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THE INTERIOR TRANSMISSION EIGENVALUE PROBLEM FOR ELASTIC WAVES IN MEDIA WITH OBSTACLES

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Abstract. In this paper, we investigate the interior transmission eigenvalue problem for elastic waves propagating outside a sound-soft or a sound-hard obstacle surrounded by an anisotropic layer. This study is motivated by the inverse problem of identifying an object embedded in an inhomogeneous media in the presence of elastic waves. Our analysis of this non-selfadjoint eigenvalue problem relies on the weak formulation of involved boundary value problems and some fundamental tools in functional analysis.

1. **Introduction.** In this work, we will study the interior transmission eigenvalue problem (ITEP) for the elastic waves propagating through an anisotropic inhomogeneous media of bounded support containing an obstacle. To set forth the problem, let D_0 and D be two open bounded domains in $\mathbf{R}^d, d = 2, 3$, with piecewise smooth boundaries ∂D_0 and ∂D. Also, we assume that $\overline{D_0} \subset D$. Let $\mathbb{C}_1 = (C^{(1)}_{ijk\ell})$ and $\mathbb{C}_2 = (C^{(2)}_{ijk\ell})$ be elasticity tensors, where $1 \le i, j, k, \ell \le d$, with real entries and each satisfies symmetry properties:

$$C^{(m)}_{ijk\ell} = C^{(m)}_{k\ell ij}, \quad C^{(m)}_{ijk\ell} = C^{(m)}_{jik\ell}, \quad (m = 1, 2).$$

In what follows, we denote $(\mathbb{A} : \mathbb{B})_{ijk\ell} = \sum_{p,q} A_{ijpq} B_{pqk\ell}$ for two tensors $\mathbb{A}, \mathbb{B}$, $A : B = \sum_{i,j} a_{ij} b_{ij}$ for two matrices A, B, and $|A|^2 = A : \overline{A}$. We are interested in

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the following ITEP:

$$\begin{cases} \nabla\cdot(\mathbb{C}_1:\nabla\mathbf{w})+k^2n\mathbf{w}=0 & \text{in } D\setminus\overline{D_0},\\ \nabla\cdot(\mathbb{C}_2:\nabla\mathbf{v})+k^2\mathbf{v}=0 & \text{in } D,\\ \mathbf{w}=\mathbf{v} \quad \text{and} \quad (\mathbb{C}_1:\nabla\mathbf{w})\nu=(\mathbb{C}_2:\nabla\mathbf{v})\nu & \text{on } \partial D,\\ \mathcal{B}\mathbf{w}=0 & \text{on } \partial D_0, \end{cases} \tag{1}$$

where ν is the unit outer normal to ∂D and ∂D_0. Here, the boundary operator $\mathcal{B}$ is given by either $\mathcal{B}\mathbf{w}=\mathbf{w}$ or $\mathcal{B}\mathbf{w}=(\mathbb{C}_1:\nabla\mathbf{w})\nu$, where

$$\begin{cases} \mathcal{B}\mathbf{w}=\mathbf{w} & \text{corresponds to the sound-soft obstacle,}\\ \mathcal{B}\mathbf{w}=(\mathbb{C}_1:\nabla\mathbf{w})\nu & \text{corresponds to the sound-hard obstacle.} \end{cases}$$

Recall that a complex number k is an interior transmission eigenvalue of (1) if there exists nontrivial $(\mathbf{w},\mathbf{v})\in(H^1(D\setminus\overline{D_0}))^d\times(H^1(D))^d$ solving (1). The main goal of this paper is to study spectral properties of ITEP (1) known to be a non-selfadjoint eigenvalue problem, such as discreteness of the eigenvalues, existence of real eigenvalues and their monotonicity properties in terms of elastic material properties and the support D_0 of the obstacle.

Originated from the inverse scattering problem, the ITEP has been studied extensively in recent years. More specifically the interior transmission eigenvalues are related to those frequencies for which it is possible to send an incident wave that doesn't scatterer. There is a vast literature available. For the sake of brevity, we will not try to exhaust all papers here. Instead, we refer to a recent monograph [7] and references therein for the detailed development of the ITEP. To put our problem into perspective, we will mention several closely related results. The discreteness (and the existence) of interior transmission eigenvalues corresponding to the acoustic media containing a sound-soft obstacle has been proved in [6]. To our best knowledge, the ITEP for (1) has not been investigated. Nonetheless, we want to mention that the ITEP for elastic inhomogeous media without the embedded obstacle D_0 has been considered in [1, 2, 9, 10, 11].

If D, $\mathbb{C}_1$, $\mathbb{C}_2$ and n are fixed, then it is clear that the shape and location of D_0 determines the distribution of interior transmission eigenvalues of (1). In principle, these eigenvalues can be determined from the scattering data [5, 22]. Since the interior transmission eigenvalues carry the geometric information of D_0, it is legitimate to consider the problem of determine information about D_0 embedded in a known inhomogeneous medium by mean of interior transmission eigenvalues. A similar problem has been studied in [20] where the knowledge of the fixed energy far field pattern was used to determine D_0. More broadly, the monotonicity properties of interior transmission eigenvalues in terms of constitutive material properties of the propagating medium have been applied to detect various type of perturbations in acoustic and electromagnetic media [8, 16, 17].The use of the interior transmission eigenvalue in this context is particularly important for anisotropic media due to the lack of uniqueness. Furthermore, the connection of ITEP with non-scattering frequencies and its application to uniqueness of the support of the media with one incident wave has been investigated in [3, 15, 18], whereas from the viewpoint of invisible cloaking, the ITEP is also studied in [19, 23].

This paper is organized as follows. In Section 2, we consider the ITEP for (1) with material contrast between $(\mathbb{C}_1,n)$ and $(\mathbb{C}_2,1)$. Also, we discuss both the sound-soft and the sound-hard embedded obstacles. We show that the set of Interior

Transmission Eigenvalues (ITE) for (1) is discrete under certain conditions. The main tool used in the proof is the analytic Fredholm theorem. Before applying the analytic Fredholm theorem, we first derive an equivalent weak formulation of (1). We then prove that the resulting sesquilinear form defines an operator of Fredholm type which depends analytically on k. In Section 3, we study the ITEP for (1) having density contrast, i.e., $n \neq 1$, with a sound-soft obstacle. We will show that in this case, the set of real interior transmission eigenvalues is discrete, and such real eigenvalues exist. The proof follows the same ideas used in Section 2. In this case we prove a monotonicity result of the smallest real eigenvalue in terms of D_0 which can, in principle, be used in the imaging of D_0 form scattering data.

2. **Elastic waves with material contrast between** $(\mathbb{C}_1, n)$ **and** $(\mathbb{C}_2, 1)$**.** This section is devoted to the investigation of the ITEP (1) in the case of different elasticity tensors. To begin, we first state assumptions on various coefficients. Assume that $\mathbb{C}_1$, $\mathbb{C}_2$ have real entries and the strong ellipticity holds, i.e., there exist constants $0 < \kappa_1 < \kappa_2$ such that

$$\begin{cases} \kappa_1 |A|^2 \leq A : \mathbb{C}_1(x) : \overline{A} \leq \kappa_2 |A|^2 & \text{for all } x \in D \setminus \overline{D_0}, \\ \kappa_1 |A|^2 \leq A : \mathbb{C}_2(x) : \overline{A} \leq \kappa_2 |A|^2 & \text{for all } x \in D, \end{cases} \tag{2}$$

and for all (complex-valued) matrix A. For isotropic media, i.e., $C_{ijk\ell} = \lambda \delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$, (2) holds whenever μ and $d\lambda + 2\mu$ are positive bounded functions.

Let $n \in L^\infty(D \setminus \overline{D_0})$ be a real-valued function with $n \geq c$ for some positive constant $c > 0$. We aim to prove the discreteness of interior transmission eigenvalues for (1). To facilitate the presentation, we define some constants:

$$\kappa^* := \sup_{x \in D \setminus \overline{D_0}} \sup_{|A|=1} \Big[A : \mathbb{C}_1(x) : \overline{A} - A : \mathbb{C}_2(x) : \overline{A} \Big],$$

$$\kappa_* := \inf_{x \in D \setminus \overline{D_0}} \inf_{|A|=1} \Big[A : \mathbb{C}_1(x) : \overline{A} - A : \mathbb{C}_2(x) : \overline{A} \Big],$$

and

$$n_* := \inf_{D \setminus \overline{D_0}} n(x), \quad n^* := \sup_{D \setminus \overline{D_0}} n(x).$$

2.1. **Sound-hard (traction-free) obstacle, i.e.,** $\mathcal{B}\mathbf{w} := (\mathbb{C}_1 : \nabla \mathbf{w})\nu = 0$ **on** ∂D_0**.** Like in most existing literature, the ITEP for (1) can be studied with the help of the variational approach. To this end, we define the Hilbert space

$$\mathcal{H}_{\text{trac}} := \left\{ \ (\mathbf{v}, \mathbf{w}) \in (H^1(D))^d \times (H^1(D \setminus \overline{D_0}))^d \ \Big| \ \mathbf{v} = \mathbf{w} \text{ on } \partial D \ \right\}.$$

We first derive the variational formula that is equivalent to (1).

Lemma 2.1. *Suppose that* $(\mathbf{v}, \mathbf{w}) \in (H^1(D))^d \times (H^1(D \setminus \overline{D_0}))^d$ *is a solution to* (1) *with* $(\mathbb{C}_1 : \nabla \mathbf{w})\nu = 0$ *on* ∂D_0*, then* $(\mathbf{v}, \mathbf{w}) \in \mathcal{H}_{\text{trac}}$ *satisfies*

$$a_k((\mathbf{v}, \mathbf{w}), (\mathbf{v}', \mathbf{w}')) = 0 \quad \textit{for all } (\mathbf{v}', \mathbf{w}') \in \mathcal{H}_{\text{trac}}, \tag{3}$$

where

$$\begin{aligned} a_k((\mathbf{v}, \mathbf{w}), (\mathbf{v}', \mathbf{w}')) = & \int_{D \setminus \overline{D_0}} \nabla \mathbf{w} : \mathbb{C}_1 : \overline{\nabla \mathbf{w}'} \, dx - \int_D \nabla \mathbf{v} : \mathbb{C}_2 : \overline{\nabla \mathbf{v}'} \, dx \\ & + k^2 \int_D \mathbf{v} \cdot \overline{\mathbf{v}'} \, dx - k^2 \int_{D \setminus \overline{D_0}} n \mathbf{w} \cdot \overline{\mathbf{w}'} \, dx. \end{aligned}$$

Conversely, if $(\mathbf{v},\mathbf{w}) \in \mathcal{H}_{\mathrm{trac}}$ *satisfies* (3)*, then such* $(\mathbf{v},\mathbf{w}) \in (H^1(D))^d \times (H^1(D\setminus\overline{D_0}))^d$ *satisfies* (1) *with* $(\mathbb{C}_1 : \nabla\mathbf{w})\nu = 0$ *on* ∂D_0*.*

Proof. We first prove the sufficiency. Let $(\mathbf{v},\mathbf{w}) \in (H^1(D))^d \times (H^1(D\setminus\overline{D_0}))^d$ be a solution to (1) with $\mathcal{B}\mathbf{w} = (\mathbb{C}_1 : \nabla\mathbf{w})\nu = 0$ on ∂D_0. Testing the first equation of (1) by $\mathbf{w}'$ gives

$$\begin{aligned} 0 &= \int_{D\setminus\overline{D_0}} (\nabla\cdot(\mathbb{C}_1 : \nabla\mathbf{w}))\cdot\overline{\mathbf{w}'}\,dx + k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx \\ &= \int_{\partial(D\setminus\overline{D_0})} (\mathbb{C}_1 : \nabla\mathbf{w})\nu\cdot\overline{\mathbf{w}'}\,dS - \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx \\ &\quad + k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx. \end{aligned}$$

Since $(\mathbb{C}_1 : \nabla\mathbf{w})\nu = 0$ on ∂D_0, we have

$$\int_{\partial D} (\mathbb{C}_1 : \nabla\mathbf{w})\nu\cdot\overline{\mathbf{w}'}\,dS - \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx + k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx = 0. \tag{4}$$

Next, we test the second equation in (1) by $\mathbf{v}'$ and obtain

$$\begin{aligned} 0 &= \int_D (\nabla\cdot(\mathbb{C}_2 : \nabla\mathbf{v}))\cdot\overline{\mathbf{v}'}\,dx + k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx \\ &= \int_{\partial D} ((\mathbb{C}_2 : \nabla\mathbf{v})\nu)\cdot\overline{\mathbf{v}'}\,dS - \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx + k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx. \end{aligned} \tag{5}$$

Since $(\mathbb{C}_1 : \nabla\mathbf{w})\nu = (\mathbb{C}_2 : \nabla\mathbf{v})\nu$ on ∂D (third equation in (1)), and $\mathbf{w}' = \mathbf{v}'$ on ∂D (definition of $\mathcal{H}_{\mathrm{trac}}$), then

$$\int_{\partial D} ((\mathbb{C}_1 : \nabla\mathbf{w})\nu)\cdot\overline{\mathbf{w}'}\,dS = \int_{\partial D} ((\mathbb{C}_2 : \nabla\mathbf{v})\nu)\cdot\overline{\mathbf{v}'}\,dS.$$

Taking the difference between (4) and (5) proves the sufficiency.

We now prove the necessity. Let $(\mathbf{v},\mathbf{w}) \in \mathcal{H}_{\mathrm{trac}}$ be a solution to (3). Firstly, we choose $\mathbf{v}' \equiv 0$, and so $\mathbf{w}' = 0$ on ∂D. Then (3) gives

$$\begin{aligned} 0 &= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx - k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx \\ &= \int_{\partial(D\setminus\overline{D_0})} ((\mathbb{C}_1 : \nabla\mathbf{w})\nu)\cdot\overline{\mathbf{w}'}\,dS - \int_{D\setminus\overline{D_0}} (\nabla\cdot(\mathbb{C}_1 : \nabla\mathbf{w}))\cdot\overline{\mathbf{w}'}\,dx \\ &\quad - k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx \\ &= -\int_{\partial D_0} ((\mathbb{C}_1 : \nabla\mathbf{w})\nu)\cdot\overline{\mathbf{w}'}\,dS - \int_{D\setminus\overline{D_0}} (\nabla\cdot(\mathbb{C}_1 : \nabla\mathbf{w}) + k^2 n\mathbf{w})\cdot\overline{\mathbf{w}'}\,dx, \end{aligned}$$

which verifies the first and last equations in (1) with $\mathcal{B}\mathbf{w} = (\mathbb{C}_1 : \nabla\mathbf{w})\nu$. Next, we choose $\mathbf{w}' \equiv 0$, and so $\mathbf{v}' = 0$ on ∂D. Then (3) implies

$$\begin{aligned} 0 &= -\int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx + k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx \\ &= -\int_{\partial D} ((\mathbb{C}_2 : \nabla\mathbf{v})\nu)\cdot\overline{\mathbf{v}'}\,dS + \int_D (\nabla\cdot(\mathbb{C}_2 : \nabla\mathbf{v}))\cdot\overline{\mathbf{v}'}\,dx + k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx \\ &= \int_D (\nabla\cdot(\mathbb{C}_2 : \nabla\mathbf{v}) + k^2\mathbf{v})\cdot\overline{\mathbf{v}'}\,dx, \end{aligned}$$

which implies the second equation in (1). Testing the first equation and the second equation in (1) by $\mathbf{w}'$ and $\mathbf{v}'$, respectively, we have

$$\int_{\partial D}((\mathbb{C}_1 : \nabla \mathbf{w})\nu) \cdot \overline{\mathbf{w}'}\, dS - \int_{\partial D}((\mathbb{C}_2 : \nabla \mathbf{v})\nu) \cdot \overline{\mathbf{v}'}\, dS = 0$$

for all $(\mathbf{v}', \mathbf{w}') \in \mathcal{H}_{\text{trac}}$. Since $\mathbf{v}' = \mathbf{w}'$ on ∂D, we conclude that

$$\int_{\partial D}((\mathbb{C}_1 : \nabla \mathbf{w})\nu - (\mathbb{C}_2 : \nabla \mathbf{v})\nu) \cdot \overline{\mathbf{v}'}\, dS = 0 \quad \text{for all } \mathbf{v}'. \tag{6}$$

Combining $\mathbf{v} = \mathbf{w}$ on ∂D and (6), we derive the third equation of (1). □

In what follows, we will use the variational formulation (3) to establish the discreteness of ITE for (1) for different media $\mathbb{C}$ and n.

2.1.1. *The case of* $\mathbb{C}_1 > \mathbb{C}_2$ *and* $n > 1$. In this section, we consider the case when $\mathbb{C}_1 - \mathbb{C}_2$ is positive and $n > 1$. We shall prove the following.

Theorem 2.2. *Assume that the convexity conditions (2) hold and $\mathbb{C}_2$ is Lipschitz continous. If κ_* satisfies*

$$\kappa_* > \frac{\kappa_2^2 - \kappa_1^2}{\kappa_1} \tag{7}$$

and $1 < n_* \le n(x) \le n^* < \infty$, then the set of interior transmission eigenvalues for (1) is discrete.

Unlike the condition $0 < n_* \le n(x) \le n^* < \infty$ assumed in [6] where the case of acoustic wave with sound-soft obstacle is considered, we assume $1 < n_* \le n(x) \le n^* < \infty$. Here we consider the homogeneous Neumann condition on ∂D_0. Thus we cannot directly apply the Poincaré inequality as used in [6]. Some of the idea in [6] have to be modified in our case.

The proof of Theorem 2.2 relies on the analytic Fredholm theorem. However, as pointed out in [6], it is not possible to show directly that the variational formulation (3) leads to an operator of Fredholm type. We shall use the concept of T-coercivity instead of the coercivity, where $T : \mathcal{H}_{\text{trac}} \to \mathcal{H}_{\text{trac}}$ is some bijective bounded linear operator.

Motivated by the idea in [6], we will introduce a suitable cut-off function. Let $\chi \in \mathcal{C}^\infty(D)$ satisfying $0 \le \chi \le 1$ in $D \setminus \overline{D_0}$, with $\chi = 1$ near ∂D and $\chi = 0$ in D_0. With this cut-off function χ, we have $\chi\mathbf{w} \in (H^1(D))^d$ and $\chi\mathbf{w}|_{\partial D} = \mathbf{w}|_{\partial D} = \mathbf{v}|_{\partial D}$ for $(\mathbf{v}, \mathbf{w}) \in \mathcal{H}_{\text{trac}}$. We define the bijective bounded linear operator (indeed $T^2 = I$) $T : \mathcal{H}_{\text{trac}} \to \mathcal{H}_{\text{trac}}$ by

$$T(\mathbf{v}, \mathbf{w}) = (-\mathbf{v} + 2\chi\mathbf{w}, \mathbf{w}).$$

Now, using T, we define a new sesquilinear form

$$
\begin{aligned}
\tag{8} &\tilde{a}_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) := a_k((\mathbf{v},\mathbf{w}),T(\mathbf{v}',\mathbf{w}')) \\
&= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx - \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla(-\mathbf{v}'+2\chi\mathbf{w}')}\,dx \\
&\quad + k^2\int_D \mathbf{v}\cdot\overline{(-\mathbf{v}'+2\chi\mathbf{w}')}\,dx - k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx \\
&= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx + \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx \\
&\quad - 2\int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla(\chi\mathbf{w}')}\,dx - k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx \\
&\quad + 2k^2\int_D \mathbf{v}\cdot\overline{\chi\mathbf{w}'}\,dx - k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx \\
&= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx + \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx \\
&\quad - 2\int_{D\setminus\overline{D_0}} ((\mathbb{C}_2 : \nabla\mathbf{v})\nabla\chi)\cdot\overline{\mathbf{w}'}\,dx - 2\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}'}\,dx \\
&\quad - k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx + 2k^2\int_{D\setminus\overline{D_0}} \mathbf{v}\cdot\overline{\chi\mathbf{w}'}\,dx \\
&\quad - k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx \\
\tag{9} &= b((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) + c_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')),
\end{aligned}
$$

where

$$
\begin{aligned}
&b((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) \\
&= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx + \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx \\
&\quad - 2\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}'}\,dx + \int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx + \int_{D\setminus\overline{D_0}} \mathbf{w}\cdot\overline{\mathbf{w}'}\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
&c_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) \\
&= -(k^2+1)\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx - \int_{D\setminus\overline{D_0}} (k^2 n+1)\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx \\
&\quad + 2k^2\int_{D\setminus\overline{D_0}} \mathbf{v}\cdot\overline{\chi\mathbf{w}'}\,dx - 2\int_{D\setminus\overline{D_0}} ((\mathbb{C}_2 : \nabla\mathbf{v})\nabla\chi)\cdot\overline{\mathbf{w}'}\,dx.
\end{aligned}
$$

Since $\chi \in \mathcal{C}_c^\infty(D\setminus D_0)$, using integration by parts gives

$$
\begin{aligned}
&-2\int_{D\setminus\overline{D_0}} ((\mathbb{C}_2 : \nabla\mathbf{v})\nabla\chi)\cdot\overline{\mathbf{w}'}\,dx \\
&= 2\int_{D\setminus\overline{D_0}} \mathbf{v}\cdot((\nabla\cdot\mathbb{C}_2)\cdot\nabla\chi)\cdot\overline{\mathbf{w}'}\,dx + 2\int_{D\setminus\overline{D_0}} \mathbf{v}\cdot\overline{((\mathbb{C}_2 : \nabla\mathbf{w}')\nabla\chi)}\,dx.
\end{aligned}
$$

We summarize the computations above in the following proposition.

Proposition 1. *The sesquilinear form $\tilde{a}_k(\bullet,\bullet)$ given in (8) can be decomposed into*

$$\tilde{a}_k(\bullet,\bullet) = b(\bullet,\bullet) + c_k(\bullet,\bullet),$$

where

$$\begin{aligned}
&b((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}'))\\
&= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'} + \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx\\
&\quad - 2\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}'}\,dx + \int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx + \int_{D\setminus\overline{D_0}} \mathbf{w}\cdot\overline{\mathbf{w}'}\,dx
\end{aligned}$$

and

$$\begin{aligned}
&c_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}'))\\
&= -(k^2+1)\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx - \int_{D\setminus\overline{D_0}} (k^2 n+1)\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx\\
&\quad + 2k^2\int_{D\setminus\overline{D_0}} \mathbf{v}\cdot\overline{\chi\mathbf{w}'}\,dx + 2\int_{D\setminus\overline{D_0}} \mathbf{v}\cdot((\nabla\cdot\mathbb{C}_2)\cdot\nabla\chi)\cdot\overline{\mathbf{w}'}\,dx\\
&\quad + 2\int_{D\setminus\overline{D_0}} \mathbf{v}\cdot\overline{((\mathbb{C}_2 : \nabla\mathbf{w}')\nabla\chi)}\,dx.
\end{aligned}$$

It is clear that for any $(\mathbf{v},\mathbf{w})\in\mathcal{H}_{\mathrm{trac}}$, $(\mathbf{v}',\mathbf{w}')\mapsto c_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}'))$ defines a bounded linear functional on $\mathcal{H}_{\mathrm{trac}}$. By Riesz's representation theorem, there exists a unique $\mathcal{C}_k : \mathcal{H}_{\mathrm{trac}}\to\mathcal{H}_{\mathrm{trac}}$ such that

$$c_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) = (\mathcal{C}_k(\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}'))_{\mathcal{H}_{\mathrm{trac}}}.$$

Then we can show that

Lemma 2.3. *If $\mathbb{C}_2$ is Lipschitz, then $\mathcal{C}_k : \mathcal{H}_{\mathrm{trac}}\to\mathcal{H}_{\mathrm{trac}}$ is bounded and compact.*

Proof. We can estimate

$$\begin{aligned}
&|(\mathcal{C}_k(\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}'))_{\mathcal{H}_{\mathrm{trac}}}|\\
&\le (|k|^2+1)\|\mathbf{v}\|_{L^2(D)}\|\mathbf{v}'\|_{L^2(D)} + (|k|^2 n^*+1)\|\mathbf{w}\|_{L^2(D\setminus\overline{D_0})}\|\mathbf{w}'\|_{L^2(D\setminus\overline{D_0})}\\
&\quad + 2|k|^2\|\mathbf{v}\|_{L^2(D)}\|\mathbf{w}'\|_{L^2(D\setminus\overline{D_0})}\\
&\quad + 2(\|\nabla\mathbb{C}_2\|_{L^\infty(D\setminus\overline{D_0})}\|\nabla\chi\|_{L^\infty(D\setminus\overline{D_0})} + \|\mathbb{C}_2\|_{L^\infty(D\setminus\overline{D_0})}\|\nabla^2\chi\|_{L^\infty(D\setminus\overline{D_0})})\\
&\qquad \times \|\mathbf{v}\|_{L^2(D)}\|\mathbf{w}'\|_{L^2(D\setminus\overline{D_0})}\\
&\quad + 2\|\mathbb{C}_2\|_{L^\infty(D\setminus\overline{D_0})}\|\nabla^2\chi\|_{L^\infty(D\setminus\overline{D_0})}\|\mathbf{v}\|_{L^2(D)}\|\nabla\mathbf{w}'\|_{L^2(D\setminus\overline{D_0})}.
\end{aligned}$$

Since $\|(\mathbf{v}',\mathbf{w}')\|_{\mathcal{H}_{\mathrm{trac}}} = \|\mathbf{v}'\|_{H^1(D)} + \|\mathbf{w}'\|_{H^1(D\setminus\overline{D_0})}$, then

$$\begin{aligned}
\|\mathcal{C}_k(\mathbf{v},\mathbf{w})\|_{\mathcal{H}_{\mathrm{trac}}} &= \sup_{\|(\mathbf{v}',\mathbf{w}')\|_{\mathcal{H}_{\mathrm{trac}}}=1} |(\mathcal{C}_k(\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}'))_{\mathcal{H}_{\mathrm{trac}}}|\\
&\le \mathsf{C}(k,\mathbb{C}_2,D,D_0)(\|\mathbf{v}\|_{L^2(D)} + \|\mathbf{w}\|_{L^2(D\setminus\overline{D_0})}),
\end{aligned}$$

that is,

$$\mathcal{C}_k : (L^2(D))^d\times(L^2(D\setminus\overline{D_0}))^d\to\mathcal{H}_{\mathrm{trac}}\quad\text{is bounded.}$$

Since

$$\iota : \mathcal{H}_{\mathrm{trac}}\hookrightarrow(L^2(D))^d\times(L^2(D\setminus\overline{D_0}))^d\quad\text{is compact,}$$

we have that $C_k\cong C_k\circ\iota : \mathcal{H}_{\mathrm{trac}}\to\mathcal{H}_{\mathrm{trac}}$ is compact. □

To show that the operator corresponding to the sesquilinear form $\tilde{a}_k$ is a Fredholm operator, it suffices to prove the coercivity of the sesquilinear form $b(\bullet,\bullet)$.

Lemma 2.4. *If κ_* satisfies (7), then $b(\bullet,\bullet)$ is coercive.*

Proof. Since $|\chi| \leq 1$, then we have

$$\begin{aligned}
&|b((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w}))| \geq \Re b((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w})) \\
&= \int_{D\setminus\overline{D_0}} \left[\nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}} - \nabla\mathbf{w} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\right] dx \\
&\quad + \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\, dx + \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}}\, dx \\
&\quad - 2\Re \int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\, dx + \|\mathbf{v}\|_{L^2(D)}^2 + \|\mathbf{w}\|_{L^2(D\setminus\overline{D_0})}^2 \\
&\geq (\kappa_* + \kappa_1)\|\nabla\mathbf{w}\|_{L^2(D\setminus\overline{D_0})}^2 + \kappa_1\|\nabla\mathbf{v}\|_{L^2(D)}^2 \\
&\quad - \kappa_2\epsilon\|\nabla\mathbf{v}\|_{L^2(D)}^2 - \frac{\kappa_2}{\epsilon}\|\nabla\mathbf{w}\|_{L^2(D\setminus\overline{D_0})}^2 + \|\mathbf{v}\|_{L^2(D)}^2 + \|\mathbf{w}\|_{L^2(D\setminus\overline{D_0})}^2 \\
&\geq \left(\kappa_* + \kappa_1 - \frac{\kappa_2}{\epsilon}\right)\|\nabla\mathbf{w}\|_{L^2(D\setminus\overline{D_0})}^2 + (\kappa_1 - \kappa_2\epsilon)\|\nabla\mathbf{v}\|_{L^2(D)}^2 \\
&\quad + \|\mathbf{v}\|_{L^2(D)}^2 + \|\mathbf{w}\|_{L^2(D\setminus\overline{D_0})}^2
\end{aligned}$$

for all $\epsilon > 0$. Note that in estimating

$$\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\, dx,$$

we have used the inequality

$$\left|\int_{D\setminus\overline{D_0}} \mathbf{A} : \mathbb{C}_2 : \overline{\mathbf{B}}\, dx\right| \leq \int_{D\setminus\overline{D_0}} \mathbf{A} : \mathbb{C}_2 : \overline{\mathbf{A}}\, dx + \int_{D\setminus\overline{D_0}} \mathbf{B} : \mathbb{C}_2 : \overline{\mathbf{B}}\, dx.$$

It is clear that (7) is equivalent to

$$\frac{\kappa_2}{\kappa_* + \kappa_1} < \frac{\kappa_1}{\kappa_2}.$$

Therefore, if we choose an ϵ satisfying

$$\frac{\kappa_2}{\kappa_* + \kappa_1} < \epsilon < \frac{\kappa_1}{\kappa_2},$$

then

$$\kappa_* + \kappa_1 - \frac{\kappa_2}{\epsilon} > 0, \qquad \kappa_1 - \kappa_2\epsilon > 0,$$

and, thus, $b(\bullet,\bullet)$ is coercive. □

By Lemma 2.3 and 2.4, we deduce that $\tilde{a}_k(\bullet,\bullet)$ described in Proposition 1 is of analytic Fredholm in k. Therefore, to prove Theorem 2.2 by the analytic Fredholm theorem, it is enough to show that $\tilde{a}_k(\bullet,\bullet)$ is coercive for some $k \neq 0$ in view of the Lax-Milgram theorem.

Lemma 2.5. *Assume that κ_* satisfies (7) and $1 < n_* \leq n(x) \leq n^* < \infty$. Then $\tilde{a}_{i\kappa}(\bullet,\bullet)$ is coercive for all sufficiently large $\kappa > 0$.*

Proof. Substituting $k = i\kappa$ into (9), we have

$$
\begin{aligned}
&|\tilde{a}_{i\kappa}((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w}))| \geq \Re\tilde{a}_{i\kappa}((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w}))\\
&= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}}\,dx - \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\,dx\\
&\quad + \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\,dx + \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}}\,dx\\
&\quad - 2\Re\int_{D\setminus\overline{D_0}} ((\mathbb{C}_2 : \nabla\mathbf{v})\nabla\chi)\cdot\overline{\mathbf{w}}\,dx - 2\Re\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\,dx\\
&\quad + \kappa^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}}\,dx - 2\Re\kappa^2\int_{D\setminus\overline{D_0}} \mathbf{v}\cdot\overline{\chi\mathbf{w}}\,dx + \kappa^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}}\,dx\\
&\geq (\kappa_* + \kappa_1)\|\nabla\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} + \kappa_1\|\nabla\mathbf{v}\|^2_{L^2(D)} + \kappa^2\|\mathbf{v}\|^2_{L^2(D)} + \kappa^2 n_*\|\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})}\\
&\quad - 2\Re\int_{D\setminus\overline{D_0}} ((\mathbb{C}_2 : \nabla\mathbf{v})\nabla\chi)\cdot\overline{\mathbf{w}}\,dx - 2\Re\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\,dx\\
&\quad - 2\Re\kappa^2\int_{D\setminus\overline{D_0}} \mathbf{v}\cdot\overline{\chi\mathbf{w}}\,dx.
\end{aligned}
$$

Note that $|\chi| \leq 1$ and $|\nabla\chi| \leq C$ for some constant C. For $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, we estimate

$$\left| -2\int_{D\setminus\overline{D_0}} ((\mathbb{C}_2 : \nabla\mathbf{v})\nabla\chi)\cdot\overline{\mathbf{w}}\,dx \right| \leq \kappa_2\epsilon_1\|\nabla\mathbf{v}\|^2_{L^2(D)} + \frac{\kappa_2 C}{\epsilon_1}\|\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})},$$

$$\left| -2\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\,dx \right| \leq \kappa_2\epsilon_2\|\nabla\mathbf{v}\|^2_{L^2(D)} + \frac{\kappa_2}{\epsilon_2}\|\nabla\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})},$$

and

$$\left| -2\kappa^2\int_{D\setminus\overline{D_0}} \mathbf{v}\cdot\overline{\chi\mathbf{w}}\,dx \right| \leq \kappa^2\epsilon_3\|\mathbf{v}\|^2_{L^2(D)} + \frac{\kappa^2}{\epsilon_3}\|\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})}.$$

Consequently, we have

$$
\begin{aligned}
&|\tilde{a}_{i\kappa}((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w}))|\\
&\geq \left(\kappa_* + \kappa_1 - \frac{\kappa_2}{\epsilon_2}\right)\|\nabla\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} + (\kappa_1 - \kappa_2(\epsilon_1+\epsilon_2))\|\nabla\mathbf{v}\|^2_{L^2(D)}\\
&\quad + \kappa^2(1-\epsilon_3)\|\mathbf{v}\|^2_{L^2(D)} + \left(\kappa^2\left(n_* - \frac{1}{\epsilon_3}\right) - \frac{\kappa_2 C}{\epsilon_1}\right)\|\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})}.
\end{aligned}
\tag{10}
$$

As in the proof of Lemma 2.4, we can choose ϵ_1, ϵ_2 such that

$$\frac{\kappa_2}{\kappa_* + \kappa_1} < \epsilon_2 < \epsilon_2 + \epsilon_1 < \frac{\kappa_1}{\kappa_2}.$$

Next, we take $\frac{1}{n_*} < \epsilon_3 < 1$. Finally, we choose $\kappa^2 > 0$ large such that

$$\kappa^2\left(n_* - \frac{1}{\epsilon_3}\right) - \frac{\kappa_2 C}{\epsilon_1} > 0.$$

By these choices of constants, we can see that all coefficients in (10) are positive and hence $\tilde{a}_{i\kappa}(\bullet,\bullet)$ is coercive for κ large. $\square$

2.1.2. *The case of* $\mathbb{C}_1 < \mathbb{C}_2$ *and* $n < 1$. In this section, we treat the case when $(\mathbb{C}_1 - \mathbb{C}_2)$ is negative and $n < 1$. Precisely, we will prove:

Theorem 2.6. *Assume that* (2) *holds and* $\mathbb{C}_1$ *is Lipschitz. If* κ^* *satisfies*

$$\kappa^* < -\frac{\kappa_2^2 - \kappa_1^2}{\kappa_1} \tag{11}$$

and $0 < n_* \leq n(x) \leq n^* < 1$, *then the set of interior transmission eigenvalues of* (1) *is discrete.*

Theorem 2.6 can be proved using the similar idea as in the proof of Theorem 2.2. Note that Lemma 2.1 remains valid. We will use the same cut-off function introduced before. Let $\chi \in \mathcal{C}^\infty(D)$ satisfy $0 \leq \chi \leq 1$ in $D \setminus \overline{D_0}$, with $\chi = 1$ near ∂D and $\chi = 0$ in D_0. In this case, we use a different bijective bounded linear operator T:

$$\begin{aligned} T : \mathcal{H}_{\rm trac} &\to \mathcal{H}_{\rm trac} \\ (\mathbf{v}, \mathbf{w}) &\mapsto (-\mathbf{v}, \mathbf{w} - 2\chi\mathbf{v}). \end{aligned}$$

Again, we consider the sesquilinear form $\tilde{a}_k$ given by

$$\begin{aligned} &\tilde{a}_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) := a_k((\mathbf{v},\mathbf{w}),T(\mathbf{v}',\mathbf{w}')) \qquad (12)\\
&= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla(\mathbf{w}' - 2\chi\mathbf{v}')}\,dx - \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla(-\mathbf{v}')}\,dx \\
&\quad + k^2 \int_D \mathbf{v}\cdot\overline{(-\mathbf{v}')}\,dx - k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{(\mathbf{w}' - 2\chi\mathbf{v}')}\,dx \\
&= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx - 2\int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla(\chi\mathbf{v}')}\,dx \\
&\quad + \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx - k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx \\
&\quad - k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx + 2k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\chi\mathbf{v}'}\,dx \\
&= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx + \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx \\
&\quad - k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx - k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx + 2k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\chi\mathbf{v}'}\,dx \\
&\quad - 2\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{v}'}\,dx - 2\int_{D\setminus\overline{D_0}} ((\mathbb{C}_1 : \nabla\mathbf{w})\nabla\chi)\cdot\overline{\mathbf{v}'}\,dx \qquad (13)\\
&= b((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) + c_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')),
\end{aligned}$$

where

$$\begin{aligned} &b((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) \\
&= \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx + \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx \\
&\quad - 2\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{v}'}\,dx \\
&\quad + \int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx + \int_{D\setminus\overline{D_0}} \mathbf{w}\cdot\overline{\mathbf{w}'}\,dx
\end{aligned}$$

and

$$\begin{aligned}
&c_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) \\
&= -(k^2+1)\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx - \int_{D\setminus\overline{D_0}}(k^2n+1)\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx \\
&\quad - 2\int_{D\setminus\overline{D_0}}((\mathbb{C}_1:\nabla\mathbf{w})\nabla\chi)\cdot\overline{\mathbf{v}'}\,dx + 2k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\chi\mathbf{v}'}\,dx.
\end{aligned}$$

Since $\nabla\chi\in\mathcal{C}_c^\infty(D\setminus D_0)$, using integration by parts, we obtain

$$\begin{aligned}
&-2\int_{D\setminus\overline{D_0}}((\mathbb{C}_1:\nabla\mathbf{w})\nabla\chi)\cdot\overline{\mathbf{v}'}\,dx \\
&= 2\int_{D\setminus\overline{D_0}}\mathbf{w}\cdot((\nabla\cdot\mathbb{C}_1)\cdot\nabla\chi)\cdot\overline{\mathbf{v}'}\,dx + 2\int_{D\setminus\overline{D_0}}\mathbf{w}\cdot\overline{((\mathbb{C}_1:\nabla\mathbf{v}')\nabla\chi)}\,dx.
\end{aligned}$$

We summarize the above computations in the following proposition.

Proposition 2. *The sesquilinear form $\tilde{a}_k(\bullet,\bullet)$ given in (12) can be decomposed into*

$$\tilde{a}_k(\bullet,\bullet) = b(\bullet,\bullet) + c_k(\bullet,\bullet),$$

where

$$\begin{aligned}
&b((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) \\
&= \int_D \nabla\mathbf{v}:\mathbb{C}_2:\overline{\nabla\mathbf{v}'}\,dx + \int_{D\setminus\overline{D_0}}\nabla\mathbf{w}:\mathbb{C}_1:\overline{\nabla\mathbf{w}'}\,dx \\
&\quad - 2\int_{D\setminus\overline{D_0}}\chi\nabla\mathbf{w}:\mathbb{C}_1:\overline{\nabla\mathbf{v}'}\,dx \\
&\quad + \int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx + \int_{D\setminus\overline{D_0}}\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx
\end{aligned}$$

and

$$\begin{aligned}
&c_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) \\
&= -(k^2+1)\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx - \int_{D\setminus\overline{D_0}}(k^2n+1)\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx \\
&\quad + 2\int_{D\setminus\overline{D_0}}\mathbf{w}\cdot((\nabla\cdot\mathbb{C}_1)\cdot\nabla\chi)\cdot\overline{\mathbf{v}'}\,dx + 2\int_{D\setminus\overline{D_0}}\mathbf{w}\cdot\overline{((\mathbb{C}_1:\nabla\mathbf{v}')\nabla\chi)}\,dx \\
&\quad + 2k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\chi\mathbf{v}'}\,dx.
\end{aligned}$$

As before, for any $(\mathbf{v},\mathbf{w})\in\mathcal{H}_{\text{trac}}$, $(\mathbf{v}',\mathbf{w}')\mapsto c_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}'))$ is a bounded linear functional on $\mathcal{H}_{\text{trac}}$. By Riesz's representation theorem, there exists a unique $\mathcal{C}_k:\mathcal{H}_{\text{trac}}\to\mathcal{H}_{\text{trac}}$ such that

$$c_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) = (\mathcal{C}_k(\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}'))_{\mathcal{H}_{\text{trac}}}.$$

Lemma 2.7. *If $\mathbb{C}_1$ is Lipschitz, then $\mathcal{C}_k:\mathcal{H}_{\text{trac}}\to\mathcal{H}_{\text{trac}}$ is bounded and compact.*

Proof. We can estimate

$$\begin{aligned}
&|(\mathcal{C}_k(\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}'))_{\mathcal{H}_{\mathrm{trac}}}| \\
&\le (|k|^2+1)\|\mathbf{v}\|_{L^2(D)}\|\mathbf{v}'\|_{L^2(D)} + (|k|^2 n^*+1)\|\mathbf{w}\|_{L^2(D\setminus\overline{D})}\|\mathbf{w}'\|_{L^2(D\setminus\overline{D})} \\
&\quad + 2(\|\nabla\mathbb{C}_1\|_{L^\infty(D\setminus\overline{D_0})}\|\nabla\chi\|_{L^\infty(D\setminus\overline{D_0})} + \|\mathbb{C}_1\|_{L^\infty(D\setminus\overline{D_0})}\|\nabla^2\chi\|_{L^\infty(D\setminus\overline{D_0})})\times \\
&\qquad \times \|\mathbf{w}\|_{L^2(D\setminus\overline{D})}\|\mathbf{v}'\|_{L^2(D)} \\
&\quad + 2(\|\mathbb{C}_1\|_{L^\infty(D\setminus\overline{D_0})}\|\nabla\chi\|_{L^\infty(D\setminus\overline{D_0})})\|\mathbf{w}\|_{L^2(D\setminus\overline{D})}\|\nabla\mathbf{v}'\|_{L^2(D)} \\
&\quad + 2|k|^2 n^*\|\mathbf{w}\|_{L^2(D\setminus\overline{D})}\|\mathbf{v}'\|_{L^2(D)}.
\end{aligned}$$

Then we have that

$$\begin{aligned}
\|\mathcal{C}_k(\mathbf{v},\mathbf{w})\|_{\mathcal{H}_{\mathrm{trac}}} &= \sup_{\|(\mathbf{v}',\mathbf{w}')\|_{\mathcal{H}_{\mathrm{trac}}}=1} |(\mathcal{C}_k(\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}'))_{\mathcal{H}_{\mathrm{trac}}}| \\
&\le C(k,\mathbb{C}_1,D,D_0)(\|\mathbf{v}\|_{L^2(D)} + \|\mathbf{w}\|_{L^2(D\setminus\overline{D})}),
\end{aligned}$$

and thus $\mathcal{C}_k : (L^2(D))^d \times (L^2(D\setminus\overline{D}))^d \to \mathcal{H}_{\mathrm{trac}}$ is bounded. Similar to Lemma 2.3, we prove the desired result. $\square$

The proof of the following lemma is similar to that of Lemma 2.4 with some necessary modifications.

Lemma 2.8. *If κ^* satisfies (11), then $b(\bullet,\bullet)$ is coercive.*

Proof. Since $|\chi|\le 1$, we can estimate

$$\begin{aligned}
&|b((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w}))| \ge \Re b((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w})) \\
&= \int_{D_0}\nabla\mathbf{v}:\mathbb{C}_2:\overline{\nabla\mathbf{v}}\,dx + \int_{D\setminus\overline{D_0}}\nabla\mathbf{v}:\mathbb{C}_2:\overline{\nabla\mathbf{v}}\,dx - \int_{D\setminus\overline{D_0}}\nabla\mathbf{v}:\mathbb{C}_1:\overline{\nabla\mathbf{v}}\,dx \\
&\quad + \int_{D\setminus\overline{D_0}}\nabla\mathbf{w}:\mathbb{C}_1:\overline{\nabla\mathbf{w}}\,dx + \int_{D\setminus\overline{D_0}}\nabla\mathbf{v}:\mathbb{C}_1:\overline{\nabla\mathbf{v}}\,dx \\
&\quad - 2\Re\int_{D\setminus\overline{D_0}}\chi\nabla\mathbf{w}:\mathbb{C}_1:\overline{\nabla\mathbf{v}}\,dx + \|\mathbf{v}\|^2_{L^2(D)} + \|\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} \\
&\ge \kappa_1\|\nabla\mathbf{v}\|^2_{L^2(D_0)} + \kappa_1\|\nabla\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} + (-\kappa^*+\kappa_1)\|\nabla\mathbf{v}\|^2_{L^2(D\setminus\overline{D_0})} \\
&\quad + \|\mathbf{v}\|^2_{L^2(D)} + \|\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} - \kappa_2\epsilon\|\nabla\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} - \frac{\kappa_2}{\epsilon}\|\nabla\mathbf{v}\|^2_{L^2(D\setminus\overline{D_0})} \\
&\ge \kappa_1\|\nabla\mathbf{v}\|^2_{L^2(D_0)} + (\kappa_1-\kappa_2\epsilon)\|\nabla\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} + \left(-\kappa^*+\kappa_1-\frac{\kappa_2}{\epsilon}\right)\|\nabla\mathbf{v}\|^2_{L^2(D\setminus\overline{D_0})} \\
&\quad + \|\mathbf{v}\|^2_{L^2(D)} + \|\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})},
\end{aligned}$$

where $\epsilon > 0$. Observe that (11) implies

$$0 < \frac{\kappa_2}{-\kappa^*+\kappa_1} < \frac{\kappa_1}{\kappa_2}.$$

So we can choose ϵ such that

$$\frac{\kappa_2}{-\kappa^*+\kappa_1} < \epsilon < \frac{\kappa_1}{\kappa_2}$$

and hence

$$\kappa_1 - \kappa_2\epsilon > 0, \quad -\kappa^*+\kappa_1-\frac{\kappa_2}{\epsilon} > 0.$$

The coercivity of b then follows immediately. $\square$

Consequently, the sesquilinear form $\tilde{a}_k(\bullet,\bullet)$ defines an analytic Fredholm operator on $\mathcal{H}_{\mathrm{trac}}$. To apply the analytic Fredholm theorem to prove Theorem 2.6, we only need to show the coercivity of $\tilde{a}_k(\bullet,\bullet)$ for some $k \neq 0$. Similar to Lemma 2.5, we can prove the following result.

Lemma 2.9. *If κ^* satisfies (11) and $0 < n_* \leq n(x) \leq n^* < 1$, then $\tilde{a}_{i\kappa}(\bullet,\bullet)$ is coercive for sufficiently large $\kappa > 0$.*

Proof. By substituting $k = i\kappa$ into (13), we have

$$\begin{aligned}
&|\tilde{a}_{i\kappa}((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w}))| \\
&\geq \Re\tilde{a}_{i\kappa}((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w})) \\
&= \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}}\, dx - \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\, dx \\
&\quad + \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_2 : \overline{\nabla\mathbf{w}}\, dx + \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}}\, dx \\
&\quad + \kappa^2 \int_D \mathbf{v}\cdot\overline{\mathbf{v}}\, dx + \kappa^2 \int_{D\setminus\overline{D_0}} n^{-1}(n\mathbf{w})\cdot\overline{(n\mathbf{w})}\, dx - 2\kappa^2\Re\int_{D\setminus\overline{D_0}} (n\mathbf{w})\cdot\overline{\chi\mathbf{v}}\, dx \\
&\quad - 2\Re\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{v}}\, dx - 2\Re\int_{D\setminus\overline{D_0}} ((\mathbb{C}_1 : \nabla\mathbf{w})\nabla\chi)\cdot\overline{\mathbf{v}}\, dx \\
&\geq (-\kappa^* + \kappa_1)\|\nabla\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} + \kappa_1\|\nabla\mathbf{v}\|^2_{L^2(D)} + \kappa^2\|\mathbf{v}\|^2_{L^2(D)} + \kappa^2\frac{1}{n^*}\|n\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} \\
&\quad - 2\Re\int_{D\setminus\overline{D_0}} ((\mathbb{C}_1 : \nabla\mathbf{w})\nabla\chi)\cdot\overline{\mathbf{v}}\, dx \\
&\quad - 2\Re\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{v}}\, dx \\
&\quad - 2\kappa^2\Re\int_{D\setminus\overline{D_0}} (n\mathbf{w})\cdot\overline{\chi\mathbf{v}}\, dx.
\end{aligned}$$

Note that $|\chi| \leq 1$ and $|\nabla\chi| \leq C$ for some constant C. For $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, we can derive

$$\left| -2\int_{D\setminus\overline{D_0}} ((\mathbb{C}_1 : \nabla\mathbf{w})\nabla\chi)\cdot\overline{\mathbf{v}}\, dx\right| \leq \epsilon_1\kappa_2\|\nabla\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} + \frac{\kappa_2 C}{\epsilon_1}\|\mathbf{v}\|^2_{L^2(D)},$$

$$\left| -2\int_{D\setminus\overline{D_0}} \chi\nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{v}}\, dx\right| \leq \kappa_2\epsilon_2\|\nabla\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} + \frac{\kappa_2}{\epsilon_2}\|\nabla\mathbf{v}\|^2_{L^2(D)},$$

and

$$\left| -2\kappa^2\int_{D\setminus\overline{D_0}} (n\mathbf{w})\cdot\overline{\chi\mathbf{v}}\, dx\right| \leq \kappa^2\epsilon_3\|n\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} + \frac{\kappa^2}{\epsilon_3}\|\mathbf{v}\|^2_{L^2(D)}.$$

Therefore, we obtain

$$\begin{aligned}
&|\tilde{a}_{i\kappa}((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w}))| \\
&\geq \Big(-\kappa^* + \kappa_1 - \kappa_2(\epsilon_1+\epsilon_2)\Big)\|\nabla\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})} + \left(\kappa_1 - \frac{\kappa_2}{\epsilon_2}\right)\|\nabla\mathbf{v}\|^2_{L^2(D)} \\
&\quad + \left(\kappa^2\left(1 - \frac{1}{\epsilon_3}\right) - \frac{\kappa_2 C}{\epsilon_1}\right)\|\mathbf{v}\|^2_{L^2(D)} + \kappa^2\left(\frac{1}{n^*} - \epsilon_3\right)\|n\mathbf{w}\|^2_{L^2(D\setminus\overline{D_0})}.
\end{aligned}$$

Since

$$\frac{\kappa_2}{-\kappa^* + \kappa_1} < \frac{\kappa_1}{\kappa_2},$$

we choose ϵ_1, ϵ_2 such that

$$\frac{\kappa_2}{-\kappa^* + \kappa_1} < \epsilon_2 < \epsilon_1 + \epsilon_2 < \frac{\kappa_1}{\kappa_2}. \tag{14}$$

With (14), taking ϵ_3 satisfying

$$\frac{1}{n^*} > \epsilon_3,$$

and κ large enough, the coercivity of $\tilde{a}_{i\kappa}$ then follows. □

2.2. **Sound-soft obstacle, i.e., $\mathcal{B}\mathbf{w} := \mathbf{w} = 0$ on ∂D_0.** Similar to the case of sound-hard obstacle, we want to derive a variational formulation equivalent to (1) with $\mathcal{B}\mathbf{w} = \mathbf{w}$. Define the Hilbert space

$$\mathcal{H}_{\rm disp} := \left\{ (\mathbf{v},\mathbf{w}) \in (H^1(D))^d \times (H^1(D\setminus\overline{D_0}))^d \;\middle|\; \begin{array}{l} \mathbf{w} = \mathbf{v} \text{ on } \partial D \\ \mathbf{w} = 0 \text{ on } \partial D_0 \end{array} \right\}.$$

Lemma 2.10. *Suppose that $(\mathbf{v},\mathbf{w}) \in (H^1(D))^d \times (H^1(D\setminus\overline{D_0}))^d$ is a solution to (1) with $\mathbf{w} = 0$ on ∂D_0. Then $(\mathbf{v},\mathbf{w}) \in \mathcal{H}_{\rm disp}$ satisfies*

$$a_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) = 0 \quad \textit{for all } (\mathbf{v}',\mathbf{w}') \in \mathcal{H}_{\rm disp}, \tag{15}$$

where

$$\begin{aligned} a_k((\mathbf{v},\mathbf{w}),(\mathbf{v}',\mathbf{w}')) = &\int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx - \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx \\ &+ k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx - k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx. \end{aligned}$$

Conversely, if $(\mathbf{v},\mathbf{w}) \in \mathcal{H}_{\rm disp}$ satisfies (15), then such $(\mathbf{v},\mathbf{w}) \in (H^1(D))^d \times (H^1(D\setminus\overline{D_0}))^d$ solves (1) with $\mathbf{w} = 0$ on ∂D_0.

The proof of Lemma 2.10 is almost identical to that of Lemma 2.1 except some minor differences. For the sake of completeness, we still present the detailed proof here.

Proof. We first prove the sufficiency. Given $(\mathbf{v},\mathbf{w}) \in (H^1(D))^d \times (H^1(D\setminus\overline{D_0}))^d$ a solution to (1) with $\mathbf{w} = 0$ on ∂D_0. Let $(\mathbf{v}',\mathbf{w}') \in \mathcal{H}_{\rm disp}$. Testing the first equation of (1) by $\mathbf{w}'$, we have

$$\begin{aligned} 0 &= \int_{D\setminus\overline{D_0}} (\nabla\cdot(\mathbb{C}_1 : \nabla\mathbf{w}))\cdot\overline{\mathbf{w}'}\,dx + k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx \\ &= \int_{\partial(D\setminus\overline{D_0})} ((\mathbb{C}_1 : \nabla\mathbf{w})\nu)\cdot\overline{\mathbf{w}'}\,dS - \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx \\ &\quad + k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx. \end{aligned}$$

Since $\mathbf{w}' = \mathbf{0}$ on ∂D_0, we obtain

$$\int_{\partial D} ((\mathbb{C}_1 : \nabla\mathbf{w})\nu)\cdot\overline{\mathbf{w}'}\,dS - \int_{D\setminus\overline{D_0}} \nabla\mathbf{w} : \mathbb{C}_1 : \overline{\nabla\mathbf{w}'}\,dx + k^2\int_{D\setminus\overline{D_0}} n\mathbf{w}\cdot\overline{\mathbf{w}'}\,dx = 0. \tag{16}$$

Next, testing the second equation of (1) by $\mathbf{v}'$ implies

$$\begin{aligned} 0 &= \int_D (\nabla\cdot(\mathbb{C}_2 : \nabla\mathbf{v}))\cdot\overline{\mathbf{v}'}\,dx + k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx \\ &= \int_{\partial D} ((\mathbb{C}_2 : \nabla\mathbf{v})\nu)\cdot\overline{\mathbf{v}'}\,dS - \int_D \nabla\mathbf{v} : \mathbb{C}_2 : \overline{\nabla\mathbf{v}'}\,dx + k^2\int_D \mathbf{v}\cdot\overline{\mathbf{v}'}\,dx. \end{aligned} \tag{17}$$

Since $(\mathbb{C}_1 : \nabla \mathbf{w})\nu = (\mathbb{C}_2 : \nabla \mathbf{v})\nu$ on ∂D and $\mathbf{w}' = \mathbf{v}'$ on ∂D, then

$$\int_{\partial D} ((\mathbb{C}_1 : \nabla \mathbf{w})\nu) \cdot \overline{\mathbf{w}'}\, dS = \int_{\partial D} ((\mathbb{C}_2 : \nabla \mathbf{v})\nu) \cdot \overline{\mathbf{v}'}\, dS.$$

Doing the subtraction between (16) and (17), we prove the implication.

Conversely, let $(\mathbf{v}, \mathbf{w}) \in \mathcal{H}_{\mathrm{disp}}$ be a solution to (15). Then $\mathbf{w} = 0$ holds by the definition of $\mathcal{H}_{\mathrm{disp}}$. Now we choose $\mathbf{v}' \equiv \mathbf{0}$, and so $\mathbf{w}' = \mathbf{0}$ on ∂D. Also, from the definition of $\mathcal{H}_{\mathrm{disp}}$, we have $\mathbf{w}' = \mathbf{0}$ on ∂D_0 as well. Then (15) gives

$$\begin{aligned}
0 &= \int_{D\setminus\overline{D_0}} \nabla \mathbf{w} : \mathbb{C}_1 : \overline{\nabla \mathbf{w}'}\, dx - k^2 \int_{D\setminus\overline{D_0}} n\mathbf{w} \cdot \overline{\mathbf{w}'}\, dx \\
&= \int_{\partial(D\setminus\overline{D_0})} ((\mathbb{C}_1 : \nabla \mathbf{w})\nu) \cdot \overline{\mathbf{w}'}\, dS - \int_{D\setminus\overline{D_0}} (\nabla \cdot (\mathbb{C}_1 : \nabla \mathbf{w})) \cdot \overline{\mathbf{w}'}\, dx \\
&\quad - k^2 \int_{D\setminus\overline{D_0}} n\mathbf{w} \cdot \overline{\mathbf{w}'}\, dx \\
&= -\int_{D\setminus\overline{D_0}} (\nabla \cdot (\mathbb{C}_1 : \nabla \mathbf{w}) + k^2 n\mathbf{w}) \cdot \overline{\mathbf{w}'}\, dx,
\end{aligned}$$

which leads to the first equation in (1). Next, we take $\mathbf{w}' \equiv \mathbf{0}$, and so $\mathbf{v}' = \mathbf{0}$ on ∂D. Then (15) implies

$$\begin{aligned}
0 &= -\int_D \nabla \mathbf{v} : \mathbb{C}_2 : \overline{\nabla \mathbf{v}'}\, dx + k^2 \int_D \mathbf{v} \cdot \overline{\mathbf{v}'}\, dx \\
&= -\int_{\partial D} ((\mathbb{C}_2 : \nabla \mathbf{v})\nu) \cdot \overline{\mathbf{v}'}\, dS + \int_D (\nabla \cdot (\mathbb{C}_2 : \nabla \mathbf{v})) \cdot \overline{\mathbf{v}'}\, dx + k^2 \int_D \mathbf{v} \cdot \overline{\mathbf{v}'}\, dx \\
&= \int_D (\nabla \cdot (\mathbb{C}_2 : \nabla \mathbf{v}) + k^2 \mathbf{v}) \cdot \overline{\mathbf{v}'}\, dx,
\end{aligned}$$

which verifies the second equation in (1). Testing the first equation in (1) by $\mathbf{w}'$ and the second equation in (1) by $\mathbf{v}'$, we deduce

$$\int_{\partial D} ((\mathbb{C}_1 : \nabla \mathbf{w})\nu) \cdot \overline{\mathbf{w}'}\, dS - \int_{\partial D} ((\mathbb{C}_2 : \nabla \mathbf{v})\nu) \cdot \overline{\mathbf{v}'}\, dS = 0.$$

In view of $\mathbf{v}' = \mathbf{w}'$ on ∂D (by the definition of $\mathcal{H}_{\mathrm{disp}}$), we have

$$\int_{\partial D} ((\mathbb{C}_1 : \nabla \mathbf{w})\nu - (\mathbb{C}_2 : \nabla \mathbf{v})\nu) \cdot \overline{\mathbf{w}'}\, dS = 0.$$

Combining this and $\mathbf{v} = \mathbf{w}$ on ∂D, we can see that the third equation of (1) holds. □

Notice that Lemma 2.10 is exactly the same as Lemma 2.1, except that the space $\mathcal{H}_{\mathrm{trac}}$ is replaced by $\mathcal{H}_{\mathrm{disp}}$. It is clear that $\mathcal{H}_{\mathrm{disp}} \subset \mathcal{H}_{\mathrm{trac}}$. Therefore, the arguments used in the case of sound-hard obstacle can be directly applied to the case of sound-soft obstacle here. Hence, analogue results as Theorem 2.2 and Theorem 2.6 hold. Precisely, we can prove the following theorems.

Theorem 2.11. *Assume* (2) *and* $\mathbb{C}_2$ *is Lipschitz. If* κ_* *satisfies* $\kappa_* > \frac{\kappa_2^2 - \kappa_1^2}{\kappa_1}$ *and* $1 < n_* \le n(x) \le n^* < \infty$, *then the set of interior transmission eigenvalues of* (1) *with* $\mathbf{w} = 0$ *on* ∂D_0 *is discrete.*

Theorem 2.12. *Assume* (2) *and* $\mathbb{C}_1$ *is Lipschitz. If* κ^* *satisfies* $\kappa^* < -\frac{\kappa_2^2 - \kappa_1^2}{\kappa_1}$ *and* $0 < n_* \le n(x) \le n^* < 1$, *then the set of interior transmission eigenvalues of* (1) *with* $\mathbf{w} = 0$ *on* ∂D_0 *is discrete.*

Remark 1. T-coercivity approach applied above has the potential to relax the assumptions on the contrasts only in a neighborhood of the boundary ∂D (see e.g. [4], [7, Section 3.2] for the acoustic case). We foresee that for our problems it is possible to obtain discreteness of interior transmission eigenvalues under sign condition on $\mathbb{C}_1 - \mathbb{C}_2$ in a neighborhood of the boundary ∂D only. However, since the main goal of our paper is to handle elastic media with obstacles and due to technicalities of the presentation, we do not develop here the details.

3. **Elastic waves with density contrast.** In the previous section, we studied the ITEP for (1) with $\mathbb{C}_1 \neq \mathbb{C}_2$ and $n \neq 1$. We showed that the set of interior transmission eigenvalues (both real and complex) of (1) is discrete. In this section, we shall consider the case where $\mathbb{C}_1 = \mathbb{C}_2 = \mathbb{C}$. Our goal is to extend the result in [6, Section 2] (with sound-soft obstacle) to the elastic waves. That is, we consider the following ITEP

$$
\begin{cases}
\nabla \cdot (\mathbb{C} : \nabla \mathbf{w}) + k^2 n \mathbf{w} = 0 & \text{in } D \setminus \overline{D_0}, \\
\nabla \cdot (\mathbb{C} : \nabla \mathbf{v}) + k^2 \mathbf{v} = 0 & \text{in } D, \\
\mathbf{w} = \mathbf{v} \quad \text{and} \quad (\mathbb{C} : \nabla \mathbf{w})\nu = (\mathbb{C} : \nabla \mathbf{v})\nu & \text{on } \partial D, \\
\mathbf{w} = 0 & \text{on } \partial D_0,
\end{cases}
\tag{18}
$$

where $\mathbb{C}$ is real-valued and satisfies the strong convexity (2). Like the result obtained in [6], in this case, we establish that the existence and discreteness of real interior transmission eigenvalues. Before proving the general result, we provide some explicit computations of eigenvalues for the case of radially symmetric system. For simplicity, we set $d = 2$, $D = B_1$, and $D_0 = B_{1/2}$. Assume that $\mathbb{C}$ is isotropic with Lamé constants λ and μ. Also, let n be a constant with $0 < n < 1$. Note that in this case $\nu = x$ on ∂D. We consider both the sound-soft and sound-hard obstacles.

Sound-soft obstacle ($\mathbf{w} = 0$ *on* ∂D_0). In this case, (1) can be explicitly written as

$$
\begin{cases}
\left.\begin{cases}
(2\mu+\lambda)\partial_1^2 w_1 + \mu \partial_2^2 w_1 + (\mu+\lambda)\partial_1\partial_2 w_2 + k^2 n w_1 = 0 \\
(\mu+\lambda)\partial_1\partial_2 w_1 + \mu\partial_1^2 w_2 + (2\mu+\lambda)\partial_2^2 w_2 + k^2 n w_2 = 0
\end{cases}\right\} & \text{in } B_1 \setminus \overline{B_{1/2}}, \\
\left.\begin{cases}
(2\mu+\lambda)\partial_1^2 v_1 + \mu \partial_2^2 v_1 + (\mu+\lambda)\partial_1\partial_2 v_2 + k^2 v_1 = 0 \\
(\mu+\lambda)\partial_1\partial_2 v_1 + \mu\partial_1^2 v_2 + (2\mu+\lambda)\partial_2^2 v_2 + k^2 v_2 = 0
\end{cases}\right\} & \text{in } B_1, \\
\left.\begin{cases}
w_1 = v_1, \quad w_2 = v_2 \\
x_1[(2\mu+\lambda)\partial_1 w_1 + \lambda\partial_2 w_2] + x_2[\mu(\partial_1 w_2 + \partial_2 w_1)] \\
\qquad = x_1[(2\mu+\lambda)\partial_1 v_1 + \lambda\partial_2 v_2] + x_2[\mu(\partial_1 v_2 + \partial_2 v_1)] \\
x_1[\mu(\partial_1 w_2 + \partial_2 w_1)] + x_2[\lambda\partial_1 w_1 + (2\mu+\lambda)\partial_2 w_2] \\
\qquad = x_1[\mu(\partial_1 v_2 + \partial_2 v_1)] + x_2[\lambda\partial_1 v_1 + (2\mu+\lambda)\partial_2 v_2]
\end{cases}\right\} & \text{on } \partial B_1, \\
w_1 = 0, \quad w_2 = 0 \quad \text{on } \partial B_{1/2}.
\end{cases}
\tag{19}
$$

We consider the radially symmetric solutions, that is, let

$$
\mathbf{w} = w(r)\mathbf{e}_r, \quad \mathbf{v} = v(r)\mathbf{e}_r,
$$

where $\mathbf{e}_r = x/r$ and $r = |x|$. Then (19) is reduced to

$$\begin{cases} r^2 w''(r) + r w'(r) + (\alpha_n^2 r^2 - 1) w(r) = 0, & 1/2 < r < 1, \\ r^2 v''(r) + r v'(r) + (\alpha_1^2 r^2 - 1) v(r) = 0, & 0 < r < 1, \\ w(1) = v(1), \quad w'(1) = v'(1), \\ w(1/2) = 0, \\ |v(0)| < \infty, \end{cases} \tag{20}$$

where

$$\alpha_n(k) = \frac{k n^{1/2}}{\sqrt{2\mu + \lambda}} \quad \text{and} \quad \alpha_1(k) = \frac{k}{\sqrt{2\mu + \lambda}}.$$

From (20), we can derive

$$v(r) = a J_1(\alpha_1 r) \quad \text{and} \quad w(r) = b J_1(\alpha_n r) + c Y_1(\alpha_n r),$$

where J_1 is the first kind Bessel function of order 1, Y_1 is the second kind Bessel function of order 1, and a, b, c are constants. In view of the boundary conditions, we have that

$$\begin{bmatrix} 0 & J_1(\alpha_n/2) & Y_1(\alpha_n/2) \\ J_1(\alpha_1) & -J_1(\alpha_n) & -Y_1(\alpha_n) \\ \alpha_1 J_1'(\alpha_1) & -\alpha_n J_1'(\alpha_n) & -\alpha_n Y_1'(\alpha_n) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0.$$

Thus, k is a transmission eigenvalue if, and only if,

$$F(k) := \det \begin{bmatrix} 0 & J_1(\alpha_n/2) & Y_1(\alpha_n/2) \\ J_1(\alpha_1) & -J_1(\alpha_n) & -Y_1(\alpha_n) \\ \alpha_1 J_1'(\alpha_1) & -\alpha_n J_1'(\alpha_n) & -\alpha_n Y_1'(\alpha_n) \end{bmatrix} = 0. \tag{21}$$

We recall the following asymptotic behavior of J_α and Y_α:

$$J_\alpha(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O(r^{-3/2}) \quad \text{as } r \to \infty,$$
$$Y_\alpha(r) = \sqrt{\frac{2}{\pi r}} \sin\left(r - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O(r^{-3/2}) \quad \text{as } r \to \infty,$$

and the differentiation formula

$$J_1'(s) = \frac{1}{2}(J_0(s) - J_2(s)) \quad \text{and} \quad Y_1'(s) = \frac{1}{2}(Y_0(s) - Y_2(s)).$$

We note that the leading term of F is nearly periodic as $k \to \infty$. Hence, we can conclude that F has infinitely many zeros can only accumulate at the infinity. Figure 1 exhibits a numerical experiment for $\mu = 1$, $\lambda = 1$, and $n = 1/2$. The first four positive eigenvalues are $k_1 = 9.8599$, $k_2 = 19.1916$, $k_3 = 26.7594$, $k_4 = 36.0246$. Note that in this example, we only consider transmission eigenvalues corresponding to radially symmetric eigenfunctions. There certainly exists non-radially symmetric eigenfunctions.

Sound-hard obstacle ($(\mathbb{C} : \nabla \mathbf{w})\nu = 0$). We can also perform similar computations in the case of sound-hard obstacle. In this case, we obtain a similar system as (19)

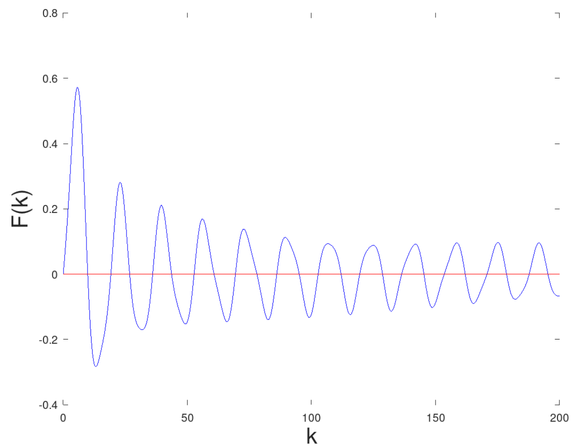

FIGURE 1. Plot of $F(k)$ in (21) with $\mu = 1$, $\lambda = 1$ and $n = 1/2$ (GNU Octave).

except the boundary condition on $\partial B_{1/2}$, i.e.,

$$\begin{cases} \left.\begin{cases} (2\mu+\lambda)\partial_1^2 w_1 + \mu\partial_2^2 w_1 + (\mu+\lambda)\partial_1\partial_2 w_2 + k^2 n w_1 = 0 \\ (\mu+\lambda)\partial_1\partial_2 w_1 + \mu\partial_1^2 w_2 + (2\mu+\lambda)\partial_2^2 w_2 + k^2 n w_2 = 0 \end{cases}\right\} & \text{in } B_1 \setminus \overline{B_{1/2}}, \\ \left.\begin{cases} (2\mu+\lambda)\partial_1^2 v_1 + \mu\partial_2^2 v_1 + (\mu+\lambda)\partial_1\partial_2 v_2 + k^2 v_1 = 0 \\ (\mu+\lambda)\partial_1\partial_2 v_1 + \mu\partial_1^2 v_2 + (2\mu+\lambda)\partial_2^2 v_2 + k^2 v_2 = 0 \end{cases}\right\} & \text{in } B_1, \\ \left.\begin{cases} w_1 = v_1, \quad w_2 = v_2 \\ x_1[(2\mu+\lambda)\partial_1 w_1 + \lambda\partial_2 w_2] + x_2[\mu(\partial_1 w_2 + \partial_2 w_1)] \\ \quad = x_1[(2\mu+\lambda)\partial_1 v_1 + \lambda\partial_2 v_2] + x_2[\mu(\partial_1 v_2 + \partial_2 v_1)] \\ x_1[\mu(\partial_1 w_2 + \partial_2 w_1)] + x_2[\lambda\partial_1 w_1 + (2\mu+\lambda)\partial_2 w_2] \\ \quad = x_1[\mu(\partial_1 v_2 + \partial_2 v_1)] + x_2[\lambda\partial_1 v_1 + (2\mu+\lambda)\partial_2 v_2] \end{cases}\right\} & \text{on } \partial B_1, \\ \left.\begin{cases} x_1[(2\mu+\lambda)\partial_1 w_1 + \lambda\partial_2 w_2] + x_2[\mu(\partial_1 w_2 + \partial_2 w_1)] = 0 \\ x_1[\mu(\partial_1 w_2 + \partial_2 w_1)] + x_2[\lambda\partial_1 w_1 + (2\mu+\lambda)\partial_2 w_2] = 0 \end{cases}\right\} & \text{on } \partial B_{1/2}. \end{cases}$$

Here, note that $\nu = 2x$ on $\partial B_{1/2}$. Thus, any solution $(w(r), v(r))$ satisfies

$$\begin{cases} r^2 w''(r) + r w'(r) + (\alpha_n^2 r^2 - 1) w(r) = 0, \quad 1/2 < r < 1, \\ r^2 v''(r) + r v'(r) + (\alpha_1^2 r^2 - 1) v(r) = 0, \quad 0 < r < 1, \\ w(1) = v(1), \quad w'(1) = v'(1), \\ 2\lambda w(1/2) + (2\mu+\lambda) w'(1/2) = 0, \\ |v(0)| < \infty. \end{cases}$$

Verifying the boundary conditions implies

$$\begin{bmatrix} 0 & f_1(k) & f_2(k) \\ J_1(\alpha_1) & -J_1(\alpha_n) & -Y_1(\alpha_n) \\ \alpha_1 J_1'(\alpha_1) & -\alpha_n J_1'(\alpha_n) & -\alpha_n Y_1'(\alpha_n) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0,$$

where

$$f_1(k) = 2\lambda J_1(\alpha_n/2) + (2\mu+\lambda)\alpha_n J_1'(\alpha_n/2),$$
$$f_2(k) = 2\lambda Y_1(\alpha_n/2) + (2\mu+\lambda)\alpha_n Y_1'(\alpha_n/2).$$

Likewise, let

$$F(k) := \det \begin{bmatrix} 0 & f_1(k) & f_2(k) \\ J_1(\alpha_1) & -J_1(\alpha_n) & -Y_1(\alpha_n) \\ \alpha_1 J_1'(\alpha_1) & -\alpha_n J_1'(\alpha_n) & -\alpha_n Y_1'(\alpha_n) \end{bmatrix}, \tag{22}$$

then k is an interior transmission eigenvalue if, and only if, k is a root of $F(k) = 0$. The existence of real eigenvalues can also be showed in the same manner. Figure 2 exhibits a numerical experiment for $\mu = 1$, $\lambda = 1$, and $n = 1/2$. The first four postive real eigenvalues are $k_1 = 3.9828$, $k_2 = 14.5026$, $k_3 = 22.6838$, $k_4 = 31.3170$.

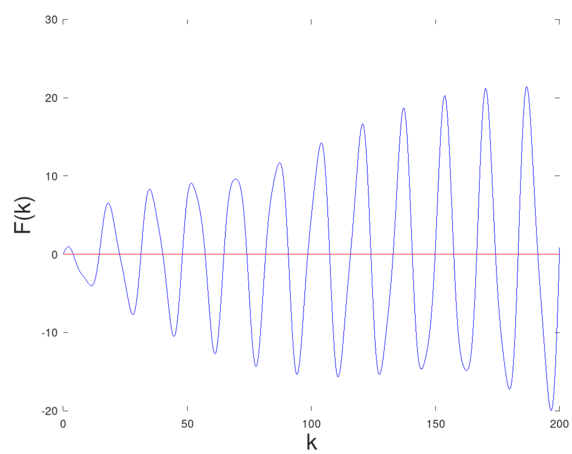

FIGURE 2. Plot of $F(k)$ in (22) with $\mu = 1$, $\lambda = 1$ and $n = 1/2$ (GNU Octave).

These experiments suggest that there exists a discrete set of real interior transmission eigenvalues for (18). We now ready to prove this statement.

3.1. **Variational formulation.** Similar to the method used in Section 2, we would like to derive a variational formula that is equivalent to (18). Define the Hilbert space

$$\mathcal{H} := \left\{ \mathbf{u} \in (H_0^1(D))^d \;\middle|\; \begin{array}{l} \nabla\cdot(\mathbb{C} : \nabla \mathbf{u}) \in (L^2(D\setminus\overline{D_0}))^d \\ (\mathbb{C} : \nabla\mathbf{u})\nu = 0 \quad \text{on } \partial D \end{array} \right\} \tag{23}$$

with norm $\|\mathbf{u}\|_{\mathcal{H}}^2 = \|\mathbf{u}\|_{H^1(D)}^2 + \|\nabla\cdot(\mathbb{C} : \nabla\mathbf{u})\|_{L^2(D\setminus\overline{D_0})}^2$.

Lemma 3.1. *Assume that* $0 \neq k \in \mathbf{R}$ *and either* $n^* < 1$ *or* $n_* > 1$*. Suppose that* $(\mathbf{v}, \mathbf{w}) \in (H^1(D))^d \times (H^1(D\setminus\overline{D_0}))^d$ *is a solution to* (18)*. Define*

$$\mathbf{u} := \begin{cases} \mathbf{w} - \mathbf{v} & \text{in } D\setminus\overline{D_0}, \\ -\mathbf{v} & \text{in } D_0. \end{cases} \tag{24}$$

Then $\mathbf{u} \in \mathcal{H}$ *and* $\mathbf{u}$ *satisfies*

$$a_k(\mathbf{u}, \mathbf{u}') = 0 \quad \textit{for all} \ \ \mathbf{u}' \in \mathcal{H}, \tag{25}$$

where

$$\begin{aligned} a_k(\mathbf{u}, \mathbf{u}') = &\int_{D \setminus \overline{D_0}} (1-n)^{-1} (\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 \mathbf{u}) \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')} \, dx \\ &+ k^2 \int_D \nabla \mathbf{u} : \mathbb{C} : \overline{\nabla \mathbf{u}'} \, dx - k^4 \int_D \mathbf{u} \cdot \overline{\mathbf{u}'} \, dx. \end{aligned}$$

Proof. Let $\tilde{\mathbf{u}} := \mathbf{w} - \mathbf{v}$ in $D \setminus \overline{D_0}$, then from (18), we have

$$\begin{cases} \nabla \cdot (\mathbb{C} : \nabla \tilde{\mathbf{u}}) + k^2 n \tilde{\mathbf{u}} = k^2 (1-n) \mathbf{v} & \text{in } D \setminus \overline{D_0}, \\ \nabla \cdot (\mathbb{C} : \nabla \mathbf{v}) + k^2 \mathbf{v} = 0 & \text{in } D, \\ \tilde{\mathbf{u}} = 0 \quad \text{and} \quad (\mathbb{C} : \nabla \tilde{\mathbf{u}}) \nu = 0 & \text{on } \partial D, \\ \tilde{\mathbf{u}} = -\mathbf{v} & \text{on } \partial D_0. \end{cases} \tag{26}$$

Testing the second equation of (26) by $\mathbf{u}' \in \mathcal{H}$, we obtain that

$$\begin{aligned} 0 &= \int_D \nabla \cdot (\mathbb{C} : \nabla \mathbf{v}) \cdot \overline{\mathbf{u}'} \, dx + k^2 \int_D \mathbf{v} \cdot \overline{\mathbf{u}'} \, dx \\ &= \int_{\partial D} ((\mathbb{C} : \nabla \mathbf{v}) \nu) \cdot \overline{\mathbf{u}'} \, dS - \int_D \nabla \mathbf{v} : \mathbb{C} : \overline{\nabla \mathbf{u}'} \, dx + k^2 \int_D \mathbf{v} \cdot \overline{\mathbf{u}'} \, dx, \end{aligned}$$

which implies

$$- \int_D \nabla \mathbf{v} : \mathbb{C} : \overline{\nabla \mathbf{u}'} \, dx + k^2 \int_D \mathbf{v} \cdot \overline{\mathbf{u}'} \, dx = 0,$$

because $\mathbf{u}' = \mathbf{0}$ on ∂D. To proceed further from the formula above, we can derive

$$\begin{aligned} 0 = &- \int_{D_0} \nabla \mathbf{v} : \mathbb{C} : \overline{\nabla \mathbf{u}'} \, dx + k^2 \int_D \mathbf{v} \cdot \overline{\mathbf{u}'} \, dx \\ &- \int_{D \setminus \overline{D_0}} \nabla \mathbf{v} : \mathbb{C} : \overline{\nabla \mathbf{u}'} \, dx \\ = &- \int_{\partial (D \setminus \overline{D_0})} \mathbf{v} \cdot \overline{((\mathbb{C} : \nabla \mathbf{u}') \nu)} \, dS - \int_{D_0} \nabla \mathbf{v} : \mathbb{C} : \overline{\nabla \mathbf{u}'} \, dx \\ &+ k^2 \int_D \mathbf{v} \cdot \overline{\mathbf{u}'} \, dx + \int_{D \setminus \overline{D_0}} \mathbf{v} \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}'))} \, dx \\ = &- \int_{\partial (D \setminus \overline{D_0})} \mathbf{v} \cdot \overline{((\mathbb{C} : \nabla \mathbf{u}') \nu)} \, dS - \int_{D_0} \nabla \mathbf{v} : \mathbb{C} : \overline{\nabla \mathbf{u}'} \, dx \\ &+ k^2 \int_{D_0} \mathbf{v} \cdot \overline{\mathbf{u}'} \, dx + \int_{D \setminus \overline{D_0}} \mathbf{v} \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')} \, dx. \end{aligned} \tag{27}$$

From the first equation of (26), it follows that

$$\begin{aligned} \mathbf{v} &= \frac{1}{k^2} (1-n)^{-1} (\nabla \cdot (\mathbb{C} : \nabla \tilde{\mathbf{u}}) + k^2 n \tilde{\mathbf{u}}) \\ &= \frac{1}{k^2} (1-n)^{-1} (\nabla \cdot (\mathbb{C} : \nabla \tilde{\mathbf{u}}) + k^2 \tilde{\mathbf{u}} + k^2 n \tilde{\mathbf{u}} - k^2 \tilde{\mathbf{u}}) \\ &= \frac{1}{k^2} (1-n)^{-1} (\nabla \cdot (\mathbb{C} : \nabla \tilde{\mathbf{u}}) + k^2 \tilde{\mathbf{u}}) + \frac{1}{k^2} (1-n)^{-1} k^2 (n-1) \tilde{\mathbf{u}} \\ &= \frac{1}{k^2} (1-n)^{-1} (\nabla \cdot (\mathbb{C} : \nabla \tilde{\mathbf{u}}) + k^2 \tilde{\mathbf{u}}) - \tilde{\mathbf{u}}. \end{aligned} \tag{28}$$

Combining (27) and (28) gives

$$
\begin{aligned}
0 = &- \int_{\partial(D\setminus\overline{D_0})} \mathbf{v}\cdot\overline{((\mathbb{C}:\nabla\mathbf{u}')\nu)}\,dS \\
&- \int_{D_0} \nabla\mathbf{v}:\mathbb{C}:\overline{\nabla\mathbf{u}'}\,dx + k^2\int_{D_0}\mathbf{v}\cdot\overline{\mathbf{u}'}\,dx \\
&+ \frac{1}{k^2}\int_{D\setminus\overline{D_0}}(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\tilde{\mathbf{u}})+k^2\tilde{\mathbf{u}})\cdot\overline{(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}')+k^2\mathbf{u}')}\,dx \\
&- \int_{D\setminus\overline{D_0}}\tilde{\mathbf{u}}\cdot\overline{(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}')+k^2\mathbf{u}')}\,dx.
\end{aligned}
$$

Multiplying the above equation by k^2, we have

$$
\begin{aligned}
0 &= \int_{D\setminus\overline{D_0}}(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\tilde{\mathbf{u}})+k^2\tilde{\mathbf{u}})\cdot\overline{(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}')+k^2\mathbf{u}')}\,dx \\
&\quad - k^2\int_{D_0}\nabla\mathbf{v}:\mathbb{C}:\overline{\nabla\mathbf{u}'}\,dx + k^4\int_{D_0}\mathbf{v}\cdot\overline{\mathbf{u}'}\,dx \\
&\quad - k^2\int_{D\setminus\overline{D_0}}\tilde{\mathbf{u}}\cdot\overline{(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}'))}\,dx - k^4\int_{D\setminus\overline{D_0}}\tilde{\mathbf{u}}\cdot\overline{\mathbf{u}'}\,dx \\
&\quad - k^2\int_{\partial(D\setminus\overline{D_0})}\mathbf{v}\cdot\overline{((\mathbb{C}:\nabla\mathbf{u}')\nu)}\,dS \\
&= \int_{D\setminus\overline{D_0}}(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\tilde{\mathbf{u}})+k^2\tilde{\mathbf{u}})\cdot\overline{(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}')+k^2\mathbf{u}')}\,dx \\
&\quad - k^2\int_{D_0}\nabla\mathbf{v}:\mathbb{C}:\overline{\nabla\mathbf{u}'}\,dx + k^4\int_{D_0}\mathbf{v}\cdot\overline{\mathbf{u}'}\,dx \\
&\quad + k^2\int_{D\setminus\overline{D_0}}\nabla\tilde{\mathbf{u}}:\mathbb{C}:\overline{\nabla\mathbf{u}'}\,dx - k^4\int_{D\setminus\overline{D_0}}\tilde{\mathbf{u}}\cdot\overline{\mathbf{u}'}\,dx \\
&\quad - k^2\int_{\partial(D\setminus\overline{D_0})}(\tilde{\mathbf{u}}+\mathbf{v})\cdot\overline{((\mathbb{C}:\nabla\mathbf{u}')\nu)}\,dS.
\end{aligned}
$$

Since $\tilde{\mathbf{u}} = -\mathbf{v}$ on ∂D_0 and $(\mathbb{C}:\nabla\mathbf{u}')\nu = \mathbf{0}$ on ∂D (definition of $\mathcal{H}$), we get that

$$
-k^2\int_{\partial(D\setminus\overline{D_0})}(\tilde{\mathbf{u}}+\mathbf{v})\cdot\overline{((\mathbb{C}:\nabla\mathbf{u}')\nu)}\,dS = 0.
$$

Define

$$
\mathbf{u} := \begin{cases} \tilde{\mathbf{u}} & \text{in } D\setminus\overline{D_0} \\ -\mathbf{v} & \text{in } D_0. \end{cases}
$$

Then it is not difficult to see that $\mathbf{u}\in\mathcal{H}$ and thus the lemma follows. $\square$

We now prove the opposite implication.

Lemma 3.2. *Assume that $0 \neq k \in \mathbf{R}$ and either $n^* < 1$ or $n^* > 1$. Suppose $\mathbf{u} \in \mathcal{H}$ satisfies (25). Then $\mathbf{u}$ satisfies*

$$\begin{cases} (\tilde{\Delta}_{\mathbb{C}} + k^2)(1-n)^{-1}(\tilde{\Delta}_{\mathbb{C}} + k^2 n)\mathbf{u} = 0 & \text{in} \quad D \setminus \overline{D_0}, \\ \nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 \mathbf{u} = 0 & \text{in} \quad D_0, \\ \Big((1-n)^{-1}(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 n \mathbf{u}) \Big)^+ = -k^2 \mathbf{u}^- & \\ \big(\mathbb{C} : \nabla \big[(1-n)^{-1}(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 n \mathbf{u}) \big] \big)^+ \nu & \text{on} \quad \partial D_0, \\ \quad = -k^2 (\mathbb{C} : \nabla \mathbf{u}^-)\nu & \end{cases} \tag{29}$$

where $\tilde{\Delta}_{\mathbb{C}} \mathbf{u} := \nabla \cdot (\mathbb{C} : \nabla \mathbf{u})$, ν is the unit outer normal to ∂D_0, and

$$\Phi^{\pm}(x) := \lim_{h \to 0_+} \Phi(x \pm h\nu) \quad \text{for} \quad x \in \partial D_0.$$

In particular, if we define

$$\mathbf{v} := \begin{cases} \dfrac{1}{k^2}(1-n)^{-1}(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 n \mathbf{u}) & \text{in} \quad D \setminus \overline{D_0}, \\ -\mathbf{u} & \text{in} \quad D_0, \end{cases} \tag{30}$$

and

$$\mathbf{w} = \mathbf{v} + \mathbf{u} \quad \text{in } D \setminus \overline{D_0}, \tag{31}$$

then $(\mathbf{v}, \mathbf{w}) \in (H^1(D))^d \times (H^1(D \setminus \overline{D_0}))^d$ solves (18).

Proof. Since $(\mathbb{C} : \nabla \mathbf{u}')\nu = 0$ on ∂D (by the definition of $\mathcal{H}$), from (25), we have

$$\begin{aligned} 0 &= \int_{D \setminus \overline{D_0}} (1-n)^{-1}(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 \mathbf{u}) \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')} \, dx \\ &\quad + k^2 \int_{\partial D} \mathbf{u} \cdot (\mathbb{C} : \overline{\nabla \mathbf{u}'})\nu \, dS - k^2 \int_D \mathbf{u} \cdot (\nabla \cdot (\mathbb{C} : \overline{\nabla \mathbf{u}'})) \, dx - k^4 \int_D \mathbf{u} \cdot \overline{\mathbf{u}'} \, dx \\ &= \int_{D \setminus \overline{D_0}} (1-n)^{-1}(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 \mathbf{u}) \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')} \, dx \\ &\quad - k^2 \int_{D \setminus \overline{D_0}} \mathbf{u} \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')} \, dx \\ &- k^2 \int_{D_0} \mathbf{u} \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')} \, dx \\ &= \int_{D \setminus \overline{D_0}} \Big[(1-n)^{-1}(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 \mathbf{u}) - k^2 \mathbf{u} \Big] \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')} \, dx \\ &\quad - k^2 \int_{D_0} \mathbf{u} \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')} \, dx \\ &= \int_{D \setminus \overline{D_0}} (1-n)^{-1}(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 n \mathbf{u}) \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')} \, dx \\ &\quad - k^2 \int_{D_0} \mathbf{u} \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')} \, dx. \end{aligned} \tag{32}$$

Moreover, using integration by parts, we can derive

$$\begin{aligned}
&\int_{D\setminus\overline{D_0}}(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u})+k^2n\mathbf{u})\cdot(\overline{\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}')})\,dx\\
&=\int_{\partial(D\setminus\overline{D_0})}(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u})+k^2n\mathbf{u})\cdot(\overline{\mathbb{C}:\nabla\mathbf{u}'})\nu\,dS\\
&\quad-\int_{D\setminus\overline{D_0}}\nabla\big[(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u})+k^2n\mathbf{u})\big]:(\overline{\mathbb{C}:\nabla\mathbf{u}'})\,dx\\
&=\int_{\partial(D\setminus\overline{D_0})}(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u})+k^2n\mathbf{u})\cdot(\overline{\mathbb{C}:\nabla\mathbf{u}'})\nu\,dS\\
&\quad-\int_{\partial(D\setminus\overline{D_0})}\Big(\big(\mathbb{C}:\nabla\big[(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u})+k^2n\mathbf{u})\big]\big)\nu\Big)\cdot\overline{\mathbf{u}'}\,dS\\
&\quad+\int_{D\setminus\overline{D_0}}\Big(\nabla\cdot\big(\mathbb{C}:\nabla\big[(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u})+k^2n\mathbf{u})\big]\big)\Big)\cdot\overline{\mathbf{u}'}\,dx.
\end{aligned}$$

In view of $\mathbf{u}'=0$ and $(\mathbb{C}:\nabla\mathbf{u}')\nu=0$ on ∂D, the formula above is reduced to

$$\begin{aligned}
&\int_{D\setminus\overline{D_0}}(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u})+k^2n\mathbf{u})\cdot(\overline{\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}')})\,dx\\
&=-\int_{\partial D_0}\Big((1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u})+k^2n\mathbf{u})\Big)^+\cdot(\overline{\mathbb{C}:\nabla\mathbf{u}'})\nu\,dS\\
&\quad+\int_{\partial D_0}\Big(\big(\mathbb{C}:\nabla\big[(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u})+k^2n\mathbf{u})\big]\big)\nu\Big)^+\cdot\overline{\mathbf{u}'}\,dS\\
&\quad+\int_{D\setminus\overline{D_0}}\Big(\nabla\cdot\big(\mathbb{C}:\nabla\big[(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u})+k^2n\mathbf{u})\big]\big)\Big)\cdot\overline{\mathbf{u}'}\,dx.
\end{aligned}\tag{33}$$

On the other hand, it is easy to see that

$$\begin{aligned}
&-k^2\int_{D_0}\mathbf{u}\cdot(\overline{\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}')})\,dx\\
&=-k^2\int_{\partial D_0}\mathbf{u}^-\cdot(\overline{\mathbb{C}:\nabla\mathbf{u}'})\nu\,dS+k^2\int_{\partial D_0}((\mathbb{C}:\nabla\mathbf{u})\nu)^-\cdot\overline{\mathbf{u}'}\,dS\\
&\quad-k^2\int_{D_0}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}))\cdot\overline{\mathbf{u}'}\,dx.
\end{aligned}\tag{34}$$

Combining (32), (33), and (34) gives

$$
\begin{aligned}
0 = &\int_{D\setminus\overline{D_0}} \Big(\nabla\cdot\big(\mathbb{C}:\nabla\big[(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}) + k^2 n\mathbf{u})\big]\big) \\
&\qquad + k^2\big[(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}) + k^2 n\mathbf{u})\big]\Big)\cdot\overline{\mathbf{u}'}\,dx \\
&- \int_{\partial D_0} \Big((1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}) + k^2 n\mathbf{u})\Big)^+ \cdot (\overline{\mathbb{C}:\nabla\mathbf{u}'})\nu\,dS \\
&+ \int_{\partial D_0} \Big(\big(\mathbb{C}:\nabla\big[(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}) + k^2 n\mathbf{u})\big]\big)\nu\Big)^+ \cdot\overline{\mathbf{u}'}\,dS \\
&- k^2\int_{D_0} (\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}) + k^2\mathbf{u})\cdot\overline{\mathbf{u}'}\,dx \\
&- k^2\int_{\partial D_0} \mathbf{u}^-\cdot(\overline{\mathbb{C}:\nabla\mathbf{u}'})\nu\,dS + k^2\int_{\partial D_0}((\mathbb{C}:\nabla\mathbf{u})\nu)^-\cdot\overline{\mathbf{u}'}\,dS \\
= &\int_{D\setminus\overline{D_0}} \Big[(\tilde{\Delta}_{\mathbb{C}} + k^2)(1-n)^{-1}(\tilde{\Delta}_{\mathbb{C}} + k^2 n)\mathbf{u}\Big]\cdot\overline{\mathbf{u}'}\,dx \\
&- k^2\int_{D_0} (\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}) + k^2\mathbf{u})\cdot\overline{\mathbf{u}'}\,dx \\
&+ \int_{\partial D_0} \Big\{\Big(\big(\mathbb{C}:\nabla\big[(1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}) + k^2 n\mathbf{u})\big]\big)\nu\Big)^+ \\
&\qquad\qquad + k^2((\mathbb{C}:\nabla\mathbf{u})\nu)^-\Big\}\cdot\overline{\mathbf{u}'}\,dS \\
&- \int_{\partial D_0} \Big\{\Big((1-n)^{-1}(\nabla\cdot(\mathbb{C}:\nabla\mathbf{u}) + k^2 n\mathbf{u})\Big)^+ + k^2\mathbf{u}^-\Big\}\cdot(\overline{\mathbb{C}:\nabla\mathbf{u}'})\nu\,dS,
\end{aligned}
$$

which established (29). Finally, from (29), it is not difficult to check that if $(\mathbf{v},\mathbf{w})$ defined by (30), (31), then it solves (18). ☐

By Lemma 3.1, we can prove that if $k\neq 0$ is an interior transmission eigenvalue of (18), then k is also an eigenvalue of the variational problem (25) provided solutions of $\nabla\cdot(\mathbb{C}:\nabla\mathbf{v}) + k^2\mathbf{v} = 0$ satisfy the *unique continuation property* (UCP).

Corollary 1. *Let k be a nonzero real number. If k is an interior transmission eigenvalue of* (18)*, then k is also an eigenvalue of the variational problem* (25) *provided the* UCP *stated above holds. Namely, for such k, there exists nontrivial $\mathbf{u}\in\mathcal{H}$ for which* (25) *holds.*

Conversely, if k is an eigenvalue of the variational problem (25)*, then k is an interior transmission eigenvalue of* (18)*.*

Remark 2. The UCP for solutions of general elasticity system remains an open problem. However, for isotropic case, the UCP holds for rather general Lamé coefficients, see [13, 24, 25]. Also, the UCP for general elliptic systems may not hold, see [21] for counterexamples. The UCP also holds for the Lamé eigenfunctions, see e.g. [14].

Proof. Let k be an interior transmission eigenvalue of (18) with corresponding eigenfunction $(\mathbf{v},\mathbf{w})\in(H^1(D))^d\times(H^1(D\setminus\overline{D_0}))^d$. In view of Lemma 3.1, we only need to show that $\mathbf{u}$ defined in (24) is nontrivial. Suppose on the contrary, such $\mathbf{u}$ is trivial. Then we have $\mathbf{w} = \mathbf{v}$ in $D \setminus \overline{D_0}$ and $\mathbf{v} = 0$ in D_0. So the UCP will imply $\mathbf{v} = 0$ in D and hence $\mathbf{w} = 0$ in $D \setminus \overline{D_0}$. This leads to a contradiction.

Conversely, for nontrivial $\mathbf{u} \in \mathcal{H}$, we only need to show that $(\mathbf{v}, \mathbf{w})$ defined in (30) and (31) are both non-trivial. Suppose on the contrary, such $(\mathbf{v}, \mathbf{w})$ is trivial. From (30), we know that $\mathbf{u} = 0$ in D_0. From (31), $\mathbf{u} = 0$ in $D \setminus \overline{D_0}$. Since $\mathbf{u} \in \mathcal{H}$, we conclude $\mathbf{u} = 0$ in D. This leads to a contradiction. We want to point out that the UCP is not needed in proving this implication. □

3.2. **Discreteness of the interior transmission eigenvalues.** Based on Lemma 3.1 and Corollary 1, we can prove the discreteness of real interior transmission eigenvalues of (18).

Theorem 3.3. *Assume that $0 < n_* \leq n(x) \leq n^* < 1$ and the* UCP *holds for the corresponding elasticity tensor $\mathbb{C}$. Then the set of real interior transmission eigenvalues of* (18) *is discrete.*

Proof. By Corollary 1, it suffices to show that (25) has at most discrete real eigenvalues. We will follow the same strategy used in Section 2. Write $a_k(\bullet, \bullet) = b_k(\bullet, \bullet) - k^4 c(\bullet, \bullet)$, where

$$
\begin{aligned}
b_k(\mathbf{u}, \mathbf{u}') = &\int_{D \setminus \overline{D_0}} (1-n)^{-1} (\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 \mathbf{u}) \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}') + k^2 \mathbf{u}')}\, dx \\
&+ k^2 \int_D \nabla \mathbf{u} : \mathbb{C} : \overline{\nabla \mathbf{u}'}\, dx + k^4 \int_D \mathbf{u} \cdot \overline{\mathbf{u}'}\, dx
\end{aligned} \tag{35}
$$

and

$$
c(\mathbf{u}, \mathbf{u}') = 2 \int_D \mathbf{u} \cdot \overline{\mathbf{u}'}\, dx. \tag{36}
$$

By Riesz's representation theorem, there exists a unique bounded linear operator $\mathcal{C} : \mathcal{H} \to \mathcal{H}$ such that

$$
c(\mathbf{u}, \mathbf{u}') = (\mathcal{C}\mathbf{u}, \mathbf{u}')_{\mathcal{H}}.
$$

We first want to show that $a_k(\bullet, \bullet)$ induces a Fredholm operator depending analytically on k.

Claim. b_k is coercive for all $k \neq 0$.

Proof. Denote $\alpha := (1 - n_*)^{-1}$. Then

$$
\begin{aligned}
b_k(\mathbf{u}, \mathbf{u}) = &\int_{D \setminus \overline{D_0}} (1-n)^{-1} (\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 \mathbf{u}) \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 \mathbf{u})}\, dx \\
&+ k^2 \int_D \nabla \mathbf{u} : \mathbb{C} : \overline{\nabla \mathbf{u}}\, dx + k^4 \|\mathbf{u}\|^2_{L^2(D)} \\
\geq &\ \alpha \|\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}) + k^2 \mathbf{u}\|^2_{L^2(D \setminus \overline{D_0})} + k^2 \kappa_1 \|\nabla \mathbf{u}\|^2_{L^2(D)} + k^4 \|\mathbf{u}\|^2_{L^2(D)} \\
\geq &\ \alpha X^2 - 2\alpha XY + (\alpha + 1) Y^2 + k^2 \kappa_1 \|\nabla \mathbf{u}\|^2_{L^2(D)} + k^4 \|\mathbf{u}\|^2_{L^2(D_0)},
\end{aligned}
$$

where $X = \|\nabla \cdot (\mathbb{C} : \nabla \mathbf{u})\|_{L^2(D\setminus\overline{D_0})}$ and $Y = k^2\|\mathbf{u}\|_{L^2(D\setminus\overline{D_0})}$. For each $\epsilon > 0$, we have the following inequality (see e.g. [6])

$$\begin{aligned}
&\alpha X^2 - 2\alpha XY + (\alpha+1)Y^2 \\
&= \epsilon\left(Y - \frac{\alpha}{\epsilon}X\right)^2 + \alpha\left(1 - \frac{\alpha}{\epsilon}\right)X^2 + (1+\alpha-\epsilon)Y^2 \\
&\geq \alpha\left(1 - \frac{\alpha}{\epsilon}\right)X^2 + (1+\alpha-\epsilon)Y^2,
\end{aligned}$$

and thus

$$\begin{aligned}
b_k(\mathbf{u},\mathbf{u}) \geq \alpha\left(1 - \frac{\alpha}{\epsilon}\right)&\|\nabla \cdot (\mathbb{C} : \nabla \mathbf{u})\|^2_{L^2(D\setminus\overline{D_0})} + (1+\alpha-\epsilon)k^4\|\mathbf{u}\|^2_{L^2(D\setminus\overline{D_0})} \\
&+ k^2\kappa_1\|\nabla\mathbf{u}\|^2_{L^2(D)} + k^4\|\mathbf{u}\|^2_{L^2(D_0)}.
\end{aligned} \tag{37}$$

Choosing $\alpha < \epsilon < \alpha + 1$ immediately shows that b_k is coercive. □

Claim. $\mathcal{C} : \mathcal{H} \to \mathcal{H}$ is compact.

Proof. Since

$$|(\mathcal{C}\mathbf{u}, \mathbf{u}')_{\mathcal{H}}| \leq 2\|\mathbf{u}\|_{L^2(D)}\|\mathbf{u}'\|_{L^2(D)},$$

we obtain

$$\|\mathcal{C}\mathbf{u}\|_{\mathcal{H}} = \sup_{\|\mathbf{u}'\|_{\mathcal{H}}=1} |(C\mathbf{u}, \mathbf{u}')_{\mathcal{H}}| \leq 2\|\mathbf{u}\|_{L^2(D)},$$

that is, $\mathcal{C} : L^2(D) \to \mathcal{H}$ is continuous. By the compactness of the embedding $\iota : \mathcal{H} \hookrightarrow L^2(D)$, we conclude that

$$\mathcal{C} \cong \mathcal{C} \circ \iota : \mathcal{H} \to \mathcal{H}$$

is compact. □

We have showed that the operator defined by a_k is of Fredholm type and depends analytically on k. To finish the proof of Theorem 3.3, we only need to show a_k is coercive for some k. Since $\mathbf{u} \in \mathcal{H} \subset (H_0^1(D))^d$, the Poincaré inequality holds, i.e.,

$$\|\mathbf{u}\|^2_{L^2(D)} \leq C\|\nabla\mathbf{u}\|^2_{L^2(D)}$$

for some positive constant C. From (37), it follows that

$$\begin{aligned}
a_k(\mathbf{u},\mathbf{u}) &= b_k(\mathbf{u},\mathbf{u}) - k^4 c(\mathbf{u},\mathbf{u}) \\
&\geq \alpha\left(1 - \frac{\alpha}{\epsilon}\right)\|\nabla \cdot (\mathbb{C} : \nabla \mathbf{u})\|^2_{L^2(D\setminus\overline{D_0})} + (1+\alpha-\epsilon)k^4\|\mathbf{u}\|^2_{L^2(D\setminus\overline{D_0})} \\
&\quad + k^2\kappa_1\|\nabla\mathbf{u}\|^2_{L^2(D)} + k^4\|\mathbf{u}\|^2_{L^2(D_0)} - 2k^4\|\mathbf{u}\|^2_{L^2(D)} \\
&\geq \alpha\left(1 - \frac{\alpha}{\epsilon}\right)\|\nabla \cdot (\mathbb{C} : \nabla \mathbf{u})\|^2_{L^2(D\setminus\overline{D_0})} + (1+\alpha-\epsilon)k^4\|\mathbf{u}\|^2_{L^2(D\setminus\overline{D_0})} \\
&\quad + k^2(\kappa_1 - 2Ck^2)\|\nabla\mathbf{u}\|^2_{L^2(D)} + k^4\|\mathbf{u}\|^2_{L^2(D_0)},
\end{aligned} \tag{38}$$

which implies that a_k is coercive for k^2 sufficiently small (of course, we still choose $\alpha < \epsilon < \alpha + 1$). Theorem 3.3 then follows from the standard analytic Fredholm theorem.

□

3.3. **Existence of real interior transmission eigenvalues.** Again, using Lemma 3.2 and Corollary 1, we can establish the existence of infinitely many real interior transmission eigenvalues of (18) when $\mathbb{C}$ is a constant elasticity tensor.

Theorem 3.4. *Assume that $\mathbb{C}$ is constant and $n(x)$ satisfies $0 < n_* \le n(x) \le n^* < 1$. Then there exist infinitely many discrete real inteior transmission eigenvalues.*

The proof of Theorem 3.4 relies the following fundamental theorem, see [6, 12].

Lemma 3.5. *Let $k \mapsto \mathcal{B}_k$ be a continuous mapping from the open interval $(0, \infty)$ to the set of self-adjoint and positive definite bounded linear operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{C}$ be a self-adjoint and non-negative compact operator on $\mathcal{H}$. If there exist two positive constants $k_0 > 0$ and $k_1 > 0$ such that*

(i) $\mathcal{B}_{k_0} - k_0^4\mathcal{C}$ is positive on $\mathcal{H}$,
(ii) $\mathcal{B}_{k_1} - k_1^4\mathcal{C}$ is non-positive on a m-dimensional subspace of $\mathcal{H}$,

then each of the equations $\lambda_j(k) = k^4$ $(j = 1, \cdots, m)$ has at least one solution in $[k_0, k_1]$, where $\lambda_j(k)$ is the j^{th} eigenvalue of $\mathcal{B}_k$ with respect to $\mathcal{C}$, i.e. $\ker(\mathcal{B}_k - \lambda_j(k)\mathcal{C}) \neq \{0\}$.

Proof of Theorem 3.4. It is easy to see that the operators $\mathcal{B}_k$ and $\mathcal{C}$ defined by sesquilinear forms b_k and c in (35) and (36), respectively, are self-adjoint and postive-definite. Estimate (38) implies that (i) of Lemma 3.5 is satisfied. It remains to verify (ii) of Lemma 3.5. We will follow the argument used in [6]. Let B_r^j, $j = 1, \cdots, M(r)$, be $M(r)$ disjoint balls of radius r in $D \setminus \overline{D_0}$. Let k_1 be the first positive transmission eigenvalue corresponding to the ITEP

$$\begin{cases} \nabla \cdot (\mathbb{C} : \nabla \mathbf{w}_j) + k^2 n^* \mathbf{w}_j = 0 & \text{in} \quad B_r^j, \\ \nabla \cdot (\mathbb{C} : \nabla \mathbf{v}_j) + k^2 \mathbf{v}_j = 0 & \text{in} \quad B_r^j, \\ \mathbf{w}_j = \mathbf{v}_j \quad \text{and} \quad (\mathbb{C} : \nabla \mathbf{w}_j)\nu = (\mathbb{C} : \nabla \mathbf{v}_j)\nu & \text{on} \quad \partial B_r^j \end{cases}$$

(see [1]). Notice that for a fixed r, k_1 is independent of the locations of B_r^j. Therefore, let $\mathbf{u}_j = \mathbf{w}_j - \mathbf{v}_j$, then $\mathbf{u}_j \in \mathcal{H}_j := \{\mathbf{u} \in (H^1(B_r^j))^d : \mathbf{u} = 0 \text{ and } (\mathbb{C} : \nabla \mathbf{u})\nu = 0 \text{ on } \partial B_r^j\}$ satisfies

$$\begin{aligned} 0 = &\int_{B_r^j} (1 - n^*)^{-1} (\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}_j) + k_1^2 \mathbf{u}_j) \cdot \overline{(\nabla \cdot (\mathbb{C} : \nabla \mathbf{u}_j) + k_1^2 \mathbf{u}_j)}\, dx \\ &+ k_1^2 \int_{B_r^j} \nabla \mathbf{u}_j : \mathbb{C} : \overline{\nabla \mathbf{u}_j}\, dx - k_1^4 \int_{B_r^j} \mathbf{u}_j \cdot \overline{\mathbf{u}_j}\, dx. \end{aligned}$$

Let $\tilde{\mathbf{u}}_j$ be the zero extension of $\mathbf{u}_j$ to the whole of D. Since $\mathbf{u}_j \in \mathcal{H}_j$, $\tilde{\mathbf{u}}_j \in \mathcal{H}$. We now define an $M(r)$-dimensional subspace of $\mathcal{H}$ by

$$\mathcal{V} := \text{span}\{\tilde{\mathbf{u}}_j\}_{j=1}^{M(r)}.$$

Observe that $\tilde{\mathbf{u}}_j$ and $\tilde{\mathbf{u}}_k$ have disjoint supports when $j \neq k$. Now, if $\mathbf{u} = \sum_{j=1}^{M(r)} \tilde{\mathbf{u}}_j \in \mathcal{V}$, then

$$\begin{aligned}
&b_{k_1}(\mathbf{u},\mathbf{u}) - k_1^4 c(\mathbf{u},\mathbf{u}) \\
&= \sum_{j=1}^{M(r)} \Bigg(\int_{D\setminus\overline{D_0}} (1-n)^{-1} |\nabla\cdot(\mathbb{C} : \nabla\tilde{\mathbf{u}}_j) + k_1^2 \tilde{\mathbf{u}}_j|^2 \, dx \\
&\qquad + k_1^2 \int_D \nabla\tilde{\mathbf{u}}_j : \mathbb{C} : \overline{\nabla\tilde{\mathbf{u}}_j} \, dx - k_1^4 \int_D \tilde{\mathbf{u}}_j \cdot \overline{\tilde{\mathbf{u}}_j} \, dx \Bigg) \\
&= \sum_{j=1}^{M(r)} \Bigg(\int_{B_r^j} (1-n)^{-1} |\nabla\cdot(\mathbb{C} : \nabla\mathbf{u}_j) + k_1^2 \mathbf{u}_j|^2 \, dx \\
&\qquad + k_1^2 \int_{B_r^j} \nabla\mathbf{u}_j : \mathbb{C} : \overline{\nabla\mathbf{u}_j} \, dx - k_1^4 \int_{B_r^j} \mathbf{u}_j \cdot \overline{\mathbf{u}_j} \, dx \Bigg) \\
&\leq \sum_{j=1}^{M(r)} \Bigg(\int_{B_r^j} (1-n^*)^{-1} |\nabla\cdot(\mathbb{C} : \nabla\mathbf{u}_j) + k_1^2 \mathbf{u}_j|^2 \, dx \\
&\qquad + k_1^2 \int_{B_r^j} \nabla\mathbf{u}_j : \mathbb{C} : \overline{\nabla\mathbf{u}_j} \, dx - k_1^4 \int_{B_r^j} \mathbf{u}_j \cdot \overline{\mathbf{u}_j} \, dx \Bigg) = 0.
\end{aligned}$$

So combining Lemma 3.5 and letting $M(r) \to \infty$, we conclude that there exist infinitely many real transmission eigenvalues. □

3.4. **Monotonicity for the first positive interior transmission eigenvalue with respect to the obstacle.** Similar to the result in [6], we also prove an interesting result connecting the monotonicity of the first positive interior transmission eigenvalue and the obstacle D_0 in (18). Let $k_1(D_0)$ be the first positive transmission eigenvalue, $\mathcal{H}(D_0)$ be the Hilbert space given in (23), and $a_k^{D_0}(\bullet,\bullet) = b_k^{D_0}(\bullet,\bullet) - k^4 c(\bullet,\bullet)$ be the sesquilinear form defined by (25).

Theorem 3.6. *Let $0 < n_* \leq n(x) \leq n^* < 1$. If $D_0 \subset D_0'$, then*

$$k_1(D_0) \geq k_1(D_0'). \tag{39}$$

Proof. Let $(\mathbf{v},\mathbf{w}) \in (H^1(D))^d \times (H^1(D\setminus\overline{D_0}))^d$ be the transmission eigenfunction corresponding to $k_1(D_0)$. Corollary 1 implies that there exists a nontrivial $\mathbf{u} \in \mathcal{H}(D_0)$ such that

$$a_{k_1(D_0)}^{D_0}(\mathbf{u},\mathbf{u}) = 0.$$

Since $D\setminus\overline{D_0'} \subset D\setminus\overline{D_0}$ and $0 < n(x) < 1$, then $\mathbf{u} \in \mathcal{H}(D_0) \subset \mathcal{H}(D_0')$ and

$$\begin{aligned}
0 = a_{k_1(D_0)}^{D_0}(\mathbf{u},\mathbf{u}) &= b_{k_1(D_0)}^{D_0}(\mathbf{u},\mathbf{u}) - (k_1(D_0))^4 c(\mathbf{u},\mathbf{u}) \\
&\geq \int_{D\setminus\overline{D_0'}} (1-n)^{-1} |\nabla\cdot(\mathbb{C} : \nabla\mathbf{u}) + (k_1(D_0))^2 \mathbf{u}|^2 \, dx \\
&\quad + (k_1(D_0))^2 \int_D \nabla\mathbf{u} : \mathbb{C} : \overline{\nabla\mathbf{u}} \, dx - (k_1(D_0))^4 \int_D \mathbf{u} \cdot \overline{\mathbf{u}'} \, dx \\
&= a_{k_1(D_0)}^{D_0'}(\mathbf{u},\mathbf{u}).
\end{aligned} \tag{40}$$

Now we want to show (39). If not, we have $k_1(D_0) < k_1(D_0')$. Recall from (38) that when k_0 is small

$$a_{k_0}^{D_0'}(\tilde{\mathbf{u}},\tilde{\mathbf{u}}) > 0, \quad \forall \; \tilde{\mathbf{u}} \in \mathcal{H}(D_0').$$

On the other hand, (40) means that

$$a^{D_0'}_{k_1(D_0)}(\mathbf{u},\mathbf{u}) \le 0 \quad \text{for some } \mathbf{u} \in \mathcal{H}(D_0').$$

Now by Lemma 3.5, there exists at least one eigenvalue in $[k_0, k_1(D_0)]$ corresponding to (18) with obstacle D_0'. This is a contradiction since $k_1(D_0')$ is the first positive eigenvalue. □

We end this paper by noting that since the first real transmission eigenvalue can be stably determined from measured scattering data (see [5], [7, Section 4.4], [22] for determination of real eigenvalues in the acoustic case, similar algorithms can be developed in elastic scattering), Theorem 3.6 could be used to obtain information about the sound-soft obstacle without knowing the (anisotropic) material properties of the media.

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