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THE IDENTIFICATION OF THIN DIELECTRIC OBJECTS FROM FAR FIELD OR NEAR FIELD SCATTERING DATA*

NOAM ZEEV† AND FIORALBA CAKONI†

Abstract. We consider the inverse scattering problem of determining the shape and the material properties of a thin dielectric infinite cylinder having an open arc as cross section from knowledge of the TM-polarized scattered electromagnetic field at a fixed frequency. We investigate two reconstruction approaches, namely the linear sampling method and the reciprocity gap functional method, using far field or near field data, respectively. Numerical examples are given showing the efficaciousness of our algorithms.

Key words. direct and inverse scattering, scattering from cracks, linear sampling method, electromagnetic scattering, reciprocity gap functional method, thin dielectric objects

AMS subject classifications. 35R30, 35Q60, 35J40, 78A25

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1. Introduction.

Important problems in nondestructive evaluation include the detection of flaws in materials in specific (typically thin) areas, as well as the determination of the integrity of thin coatings. We refer the reader to the special issue of Inverse Problems [22] for a detailed account on different approaches to these problems using electromagnetic waves. In particular, considerable work has been done on inversion schemes using eddy-current approximation measurements to detect the presence of thin anomalies [3], [13], [29]. In related work [4], [24], the authors use electrostatic and electromagnetic measurements, respectively, to detect the shape of a thin target. In addition to the shape, it is of course desirable to obtain information on the material properties of the target. In this paper we show the applicability of qualitative methods in inverse scattering [7] to these problems. In particular, we investigate the inverse problem of using far field or near field time harmonic electromagnetic measurements to determine the shape and information about the thickness and physical properties of a thin dielectric film embedded in a known inhomogeneous background. Such problems arise in the study of optical devices in communication networks [27] (typical structures of this type can be found in [14]) or in the detection of thin air pockets inside structures [29]. In this work we assume that the obstacle is a thin dielectric right cylinder whose properties depend only on the cross section of the cylinder and that the incident electromagnetic field is E-polarized. After factoring out the term $e^{-i\omega t}$, where ω is the fixed frequency, the only nonzero component u of the total electric field satisfies the Helmholtz equation

$$\Delta u + k^2 n(x) u = 0$$

in the exterior of the cylinder, where the complex valued function $n(x)$ is the index of refraction of the background medium which satisfies $\mathrm{Re}\, n > 0$ and $\mathrm{Im}\, n \geq 0$. Difficulties arise in computing the total field inside the thin dielectric obstacle due to

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†Department of Mathematical Sciences, University of Delaware, Newark, DE 19716-2553 (zeev@math.udel.edu, cakoni@math.udel.edu). The research of the second author was supported in part by the U.S. Air Force Office of Scientific Research under grant FA9550-05-1-0127.

the fact that the index of refraction and the thickness of the obstacle are of different scales. The direct scattering problem for a thin dielectric structure was studied in [1] and [27] where a perturbation approach was used to approximate the solution by solving a sequence of integral equations. Alternatively, based on asymptotic analysis with respect to the thickness of the obstacle (see [16]), a first approximate model of the wave propagation inside the obstacle is to replace the obstacle by an infinite cylinder having an open arc in $\mathbb{R}^2$ as its cross section and the interior field by an appropriate boundary condition on the arc. Both the perturbation method and the arc approximation model are based on asymptotic analysis of the exact model with respect to the thickness, and they therefore compute an approximation to the total field. The error analysis of the forward problem for the perturbation technique can be found in [1] and [27], and for the arc approximation model in [16].

Our analysis of the inverse problem uses the arc approximation model. Note that the physical properties and the thickness of the thin dielectric obstacle appear now as a boundary coefficient. We remark that this model is well suited to our inversion algorithm, especially since the noniterative inversion methods such as the linear sampling method and the reciprocity gap functional method are able to reconstruct boundary coefficients in addition to the support. More specifically, let h be the thickness and Γ the cross section of the mean surface of the dielectric medium. Assuming that the interior magnetic field is approximated up to $O(h)$ error, whereas the interior electric field is approximated up to $O(h^2)$ error, the following boundary conditions on the open arc Γ are obtained for the component u of the total electric field [15]:

$$\left[\frac{\partial u}{\partial \nu}\right] = 0 \qquad \text{and} \qquad [u] - i\lambda \frac{\partial u^+}{\partial \nu} = 0 \qquad \text{on } \Gamma,$$

where $u^{\pm}(x) = \lim_{h\to 0^+} u(x \pm h\nu)$ and $\frac{\partial u^{\pm}}{\partial \nu}(x) = \lim_{h\to 0^+} \nu \cdot \nabla u(x \pm h\nu)$ for $x \in \Gamma$, $[u] := u^+ - u^-$ and $\left[\frac{\partial u}{\partial \nu}\right] := \frac{\partial u^+}{\partial \nu} - \frac{\partial u^-}{\partial \nu}$ are the respective jumps across Γ, and the dimensionless positive valued function $\lambda > \lambda_0 > 0$ involves electric permittivity and magnetic permeability of the dielectric medium and the background as well as the thickness h and frequency ω. Note that the above condition can fail at the tips of the crack. Here we assume that $\Gamma \subset \mathbb{R}^2$ is a *simple piecewise smooth arc*, i.e., $\Gamma = \{\rho(s) : s \in [s_0, s_1]\}$, where the mapping $\rho : [s_0, s_1] \to \mathbb{R}^2$ is one-to-one, continuous, and piecewise smooth. The normal vector ν pointing to the right side of Γ is defined everywhere except at a finite number of points on Γ.

Hence we arrive at the following boundary value problem for the scattered field u^s due to an incident field u^i scattered by the crack Γ:

$$\Delta u^s + k^2 n(x) u^s = 0 \qquad \text{in } \mathbb{R}^2 \setminus \overline{\Gamma}, \tag{1.1}$$

$$\left[\frac{\partial (u^s + u^i)}{\partial \nu}\right] = 0 \qquad \text{on } \Gamma, \tag{1.2}$$

$$\left[(u^s + u^i)\right] - i\lambda \frac{\partial (u^s + u^i)^+}{\partial \nu} = 0 \qquad \text{on } \Gamma, \tag{1.3}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{1.4}$$

where the Sommerfeld radiation condition (1.4) is satisfied uniformly in $\hat{x} = x/|x|$ with $r = |x|$. Here we assume that, in general, the positive index of refraction $n(x) > 0$ for the background medium satisfies $n(x) = 1$ outside a large ball containing the crack and k is the wave number in the air. In this study the incident field can be a plane wave or the field generated by a point source, and this will become precise later.

The main concern of this paper is to solve the inverse problem of determining the shape Γ and some information on λ from measured far field or near field scattered data. In order to develop the mathematical tools to study the inverse problem, in the next section we investigate the well-posedness of the direct scattering problem (1.1)–(1.4). We apply a boundary integral equation method to obtain a Fredholm first kind integral equation on Γ for the scattered field. In section 3 we formulate and solve the inverse scattering problem using far field scattering data due to plane waves as incident fields. For simplicity in this section we assume that the crack is embedded in a homogeneous background; i.e., $n = 1$ everywhere. We apply the *linear sampling method*, which was first introduced in [6] for the case of an obstacle with empty interior, to determine the shape of the crack. After the reconstruction of Γ (without making a priori use of the boundary condition) we use the solution of the far field equation to reconstruct λ as well. For information on other solution methods for the inverse scattering problem for Dirichlet or Neumann cracks from far field data, we refer the reader to [2], [17], [18], and [21]. In section 4 we consider the case when the crack is embedded in a known inhomogeneous background and the data is the scattered field measured on a closed curve surrounding the crack due to a point source as incident field. We modify the reciprocity gap functional method which up to now has been developed only for obstacles with nonempty interior [8], [12]. The last section of this paper is dedicated to numerical implementation with examples of both algorithms for solving the inverse problem. We note that the solution of the inverse problem in both cases is based on solving an ill-posed linear equation whose right-hand side involves normal derivatives with respect to the unknown crack. We propose a new approach to deal with this difficulty which was left as an open question in [6].

2. The solution of the direct scattering problem. In order to formulate the above scattering problems more precisely we need to properly define the trace spaces on Γ. To this end we extend the arc Γ to an arbitrary piecewise smooth, simply connected, closed curve ∂D enclosing a bounded domain D such that the normal vector ν on Γ coincides with the outward normal vector on ∂D, which we again denote by ν. The classical reference for the trace spaces is [23], and the notation there is different from those in [25]. However, in this work we use the notation in [25], because this is our main reference for the potential theory needed here. If $H^1_{loc}(\mathbb{R}^2)$, $L^2(\partial D)$, $H^{\frac{1}{2}}(\partial D)$, and $H^{-\frac{1}{2}}(\partial D)$ denote the usual Sobolev spaces, we define the following spaces:

$$\begin{aligned}
L^2(\Gamma) &:= \{u|_\Gamma : u \in L^2(\partial D)\},\\
H^{\frac{1}{2}}(\Gamma) &:= \{u|_\Gamma : u \in H^{\frac{1}{2}}(\partial D)\},\\
\tilde{H}^{\frac{1}{2}}(\Gamma) &:= \{u \in H^{\frac{1}{2}}(\Gamma) : \operatorname{supp} u \subseteq \overline{\Gamma}\}.
\end{aligned}$$

In other words, $\tilde{H}^{\frac{1}{2}}(\Gamma)$ contains functions $u \in H^{\frac{1}{2}}(\Gamma)$ such that their extension by zero to the whole boundary ∂D is in $H^{\frac{1}{2}}(\partial D)$ (Theorem 3.33 in [25]). (For the reader's convenience we remark that $\tilde{H}^{\frac{1}{2}}(\Gamma)$ coincides with the space $H^{\frac{1}{2}}_{00}(\Gamma)$ introduced by Lions and Magenes (see [23, p. 66]).) Now we denote by $H^{-\frac{1}{2}}(\Gamma)$ the dual space of $\tilde{H}^{\frac{1}{2}}(\Gamma)$ and by $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ the dual space of $H^{\frac{1}{2}}(\Gamma)$. Hence we have the chain

$$\mathcal{D}(\Gamma) \subset \tilde{H}^{\frac{1}{2}}(\Gamma) \subset H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma) \subset \tilde{H}^{-\frac{1}{2}}(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma) \subset \mathcal{D}'(\Gamma),$$

where $\mathcal{D}(\Gamma) := C_0^\infty(\Gamma)$. We note that $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ can also be identified with $H_{\overline{\Gamma}}^{-\frac{1}{2}}(\partial D) := \{u \in H^{-\frac{1}{2}}(\partial D) : \ \operatorname{supp} u \subset \overline{\Gamma}\}$ (see Theorem 3.29 in [25]).

The scattering problem (1.1)–(1.4) is a particular case of the following boundary value problem: Let $n(x)$ be a piecewise smooth complex valued function with piecewise continuous jump discontinuities such that $\operatorname{Re} n > 0$, $\operatorname{Im} n \geq 0$, and $n(x) = 1$ outside a large enough ball, whereas λ is a piecewise smooth function on Γ such that $\lambda(x) > \lambda_0 > 0$. Given $f \in H^{-\frac{1}{2}}(\Gamma)$ and $h \in H^{-\frac{1}{2}}(\Gamma)$, find $v \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{\Gamma})$ satisfying

$$\Delta v + k^2 n(x) v = 0 \qquad \text{in } \mathbb{R}^2 \setminus \overline{\Gamma}, \tag{2.1}$$

$$\left[\frac{\partial v}{\partial \nu}\right] = f \qquad \text{on } \Gamma, \tag{2.2}$$

$$[v] - i\lambda \frac{\partial v^+}{\partial \nu} = h \qquad \text{on } \Gamma, \tag{2.3}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial v}{\partial r} - ikv\right) = 0. \tag{2.4}$$

Theorem 2.1. *The problem* (2.1)–(2.4) *has at most one solution.*

Proof. Denote by B_R a sufficiently large ball with radius R containing $\overline{D}$ and by ∂B_R its boundary. Let v be a solution of (2.1)–(2.4) with $f = h = 0$. Obviously $v \in H^1(B_R \setminus \overline{D}) \cup H^1(D)$ satisfies the Helmholtz equation in $B_R \setminus \overline{D}$ and D and the following transmission conditions on the complementary part $\partial D \setminus \overline{\Gamma}$ of ∂D:

$$v^+ = v^- \qquad \text{and} \qquad \frac{\partial v^+}{\partial \nu} = \frac{\partial v^-}{\partial \nu} \qquad \text{on } \partial D \setminus \overline{\Gamma}, \tag{2.5}$$

where the $+$ denotes the limit approaching ∂D from inside D and $-$ the limit approaching ∂D from outside of D. An application of Green's formula for u and $\overline{u}$ in D and $B_R \setminus \overline{D}$ and using the transmission conditions (2.5) yields

$$\begin{aligned}\int_{\partial B_R} v\frac{\partial \bar{v}}{\partial \nu} dx = &\int_{B_R\setminus\bar{D}} |\nabla v|^2\,dx + \int_D |\nabla v|^2\,dx - k^2 \int_{B_R\setminus\bar{D}} \overline{n}\,|v|^2\,dx \\ &- k^2\int_D \overline{n}\,|v|^2\,dx + \int_\Gamma [v]\,\frac{\partial \bar{v}}{\partial \nu} dx.\end{aligned}$$

Using the boundary conditions (2.2)–(2.3), we now obtain

$$\begin{aligned}\int_{\partial B_R} v\frac{\partial \bar{v}}{\partial \nu} dx = &\int_{B_R\setminus\bar{D}} |\nabla v|^2\,dx + \int_D |\nabla v|^2\,dx - k^2 \int_{B_R\setminus\bar{D}} \overline{n}\,|v|^2\,dx \\ &- k^2\int_D \overline{n}\,|v|^2\,dx + i\int_\Gamma \lambda \left|\frac{\partial v}{\partial \nu}\right|^2 dx.\end{aligned} \tag{2.6}$$

Since $\lambda > 0$ and $\operatorname{Im} \overline{n} \leq 0$, we conclude that

$$\operatorname{Im}\left(\int_{\partial B_R} v\frac{\partial \bar{v}}{\partial \nu} dx\right) \geq 0,$$

whence from [10, Theorem 2.12] and a unique continuation argument we obtain that $v = 0$ in $\mathbb{R}^2 \setminus \overline{\Gamma}$. ∎

Theorem 2.2. *The problem* (2.1)–(2.4) *has a unique solution* v *which satisfies*

$$\|v\|_{H^1(B_R\setminus\overline{\Gamma})} \leq C\left(\|f\|_{H^{-\frac{1}{2}}(\Gamma)} + \|h\|_{H^{-\frac{1}{2}}(\Gamma)}\right), \quad x \in \mathbb{R}^2 \setminus \overline{\Gamma}, \tag{2.7}$$

where the positive constant C depends on R but not on f and h.

Proof. First we note that if $v \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{\Gamma})$ is a solution to (2.1)–(2.4), then $[v] \in H^{\frac{1}{2}}(\partial D)$ and $\left[\frac{\partial v}{\partial \nu}\right] \in H^{-\frac{1}{2}}(\partial D)$. Now by local regularity for solutions of the Helmholtz equation we have that $v \in C^\infty$ away from Γ, whence $[v] = \left[\frac{\partial v}{\partial \nu}\right] = 0$ on $\partial D \setminus \overline{\Gamma}$. Therefore $[v] \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ and $\left[\frac{\partial v}{\partial \nu}\right] \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$. Let $\mathbb{G}(x,y)$ be the radiating Green function of the background medium that satisfies

$$\Delta \mathbb{G}(x,\, y) + k^2 n(x) \mathbb{G}(x,\, y) = \delta(x-y). \tag{2.8}$$

From the Green representation formula (see [25]) we have

$$v(x) = \begin{cases} \displaystyle\int_{\partial D} \frac{\partial v^+(y)}{\partial \nu_y} \mathbb{G}(x,y) ds_y - \int_{\partial D} v^+(y) \frac{\partial}{\partial \nu_y} \mathbb{G}(x,y) ds_y, & x \in D, \\ \displaystyle -\int_{\partial D} \frac{\partial v^-(y)}{\partial \nu_y} \mathbb{G}(x,y) ds_y + \int_{\partial D} v^-(y) \frac{\partial}{\partial \nu_y} \mathbb{G}(x,y) ds_y, & x \in \mathbb{R}^2 \setminus \overline{D}, \end{cases} \tag{2.9}$$

where D is the region bounded by the extension ∂D of Γ, and the $+$ sign denotes the limit approaching ∂D from inside D whereas $-$ denotes the limit approaching ∂D from outside of D. Using the jump relations of the single- and double-layer potentials across the boundary ∂D [25], eliminating the integrals over $\partial D \setminus \overline{\Gamma}$, and using the boundary conditions (2.2)–(2.3), we obtain that the jump $[v]$ satisfies

$$\left(\frac{i}{\lambda} I + T_\Gamma \right) [v] = \frac{i}{\lambda} h + \left(K'_\Gamma - \frac{I}{2} \right) f, \tag{2.10}$$

where the operators $K'_\Gamma : \tilde{H}^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and $T_\Gamma : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are defined by

$$(K'_\Gamma \psi)(x) := \int_\Gamma \psi(y) \frac{\partial}{\partial \nu_x} \mathbb{G}(x,y)\, ds_y \quad \text{for } x \in \Gamma,$$

$$(T_\Gamma \psi)(x) := \frac{\partial}{\partial \nu_x} \int_\Gamma \psi(y) \frac{\partial}{\partial \nu_y} \mathbb{G}(x,y)\, ds_y \quad \text{for } x \in \Gamma,$$

respectively. If (2.10) can be solved for $[v]$, then from the boundary conditions we know $\partial v^+/\partial \nu$ and $\partial v^-/\partial \nu$. Furthermore, it is easy to see that

$$\frac{1}{2}(v^+ + v^-) = -S_\Gamma \left[\frac{\partial u}{\partial \nu} \right] + K_\Gamma [u] \quad \text{on } \Gamma, \tag{2.11}$$

where now $S_\Gamma : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is defined by

$$(S_\Gamma \psi)(x) := \int_\Gamma \psi(y)\, \mathbb{G}(x,y)\, ds_y \quad \text{for } x \in \Gamma.$$

Hence the knowledge of $[v] = v^+ - v^-$ and (2.11) determines v^+ and v^- on Γ and therefore the solution v from the Green representation formula (2.9). To solve (2.10) we observe that $I : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a compact operator due to Rellich's embedding theorem and that T can be written as a sum of a coercive operator and a compact operator (see Theorems 7.8 and 7.10 in [25]). Hence, since $\lambda(x) > \lambda_0 > 0$, we conclude that $(\frac{i}{\lambda}I + T_\Gamma) : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a Fredholm operator of index zero. Therefore, it suffices to prove only the injectivity of $\frac{i}{\lambda}I + T_\Gamma$. To this end let $\xi \in \tilde{H}^{1/2}(\Gamma)$ satisfy

$$\left(\frac{i}{\lambda}I + T_\Gamma\right)\xi = 0,$$

and define the following potential:

$$w(x) = \int_\Gamma \xi(y)\frac{\partial}{\partial \nu_y}\mathbb{G}(x,y)ds_y \quad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Gamma}. \tag{2.12}$$

Approaching Γ and using the jump relations for the double layer potential, we obtain

$$\frac{\partial w^+}{\partial \nu} = \frac{\partial}{\partial \nu_x}\int_\Gamma \xi(y)\frac{\partial}{\partial \nu_y}\mathbb{G}(x,y)ds_y = T_\Gamma(\xi),$$

$$[w] = \xi \qquad \text{and} \qquad \left[\frac{\partial w}{\partial \nu}\right] = 0.$$

Hence we have

$$[w] - i\lambda\frac{\partial w^+}{\partial \nu} = \xi - i\lambda T_\Gamma \xi = -i\lambda\left(\frac{i}{\lambda}I + T_\Gamma\right)\xi = 0.$$

Therefore w defined by (2.12) satisfies (2.1)–(2.4) with $f = h = 0$, and from Theorem 2.1, $w = 0$ in $\mathbb{R}^2 \setminus \overline{\Gamma}$, which finally implies $[w] = \xi = 0$. This ends the proof. ∎

3. Reconstruction of the crack from far field data. In this section we assume that the thin dielectric film is embedded in a homogeneous background and the measurements are made from far away. In addition, we assume that the incident field is a time harmonic plane wave given by $u^i := e^{ikx\cdot d}$ for $x \in \mathbb{R}^2$, where the unit vector $d \in S := \{x \in \mathbb{R}^2 : |x| = 1\}$ is the incident direction. In this setting the scattered field u^s satisfies (2.1)–(2.4) with $n(x) = 1$, $f := -[\frac{\partial e^{ikx\cdot d}}{\partial \nu}] = 0$, and $h := -[e^{ikx\cdot d}] + i\lambda\frac{\partial e^{ikx\cdot d}}{\partial \nu} = i\lambda\frac{\partial e^{ikx\cdot d}}{\partial \nu}$. Note that $\mathbb{G}(x,y)$ is now the fundamental solution of the Helmholtz equation $\Phi(x,y) := \frac{i}{4}H_0^{(1)}(k|x-y|)$ with $H_0^{(1)}$ being the Hankel function of the first kind of order zero. It is shown in [10] that the scattered field, which now depends also on d, has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}}u_\infty(\hat{x},d) + O(r^{-3/2}), \tag{3.1}$$

where u_∞ is the *far field pattern* of the scattered wave u^s, $\hat{x} = x/|x|$, and $r = |x|$.

The *inverse scattering problem* that we will consider in this section of our paper is to determine Γ and λ from a knowledge of $u_\infty(\hat{x},d)$ for $\hat{x}$ and d on the unit circle S. Using [5], one can easily generalize the following analysis for the case of limited aperture data, i.e., for $\hat{x}, d \in S_0 \subset S$. For the unique determination of Γ and λ from the above data, see [28] (see also [7], [10]). We will use the *linear sampling method* to solve this inverse problem [6]. To this end, we define the *far field operator* $F : L^2(S) \to L^2(S)$ by

$$(Fg)(\hat{x}) := \int_S u_\infty(\hat{x},d)g(d)\,ds(d) \tag{3.2}$$

and consider the *far field equation*

$$Fg = \Phi^e_\infty, \tag{3.3}$$

where Φ^e_∞ is the far field pattern of a suitable solution (to be defined later) to the scattering problem. The aim is to characterize the crack Γ by the behavior of an approximate solution g of the far field equation (3.3). We recall that a Herglotz wave function is a solution of the Helmholtz equation in $\mathbb{R}^2$ of the form

$$v_g(x) := \int_S g(d) e^{ikx\cdot d} ds(d), \tag{3.4}$$

where $g \in L^2(S)$ is the *kernel* of v_g. By superposition we have the following relation:

$$(Fg) = \mathcal{B}(i\lambda \mathcal{H} g),$$

where $\mathcal{H} : L^2(S) \to H^{-\frac{1}{2}}(\Gamma)$ is defined by

$$\mathcal{H} g := \frac{\partial v_g}{\partial \nu} \tag{3.5}$$

and $\mathcal{B} : H^{-\frac{1}{2}}(\Gamma) \to L^2(S)$ takes $h \in H^{-\frac{1}{2}}(\Gamma)$ to the far field pattern u_∞ of the solution to (2.1)–(2.4) with $n(x) = 1$, $f := 0$, and h. For $\beta \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ we construct the double layer potential

$$\mathcal{D}(\beta)(x) := \int_\Gamma \beta(y) \frac{\partial}{\partial \nu_y} \Phi(x, y) ds(y),$$

which has as far field pattern $\gamma \mathcal{F} \beta$, where

$$\mathcal{F}\beta := \int_\Gamma \beta(y) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu_y} ds(y)$$

and $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$. The calculation

$$\int_S g(\hat{y}) \int_\Gamma \beta(x) \frac{\partial}{\partial \nu} e^{-ikx\cdot\hat{y}} ds(x) d(\hat{y}) = \int_\Gamma \beta(x) \int_S g(\hat{y}) \frac{\partial}{\partial \nu} e^{-ikx\cdot\hat{y}} ds(\hat{y}) ds(x)$$

shows that $\mathcal{H}g(-\hat{y})$ is the transpose of $\mathcal{F}\beta$ in the duality pairing between $\tilde{H}^{\frac{1}{2}}(\Gamma)$, $H^{-\frac{1}{2}}(\Gamma)$ and $L^2(S)$, $L^2(S)$, respectively, where $\mathcal{H} : L^2(S) \to H^{-\frac{1}{2}}(\Gamma)$ and $\mathcal{F} : \tilde{H}^{\frac{1}{2}}(\Gamma) \to L^2(S)$.

LEMMA 3.1. *The compact operators $\mathcal{F} : \tilde{H}^{\frac{1}{2}}(\Gamma) \to L^2(S)$ and $\mathcal{H} : L^2(S) \to H^{-\frac{1}{2}}(\Gamma)$ are injective and have dense range, provided that there does not exist a nontrivial Herglotz wave function such that its normal derivative vanishes on* Γ.

Proof. From the above and Lemma 2.10 in [25] it suffices to show that both $\mathcal{F}$ and $\mathcal{H}$ are injective operators. To this end, if $\mathcal{F}(\beta) = 0$, then $\mathcal{D}\beta = 0$ in $\mathbb{R}^2 \setminus \overline{\Gamma}$, which implies $\beta := -[\mathcal{D}\beta] = 0$ from the jump relation. Next, the assumption of the theorem guarantees that $\mathcal{H}$ is also injective. Note that injectivity of $\mathcal{F}$ and consequently the denseness of the range of $\mathcal{H}$ do not require the assumption stated in the lemma. ∎

We remark that, from the above, the far field operator fails to be injective and have dense range if Γ is such that there exists a nontrivial Herglotz wave function with vanishing normal derivative on Γ. An instance of this situation is if Γ is part of a circle of radius r such that kr is a zero of J_1 Bessel function. It is interesting to notice that in the case of the exact model (namely, the obstacle D is a region with nonempty interior), the far field operator fails to be injective and have dense range if the wave number k is a transmission eigenvalue for D with eigenfunction being a Herglotz function (for details on transmission eigenvalues, see [7]). (Note that the injectivity and the denseness of the range of the far field operator are typically needed in most of the inversion schemes in order to use Tikhonov regularization technique.)

From the above analysis and the jump relations applied to the double layer potential $\mathcal{D}$ we see that

$$\mathcal{F}\beta = \gamma^{-1}\mathcal{B}\left(I - i\lambda T_\Gamma\right)\beta,$$

which implies the following factorization of the far field operator:

$$Fg = \gamma\mathcal{F}\left(I - i\lambda T_\Gamma\right)^{-1}\left(i\lambda\mathcal{H}g\right), \qquad g \in L^2(S). \tag{3.6}$$

LEMMA 3.2. *For any simple piecewise smooth arc L and $\beta_L \in \tilde{H}^{\frac{1}{2}}(L)$ we define $\Phi_\infty^L \in L^2(S)$ by*

$$\Phi_\infty^L(\hat{x}) := \int_L \beta_L(y)\frac{\partial}{\partial\nu_y}e^{-ik\hat{x}\cdot y}ds_y. \tag{3.7}$$

Then $\Phi_\infty^L(\hat{x}) \in \mathrm{Range}(\mathcal{F})$ if and only if $L \subset \Gamma$.

Proof. First assume that $L \subset \Gamma$. Then since $\tilde{H}^{\frac{1}{2}}(L) \subset \tilde{H}^{\frac{1}{2}}(\Gamma)$ it follows directly from the definition of $\mathcal{F}$ that $\Phi_\infty^L(\hat{x}) \in \mathrm{Range}(\mathcal{F})$.

Now let $L \not\subset \Gamma$ and assume, on the contrary, that $\Phi_\infty^L(\hat{x}) \in \mathrm{Range}(\mathcal{F})$; i.e., there exists $\beta \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ such that

$$\Phi_\infty^L(\hat{x}) = \int_\Gamma \beta(y)\frac{\partial}{\partial\nu_y}e^{-ik\hat{x}\cdot y}ds_y.$$

Hence by Rellich's lemma and the unique continuation principle we have that the potentials

$$\Phi^L(x) = \int_L \beta_L(y)\frac{\partial}{\partial\nu_y}\Phi(x,y)ds_y \qquad \text{and} \qquad \mathcal{D}(x) = \int_\Gamma \beta(y)\frac{\partial}{\partial\nu_y}\Phi(x,y)ds_y$$

coincide in $\mathbb{R}^2 \setminus (\overline{\Gamma} \cup \overline{L})$. Now let $x_0 \in L$, $x_0 \notin \Gamma$, and let $B_\epsilon(x_0)$ be a small ball with center at x_0 such that $B_\epsilon(x_0) \cap \Gamma = \emptyset$. Hence $\mathcal{D}$ is analytic in $B_\epsilon(x_0)$, while Φ^L has a singularity at x_0, which is a contradiction. This proves the lemma. ∎

Now using Lemma 3.2, the regularization theory for $\mathcal{F}\beta = \Phi_\infty^L$, and the fact that $(i\lambda T_\Gamma - I)\,\beta$ can be approximated by $i\lambda\mathcal{H}g$ in $H^{-\frac{1}{2}}(\Gamma)$, we have the following result for the solution of the *far field equation*:

$$(Fg)(\hat{x}) = \gamma\Phi_\infty^L(\hat{x}), \qquad \hat{x} \in S, \tag{3.8}$$

which is the basis of the linear sampling method for reconstructing Γ (cf. Theorem 8.45 of [6]).

THEOREM 3.3. *Assume that Γ is a simple piecewise smooth arc and that there does not exist any nontrivial Herglotz wave function having vanishing normal derivative on Γ. Then if F is the far field operator corresponding to* (2.1)–(2.4) *with* $n(x) = 1$,

$f := 0$, and $h := i\lambda \frac{\partial e^{ikx\cdot d}}{\partial \nu}$, the following are true:

1. *If $L \subset \Gamma$, then for every $\epsilon > 0$ there exists a solution $g_\epsilon^L \in L^2(S)$ of the inequality*

$$\|Fg_\epsilon^L - \gamma \Phi_\infty^L\|_{L^2(S)} < \epsilon$$

such that $\mathcal{H}_{g_\epsilon^L}$ converges to a well-defined function in $H^{-\frac{1}{2}}(\Gamma)$.

2. *If $L \not\subset \Gamma$, then for every $\epsilon > 0$ all $g_\epsilon^L \in L^2(S)$ satisfying*

$$\|Fg_\epsilon^L - \gamma \Phi_\infty^L\|_{L^2(S)} < \epsilon$$

are such that

$$\lim_{\epsilon \to 0} \|g_\epsilon^L\|_{L^2(S)} = \infty \quad \text{and} \quad \lim_{\epsilon \to 0} \|\mathcal{H}_{g_\epsilon^L}\|_{H^{-\frac{1}{2}}(\Gamma)} = \infty,$$

where $\mathcal{H}_{g_\epsilon^L}$ is defined by (3.5) with $v_{g_\epsilon^L}$ being the Herglotz wave function with kernel g_ϵ^L.

Remark 3.1. From the above analysis we notice that, for $L \subset \Gamma$, $i\lambda \mathcal{H}_{g_\epsilon^L}$ approximates $(I - i\lambda T_\Gamma)\beta_L$, where g_ϵ^L is the approximate solution of (3.8). In particular, in the $H^{-\frac{1}{2}}(\Gamma)$-norm we have

$$\frac{\partial v_{g_\epsilon^L}}{\partial \nu} \approx \frac{i\beta_L}{\lambda} - T_\Gamma \beta_L, \qquad L \subset \Gamma \quad \text{and} \quad \beta_L \in C_0^\infty(L), \tag{3.9}$$

which can be used to recover λ, provided that (a reconstruction of) Γ is now known (e.g., by using the linear sampling method based on Theorem 3.3).

4. Reconstruction of the crack from near field data. We now assume that the dielectric thin film is embedded in a known inhomogeneous medium with index of refraction $n(x)$ that satisfies the assumptions stated in section 2. The incident field is a point source given by $\Phi(x, x_0, k_s) := \frac{i}{4} H_0^{(1)}(k_s|x - x_0|)$ located at a point x_0 outside a bounded region Ω surrounding the crack and $k_s^2 = k^2 n(x_0)$ (see Figure 4.1). In this case the scattered field u^s satisfies (2.1)–(2.4) with $f := -\big[\frac{\partial \Phi(\cdot, x_0, k_s)}{\partial \nu}\big] = 0$ and $h := -\left[\Phi(\cdot, x_0, k_s)\right] + i\lambda \frac{\partial \Phi(\cdot, x_0, k_s)}{\partial \nu} = i\lambda \frac{\partial \Phi(\cdot, x_0, k_s)}{\partial \nu}$. Note that u^s is the sum of the scattered field due to the crack and the scattered field due to the medium. Let $u(\cdot, x_0) = u^s(\cdot, x_0) + \Phi(\cdot, x_0, k_s)$ denote the total field, let Λ be a closed curve containing Ω, and suppose that in a neighborhood of Λ the index of refraction $n(x)$ is constant and $k^2 n = k_s^2$. Note that Λ can be part of a closed analytic curve, and by an analycity argument the following analysis holds true as well. For technical reasons we write $u(\cdot, x_0) = u_c^s(\cdot, x_0) + \mathbb{G}(\cdot, x_0)$, where $u_c^s(\cdot, x_0)$ is the scattered field due to the crack and $\mathbb{G}(\cdot, x_0)$ is the total field due to the medium or the background Green's function satisfying (2.8).

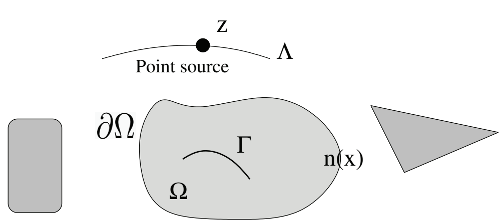


FIG. 4.1. *Geometry for the reciprocity gap functional method.*

The *inverse scattering problem* we are interested in now is to determine the crack Γ from a knowledge of $u(\cdot, x_0)$ and $\frac{\partial u(\cdot,x_0)}{\partial \nu}$ on the boundary $\partial\Omega$ for all point sources located at any $x_0 \in \Lambda$. These measurements in the case of Maxwell's equations correspond to measuring the tangential components of the electric and magnetic fields. To solve this inverse scattering problem we adapt the *reciprocity gap functional* method first introduced [12] for obstacles with nonempty interior. To this end, let

$$\mathbb{H}(\Omega) := \left\{ w \in H^1(\Omega) : \ \Delta w + k^2 n(x) w = 0 \quad \text{in } \Omega \right\}.$$

In a similar way as in Lemma 3.1 (see also the proof of Lemma 4.4 in [8]) it can be shown that the set

$$\left\{ (\mathcal{S}\varphi)(y) := \int_\Sigma \varphi(x)\Phi(x,y)\, ds_x \quad \text{for } \varphi \in L^2(\Sigma) \right\} \tag{4.1}$$

is dense in $\mathbb{H}(\Omega)$, where Σ is a open curve outside Ω and $\Phi(x, y)$ is the radiating fundamental solution of $\Delta u + k^2 \tilde{n}(x)u = 0$, where $\tilde{n}(x) = n(x)$ for $x \in \Omega$ and $\tilde{n}(x) = 1$ for $x \in \mathbb{R}^2 \setminus \Omega$ (or any other convenient extension). In particular, if $n(x) = n_0$ is constant in Ω, then $\Phi(x, y)$ is simply $\frac{i}{4} H_0^{(1)}(k\sqrt{n_0}|x - x_0|)$.

We define the *reciprocity gap operator* $\mathcal{R} : \mathbb{H}(\Omega) \to L^2(\Lambda)$ by

$$\mathcal{R}(w)(x_0) = \int_{\partial\Omega} \left(u(\cdot, x_0)\frac{\partial w}{\partial \nu} - w\frac{\partial u(\cdot, x_0)}{\partial \nu} \right) ds, \qquad x_0 \in \Lambda,\ w \in \mathbb{H}(\Omega). \tag{4.2}$$

THEOREM 4.1. *The compact operator $\mathcal{R} : \mathbb{H}(\Omega) \to L^2(\Lambda)$ is injective and has dense range, provided that there does not exist any $w \in \mathbb{H}(\Omega)$ such that $\partial w/\partial\nu = 0$ on Γ.*

Proof. If $\mathcal{R}(w) = 0$, applying Green's second identity and using the zero boundary condition for $u(\cdot, x_0)$, it is easy to see that

$$0 = \mathcal{R}(w)(x_0) = -i \int_\Gamma \lambda \frac{\partial u(\cdot, x_0)}{\partial \nu}\frac{\partial w}{\partial \nu} ds.$$

Noting that from the boundary condition $\lambda\partial u(\cdot, x_0)/\partial\nu \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ it suffices to show that $\lambda\partial u(\cdot, x_0)/\partial\nu$ for $x_0 \in \Lambda$ are dense in $\tilde{H}^{\frac{1}{2}}(\Gamma)$. Indeed, this fact implies that $\partial w/\partial\nu = 0$, which contradicts the assumption of the theorem. To prove the denseness property let $\psi \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$0 = \int_\Gamma \lambda(x)\psi(x)\frac{\partial u(x, x_0)}{\partial \nu}\, ds_x \qquad \text{for all } x_0 \in \Lambda,$$

where the integral is understood in the sense of duality pairing, and let $w \in H^1(\mathbb{R}^2 \setminus \overline{\Gamma})$ be the solution of (2.1)–(2.4) with $f := 0$ and $h := \lambda\psi$. Applying Green's second identity to w and u_c^s (note that both satisfy $\Delta u + k^2 nu = 0$ outside Γ) and the boundary condition for $u(\cdot, x_0)$, we obtain that

$$\begin{aligned}
0 &= \int_\Gamma \lambda(x)\psi(x)\frac{\partial u(x,x_0)}{\partial \nu}\,ds_x = \int_\Gamma \left([w] - i\lambda\frac{\partial w}{\partial \nu}\right)\frac{\partial\left(u_c^s + \mathbb{G}(\cdot,x_0)\right)}{\partial \nu}\,ds \\
&= \int_\Gamma \left([u_c^s] - i\lambda\frac{\partial u_c^s}{\partial \nu}\right)\frac{\partial w}{\partial \nu}\,ds + \int_\Gamma \left([w] - i\lambda\frac{\partial w}{\partial \nu}\right)\frac{\partial \mathbb{G}(\cdot,x_0)}{\partial \nu}\,ds \\
&= -\int_\Gamma \left([\mathbb{G}(\cdot,x_0)] + i\lambda\frac{\partial \mathbb{G}(\cdot,x_0)}{\partial \nu}\right)\frac{\partial w}{\partial \nu}\,ds + \int_\Gamma \left([w] - i\lambda\frac{\partial w}{\partial \nu}\right)\frac{\partial \mathbb{G}(\cdot,x_0)}{\partial \nu}\,ds \\
&= \int_\Gamma [w]\frac{\partial \mathbb{G}(\cdot,x_0)}{\partial \nu}\,ds \qquad \text{for all } x_0 \in \Lambda.
\end{aligned}$$

Hence $P(x_0) = \int_\Gamma [w(x)]\frac{\partial \mathbb{G}(x,x_0)}{\partial \nu}\,ds_x$ is a radiating solution as a function of x_0 which vanishes on Λ, which implies that $P(x_0) = 0$ outside the domain bounded by Λ and consequently in $\mathbb{R}^2 \setminus \overline{\Gamma}$ by unique continuation. Hence, using the jump relations, we have that $[w] = 0$ on Γ, which together with $[\partial w/\partial \nu] = 0$ on Γ implies $w = 0$ and consequently $\psi = 0$.

Next we show that $\mathcal{R}$ has dense range. Let $\alpha \in L^2(\Lambda)$ be such that $(\mathcal{R}w, \overline{\alpha})_{L^2(\Lambda)} = 0$ for all $w \in \mathbb{H}(\Omega)$. The bilinearity of $\mathcal{R}$ implies that

$$(\mathcal{R}w, \overline{\alpha})_{L^2(\Lambda)} = \int_{\partial\Omega}\left(Q\frac{\partial w}{\partial \nu} - w\frac{\partial Q}{\partial \nu}\right)ds = -i\int_\Gamma \lambda\frac{\partial Q}{\partial \nu}\frac{\partial w}{\partial \nu}ds = 0 \tag{4.3}$$

for all $w \in \mathbb{H}(\Omega)$, where

$$Q(x) = \int_\Lambda \alpha(x_0)u(x,x_0)\,ds(x_0) = \int_\Lambda \alpha(x_0)u_c^s(x,x_0)\,ds(x_0) + \int_\Lambda \alpha(x_0)\mathbb{G}(x,x_0)\,ds(x_0).$$

Hence (4.3) implies that $\partial Q/\partial \nu = 0$ on Γ since obviously the set $\{\partial w/\partial \nu : \; w \in \mathbb{H}(\Omega)\}$ is dense in $H^{-\frac{1}{2}}(\Gamma)$ and Q is smooth near Γ. Since $Q \in \mathbb{H}(\Omega)$, from the assumption we can conclude that $Q = 0$ in Ω and therefore by unique continuation in the domain bounded by Λ. Since Q is continuous across Λ we conclude that Q is a radiating solution and is zero on Λ which implies that $Q = 0$. Finally by the jump relation of the normal derivative of single layer potential we finally have that $\alpha = 0$, which ends the proof. ∎

Now we have all the ingredients to describe a sampling algorithm to determine Γ without knowing λ. Let L be an open arc in Ω and consider

$$\Phi^L(x) = \int_L \beta_L(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y, \qquad \beta_L \in \tilde{H}^{\frac{1}{2}}(L),$$

where $\Phi(x,y)$ satisfies

$$\Delta\Phi(x,y) + k^2\tilde{n}(x)\Phi(x,y) = \delta(x,y)$$

and again $\tilde{n}(x) = n(x)$ for $x \in \Omega$ and $\tilde{n}(x) = 1$ for $x \in \mathbb{R}^2 \setminus \Omega$ (or any other convenient extension). Note that we need to know only the index of refraction of the background medium inside Ω, and this is the strength of this method compared to the linear sampling method. Again if $n(x) = n_0$ is constant in Ω, one can choose $\Phi(x,y) := \frac{i}{4}H_0^{(1)}(k\sqrt{n_0}|x-x_0|)$. The proposed algorithm consists of seeking for each open arc $L \subset \Omega$ a solution $\varphi \in L^2(\Sigma)$ to the first kind ill-posed linear equation

$$\mathcal{R}(\mathcal{S}\varphi)(x_0) = \mathcal{R}(\Phi^L)(x_0), \qquad x_0 \in \Lambda. \tag{4.4}$$

Note that in order to guarantee that $\mathcal{RS}$ is injective and has dense range from the proof of Theorem 4.1 it suffices to assume that there does not exist any potential in (4.1) with vanishing normal derivative on Γ. This condition is similar to the condition we imposed on the Herglotz functions in section 3, and it excludes some special types of cracks.

Note also that in (4.4), $\mathcal{S}\varphi$ can be replaced with any one parameter dense family of functions in $\mathbb{H}(\Omega)$. We refer the reader to [26] for a variational approach to constructing such a family. In particular, if $n(x)$ is constant in Ω, one could use the corresponding Herglotz wave functions using the result of [11].

Now, if $L \subset \Gamma$, from the proof of the first part of Theorem 4.1 we see that (4.4) has a unique solution if and only if $\frac{\partial \mathcal{S}\varphi}{\partial\nu} = \frac{\partial \Phi^L}{\partial \nu}$ on Γ. This can only be satisfied approximately for some $\varphi \in L^2(\Sigma)$ since the set $\left\{\partial\mathcal{S}\varphi/\partial\nu : \varphi \in L^2(\Sigma)\right\}$ is dense in $H^{-\frac{1}{2}}(\Gamma)$.

Next, if $L \not\subset \Gamma$, we can find φ_ϵ such that

$$\|\mathcal{R}(\mathcal{S}\varphi_\epsilon) - \mathcal{R}(\Phi^L)\|_{L^2(\Lambda)} < \epsilon$$

and $\|\partial\mathcal{S}\varphi_\epsilon/\partial\nu\|_{H^{-\frac{1}{2}}(\Gamma)} < C$. Since $u(\cdot,x_0) = u_c^s(\cdot,x_0) + \mathbb{G}(\cdot,x_0)$, using Green's formula we obtain

$$\begin{aligned}
\mathcal{R}(\Phi^L)(x_0) &= \int_{\partial\Omega}\left(u_c^s(x,x_0)\frac{\partial\Phi^L(x)}{\partial\nu} - \Phi_L(x)\frac{\partial u_c^s(x,x_0)}{\partial\nu}\right)ds_x \\
&\quad + \int_{\partial\Omega}\left(\mathbb{G}(x,x_0)\frac{\partial\Phi^L(x)}{\partial\nu} - \Phi_L(x)\frac{\partial\mathbb{G}(x,x_0)}{\partial\nu}\right)ds_x \\
&= w(x_0) + \int_L \beta_L(y)\frac{\partial}{\partial\nu_y}\int_{\partial\Omega}\left(\mathbb{G}(x,x_0)\frac{\partial\Phi(x,y)}{\partial\nu} - \Phi(x,y)\frac{\partial\mathbb{G}(x,x_0)}{\partial\nu}\right)ds_x\,ds_y \\
&= w(x_0) + \int_L \beta_L(y)\frac{\partial}{\partial\nu_y}\mathbb{G}(x_0,y)ds_y,
\end{aligned}$$

where $w(x_0)$ is a solution to $\Delta_{x_0}w + k^2 n(x_0)w = 0$. On the other hand,

$$\mathcal{R}(\mathcal{S}\varphi_\epsilon)(x_0) = -i\int_\Gamma \lambda\frac{\partial u(x,x_0)}{\partial\nu}\frac{\partial\mathcal{S}\varphi_\epsilon}{\partial\nu}ds.$$

Since $\|\partial\mathcal{S}\varphi_\epsilon/\partial\nu\|_{H^{-\frac{1}{2}}(\Gamma)} < C$, we can assume that there exists a sequence such that $\lim_{\epsilon\to 0}\partial\mathcal{S}\varphi_\epsilon/\partial\nu = \theta \in H^{-\frac{1}{2}}(\Gamma)$ weakly, whence

$$\lim_{\epsilon\to 0}\mathcal{R}(\mathcal{S}\varphi_\epsilon)(x_0) = -i\int_\Gamma \lambda\frac{\partial u(x,x_0)}{\partial\nu}\,\theta(x)ds_x.$$

Hence

$$
\begin{aligned}
&-i\int_\Gamma \lambda \frac{\partial u_c^s(x,x_0)}{\partial \nu}\,\theta(x)ds_x - i\int_\Gamma \lambda \frac{\partial \mathbb{G}(x,x_0)}{\partial \nu}\,\theta(x)ds_x \\
&= w(x_0) + \int_L \beta_L(x)\frac{\partial}{\partial \nu_x}\mathbb{G}(x_0,x)ds_x. \qquad (4.5)
\end{aligned}
$$

Since the first term on both sides can be extended as solution to $\Delta_{x_0} w + k^2 n(x_0) w = 0$ outside the domain bounded by Λ, we deduce by uniqueness and the unique continuation principle that (4.5) holds in $\mathbb{R}^2 \setminus \overline{\Gamma} \cup \overline{L}$. Now we arrive at a contradiction since for $x_0 \in L$, $x_0 \notin \Gamma$, and $B_\epsilon(x_0)$ a small ball with center at x_0 such that $B_\epsilon(x_0) \cap \Gamma = \emptyset$ the left-hand side is analytic whereas the right-hand side has a singularity at x_0.

The above analysis has proven the following main theorem of this section, which is the basis of the linear sampling method based on the reciprocity gap functional for determining Γ.

THEOREM 4.2. *Assume that Γ is simple piecewise smooth arc and that there does not exist any potential in* (4.1) *having zero normal derivative on Γ. Then if $u(\cdot, x_0)$ is the total field corresponding to* (2.1)–(2.4) *with $f := 0$ and $h := i\lambda \frac{\partial \Phi(\cdot, x_0, k_s)}{\partial \nu}$, the following are true:*

1. *If $L \subset \Gamma$, then for every $\epsilon > 0$ there exists a $\varphi_\epsilon^L \in L^2(\Sigma)$ satisfying*

$$\|\mathcal{R}(\mathcal{S}\varphi_\epsilon^L) - \mathcal{R}(\Phi^L)\|_{L^2(\Lambda)} < \epsilon$$

such that $\frac{\partial \mathcal{S}\varphi_\epsilon^L}{\partial \nu}$ converges to $\frac{\partial \Phi^L}{\partial \nu}$ in $H^{-\frac{1}{2}}(\Gamma)$.

2. *If $L \not\subset \Gamma$, then for every $\epsilon > 0$ any $\varphi_\epsilon^L \in L^2(\Sigma)$ satisfying*

$$\|\mathcal{R}(\mathcal{S}\varphi_\epsilon^L) - \mathcal{R}(\Phi^L)\|_{L^2(\Lambda)} < \epsilon$$

is such that

$$\lim_{\epsilon \to 0} \|\varphi_\epsilon^L\|_{L^2(\Sigma)} = \infty \quad \textit{and} \quad \lim_{\epsilon \to 0} \left\| \frac{\partial \mathcal{S}\varphi_\epsilon^L}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\Gamma)} = \infty,$$

where $\mathcal{S}\varphi_\epsilon^L$ is defined by (4.1).

5. Numerical examples.

5.1. The linear sampling method. In this section we will give some results of numerical experiments for identifying cracks based on the theory developed in section 3. The far field data we use are synthetic, but corrupted by random noise added pointwise to the measurements. The forward problem is numerically solved using a quadrature method for the first kind hypersingular integral equation (2.10) as developed by Kress and co-authors in [9], [19], and [20]. This method seems well suited to our problem since we need to invert a hypersingular integral operator. This claim is validated by our numerical results which show a convergence rate similar to that in [9]. However, the exact order of singularity of the solution of the forward problem at the tips of the crack still remains to be studied. The computed far field data is obtained as a trigonometric series $u_\infty = \sum_{n=-N}^{N} u_{\infty,n}\exp(in\theta)$. We then add random noise to the Fourier coefficients of u_∞ to obtain the approximate far field pattern $u_{\infty,a} = \sum_{n=-N}^{N} u_{\infty,a,n}\exp(in\theta)$, where $u_{\infty,a,n} = u_{\infty,n}(1 + \epsilon\chi_n)$ with χ_n a random variable in $[-1, 1]$ ($\epsilon = 0.05$ in our examples). We remark that this random noise is rather "special," since the far field data polluted in this way remains the far field of some radiating solution to the Helmholtz equation, as the decay rate of the Fourier coefficients is not modified. This may not be the case for the noise in measured data. Also, it would be interesting to test our inversion method using simulated data computed from the exact model of the forward problem. Unfortunately, we do not have available a forward code to do so. Notice that we do not commit any inverse crime since the method for solving the direct problem and the method for solving the inverse problem are completely different.

The inversion scheme is based on solving the following ill-posed first kind equation:

$$\int_S u_\infty(\hat{x}, d) g(d)\, ds(d) = \gamma \int_L \beta_L(y) \frac{\partial}{\partial \nu_y} e^{-ik\hat{x}\cdot y} ds_y \tag{5.1}$$

for an arbitrary open arc L and $\beta_L \in \tilde{H}^{\frac{1}{2}}(L)$. Then the crack can be reconstructed by using the fact that if $L \subset \Gamma$, we can find a bounded $g \in L^2(S)$ that satisfies (5.1) with discrepancy ϵ, whereas if $L \not\subset \Gamma$, all approximate solutions to (5.1) are unbounded. In this study we search for the crack by taking L to be a small segment centered at a sampling point z with unit normal vector n_z and β_L a sequence that converges to $\delta(z)$. Thus, in the limiting case, (5.1) is replaced by

$$\int_{-\pi}^{\pi} u_\infty(\hat{x}, \theta) g_{z,n_z}(\theta)\, d\theta = -ik\gamma\, n_z \cdot \hat{x}\, e^{-ik\hat{x}\cdot z}, \qquad \hat{x} \in S,\ z \in \mathbb{R}^2,\ n_z \in S. \tag{5.2}$$

Now, if $z \in \Gamma$ and n_z coincides with the normal vector to Γ at z, then we can find a bounded $g_{z,n_z} \in L^2(S)$ that approximately solves (5.2). Otherwise all such $g_{z,n_z} \in L^2(S)$ are unbounded.

In order to solve (5.2) we use Tikhonov regularization and the Morozov discrepancy principle to deal with the severe ill-posedness of this equation. In particular, using the above expression for $u_{\infty,a}$, (5.2) is rewritten as an ill-conditioned matrix equation for the Fourier coefficients of g, which we write in the form

$$A g_{z,\theta_z} = f_{z,\theta_z}, \qquad \text{where } \theta_z = n_z \cdot \hat{x} \in [-\pi, \pi]. \tag{5.3}$$

As already noted, this equation needs to be regularized. To do this, we begin by computing the singular value decomposition of A, i.e., $A = U\Lambda V^*$, where U and V are unitary and Λ is real diagonal with $\Lambda_{l,l} = \sigma_l,\ 1 \le l \le n$, where σ_l are the singular values of A. The solution of (5.3) is then equivalent to solving $\Lambda V^* g_{z,\theta_z} = U^* f_{z,\theta_z}$. Now letting $\rho_{z,\theta_z} = (\rho_1, \rho_2, \ldots, \rho_n)^\top = U^* f_{z,\theta_z}$, the Tikhonov regularization of (5.3) leads to the problem of solving

$$\min_{g_{z,\theta_z} \in \mathbb{R}^n} \|\Lambda V^* g_{z,\theta_z} - \rho_{z,\theta_z}\|_{l^2}^2 + \alpha \|g_{z,\theta_z}\|_{l^2}^2,$$

where $\alpha > 0$ is the regularization parameter [6]. The numerical procedure for locating the crack is the following: we consider a uniform grid of sampling points $\{z_i\}_{i=1,\ldots,N}$ in the probing area, and for each sampling point z_i we choose a finite number of angles $\theta^j_{z_i}$, $j = 1, \ldots, M$, and compute

$$\mathcal{G}(z_i, \theta^j_{z_i}) = \|f_{z_i,\theta^j_{z_i}}\|_{\ell^2} / \|g_{z,\theta^j_{z_i}}\|_{\ell^2}.$$

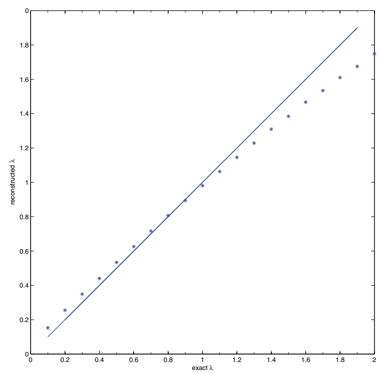


FIG. 5.1. *Here we show the reconstruction of different values of λ for the curved crack shown in Figure* 5.3(a). *The reconstructed values are shown by the little stars. The true reconstruction would be on the solid line.*

It is expected that $\mathcal{G}(z_i, \theta^j_{z_i})$ becomes relatively large if z_i is a point of Γ and $\theta^j_{z_i}$ is (near) the angle corresponding to the tangent line to Γ at z_i. In all our reconstruction examples we plot the indicator function

$$\mathbb{G}(z_i) = \sum_{j=1}^{M} \mathcal{G}(z_i, \theta^j_{z_i}) \qquad \text{for all sampling points } z_i \text{ on the grid.}$$

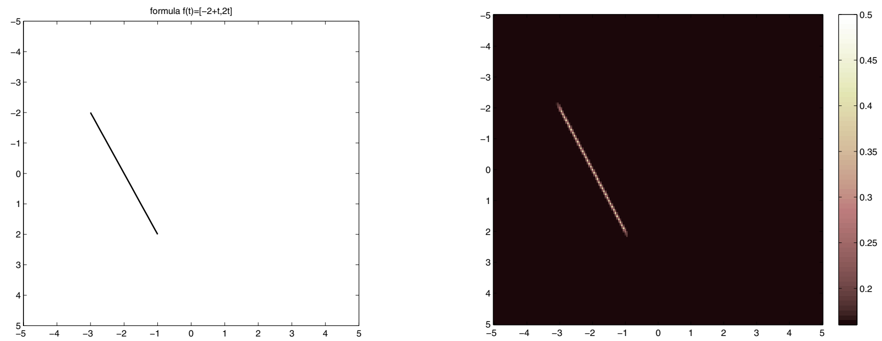
We found out that one obtains similar plots when taking the maximum over all $\theta^j_{z_i}$ for every fixed z_i instead of the summation. However, further investigation is needed to construct an indicator function that better captures the effect of the angles θ_z.

Having reconstructed Γ, it is possible to use (3.9) to reconstruct λ. In this work we have not investigated the best numerical strategy to implement (3.9) in the case when λ is a function. Here we present some preliminary results in the simplest case when λ is a constant. To this end it is natural to consider the imaginary part of (3.9). Hence the reconstruction formula is based on

$$\mathrm{Im}\frac{\partial v_{g^L_\epsilon}}{\partial \nu}(x) = \frac{1}{\lambda}\beta_L(x) - \mathrm{Im}(T_\Gamma \beta_L)(x), \qquad x \in \Gamma,$$

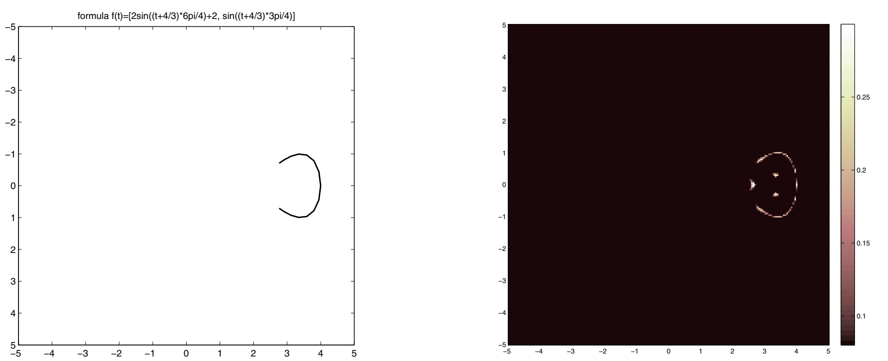
where $\beta_L \in C_0^\infty(L)$. (Note that the imaginary part of the potential T_Γ is an operator with a smooth kernel.) Now if λ is constant, we fix a point in $z \in \Gamma$ and the normal vector n_z to the crack at z and let g_{z,n_z} be the corresponding solution of the discrete far field equation. Then $\mathrm{Im}\frac{\partial v_{g_{z,n_z}}(z)}{\partial n_z}$ is approximately $1/\lambda A_z + B_z$, where A_z and B_z do not depend on λ. Hence, it is possible to avoid unstable computation of A_z and B_z, by computing the Herglotz wave function for two values of λ. Our preliminary examples show that the determination of A_z and B_z is robust. An example of reconstruction of λ based on this approach is shown in Figure 5.1. However, we note that more numerical study is needed to make the reconstruction formula for λ more practical especially for nonconstant λ. The information on λ is useful since it contains knowledge about the thickness of the dielectric object as well as its physical properties.

The numerical examples presented here consist of using the linear sampling method to reconstruct the shape of the dielectric crack, as explained above, for the following



(a) (b)

Fig. 5.2. *Panel* (a) *shows the exact crack and panel* (b) *the reconstruction using the linear sampling method. The wave number is* $k = 3$.



(a) (b)

Fig. 5.3. *Panel* (a) *shows the exact crack and panel* (b) *the reconstruction using the linear sampling method. The wave number is* $k = 3$.

open arcs:

$$\Gamma := \{-2 + s, 2s : -1 \leq s \leq 1\},$$

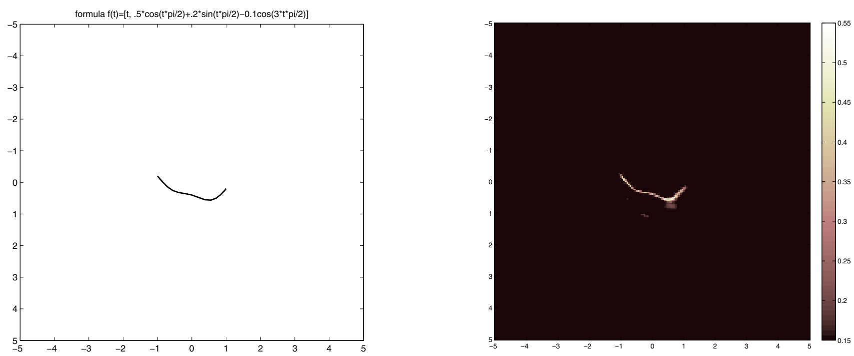
shown in Figure 5.2(a);

$$\Gamma := \left\{2\sin\left(\frac{3\pi}{2}s\right) + 2, \sin\left(\frac{3\pi}{2}s + \pi\right) : -1 \leq s \leq 1\right\},$$

shown in Figure 5.3(a); and

$$\Gamma := \left\{s, 0.5\cos\frac{\pi s}{2} + 0.2\sin\frac{\pi s}{2} - 0.1\cos\frac{3\pi s}{2} : -1 \leq s \leq 1\right\},$$

shown in Figure 5.4(a). The respective reconstructions are shown in Figures 5.2(b), 5.3(b), and 5.4(b). In all reconstructions we keep $k = 3$ and the noise level 5%.



(a) (b)

Fig. 5.4. *Panel* (a) *shows the exact crack and panel* (b) *the reconstruction using the linear sampling method. The wave number is* $k = 3$.

5.2. The reciprocity gap functional method. Here we assume that $n(x) = n_0$ for $x \in \overline{\Omega}$, where n_0 is a complex constant and $n(x) = 1$ outside Ω. Similarly to the case of the linear sampling method, we take L to be a small segment centered at a sampling point z with unit normal vector n_z and β_L a sequence converging to $\delta(z)$. Hence, in the same manner as for the linear sampling method, we can write (4.4) as

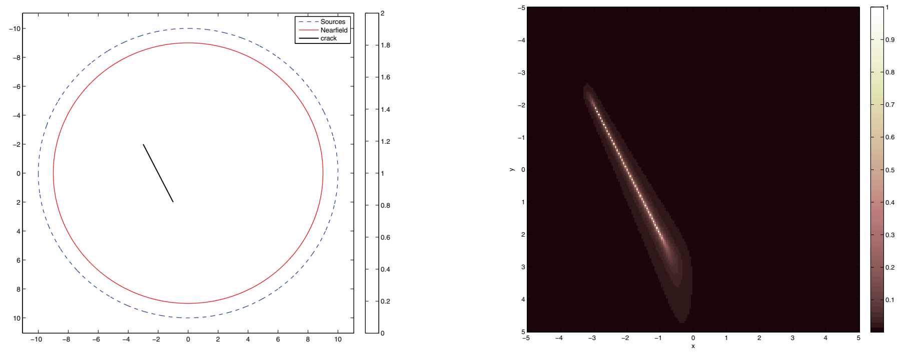
$$A\varphi_{z,n_z}(x_0) = f_{z,n_z}(x_0), \qquad x_0 \in \Lambda, \tag{5.4}$$

where $A : L^2(\Sigma) \to L^2(\Lambda)$ is the integral operator with kernel

$$K(x, x_0) := \mathcal{R}(H_0^{(1)}(k\sqrt{n_0}|x - (\cdot)|))(x_0)$$

and $f_{z,n_z}(x_0) = \mathcal{R}(ikn_z \cdot \nabla H_0^{(1)}(k\sqrt{n_0}|z - (\cdot)|))(x_0)$. Equation (5.4) is an ill-posed linear integral equation and is solved in the same way as explained in section 5.1. Similarly, the approximate solution φ_{z,n_z} is then used to identify the crack.

We end by showing an example of reconstruction for a linear crack embedded in a homogeneous medium surrounded by a circle with $n_0 = 1.5$. The data are measured on the upper half of a bigger circle. We are limited here to small contrast for the host medium, due only to the lack of forward data; it is not a limitation of the reciprocity gap functional method. We computed the data by adding the scattered field due to a crack embedded in a homogeneous media with $n = 1$ in $\mathbb{R}^2$ and the scattered field due to the disk with $n = 1.9$; i.e., we are ignoring multiple scattering effects. The example presented in Figure 5.5 indicates reliable performance of the RGF (reciprocity gap functionals) method based on sampling. Certainly, more numerical experiments are needed to validate it, including the case of absorbing background and limited aperture measurements.



(a) (b)

Fig. 5.5. *Panel* (a) *shows the configuration of the example. The crack is the black solid line embedded inside the red solid circle $\partial\Omega$ corresponding to $n_0 = 1.9$. The sources are paced on the upper half of the blue dashed circle denoted by Σ. The measurements are made on the red circle $\partial\Omega$. Panel* (b) *shows the reconstructed crack using the reciprocity gap functional method. A zoom of the area inside the red circle containing the crack is shown. The wave number is $k = 5$.*

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