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# A NOTE ON TRANSMISSION EIGENVALUES IN ELECTROMAGNETIC SCATTERING THEORY

Fioralba Cakoni

Department of Mathematics
Rutgers University
Piscataway, NJ 08854-8019, USA

Shixu Meng

Institute of Applied Mathematics
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100190, China

Jingni Xiao*

Department of Mathematics
Rutgers University
Piscataway, NJ 08854-8019, USA

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Abstract. This short note was motivated by our efforts to investigate whether there exists a half plane free of transmission eigenvalues for Maxwell's equations. This question is related to solvability of the time domain interior transmission problem which plays a fundamental role in the justification of linear sampling and factorization methods with time dependent data. Our original goal was to adapt semiclassical analysis techniques developed in [21, 23] to prove that for some combination of electromagnetic parameters, the transmission eigenvalues lie in a strip around the real axis. Unfortunately we failed. To try to understand why, we looked at the particular example of spherically symmetric media, which provided us with some insight on why we couldn't prove the above result. Hence this paper reports our findings on the location of all transmission eigenvalues and the existence of complex transmission eigenvalues for Maxwell's equations for spherically stratified media. We hope that these results can provide reasonable conjectures for general electromagnetic media.

## 1. Introduction.

The transmission eigenvalue problem is intrinsic to scattering theory for inhomogeneous media. This eigenvalue problem appeared first in [13, 16], in connection with injectivity of the relative scattering operator. Transmission eigenvalues are related to interrogating frequencies for which there is an incident field that doesn't scatterer by the medium. The transmission eigenvalue problem has a deceptively simple formulation, namely two elliptic PDEs in a bounded domain

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* Corresponding author: jingni.xiao@rutgers.edu.

(one governs the wave propagation in the scattering medium and the other in the background that occupies the support of the medium) that share the same Cauchy data on the boundary, but presents a perplexing mathematical structure. In particular, it is a non-selfadjoint eigenvalue problem for a non-strongly elliptic operator. Hence the investigation of its spectral properties becomes challenging. For a comprehensive discussion on the transmission eigenvalue problem and its application to inverse acoustic and electromagnetic scattering theory we refer the reader to the monographs [3, 4].

We now present the mathematical formulation of the transmission eigenvalue problem for a dielectric inhomogeneity occupying a bounded simply connected region $D \subset \mathbb{R}^3$ with piece-wise smooth boundary $\partial D$ and outward normal vector $\nu$. For $x \in D$, let $\mu_0(x), \varepsilon_0(x)$ and $\mu(x), \varepsilon(x)$ denote magnetic permeability and electric permittivity of the background and inhomogeneity, respectively, which in general are positive definite matrix valued functions with $L^\infty(D)$ entries. The transmission eigenvalue problem is formulated as finding $\omega \in \mathbb{C}$ and nonzero fields $E_0$, $H_0$ and $E$, $H$ (corresponding to electromagnetic fields in the background and inhomogeneity, respectively) such that

$$\begin{cases} \mathrm{curl} E_0 - \mathtt{i}\omega\mu_0 H_0 = 0, \quad \mathrm{curl} H_0 + \mathtt{i}\omega\varepsilon_0 E_0 = 0, & \\ \quad\mathrm{curl} E - \mathtt{i}\omega\mu H = 0, \quad \mathrm{curl} H + \mathtt{i}\omega\varepsilon E = 0, & \text{in } D, \\ \quad\quad \nu \times (E - E_0) = \nu \times (H - H_0) = 0, & \text{on } \partial D, \end{cases} \tag{1}$$

Such values of $\omega \in \mathbb{C}$ are called *transmission eigenvalues* and the nonzero fields $E_0$, $H_0$ and $E$, $H$ are called the corresponding *transmission eigenfunctions*. Note that in the context of scattering theory, the eigenvalue parameter $\omega$, if real, corresponds to the interrogating frequency. The spectral properties of the transmission eigenvalue problem (1) essentially depend on the assumptions on the contrasts $\epsilon - \epsilon_0$ and $\mu - \mu_0$. Spectral questions central to the inverse scattering theory include (see, [3]): *discreteness* of the spectrum that is closely related to the determination of the support of inhomogeneity from scattering data using linear sampling and factorization methods, *location* of transmission eigenvalues in the complex plane that is essential to the development of the time domain linear sampling method and the *existence* of transmission eigenvalues as well as the *accurate determination* of real transmission eigenvalues from scattering data, which has became important since real transmission eigenvalues can be used to obtain information about the material properties of the scattering media.

The structure of spectrum of the transmission eigenvalue problem is better understood in the case of scalar inhomogeneous Helmholtz equations. The vector electromagnetic transmission eigenvalue problem presents additional complications and several questions still remain open. The state-of-the-art results on discreteness of the spectrum are presented in [8] (see references therein for additional results on discreteness) for anisotropic media under optimal assumptions on regularity and the sign of the contrasts $\epsilon - \epsilon_0$ and $\mu - \mu_0$. The completeness of eigenfunctions is proven in [14] for scalar valued function $\epsilon(x)$ and $\epsilon_0 = \mu_0 = \mu = 1$, such that $\epsilon(x) - 1$ is one sign near the boundary $\partial D$. If $\epsilon(x) - 1$ is one sign in the entire $D$, the existence of an infinite set of real eigenvalues is proven [5, 6]. The connection of transmission eigenvalues with non-scattering frequencies for Maxwell's equations and inhomogeneities with corners is discussed in [1]. The location of transmission eigenvalues in the complex plane for Maxwell's equations is not well-studied. The best to date result for Maxwell's equations is presented in [8], where more precisely it is shown that transmission eigenvalues lie inside any arbitrary small wedge around the real and imaginary axis. In contrast, for the scalar case of Helmholtz equation, much finer results on the location of transmission eigenvalues are available [15, 20, 21, 22, 23]. More importantly, for the Helmholtz equation, Vodev [23] is the first to show that for inhomogeneities with smooth support and coefficients, there are no transmission eigenvalues outside a horizontal strip around the real axis, i.e. all transmission eigenvalues have imaginary part uniformly bounded, under certain conditions for the contrasts of the medium. This result is fundamental to prove solvability of time dependent interior transmission problem via Fourier-Laplace technique, which is the fundamental ingredient in the mathematical justification of linear sampling and factorization methods for reconstructing the support of an inhomogeneity with time dependent data [7].

This short note addresses the location of electromagnetic transmission eigenvalues in the complex plane motivated by desire to eventually study the solvability of time dependent interior transmission problem for Maxwell's equations. In the process of trying to find whether there are possible combination of $\epsilon, \epsilon_0$ and $\mu, \mu_0$ for which transmission eigenvalues lie in a strip, we noticed that there are no conditions on these coefficients that can bring the transmission eigenvalue problem to the form for which one can adapt high frequency estimates of Vodev in [23], which allow for showing that the imaginary part of transmission eigenvalues is uniformly bounded. More specifically, restricting ourselves to spherically symmetric isotropic media, we show that the set of transmission eigenvalues for Maxwell's equations can be divided into two subsets, each corresponding to a transmission eigenvalue problem for Helmholtz equations. Unfortunately, for a given set of parameters $\epsilon, \epsilon_0$ and $\mu, \mu_0$ it is not possible to satisfy for both of these transmission eigenvalue problems for Helmholtz equations the assumptions in [23] to prove that the eigenvalues lie in a strip; See, Section 3.1. Our goal is simply to make this fact available hoping that it will motivate a correct way to approach the problem of the location of transmission eigenvalues for Maxwell's equations. We also prove the existence of complex electromagnetic transmission eigenvalues for the constant coefficient case in Section 3.2.

## 2. Preliminaries.

In this paper we consider spherically stratified media. Let $D = B_R \subset \mathbb{R}^3$ be a ball of radius $R$ centered at the origin. Let the electromagnetic coefficients $\epsilon(r)$ and $\mu(r)$ be functions of the radial variable $r := |x|$. For the reader convenience, we include some basic expansion formulas based on separation of variables for solutions of Maxwell's equations with non-constant radially symmetric coefficients.

Let $\hat{x} := x/|x|$ be the angular variable living on the unit sphere $\mathbb{S}^2$, and denote by $Y_\ell^m(\hat{x})$ with $-\ell \le m \le \ell$, $\ell = 0, 1, \dots$, the spherical harmonics which form an orthonormal set in $L^2(\mathbb{S}^2)$. We set

$$U_\ell^m(\hat{x}) := \frac{1}{\sqrt{\ell(\ell+1)}} \mathrm{Grad} Y_\ell^m(\hat{x}) \qquad \text{and} \qquad V_\ell^m(\hat{x}) := \hat{x} \times U_\ell^m(\hat{x})$$

where Grad denotes the surface gradient operator on the unit sphere $\mathbb{S}^2$. It is known that $\{U_\ell^m, V_\ell^m;\ -\ell \le m \le \ell, \ell \in \mathbb{N}_+\}$ form a complete orthonormal basis in the tangent space $L^2_t(\mathbb{S}^2) := \{F \in L^2(\mathbb{S}^2; \mathbb{C}^3);\ \hat{x} \cdot F = 0\}$ (see [9, 17]). Given $\ell \in \mathbb{N}$ and a constant $k$, we also introduce the "generalized" spherical Bessel functions

$j_\ell^E = j_\ell^E(r)$ and $j_\ell^H = j_\ell^H(r)$. They are defined as solutions of the ODEs

$$\varepsilon r \left( \frac{1}{\varepsilon} \left( r j_\ell^E \right)' \right)' + \left( \omega^2 \varepsilon \mu r^2 - \ell(\ell+1) \right) j_\ell^E = 0 \tag{2}$$

and

$$\mu r \left( \frac{1}{\mu} \left( r j_\ell^H \right)' \right)' + \left( \omega^2 \varepsilon \mu r^2 - \ell(\ell+1) \right) j_\ell^H = 0 \tag{3}$$

which behave like spherical Bessel function $j_\ell$ as $r \to 0$, i.e.,

$$\lim_{r\to 0} r^{-\ell} j_\ell^E(r) = \lim_{r\to 0} r^{-\ell} j_\ell^H(r) = \frac{\sqrt{\pi} k^\ell}{2^{\ell+1}\Gamma(\ell+3/2)}.$$

Hereinafter, "$'$" represents the derivative with respect to $r$.

**Remark 1.** *In the case when $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$, which are both constants, the functions $j_\ell^E$ and $j_\ell^H$ are the spherical Bessel functions. In particular, $j_\ell^E(r) = j_\ell^H(r) = j_\ell(kr)$, where $k = \omega\sqrt{\varepsilon_0 \mu_0}$ (see,* [17, Theorem 2.48]*).*

**Lemma 2.1.** *Let $\varepsilon = \varepsilon(r)$ and $\mu = \mu(r)$ with $r = |x|$. Then for any $\ell \in \mathbb{N}_+$ and $-\ell \le m \le \ell$, the two pairs of vector fields*

$$E_\ell^H(x) = \mathrm{curl}\left( x j_\ell^H Y_\ell^m(\hat{x}) \right), \qquad H_\ell^H(x) = \frac{1}{\mathtt{i}\omega\mu} \mathrm{curlcurl}\left( x j_\ell^H Y_\ell^m(\hat{x}) \right)$$

*and*

$$E_\ell^E(x) = -\frac{1}{\mathtt{i}\omega\varepsilon} \mathrm{curlcurl}\left( x j_\ell^E Y_\ell^m(\hat{x}) \right), \qquad H_\ell^E(x) = \mathrm{curl}\left( x j_\ell^E Y_\ell^m(\hat{x}) \right)$$

*both satisfy Maxwell's equations* $\mathrm{curl}E - \mathtt{i}\omega\mu H = 0$, $\mathrm{curl}H + \mathtt{i}\omega\varepsilon E = 0$.

*Proof.* Given $f = f(r)$ and $\ell \in \mathbb{N}_+$, by straightforward computations we have

$$\begin{aligned} \mathrm{curl}\left( f(r) Y_\ell^m(\hat{x})\hat{x} \right) &= -\sqrt{\ell(\ell+1)}\, \frac{f(r)}{r}\, V_\ell^m(\hat{x}), \\ \mathrm{curl}\left( f(r) U_\ell^m(\hat{x}) \right) &= \frac{(r f(r))'}{r}\, V_\ell^m(\hat{x}), \end{aligned} \tag{4}$$

and

$$\begin{aligned} \mathrm{curl}\left( f(r) V_\ell^m(\hat{x}) \right) &= -\sqrt{\ell(\ell+1)}\, \frac{f(r)}{r}\, Y_\ell^m(\hat{x})\, \hat{x} - \frac{(r f(r))'}{r}\, U_\ell^m(\hat{x}) \\ &= -\frac{1}{\sqrt{\ell(\ell+1)}}\, \mathrm{curl}\,\mathrm{curl}\left( r f(r) Y_\ell^m(\hat{x})\hat{x} \right). \end{aligned} \tag{5}$$

As a consequence,

$$E_\ell^H(x) = -\sqrt{\ell(\ell+1)}\, j_\ell^H(r)\, V_\ell^m(\hat{x}),$$

and

$$H_\ell^H(x) = \frac{\ell(\ell+1)}{\mathtt{i}\omega\mu}\, \frac{j_\ell^H(r)}{r}\, Y_\ell^m(\hat{x})\, \hat{x} + \frac{\sqrt{\ell(\ell+1)}}{\mathtt{i}\omega\mu}\, \frac{\left( r j_\ell^H(r) \right)'}{r}\, U_\ell^m(\hat{x}).$$

Hence,

$$\begin{aligned} &\frac{\mathtt{i}\omega\mu}{\sqrt{\ell(\ell+1)}} \left( \mathrm{curl} H_\ell^H + \mathtt{i}\omega\varepsilon E_\ell^H \right) \\ &= \left( -\ell(\ell+1) \frac{j_\ell^H(r)}{r^2} + \frac{\mu}{r} \left( \frac{\left( r j_\ell^H(r) \right)'}{\mu} \right)' + \omega^2 \varepsilon\mu j_\ell^H(r) \right) V_\ell^m(\hat{x}) = 0, \end{aligned}$$

where the second equality is due to the ODE (3) for $j_\ell^H$. With the same argument but using the ODE (2) for $j_\ell^E$ one can show that $\mathrm{curl}E_\ell^E - \mathtt{i}\omega\mu H_\ell^E = 0$. The other two equations, namely, $\mathrm{curl}E_\ell^H - \mathtt{i}\omega\mu H_\ell^H = 0$ and $\mathrm{curl}H_\ell^E + \mathtt{i}\omega\varepsilon E_\ell^E = 0$, are readily verified from the definitions of $E_\ell^H, H_\ell^H, E_\ell^E$ and $H_\ell^E$. □

**Lemma 2.2.** *Given $\varepsilon = \varepsilon(r)$ and $\mu = \mu(r)$ with $r = |x|$, let $E$ and $H$ in $C^1(B_R, \mathbb{C}^3)$ be a pair of solutions to Maxwell's equations $\mathrm{curl}E - \mathtt{i}\omega\mu H = 0$, $\mathrm{curl}H + \mathtt{i}\omega\varepsilon E = 0$ in $B_R$. Then there is a unique choice of coefficients $\{A_\ell^m, B_\ell^m \in \mathbb{C};\ \ell \in \mathbb{N}_+, -\ell \le m \le \ell\}$ such that*

$$
\begin{aligned}
E(x) = &\frac{1}{\mathtt{i}\omega\varepsilon}\sum_{\ell\ge1,\,|m|\le\ell} A_\ell^m\left(\sqrt{\ell(\ell+1)}\,\frac{j_\ell^E(r)}{r}\,Y_\ell^m(\hat{x})\,\hat{x} + \frac{\big(rj_\ell^E(r)\big)'}{r}\,U_\ell^m(\hat{x})\right)\\
&+\sum_{\ell\ge1,\,|m|\le\ell} B_\ell^m\, j_\ell^H(r)\,V_\ell^m(\hat{x})\\
=&\frac{1}{\mathtt{i}\omega\varepsilon}\sum_{\ell\ge1,\,|m|\le\ell}\frac{A_\ell^m}{\sqrt{\ell(\ell+1)}}\,\mathrm{curl}\,\mathrm{curl}\left(xj_\ell^E(r)Y_\ell^m(\hat{x})\right)\\
&-\sum_{\ell\ge1,\,|m|\le\ell}\frac{B_\ell^m}{\sqrt{\ell(\ell+1)}}\,\mathrm{curl}\left(xj_\ell^H(r)Y_\ell^m(\hat{x})\right),
\end{aligned}
\tag{6}
$$

*and*

$$
\begin{aligned}
H(x) = &-\frac{1}{\mathtt{i}\omega\mu}\sum_{\ell\ge1,\,|m|\le\ell} B_\ell^m\left(\sqrt{\ell(\ell+1)}\,\frac{j_\ell^H(r)}{r}\,Y_\ell^m(\hat{x})\,\hat{x} + \frac{\big(rj_\ell^H(r)\big)'}{r}U_\ell^m(\hat{x})\right)\\
&+\sum_{\ell\ge1,\,|m|\le\ell} A_\ell^m\, j_\ell^E(r)\,V_\ell^m(\hat{x})\\
=&-\frac{1}{\mathtt{i}\omega\mu}\sum_{\ell\ge1,\,|m|\le\ell}\frac{B_\ell^m}{\sqrt{\ell(\ell+1)}}\,\mathrm{curl}\,\mathrm{curl}\left(xj_\ell^H(r)Y_\ell^m(\hat{x})\right)\\
&-\sum_{\ell\ge1,\,|m|\le\ell}\frac{A_\ell^m}{\sqrt{\ell(\ell+1)}}\,\mathrm{curl}\left(xj_\ell^E Y_\ell^m(\hat{x})\right),
\end{aligned}
\tag{7}
$$

*where both series converge uniformly in every ball $B_{R'}(0)$ for $R' < R$.*

*Proof.* The vector fields $E$ and $H$ can be written as (see, [17, Corollary 2.47])

$$E(x) = a_0(r)\hat{x} + \sum_{\ell\ge1,\,|m|\le\ell}\left(a_\ell^m(r)Y_\ell^m(\hat{x})\hat{x} + b_\ell^m(r)U_\ell^m(\hat{x}) + c_\ell^m(r)V_\ell^m(\hat{x})\right)$$

and

$$H(x) = \alpha_0(r)\hat{x} + \sum_{\ell\ge1,\,|m|\le\ell}\left(\alpha_\ell^m(r)Y_\ell^m(\hat{x})\hat{x} + \beta_\ell^m(r)U_\ell^m(\hat{x}) + \gamma_\ell^m(r)V_\ell^m(\hat{x})\right).$$

Applying (4), (5) and Maxwell's equations to the spherical expansion of $E$ and $H$ yields $a_0 = \alpha_0 \equiv 0$ and

$$\begin{cases} -\dfrac{1}{\sqrt{\ell(\ell+1)}}\dfrac{a_\ell^m(r)}{r} + \dfrac{(rb_\ell^m(r))'}{r} - \mathtt{i}\omega\mu\,\gamma_\ell^m(r) = 0, \\ \sqrt{\ell(\ell+1)}\,c_\ell^m(r) + \mathtt{i}\omega\mu\,r\,\alpha_\ell^m(r) = 0, \\ (rc_\ell^m(r))' + \mathtt{i}\omega\mu\,r\,\beta_\ell^m(r) = 0, \end{cases}$$

$$\begin{cases} -\dfrac{1}{\sqrt{\ell(\ell+1)}}\dfrac{\alpha_\ell^m(r)}{r} + \dfrac{(r\beta_\ell^m(r))'}{r} + \mathtt{i}\omega\varepsilon\,c_\ell^m(r) = 0, \\ \sqrt{\ell(\ell+1)}\,\gamma_\ell^m(r) - \mathtt{i}\omega\varepsilon\,r\,a_\ell^m(r) = 0, \\ (r\gamma_\ell^m(r))' - \mathtt{i}\omega\varepsilon\,r\,b_\ell^m(r) = 0, \end{cases}$$

for $l \in \mathbb{N}_+$. As a consequence, one can obtain that $\gamma_\ell^m(r)$ and $c_\ell^m(r)$ solve the ODEs, respectively, (2) and (3). As smooth solutions, there are constants $A_\ell^m$ and $B_\ell^m$ such that $\gamma_\ell^m(r) = A_\ell^m j_\ell^E$ and $c_\ell^m(r) = B_\ell^m j_\ell^H$. The expressions for $a_\ell^m, b_\ell^m, \alpha_\ell^m$ and $\beta_\ell^m$ can then be seen from the second and the third line of the above set of six equations. We have now verified the first forms in the expressions (6) and (7) for, respectively, $E$ and $H$. The second forms can be then observed from (4) and (5).

We are left to show the uniqueness of the coefficients and the uniform convergence of the series. Let $\tilde R \in (0, R]$ be such that $j_\ell^E(\tilde R) \neq 0$ and $j_\ell^H(\tilde R) \neq 0$. Then the uniqueness of the coefficients $A_\ell^m$ and $B_\ell^m$ can be observed by taking the $L^2(\mathbb{S}^2)$ inner product of, respectively, $E(\tilde R\hat x)$ and $H(\tilde R\hat x)$ with $V_\ell^m(\hat x)$. In particular, we have

$$A_\ell^m j_\ell^E(\tilde R) = \Big(H(\tilde R\,\cdot), V_\ell^m\Big)_{L^2(\mathbb{S}^2)} \qquad \text{and} \qquad B_\ell^m j_\ell^E(\tilde R) = \Big(E(\tilde R\,\cdot), V_\ell^m\Big)_{L^2(\mathbb{S}^2)}.$$

As for the uniform convergence, notice that we have also

$$\begin{cases} A_\ell^m \dfrac{j_\ell^E(\tilde R)}{\mathtt{i}\omega\varepsilon(\tilde R)\tilde R} = \dfrac{1}{\sqrt{\ell(\ell+1)}}\Big(E(\tilde R\,\cdot), \hat x\,Y_\ell^m\Big)_{L^2(\mathbb{S}^2)}, \\ A_\ell^m \dfrac{\left(rj_\ell^E\right)'|_{\tilde R}}{\mathtt{i}\omega\varepsilon(\tilde R)\tilde R} = \Big(E(\tilde R\,\cdot), U_\ell^m\Big)_{L^2(\mathbb{S}^2)}, \end{cases}$$

and

$$\begin{cases} B_\ell^m \dfrac{j_\ell^H(\tilde R)}{\mathtt{i}\omega\mu(\tilde R)\tilde R} = \dfrac{1}{\sqrt{\ell(\ell+1)}}\Big(H(\tilde R\,\cdot), \hat x\,Y_\ell^m\Big)_{L^2(\mathbb{S}^2)}, \\ B_\ell^m \dfrac{\left(rj_\ell^H\right)'_{\tilde R}}{\mathtt{i}\omega\mu(\tilde R)\tilde R} = \Big(H(\tilde R\,\cdot), U_\ell^m\Big)_{L^2(\mathbb{S}^2)}. \end{cases}$$

Assume further that $\tilde R > R'$ and $\left(rj_\ell^E\right)'|_{\tilde R}$, $\left(rj_\ell^H\right)'|_{\tilde R} \neq 0$. Then for any $r \in (0, R')$ we have

$$\begin{aligned} E(r\hat x) = {} & \frac{\varepsilon(\tilde R)}{\varepsilon(r)} \sum_{\ell\geq 1,\,|m|\leq \ell} \frac{j_\ell^E(r)/r}{j_\ell^E(\tilde R)/\tilde R}\Big(E(\tilde R\,\cdot), \hat x\,Y_\ell^m\Big)_{L^2(\mathbb{S}^2)} Y_\ell^m(\hat x)\hat x \\ & + \frac{\varepsilon(\tilde R)}{\varepsilon(r)} \sum_{\ell\geq 1,\,|m|\leq \ell} \frac{\left(rj_\ell^E\right)'/r}{\left(rj_\ell^E\right)'|_{\tilde R}/\tilde R}\Big(E(\tilde R\,\cdot), U_\ell^m\Big)_{L^2(\mathbb{S}^2)} U_\ell^m(\hat x) \\ & + \sum_{\ell\geq 1,\,|m|\leq \ell} \frac{j_\ell^H(r)}{j_\ell^H(\tilde R)}\Big(E(\tilde R\,\cdot), V_\ell^m\Big)_{L^2(\mathbb{S}^2)} V_\ell^m(\hat x), \end{aligned}$$

and an analogous expansion for $H(r\hat{x})$. The proof of the uniform convergence of (6) and (7) can be then complete by following the lines of [17, Thereom 2.48]. Here we skip the details to avoid repetition. $\square$

Lemma 2.2 enables us to use separation of variable to obtain explicit expressions for the transmission eigenvalues and eigenfunction in the case of spherically symmetric electromagnetic material.

3. **The Transmission Eigenvalue Problem.** In this section, we consider the transmission eigenvalue problem (1) for spherically stratified media. We shall build a connection between the Maxwell transmission eigenvalues and the Helmholtz ones. In addition, we prove the existence of infinitely many (complex) transmission eigenvalues for the Maxwell system.

Let the inhomogeneity be of support $D := B_R$, and let electric permittivity $\epsilon(r)$ and magnetic permeability $\mu(r)$ be both radial scalar functions in $C^1([0,\, R])$. Let the corresponding coefficients $\varepsilon_0, \mu_0 \in \mathbb{R}_+$ in the background be constants. Here, we assume homogeneous background for simplicity of presentation; a similar discussion can be carried over for spherically stratified background as well. In this case the transmission eigenvalue problem reads:

$$\begin{cases} \mathrm{curl} E_0 - \mathtt{i}\omega\mu_0 H_0 = 0, \quad \mathrm{curl} H_0 + \mathtt{i}\omega\varepsilon_0 E = 0, & \\ \mathrm{curl} E - \mathtt{i}\omega\mu(r) H = 0, \quad \mathrm{curl} H + \mathtt{i}\omega\varepsilon(r) E = 0, & \text{for } |x| < R, \\ \qquad \nu \times (E - E_0) = \nu \times (H - H_0) = 0, & \text{for } |x| = R. \end{cases} \tag{8}$$

Note that here the outward normal vector coincides with $\nu := \hat{x}$. For simplicity of presentation we introduce the radial functions $p_\mu^2(r)$ and $q_\varepsilon^2(r)$ related to $\mu(r)$ and $\epsilon(r)$, respectively,

$$\mu(r) = \mu_0\, p_\mu^2(r) \qquad \text{and} \qquad \varepsilon(r) = \varepsilon_0\, q_\varepsilon^2(r).$$

Furthermore, for $\omega \in \mathbb{C}$ we denote $k = \omega\sqrt{\varepsilon_0\mu_0}$.

We can now separate variables in (8). From Lemma 2.2 we have the series expansions (6) for $E$ and (7) for $H$. Also from Remark 1 we have the following expressions for $E_0$ and $H_0$

$$E_0 = \frac{1}{\mathtt{i}\omega\varepsilon_0} \sum_{\ell\geq 1,\, |m|\leq \ell} a_\ell^m \left( \sqrt{\ell(\ell+1)}\, \frac{j_\ell(kr)}{r}\, Y_\ell^m(\hat{x})\, \hat{x} + \frac{(r j_\ell(kr))'}{r}\, U_\ell^m(\hat{x}) \right) + \sum_{\ell\geq 1,\, |m|\leq \ell} b_\ell^m\, j_\ell(kr)\, V_\ell^m(\hat{x}),$$

and

$$H_0 = -\frac{1}{\mathtt{i}\omega\mu_0} \sum_{\ell\geq 1,\, |m|\leq \ell} b_\ell^m \left( \sqrt{\ell(\ell+1)}\, \frac{j_\ell(kr)}{r}\, Y_\ell^m(\hat{x})\, \hat{x} + \frac{(r j_\ell(kr))'}{r}\, U_\ell^m(\hat{x}) \right) + \sum_{\ell\geq 1,\, |m|\leq \ell} a_\ell^m\, j_\ell(kr)\, V_\ell^m(\hat{x}).$$

Applying the boundary conditions we see that (8) admits nontrivial solutions if and only if

$$\det \mathcal{A}_\ell(k) \det \mathcal{B}_\ell(k) = 0$$

for at least one $l \in \mathbb{N}$, where

$$\mathcal{A}_\ell(k) := \begin{pmatrix} j_\ell(kR) & j_\ell^E(R) \\ (rj_\ell(kr))'\big|_R & q_\varepsilon^{-2}\left(rj_\ell^E(r)\right)'\Big|_R \end{pmatrix}$$

$$\mathcal{B}_\ell(k) := \begin{pmatrix} j_\ell(kR) & j_\ell^H(R) \\ (rj_\ell(kr))'\big|_R & p_\mu^{-2}\left(rj_\ell^H(r)\right)'\Big|_R \end{pmatrix}.$$

Hence, $k \in \mathbb{C}$ (and the corresponding $\omega \in \mathbb{C}$ for the Maxwell equations) is a *transmission eigenvalue* if and only if either $\det \mathcal{A}_\ell(k) = 0$ or $\det \mathcal{B}_\ell(k) = 0$ for some $\ell \in \mathbb{N}_+$, Our goal in the rest of the paper is to analyze the zeros of $\det \mathcal{A}_\ell(k)$ and $\det \mathcal{B}_\ell(k)$ which are entire functions of $k \in \mathbb{C}$.

3.1. **The Maxwell and Helmholtz Transmission Eigenvalues.** Next we aim to associate the zeros of each of the determinants $\det \mathcal{A}_\ell(k)$ and $\det \mathcal{B}_\ell(k)$ with transmission eigenvalues associated with the Helmholtz equation. The eigenvalue problem for the Helmholtz equation is better understood and the hope is that such a connection could shed light into the structure of transmission eigenvalues for Maxwell's equations. We start by formulating the transmission eigenvalue problem for the scalar Helmholtz equation. In the general configuration, one look for values of $k \in \mathbb{C}$ and two nonzero functions $u$ and $u_0$ satisfying

$$\begin{cases} \nabla \cdot c_0(x)\nabla u_0 + k^2 n_0(x) u_0 = 0, \quad \nabla \cdot c(x)\nabla u + k^2 n(x) u = 0, & \text{in } D, \\ u - u_0 = c\dfrac{\partial u}{\partial \nu} - c_0 \dfrac{\partial u_0}{\partial \nu} = 0, & \text{on } \partial D. \end{cases} \tag{9}$$

We recall that this is a non-selfadjoint eigenvalue problem and complex eigenvalues are proven to exist for spherically symmetric media [10, 11, 18], and for some cases of general media in [21]. The location of transmission eigenvalues of (9), which is the main concern for us, has been studied in [20, 21, 22, 23] for $C^\infty$-smooth media. We are particularly interested in media for which the imaginary part of transmission eigenvalues remain bounded. Having transmission eigenvalues lie in a strip, allows in [7] for the application of Fourier-Laplace techniques to prove solvability of the time dependent interior transmission problem and to justify the time domain linear sampling methods for solving the inverse support problem for inhomogeneous media. We summarize the main results on transmission eigenvalues free zones in the complex plane corresponding to (9), in order to make a reference later on in the paper for the case of electromagnetic transmission eigenvalues. The following holds true under the assumption that $c_0, c, n_0, n$ and $\partial D$ are in $C^\infty$

1. *The case of* $c_0(x) \equiv c(x)$ *in* $D$. If $n_0(x) \neq n(x)$ on $\partial D$, then there exists a constant $\alpha > 0$ such that there are no transmission eigenvalues in the region [21, 23]
$$\{k \in \mathbb{C} : |\Im(k)| \geq \alpha\}.$$
Note that this estimate is optimal since as we mentioned before complex transmission eigenvalues do exist for spherically stratified media [10].
2. *The case of* $c_0(x) \neq c(x)$ *on* $\partial D$. In this case the discreteness of transmission eigenvalues is proven provided that [19] $(c_0(x)n_0(x) - c(x)n(x)) \neq 0$ on $\partial D$ or that [3, Section 3.2] $c_0(x) \neq c_1(x)$ for all $x \in D$. Then the location of the transmission eigenvalues depends on the sign of $(c_0(x) - c(x))(c_0(x)n_0(x) - c(x)n(x))$ on $\partial D$.

*Sub-case (a).* In particular, if

$$(c_0(x) - c(x))(c_0(x)n_0(x) - c(x)n(x)) < 0 \text{ on } \partial D, \tag{10}$$

then [23] there are no transmission eigenvalues in

$$\{k \in \mathbb{C} : |\Re(k)| > 1, \ |\Im(k)| \geq C\}.$$

However, it is clarified in [21, Remark 4] that, there exist infinitely many transmission eigenvalues near the imaginary axis in this case. Therefore the imaginary part of all transmission eigenvalues is not uniformly bounded (note that the transmission eigenvalues form a discrete set in the complex plane with infinity as the only accumulation point). Moreover, there exist also infinitely many transmission eigenvalues around the real axis (see, [21, Remark 5]). Thus it is not possible to find a half space free of transmission eigenvalues.

*Sub-case (b).* Next if

$$(c_0(x) - c(x))(c_0(x)n_0(x) - c(x)n(x)) > 0 \text{ on } \partial D \tag{11}$$

there are no transmission eigenvalues in (see [23])

$$\left\{k \in \mathbb{C} : |\Re(k)| > 1, \, |\Im(k)| \geq \alpha_\epsilon (\Re(k))^{1/2+\epsilon}\right\}$$

for some constant $\alpha_\epsilon > 0$, and $0 < \epsilon \ll 1$. If in addition $n(x)/c(x) \neq n_0(x)/c_0(x)$, the above estimate can be improved to

$$\{k \in \mathbb{C} : |\Re(k)| > 1, \, |\Im(k)| \geq \alpha \log(\Re(k) + 1)\}.$$

Nevertheless the imaginary part of transmission eigenvalues in this estimate still can grow. In this case however there are no transmission eigenvalues in a neighborhood of the imaginary axis (see [21]), thus it remains open whether the imaginary part of the eigenvalues is in fact uniformly bounded.

**Remark 2.** *Summarizing the above results, we remark that up-to-date only for Case 1, it is proven that transmission eigenvalues corresponding to the scalar problem (9) have imaginary part uniformly bounded, i.e. they lie in a horizontal strip around the real axis.*

Now, we return to the case of spherically symmetric media with the aim to identify a scalar transmission eigenvalue problem for the Helmholtz equation whose transmission eigenvalues coincide with the zeros of $\det \mathcal{A}_\ell(k)$ and $\det \mathcal{B}_\ell(k)$. To this end, let the coefficients $c(r)$ and $n(r)$ be radially symmetric functions. We consider solutions to

$$\nabla \cdot c(r)\nabla u + k^2 n(r) u = 0, \qquad \text{for } |r| < R$$

which are of the form

$$u = \sum_{l\geq 0,\, |m|\leq \ell} d_\ell^m j_\ell^u(r) r Y_\ell^m(\hat{x}), \tag{12}$$

where the functions $j_\ell^u$ satisfy

$$\frac{1}{cr}\left(r^2 c\,(r j_\ell^u)'\right)' + \left(k^2 \frac{n}{c} r^2 - \ell(\ell+1)\right) j_\ell^u = 0$$

with the additional assumption that $j_\ell^u$ behaves like spherical Bessel function $j_\ell$ as $r \to 0$, i.e.

$$\lim_{r\to 0} r^{-\ell} j_\ell^u(r) = \frac{\sqrt{\pi}\, k^\ell}{2^{\ell+1}\,\Gamma(\ell + 3/2)}. \tag{13}$$

We note here that the expansion (12) has an extra factor $r$, compared to the usual expansion of solutions to Helmholtz equation. This is for later use where $c(r)$ and $n(r)$ are of "order" $r^{-2}$; see for example, (14). The essential purpose of the "unusual" expansion (12) is to form particular scalar interior transmission problems, for which the eigenvalues and eigenfunctions are closely related to those of electromagnetic interior transmission problems; see, Lemma 3.1.

Recalling that $j_\ell^E$ and $j_\ell^H$ solve (2) and (3), respectively, we observe that

$$u^E(x) := \sum_{\ell\geq 1,\, |m|\leq \ell} d_\ell^m\, j_\ell^E(r)\, r\, Y_\ell^m(\hat{x}) \quad \text{and} \quad u^H(x) := \sum_{\ell\geq 1,\, |m|\leq \ell} d_\ell^m\, j_\ell^H(r)\, r\, Y_\ell^m(\hat{x})$$

satisfies

$$\nabla\cdot\frac{1}{q_\varepsilon^2 r^2}\nabla u^E + k^2\,\frac{p_\mu^2}{r^2}\,u^E = 0 \qquad \text{and} \qquad \nabla\cdot\frac{1}{p_\mu^2 r^2}\nabla u^H + k^2\,\frac{q_\varepsilon^2}{r^2}\,u^H = 0, \tag{14}$$

respectively. In particular, if $j_\ell(kr)$ are the spherical Bessel functions of order $\ell$ we have that

$$u_0(x) := \sum_{\ell\geq 1,\, |m|\leq \ell} d_\ell^m\, j_\ell(kr)\, r\, Y_\ell^m(\hat{x})$$

satisfy

$$\nabla\cdot\frac{1}{r^2}\nabla u_0 + k^2\frac{1}{r^2}u_0 = 0.$$

Note that from (13) the solutions $u^E$, $u^H$ and $u_0$ of the above ODEs are regular at $r = 0$. More precisely they satisfy

$$r^{-2}\,u^E(x) = O(1), \qquad r^{-2}\,u^H(x) = O(1), \qquad r^{-2}\,u_0(x) = O(1), \qquad \text{as } r\to 0.$$

Now we consider the spherical symmetric case of the transmission eigenvalue problem (9):

$$\begin{cases} \nabla\cdot c_0(r)\nabla u_0 + k^2 n_0(r)u_0 = 0, \quad \nabla\cdot c(r)\nabla u + k^2 n(r)u = 0, \quad \text{for } r < R, \\ \qquad u - u_0 = c\dfrac{\partial u}{\partial r} - c_0\dfrac{\partial u_0}{\partial r} = 0, \qquad\qquad \text{for } r = R. \end{cases} \tag{15}$$

Then, by the above arguments we have the following lemma:

**Lemma 3.1.** *Suppose that $k$ is an eigenvalue of Maxwell's problem (8) with $\det\mathcal{A}_\ell = 0$ for some $l\in\mathbb{N}_+$. Then $k$ is also an eigenvalue of the scalar problem (15) with*

$$c(r) = \frac{1}{q_\varepsilon^2 r^2}, \quad n(r) = \frac{p_\mu^2}{r^2} \qquad \textit{and} \qquad c_0(r) = \frac{1}{r^2}, \quad n_0(r) = \frac{1}{r^2} \tag{16}$$

*with corresponding eigenfunctions of the form*

$$\begin{cases} u^E(x) \;=\; \displaystyle\sum_{\ell\geq 1,\, \det\mathcal{A}_\ell = 0,\, |m|\leq \ell} d_\ell^m\, j_\ell^E(r)\, r\, Y_\ell^m(\hat{x}), \\ u_0(x) \;=\; \displaystyle\sum_{\ell\geq 1,\, \det\mathcal{A}_\ell = 0,\, |m|\leq \ell} \delta_\ell^m\, j_\ell(kr)\, r\, Y_\ell^m(\hat{x}). \end{cases}$$

*Analogously, if $k$ is an eigenvalue for (8) with $\det\mathcal{B}_\ell = 0$ and some $l\in\mathbb{N}_+$, then $k$ is also an eigenvalue for (15) with*

$$c(r) = \frac{1}{p_\mu^2 r^2}\,, \quad n(r) = \frac{q_\varepsilon^2}{r^2} \qquad \textit{and} \qquad c_0(r) = \frac{1}{r^2}\,, \quad n_0(r) = \frac{1}{r^2} \tag{17}$$

*with the corresponding eigenfunctions*

$$\begin{cases} u^H(x) \;=\; \displaystyle\sum_{\ell\geq 1,\, \det \mathcal{B}_\ell = 0,\, |m|\leq \ell} d_\ell^m\, j_\ell^H(r)\, r\, Y_\ell^m(\hat{x}), \\ u_0(x) \;=\; \displaystyle\sum_{\ell\geq 1,\, \det \mathcal{A}_\ell = 0,\, |m|\leq \ell} \delta_\ell^m\, j_\ell(kr)\, r\, Y_\ell^m(\hat{x}). \end{cases}$$

*All above sums are finite since there are only finitely many linearly independent eigenfunctions.*

**Remark 3.** *From Lemma 3.1 we see that, for any choice of electromagnetic parameters $\epsilon_0, \mu_0$ and $\epsilon, \mu$, with contrasts $q_\varepsilon, p_\mu$ such that $q_\varepsilon \neq p_\mu$ and $(1-q_\varepsilon)(1-p_\mu) \neq 0$, the set of transmission eigenvalues contains a subset of transmission eigenvalues corresponding to the scalar Helmholtz equation of Case 2(a) for which the imaginary part of the eigenvalues is known to grow. If either $q_\varepsilon = 1$ or $p_\mu = 1$, a subset of transmission eigenvalues corresponds to the scalar Helmholtz equation of the case $2(b)$ for which the eigenvalue free zone is not known to be optimal whereas the remaining transmission eigenvalues lie in a horizontal strip. Thus this case remains still open. In the special case of $q_\varepsilon = p_\mu$ for $r = R$, we have that*

$$c\, n - c_0\, n_0 \equiv 0 \qquad \text{in } B_R$$

*in both scalar transmission eigenvalue problems appearing in Lemma 3.1 (which coincide in this case), and no results on the location of transmission eigenvalues are available in general in this case.*

Lemma 3.1 connects the set of electromagnetic transmission eigenvalues to sets of transmission eigenvalues for scalar Helmholtz equations. In our discussion the location of transmission eigenvalues for Maxwell's equations is given in reference to known results in the scalar case, which is based on the observation that electromagnetic transmission eigenvalues form a *subset* of transmission eigenvalues for two scalar Helmholtz eigenvalue problems. Notice that the excluded scalar transmission eigenvalues are those that correspond to zeros of zeroth order determinants involving $j_0^E$ and $j_0^H$ not appearing in Lemma 2.2, i.e. scalar transmission eigenvalues with spherically stratified eigenfunctions. Now the question arises: what is the location of the zeros of determinants involving $j_0^E$ and $j_0^H$, which do not yield electromagnetic transmission eigenvalues? Next we show that the excluded set lies in a strip about the real axis, thus it is not "responsible" for the blow up of the imaginary part of Maxwell's transmission eigenvalues. More precisely, we show that

**Lemma 3.2.** *Consider the transmission eigenvalue problem (15) with coefficients given by (16) or (17). Assume that*

$$\delta := \frac{1}{R}\int_0^R q_\epsilon(\rho)\, q_\mu(\rho)\, d\rho \neq 1. \tag{18}$$

*Then the (complex) eigenvalues corresponding to the eigenfunction $u_0 = j_0(kr)r$ all lie inside a horizontal strip in the complex plane.*

*Proof.* Notice that we can still define $j_0^E$ and $j_0^H$ from ordinary differential equations (2) and (3), respectively, by taking $l = 0$. Moreover, in this setting the eigenfunctions $u$ in (15) with the coefficients (16) and (17) are still given by, respectively,

$$u^E(x) := j_0^E(r)\, r \qquad \text{and} \qquad u^H(x) := j_0^H(r)\, r.$$

It suffices to prove the result for the case (16), namely, for the eigenfunction $u^E$, as exactly the same analysis holds for $u^H$ by interchanging $q_\epsilon$ and $q_\mu$.
Let $y = j_0^E(r)r$. Then $y$ satisfies the ordinary differential equation

$$y'' + \left(k^2 q_\varepsilon^2 p_\mu^2 - q_\varepsilon \left(\frac{1}{q_\varepsilon}\right)''\right) y = 0.$$

In order to determine uniquely a regular solution at the origin we introduce initial conditions

$$y(0) = 0, \qquad y'(0) = 1.$$

Now applying Liouville's transformation yields

$$z(\xi) := \sqrt{q_\varepsilon(r)p_\mu(r)}\, y(r) \qquad \text{with} \quad \xi = \xi(r) = \int_0^r q_\varepsilon(\rho)p_\mu(\rho)\, d\rho.$$

It is then obtained that this function $z$ satisfies

$$z'' + \big(k^2 - w(\xi)\big)\, z = 0, \qquad\qquad z(0) = 0, \quad z'(0) = \frac{1}{\sqrt{q_\varepsilon(0)p_\mu(0)}}, \tag{19}$$

where

$$w(\xi) = w(\xi(r)) = \frac{q_\varepsilon \left(\frac{1}{\sqrt{q_\varepsilon}}\right)'' - \sqrt{q_\varepsilon p_\mu}\left(\frac{1}{\sqrt{q_\varepsilon p_\mu}}\right)''}{(p_\mu q_\varepsilon)^2}.$$

Exactly the same as in the proof of Theorem 8.13 in [9], we rewrite (19) as the Volterra integral equation

$$z(\xi) = \frac{\sin(k\xi)}{k\sqrt{q_\varepsilon(0)p_\mu(0)}} + \frac{1}{k}\int_0^\xi z(\eta)\, w(\eta)\, \sin k(\eta - \xi)\, d\eta.$$

Then by the method of successive approximation we have that

$$z(\xi) = \frac{\sin k\xi}{k\sqrt{q_\varepsilon(0)p_\mu(0)}} + O\left(\frac{1}{k^2}\right) e^{|\Im k|} \quad \text{and} \quad z'(\xi) = \frac{\cos k\xi}{\sqrt{q_\varepsilon(0)p_\mu(0)}} + O\left(\frac{1}{k}\right) e^{|\Im k|},$$

as $|k| \to \infty$. As a consequence we have

$$y(r) = \frac{1}{k\sqrt{q_\varepsilon(0)p_\mu(0)}\sqrt{q_\varepsilon(r)p_\mu(r)}} \sin\left(k\int_0^r q_\varepsilon(\rho)p_\mu(\rho)d\rho\right) + O\left(\frac{1}{k^2}\right) e^{|\Im k|},$$

and

$$y'(r) = \sqrt{\frac{q_\varepsilon(r)p_\mu(r)}{q_\varepsilon(0)p_\mu(0)}} \cos\left(k\int_0^r q_\varepsilon(\rho)p_\mu(\rho)d\rho\right) + O\left(\frac{1}{k}\right) e^{|\Im k|},$$

as $|k| \to \infty$. The corresponding transmission eigenvalues of (15) with coefficients given by (16) are the zeros of

$$\mathcal{D}(k) := \begin{pmatrix} j_0(kR)R & y(R) \\ (rj_0(kr))'\big|_R & q_\varepsilon^{-2}(R)y'(R) \end{pmatrix} = 0.$$

Since $j_0(kr) = \sin kr/(kr)$ and from the above asymptotic of $y(r)$ and $y'(r)$ we have that all these zeros satisfy

$$\begin{aligned} 0 = &\frac{1}{q_\varepsilon^2(R)}\sqrt{\frac{q_\varepsilon(R)p_\mu(R)}{q_\varepsilon(0)p_\mu(0)}}\, \sin(kR)\, \cos\left(k\int_0^R q_\varepsilon(\rho)p_\mu(\rho)d\rho\right) \\ &+ \frac{1}{\sqrt{q_\varepsilon(0)p_\mu(0)}\sqrt{q_\varepsilon(R)p_\mu(R)}} \cos(kR)\sin\left(k\int_0^R q_\varepsilon(\rho)p_\mu(\rho)d\rho\right) + O\left(\frac{1}{k}\right) e^{|\Im k|}. \end{aligned}$$

Obviously, the main term of the above expression can be recasted as

$$\begin{aligned} &A\,\sin(kR)\,\cos(k\delta R) + B\,\cos(kR)\,\sin(k\delta R)\\ &= (A+B)\left[\sin(k(1+\delta)R) - K\sin(k(\delta-1)R)\right], \end{aligned}$$

where $\delta$ is given by (18), and

$$A := \frac{1}{2q_\varepsilon^2(R)}\sqrt{\frac{q_\varepsilon(R)p_\mu(R)}{q_\varepsilon(0)p_\mu(0)}}, \quad B := \frac{1}{2\sqrt{q_\varepsilon(0)p_\mu(0)}\sqrt{q_\varepsilon(R)p_\mu(R)}}, \quad K := \frac{A-B}{A+B}.$$

To fix our idea let us first assume that $\delta > 1$. Thus we look for the zeros $z := x + \mathrm{i}y \in \mathbb{C}$ of the following expression, where we let $\alpha := \frac{1+\delta}{\delta-1} > 1$ and $z := k(\delta-1)R$,

$$0 = \sin(\alpha z) - K\sin(z) + O\left(\frac{1}{z}\right)e^{|\Im z|}. \tag{20}$$

Our claim is that the modulus of the imaginary part $y$ of the zeros $z = x + \mathrm{i}y$ of (20) can not be arbitrary large. If so, then there is a sequence $\{z_n\}$ with $y_n > 0$ ($y_n < 0$) such that $y_n \to +\infty$ ($y_n \to -\infty$). For this sequence, if $y_n > 0$, we have

$$e^{(\alpha-1)y_n}\left[e^{\mathrm{i}\alpha x_n}e^{-2\alpha y_n} - e^{-\mathrm{i}\alpha x_n}\right] = K\left[e^{\mathrm{i}x_n}e^{-2y_n} - e^{-\mathrm{i}x_n} + O\left(\frac{1}{x_n + \mathrm{i}y_n}\right)\right].$$

Since $\alpha > 1$ the modulus of the lefthand side goes to $\infty$ as $n \to \infty$ whereas the righthand side remains bounded. Obviously the same holds if $y_n \to -\infty$. In the case of $\delta < 1$ the above argument work exactly in the same way if we set $\alpha := \frac{1+\delta}{1-\delta} > 1$ and $z := k(1-\delta)R$.

□

3.2. **On the Existence of Complex Electromagnetic Transmission Eigenvalues.** Do complex transmission eigenvalues for Maxwell's equations for spherically stratified media exist? Nothing is known on the existence of complex transmission eigenvalues for Maxwell's equations. In this section we provide an answer in the particular case of the homogeneous media, i.e. for constant $\epsilon > 0$ and $\mu > 0$ and $D := B_R$, by investigating directly the zeros of the involved determinants. We remark that as per Vodev's result in [21] for the scalar case with coefficient satisfying (10), there exist infinitely many complex transmission eigenvalues in a neighborhood of the imaginary axis, thus from the above discussion if both $q_\epsilon \neq 1$ and $q_\mu \neq 1$ there exist infinitely many transmission eigenvalues about the imaginary axis for the spherically stratified media. For the scalar spherically symmetric media governed by $\Delta u + k^2 n(r)u = 0$ the question of the existence of complex transmission eigenvalues is extensively addressed in [10, 11, 12, 18] (see also [9, Chapter 10]), by analyzing the zeroth order determinant (i.e. transmission eigenvalues corresponding to radial eigenfunctions). As discussed above the zeroth order determinants do not yield transmission eigenvalues for Maxwell's equations, thus their result cannot be transferred to our case of Maxwell's equations. We remark, that the existence of complex transmission eigenvalues for general media governed by $\Delta u + k^2 n(x)u = 0$ is still open.

To show an example that complex transmission eigenvalues exist for the electromagnetic problem, we consider homogeneous electromagnetic spherical media $D := B_R$ and constant $\epsilon > 0$ and $\mu > 0$, and look for complex zeros of $\det \mathcal{A}_\ell(k)$ which in this case becomes

$$\mathcal{A}_\ell(k) := \begin{pmatrix} j_\ell(kR) & j_\ell(kq_\epsilon q_\mu R) \\ (rj_\ell(kr))'\big|_R & q_\varepsilon^{-2}\,(rj_\ell(kq_\epsilon q_\mu r))'\big|_R \end{pmatrix}$$

where $q_\epsilon^2 := \epsilon/\epsilon_0$ and $q_\mu^2 := \mu/\mu_0$. For a fixed $\ell \in \mathbb{N}$ and large enough $z \in \mathbb{C}$, using the asymptotic formula for the spherical Bessel functions

$$j_\ell(z) = \frac{1}{z}\sin\left(z - \frac{\ell\pi}{2}\right) + e^{|\Im z|}\, O\left(\frac{1}{z^2}\right),$$
$$(z\, j_\ell(z))' = \cos\left(z - \frac{\ell\pi}{2}\right) + e^{|\Im z|}\, O\left(\frac{1}{z}\right),$$

we obtain

$$\begin{aligned}\det \mathcal{A}_\ell(k) =& \frac{1}{k^2 R q_\epsilon q_\mu}\left[\frac{1}{q_\epsilon^2}\sin\left(kR - \frac{\ell\pi}{2}\right)\cos\left(kq_\epsilon q_\mu R - \frac{\ell\pi}{2}\right)\right.\\ &\left. - \cos\left(kR - \frac{\ell\pi}{2}\right)\sin\left(kq_\epsilon q_\mu R - \frac{\ell\pi}{2}\right)\right] + e^{|\Im k|}\, O\left(\frac{1}{k^3}\right).\end{aligned}$$

Thus

$$k^2 R\, q_\epsilon\, q_\mu e^{-|\Im k|}\det \mathcal{A}_\ell(k) = e^{-|\Im k|} T(k) + O\left(\frac{1}{k}\right) \tag{21}$$

where

$$\begin{aligned}T(k) &= \frac{1}{q_\epsilon^2}\sin(kR - \frac{\ell\pi}{2})\cos(kq_\epsilon q_\mu R - \frac{\ell\pi}{2}) - \cos(kR - \frac{\ell\pi}{2})\sin(kq_\epsilon q_\mu R - \frac{\ell\pi}{2})\\ &= \frac{1+q_\epsilon^2}{2q_\epsilon^2}\left[(-1)^\ell \frac{1-q_\epsilon^2}{1+q_\epsilon^2}\sin\left(k(1+q_\epsilon \mu_\epsilon)R\right) + \sin\left(k(1-q_\epsilon q_\mu)R\right)\right].\end{aligned}$$

To study the zeros of $T(k)$ it suffices to study the zeros of

$$f(\kappa) = \frac{1-q_\epsilon^2}{1+q_\epsilon^2}\sin\left((q_\epsilon q_\mu + 1)\kappa\right) + (-1)^\ell \sin\left((1-q_\epsilon q_\mu)\kappa\right) \tag{22}$$

where $\kappa := kR$. The zeros of this function have been studied in [10, 18] (see also [2, Chapter 9] and [9, Chapter 10]). In particular, provided that

$$\frac{1+q_\epsilon^2}{|1-q_\epsilon^2|} > \frac{q_\epsilon q_\mu + 1}{q_\epsilon q_\mu - 1} \quad \text{if } q_\epsilon q_\mu > 1 \quad \text{or} \quad \frac{1+q_\epsilon^2}{|1-q_\epsilon^2|} > \frac{1+q_\epsilon q_\mu}{1-q_\epsilon q_\mu} \quad \text{if } q_\epsilon q_\mu < 1, \tag{23}$$

the function $f(\kappa)$, and consequently $T(k)$, has infinitely many complex zeros and infinitely real zeros, and all zeros lie in a strip around real axis. For the proof we refer the reader to the [9, Theorem 10.22 and Lemma 10.23]) where we use $A := (-1)^\ell(1+q_\epsilon^2)/(1-q_\epsilon^2)$ and $\delta := q_\epsilon q_\mu$. Now to prove that under the assumption (23) each of $\det \mathcal{A}_\ell(k)$ has infinitely many complex transmission eigenvalues we use (21) to conclude that

$$|k^2 R\, q_\epsilon q_\mu \det \mathcal{A}_\ell(k) - T(k)| < |T(k)|$$

for $k$ with $|k|$ sufficiently large and lying on the periodic arrays of Jordan curves surrounding each of the complex zeros of $T(k)$. Then Rouche's theorem implies that $\det \mathcal{A}_\ell(k)$ has a complex zero inside each of the Jordan curves in the periodic array.

We obtain the same type of result for the zeros of $\det \mathcal{B}_\ell(k)$ by interchanging $q_\epsilon$ and $q_\mu$ in (23).

Finally, we can obtain the existence of transmission eigenvalues in the degenerate case of $q_\epsilon = q_\mu$ by studying directly the zeros of $\det \mathcal{A}_\ell(k)$. This case is not covered by the general theory of the transmission eigenvalues as noted in Remark 3. To this end, as discussed above it suffices to study the zeros of $f(\kappa)$ given by (22). Using [2, Theorem 9.7] where we let $n_0 := q_\epsilon^2$ (note that in the proof there one needs to make a modification for $\ell$ odd, namely the zeros of $f(\tau)$ are the critical points of

the quotient of cosines instead of sines, but this does change the end result) we can conclude that $f(\kappa)$ and consequently $T(k)$ has infinitely many real and complex zeros if $q_\epsilon^2$ is not an integer or reciprocal of an integer. In the proof of this result, it is easy to see the construction of a periodic array of Jordan curves surrounding the complex zeros of $T(k)$ and the reasoning above implies the existence of infinitely many transmission eigenvalues. In particular if $q_\epsilon^2$ is an integer or reciprocal of an integer, [2, Theorem 9.7] states that the zeros of $f(\kappa)$ are all real and hence we cannot conclude anything from the asymptotic formula (21).
Rewriting (23) equivalently, we summarizing the above results in the following theorem.

**Theorem 3.3.** *Assume that $D := B_R$, $\epsilon_0, \mu_0$ and $\epsilon, \mu$ are all positive constants such that $q_\epsilon q_\mu \neq 1$, where $q_\epsilon^2 := \epsilon/\epsilon_0$ and $q_\mu^2 := \mu/\mu_0$. Then there exist infinitely many real transmission eigenvalues. In addition, if $q_\epsilon \neq q_\mu$ and one of the following conditions is satisfied*

$$(q_\epsilon - 1)(q_\mu - 1) \geq 0, \quad (q_\epsilon q_\mu^3 - 1)(q_\mu^2 - 1) < 0, \quad (q_\mu q_\epsilon^3 - 1)(q_\epsilon^2 - 1) < 0$$

*or $q_\epsilon = q_\mu$ and $q_\epsilon^2$ is an integer or reciprocal of an integer, then there exist infinitely many complex transmission eigenvalues.*

4. **Conclusion.** In this short note we display via the example of spherically stratified media that for the Maxwell's equations, there is no combination of the electromagnetic coefficients for which one can use the tools in [21] and [23] (the best result up-to-date on the location of transmission eigenvalues for the scalar case) to prove that set of the imaginary part of all the transmission eigenvalues is bounded. More specifically if $(\epsilon(R) - \epsilon_0)(\mu(R) - \mu_0) \neq 0$, then the imaginary part of all transmission eigenvalues can not be uniformly bounded. The case when either $\epsilon(R) = \epsilon_0$ or $\mu(R) = \mu_0$ is still open. In particular the latter brings up the *open problem*: Do all transmission eigenvalues corresponding to

$$\begin{cases} \nabla \cdot c_0(x)\nabla u_0 + k^2 u_0 = 0, \quad \nabla \cdot c(x)\nabla u + k^2 u = 0, & \text{in } D, \\ u - u_0 = c\dfrac{\partial u}{\partial \nu} - c_0\dfrac{\partial u_0}{\partial \nu} = 0, & \text{on } \partial D. \end{cases}$$

with $c \neq c_0$ on $\partial D$ lie in a horizontal strip? A positive or negative answer to this question would provide the mathematical path forward to understanding the location of the transmission eigenvalues for Maxwell's equations with contrast in only one parameter, to clarifying the solvibility of time domain electromagnetic interior transmission problem, and eventually to justifying linear sampling and factorization methods in electromagnetism with time dependent data.

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*E-mail address*: `fc292@math.rutgers.edu`
*E-mail address*: `shixumeng@amss.ac.cn`
*E-mail address*: `jingni.xiao@rutgers.edu`