(\tau)\, d\tau, \quad t \in [0, 2\pi]. \tag{3.17}$$

(For the notation $a^\perp$ we refer to p. 7). The operators $d\widetilde{S}_{cm}[\chi, z_c; \zeta]$ and $d\widetilde{K}_{cm}[\eta, z_c; \zeta]$ denote the Fréchet derivative with respect to z_c in the direction ζ of $\widetilde{S}_{cm}\chi$ and $\widetilde{K}_{cm}\eta$, respectively, and are obtained by differentiating their smooth kernels with respect to z_c

$$\begin{aligned} d\widetilde{S}_{cm}[\psi, z_c; \zeta](t) = &\frac{1}{2\pi}\int_0^{2\pi} \frac{[z_m(t) - z_c(\tau)]\cdot\zeta(\tau)}{|z_m(t) - z_c(\tau)|^2}\, |z_c'(\tau)|\psi(\tau)d\tau \\ &+ \frac{1}{2\pi}\int_0^{2\pi} \ln\frac{1}{|z_m(t) - z_c(\tau)|}\,\psi(\tau)\,\frac{z_c'(\tau)\cdot\zeta'(\tau)}{|z_c'(\tau)|}\,d\tau \end{aligned}$$

and

$$\begin{aligned} d\widetilde{K}_{cm}[\psi, z_c; \zeta](t) = &\frac{1}{2\pi}\int_0^{2\pi} \frac{[\zeta'(\tau)]^\perp\cdot[z_m(t) - z_c(\tau)] - [z_c'(\tau)]^\perp\cdot\zeta(\tau)}{|z_m(t) - z_c(\tau)|^2}\,\psi(\tau)d\tau \\ &+ \frac{1}{\pi}\int_0^{2\pi} \frac{[z_c'(\tau)]^\perp\cdot[z_m(t) - z_c(\tau)][z_c(t) - z_c(\tau)]\cdot\zeta(\tau)}{|z_m(t) - z_c(\tau)|^4} \end{aligned}$$

for $t \in [0, 2\pi]$. In view of these notations, the parametrized version of the field equation (3.13) and the linearized version of the data equation (3.12) now read

$$\widetilde{S}_{cc}\,\frac{1}{|z_c'|}\,\frac{d}{dt}\,\frac{\mu}{|z_c'|}\,\frac{d}{dt}\,\eta - \lambda S_c c\eta - \widetilde{K}_{cc}\eta - \eta = \widetilde{K}_{mc}f - \widetilde{S}_{mc}g \tag{3.18}$$

and

$$\widetilde{S}_{cm}\chi - \widetilde{K}_{cm}\eta + d\widetilde{S}_{cm}[\chi, z_c; \zeta] - d\widetilde{K}_{cm}[\eta, z_c; \zeta] = f + \widetilde{K}_{mm}f - \widetilde{S}_{mm}g \tag{3.19}$$

where χ in (3.19) is given by

$$\chi = \frac{1}{|z_c'|}\,\frac{d}{dt}\,\frac{\mu}{|z_c'|}\,\frac{d\eta}{dt} - \lambda\eta. \tag{3.20}$$

For convenience, we identified f, g, μ and λ with $f\circ z_m$, $g\circ z_m$, $\mu\circ z_c$ and $\lambda\circ z_c$, respectively.

Solving the inverse shape problem can be summarized by the following algorithm: Let (f, g) be a given pair of Cauchy data and let the impedance functions μ and λ be known.

(i) We start with an initial guess for the unknown boundary Γ_c parametrized by z_c. Then we solve the well-posed field equation (3.18) to find η and then compute the corresponding χ via (3.20).

(ii) With η and χ from step 1 inserted, we solve the ill-posed linearized data equation (3.19) for ζ using Tikhonov regularization. The update $z_c{+}\zeta$ for the parametrization is then used in step 1 for the next iteration.

(iii) The first and second step are repeated until a suitable stopping criterion is satisfied.

For the numerical solution of the field equation (3.18) we use the concepts explained in Subsection 2.1 with obvious modifications. In particular, the diagonal values of the smooth kernels of K_{cc} are obtained analogous to (2.22). For the discretization of the right-hand side of (3.18) the integral operators have smooth kernels and can be efficiently approximated via the composite rectangular rule. The latter also applies to the integral operators in the data equation (3.19) whereas for its right-hand side now the more sophisticated logarithmic quadrature rules of Subsection 2.1 are needed again.

The following numerical examples are intended as proof of concept and not as indications of an already fully developed method. In particular, the regularization parameters and the number of iterations are chosen by trial and error instead of, for example, a discrepancy principle. In all examples the data were obtained by the integral equations discussed in Section 2 with $n = 64$. Note that the boundary integral equations for creating the data is obtained via the potential approach whereas the integral equations in the inverse algorithm are based on Green's formula and thus committing an inverse crime is avoided.

In principle, the parameterization of the update ζ obtained from (3.19) is not unique. To cope with this ambiguity, one possibility is to allow only parameterizations of the form

$$z_c(t) = r(t)(\cos t, \sin t), \quad 0 \le t \le 2\pi, \tag{3.21}$$

with a non-negative function r representing the radial distance of Γ_c from the origin. Consequently, the perturbations are of the form

$$\zeta(t) = q(t)(\cos t, \sin t), \quad 0 \le t \le 2\pi, \tag{3.22}$$

with a real function q. In the approximations we assume r and its update q to have the form of a trigonometric polynomial of degree J, in particular,

$$q(t) = \sum_{j=0}^{J} a_j \cos jt + \sum_{j=1}^{J} b_j \sin jt. \tag{3.23}$$

For all examples the impedance functions are given by (2.25) and the measurement curve Γ_m is the circle with the representation $z_m(t) = 0.9\,(\cos t, \sin t)$ for $t \in [0, 2\pi]$. The number of quadrature points is 64 on each curve, i.e., $n = 32$. The degree of the polynomials (3.23) is always chosen as $J = 6$. The regularization parameter for an H^2 regularization of the linearized data equation (3.19) is $\alpha = 0.9^p$ for p-th iteration step. For the perturbed data, random noise is added point wise and the relative error is with respect to the L^2 norm. The iterations are started with an initial guess given by a circle of radius 0.8 centered at the origin.

In the figures the exact Γ_c is given as dotted (magenta) curve and the reconstruction as full (red) curve. The measurement curve Γ_m is dashed-dotted (green) and the initial guess dashed (blue).

Fig. 1 shows the results of the first example with Γ_c given by the ellipse with parameterization

$$z_c(t) = (0.6 \cos t, 0.4 \sin t), \quad t \in [0, 2\pi], \tag{3.24}$$

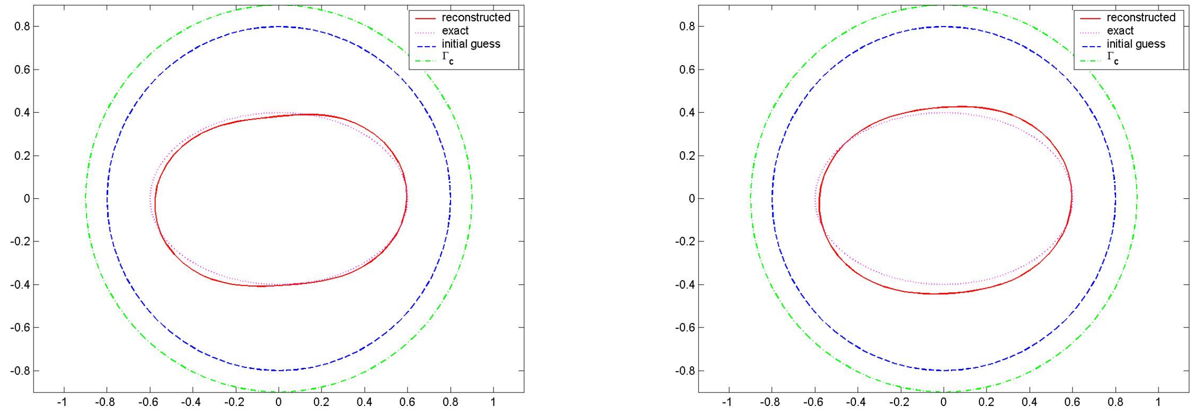

Figure 1. Reconstruction of the ellipse (3.24) for exact data after 10 iterations (left) and for 3% noise after 5 iterations (right).

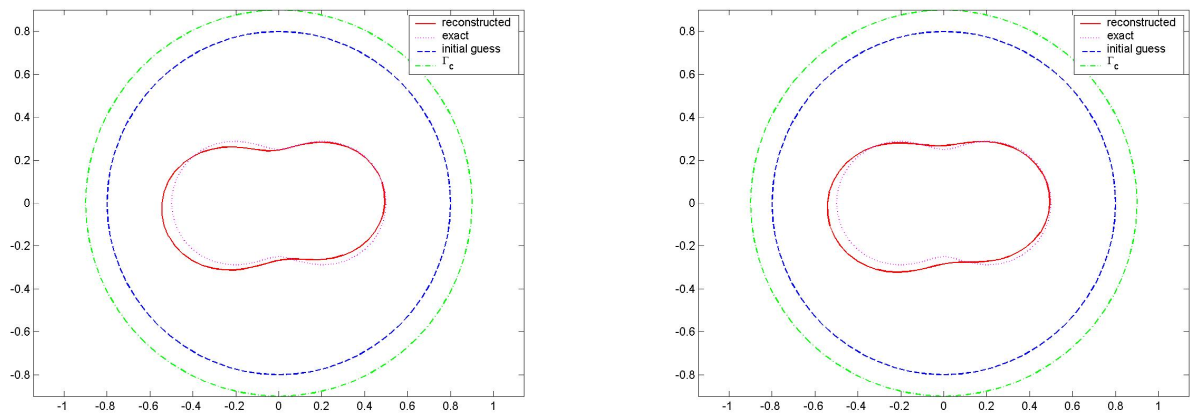

Figure 2. Reconstruction of the peanut (3.25) for exact data after 20 iterations (left) and for 3% noise after 10 iterations (right).

and the reconstruction is from one Cauchy pair for the Dirichlet data $f(z_m(t)) = 1$. For the second example Γ_c is chosen as a peanut with representation

$$z_c(t) = 0.5\sqrt{\cos^2 + 0.25\sin^2 t}\,(\cos t, \sin t), \quad 0 \le t \le 2\pi, \tag{3.25}$$

and the Dirichlet data by $f(z_m(t)) = 1 + \sin^2 t$. The results are shown in Fig. 2. Finally for a third example Γ_c is the kite (2.24) with Dirichlet data $f(z_m(t)) = 1 + \cos^2 t$ and the results are shown in Fig. 3.

3.3. Solution of the inverse impedance problem

Turning to the solution of the inverse impedance problem, we note that we can understand (3.12)–(3.13) also as basis for the inverse coefficients problem by considering

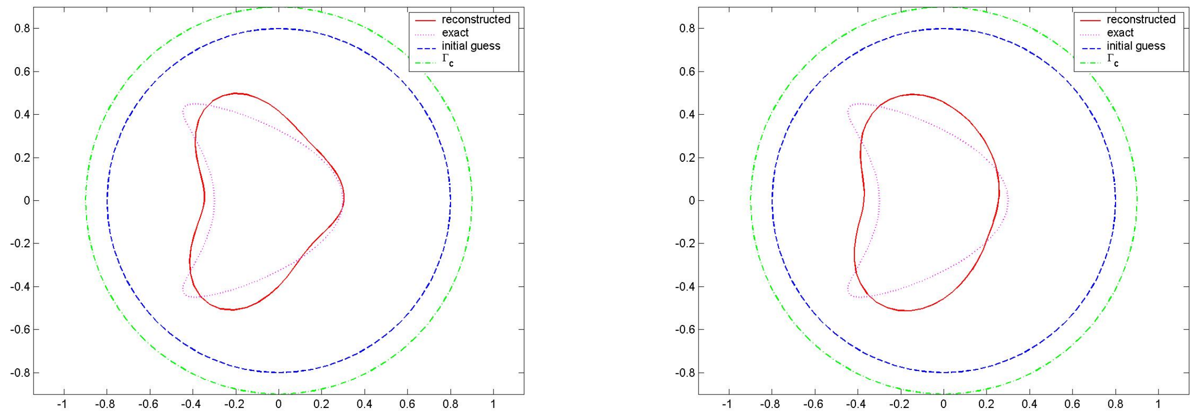

Figure 3. Reconstruction of the kite (2.24) for exact data after 30 iterations (left) and for 1% noise after 10 iterations (right).

μ, λ and η as the unknowns. For this we recall the definition (3.8) of χ and, given Γ_c and Cauchy pair f, g on Γ_m we rewrite the data and the field equation in the form

$$S_{cm}\chi - K_{cm}\eta = f + K_{mm}f - S_{mm}g \tag{3.26}$$

and

$$S_{cc}\chi - \eta - K_{cc}\eta = K_{mc}f - S_{mc}g \tag{3.27}$$

and interpret them as a system for the two unknowns η and χ, that is, for the boundary values $\eta = u|_{\Gamma_c}$ and the normal derivative $\chi = \partial_\nu u|_{\Gamma_c}$. After solving these ill-posed equations for η and χ by Tikhonov regularization we can obtain the impedance coefficients λ and μ from the generalized impedance boundary condition, i.e., from the relation (3.8).

The following theorem indicates that the prerequisites for the application of the Tikhonov regularization for the system (3.26)–(3.27) are satisfied in the L^2 setting, which is the appropriate setting for the ill-posed equations.

Theorem 3.4 *The operator* $\mathbb{L} : L^2(\Gamma_c) \times L^2(\Gamma_c) \to L^2(\Gamma_m) \times L^2(\Gamma_c)$ *given by*

$$\mathbb{L} := \begin{pmatrix} S_{cm} & -K_{cm} \\ \\ S_{cc} & -I - K_{cc} \end{pmatrix}$$

is injective and has dense range.

Proof. Assume that (χ, η) is in the null space of $\mathbb{L}$ and define

$$u(x) := \int_{\Gamma_c} \left\{ \Phi(x,y)\chi(y) - \frac{\partial \Phi(x,y)}{\partial \nu(y)}\,\eta(y) \right\} ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma_c.$$

Then $u|_{\Gamma_m} = S_{cm}\chi - K_{cm}\eta = 0$. Hence, based on the geometric Assumption G, proceeding as in the proof of Theorem 3.16 in [14] to cope with the logarithmic behavior of the single-layer potential at infinity it can be seen that $u = 0$ in the exterior of Γ_m and consequently, by analyticity, $u = 0$ in the exterior of Γ_c. By the mapping properties of the double- and single-layer operators the second equation $\eta + K_{cc}\eta = S_{cc}\chi$ implies that $\eta \in C(\Gamma_c)$ and consequently u is continuous in the closure of the interior of Γ_c with $u_- = 0$ on Γ_c. Therefore $u = 0$ in all of $\mathbb{R}^2$ and the jump relations and injectivity of S_{cc} imply that $\eta = \chi = 0$. Thus injectivity of $\mathbb{L}$ is established.

The adjoint operator $\mathbb{L} : L^2(\Gamma_m) \times L^2(\Gamma_c) \to L^2(\Gamma_c) \times L^2(\Gamma_c)$ of $\mathbb{L}$ is given by

$$\mathbb{L}^* := \begin{pmatrix} S_{mc} & S_{cc} \\ \\ -K'_{mc} & -I - K'_{cc} \end{pmatrix}.$$

Assume that (φ_m, φ_c) is in the null space of $\mathbb{L}^*$ and define

$$v(x) := \int_{\Gamma_m} \Phi(x,y)\varphi_m(y)\,ds(y) + \int_{\Gamma_c} \Phi(x,y)\varphi_c(y)\,ds(y) \quad x \in \mathbb{R}^2.$$

Then $v|_{\Gamma_c} = S_{mc}\varphi_m + S_{sc}\varphi_c = 0$ and therefore $v = 0$ in the interior of Γ_c. From the second equation $K'_{mc}\varphi_m + K'_{cc}\varphi_c + \varphi_c = 0$ as above we first conclude that $\varphi_c \in C(\Gamma_c)$ and then, from the jump relations, that $\partial_\nu v_- = 0$ on Γ_c. Hence, $\varphi_c = 0$ since the normal derivative of v vanishes on both sides of Γ_c. Now analyticity implies that $v = 0$ in the interior of Γ_m and consequently $S_{mm}\varphi_m = 0$ whence $\varphi_m = 0$ follows. Therefore $\mathbb{L}^*$ is injective and hence $\mathbb{L}$ has dense range. □

The uniqueness result of Theorem 3.1 suggests that we need three linearly independent Cauchy pairs to reconstruct λ and μ. Solving the ill-posed data equations (3.26)–(3.27) for three Cauchy pairs we obtain density pairs (η_1, χ_1), (η_2, χ_2) and (η_3, χ_3), that is, Cauchy pairs on Γ_c. From the latter it is possible to pursue the reconstruction steps from the proof of Theorem 3.1. However, numerical examples suggest that the reconstructions of μ and λ are more stable if the differential form of the boundary condition (2.15) is replaced by a non-local integrated version. More specifically, for a fixed $x_0 \in \Gamma_c$ and letting σ denote the arc length between x_0 and x, two integrations of (2.15) over Γ_c from x_0 to $x \in \Gamma_c$ yield

$$\mathcal{I}^2\left\{\lambda u + \frac{\partial u}{\partial \nu}\right\} - \mathcal{I}\left\{u\frac{d\mu}{ds}\right\} + u\mu - u(x_0)\mu(x_0) + C_1\sigma + C_2 = 0 \qquad \text{on } \Gamma_c \tag{3.28}$$

where $\mathcal{I}$ denotes integration over Γ_c from x_0 to x and C_1 and C_2 are constants to be determined.

We approximate the unknown (parameterized) impedance functions λ and μ by trigonometric polynomials of degree J. Given the three Cauchy pairs on Γ_c, obtained from solving (3.26) and (3.27) for the three data pairs, we collocate the parametrized version of (3.28) for each pair at the $2n$ collocation points $t_j = j\pi/n$, $j = 1, \dots, 2n$. The resulting linear system of $6n$ equations for the $2(2J+1)$ Fourier coefficients of λ_{approx} and μ_{approx} and the 6 integration constants in (3.28) for the three pairs then can be solved in the least squares sense.

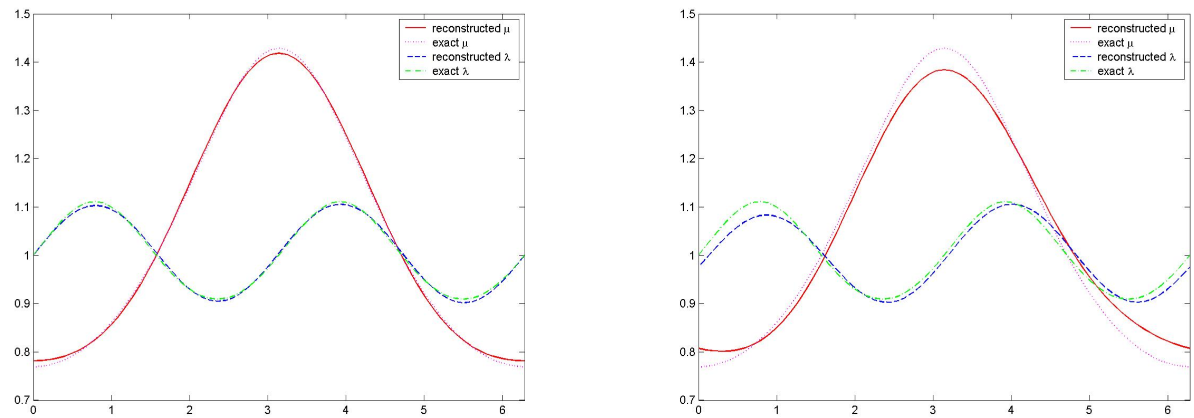

Figure 4. Reconstruction of impedance functions for the ellipse (3.24) for exact data (left) and 2% noise (right).

For all examples the impedance functions are given by (2.25) and the measurement curve Γ_m is the circle with radius 0.9 centered at the origin. The Dirichlet values for the three Cauchy data are chosen as

$$f_q(z_m(t)) = |z_m(t) - y_q|^2, \quad t \in [0, 2\pi],$$

with $y_q = (\cos 2q\pi/3, \sin 2q\pi/3)$ for $q = 1, 2, 3$. The number of quadrature points is 64 on each curve, i.e., $n = 32$. The degree of the polynomials for the approximation of the impedance function is always chosen as $J = 2$. For the perturbed data, random noise is added point wise and the relative error is with respect to the L^2 norm. The regularization parameters α for the L^2 Tikhonov regularization of the system (3.26)–(3.27) and β for the L^2 penalty on the Fourier coefficients and the integration constants in the least squares solution are chosen by trial and error.

In the figures the exact μ is given as dotted (magenta) curve and the reconstruction as full (red) curve, the exact λ is dashed-dotted (green) and the reconstruction dashed (blue).

Fig. 4 shows the reconstruction for the ellipse (3.24) as interior curve Γ_c with the regularization parameters $\alpha_{\text{exact}} = 10^{-10}$ and $\alpha_{\text{noise}} = 10^{-6}$ for the integral equation and $\beta_{\text{exact}} = 0$ and $\beta_{\text{noise}} = 10^{-5}$ in the least squares problem. The reconstructions in Fig. 5 are for the peanut (3.25) with $\alpha_{\text{exact}} = 10^{-10}$, $\alpha_{\text{noise}} = 10^{-5}$, $\beta_{\text{exact}} = 0$ and $\beta_{\text{noise}} = 10^{-2}$. Finally Fig. 6 shows the reconstructions for the kite (2.24) with $\alpha_{\text{exact}} = 10^{-10}$, $\alpha_{\text{noise}} = 10^{-5}$, $\beta_{\text{exact}} = 0$ and $\beta_{\text{noise}} = 10^{-1}$. In general, further numerical example indicate that the simultaneous reconstruction of both impedance functions is very sensitive to noise.

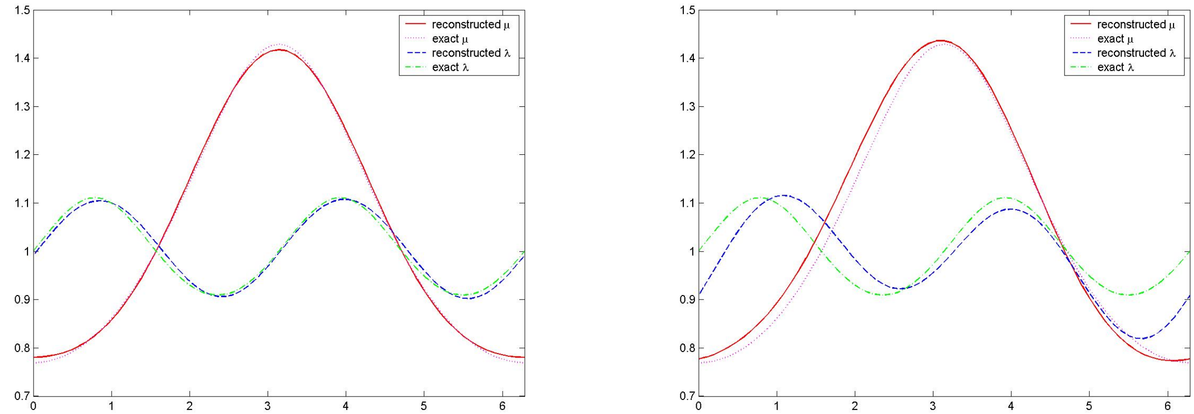

Figure 5. Reconstruction of impedance functions for the peanut (3.25) for exact data (left) and 1% noise (right).

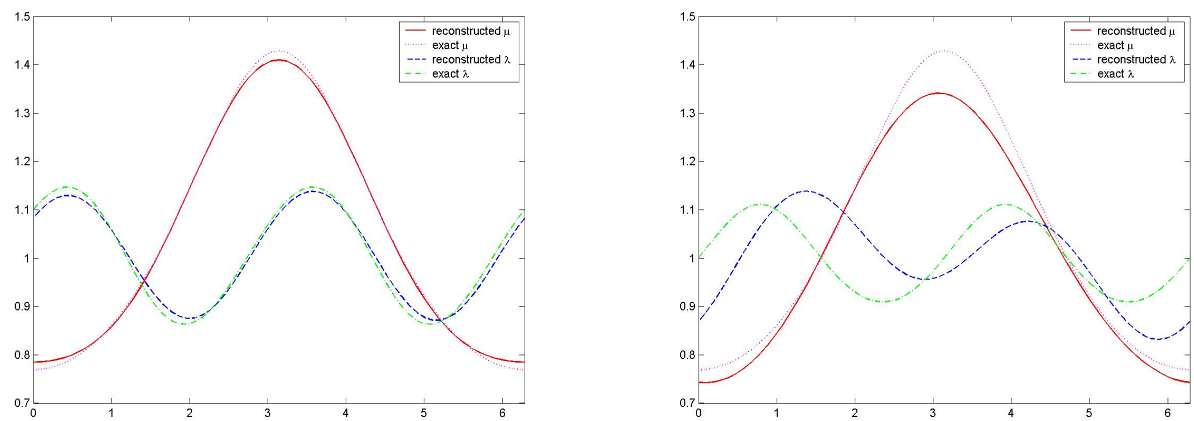

Figure 6. Reconstruction of impedance functions for the kite (2.24) for exact data (left) and 1% noise (right).

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