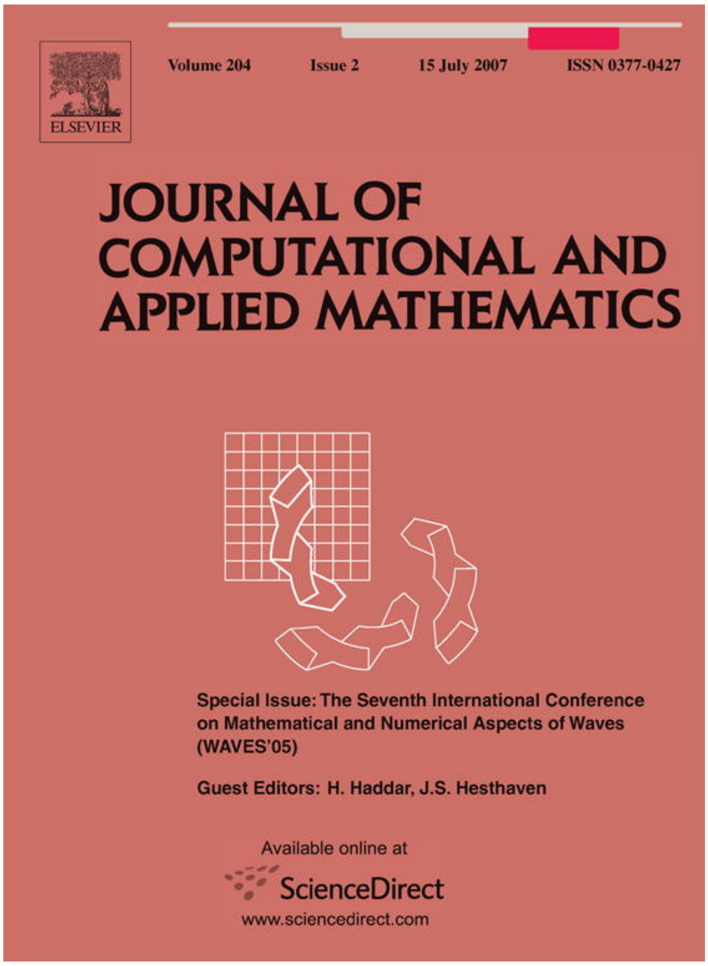
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Recent developments in the qualitative approach to inverse electromagnetic scattering theory ☆

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Abstract

We consider the inverse scattering problem of determining *both* the shape and some of the physical properties of the scattering object from a knowledge of the (measured) electric and magnetic fields due to the scattering of an incident time-harmonic electromagnetic wave at fixed frequency. We shall discuss the *linear sampling method* for solving the inverse scattering problem which does not require any a priori knowledge of the geometry and the physical properties of the scatterer. Included in our discussion is the case of partially coated objects and inhomogeneous background. We give references for numerical examples for each problem discussed in this paper.

Keywords: Electromagnetic inverse scattering; Inverse scattering problem; Linear sampling method; Mixed boundary value problem

1. Introduction

The field of inverse electromagnetic scattering theory has drawn increased attention in recent years due to its importance in many areas of science and technology. The aim of research in this field has been to not only detect but also to identify unknown objects through the use of electromagnetic waves. However, the remarkable progress that has been achieved to date depends on having certain a priori information on the physical properties of the scattering objects such as being able to ignore multiple scattering effects, knowing that the index of refraction is a small perturbation of a known background medium, assumptions on the electrical properties of the object, etc. In particular, until few years ago, essentially all existing algorithms for target identification were based on either a weak scattering approximation or on the use of nonlinear optimization techniques. Although nonlinear optimization techniques avoid the incorrect modeling assumptions of weak scattering approximations, for many practical applications such approaches require a priori information that may not be available. Hence in recent years alternative methods for imagining have been developed which avoid incorrect model assumptions but, as opposed to nonlinear optimization techniques, only seek limited information about the scattering object and do not rely on any a priori knowledge of the geometry and physical properties of the scatterer. Such methods come under the general title of *qualitative methods in inverse scattering theory*. Examples of such approaches are the linear sampling method [6,5,13,14], the factorization method [21–23], the

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method of singular sources [28], the range test method for finitely many incident waves [26,29], etc. In this paper we will survey the recent developments of the *linear sampling method* for solving the inverse electromagnetic scattering problem. The recent monograph by Cakoni and Colton [6] presents a general introduction to the linear sampling method for scalar problems and [5] provides a survey of some open problems in the area.

The inverse scattering problem we consider in this paper is to determine the shape and some of the physical properties of an obstacle from a knowledge of the scattered field due to the scattering of incident time-harmonic electromagnetic waves at fixed frequency. In many applications the scattering object is a composite material such that parts of the scatterer have different electrical properties e.g., a thin coating of a material is put on part of the boundary of the object. Scattering problems in such situations lead to the investigation of mixed boundary value problems for Maxwell's equations [7,8]. In general, it is not known a priori whether the object is coated and if so what is the extent of the coating. Hence, in order to identify the target from measured far field data, it is necessary to determine both the shape of the scatterer and whether or not the scatterer is coated. Mathematically, this means determining both the shape and a coefficient in a boundary condition, neither one of which is known a priori. In this paper we shall show how to resolve such problems through the use of the linear sampling method.

The plan of the paper is as follows. In the next section we introduce the main mathematical ideas of the linear sampling method for the simple case of electromagnetic scattering by a perfect conductor. In particular we will provide a mathematical justification of the method for the practical case of limited aperture scattering data. In Section 3 we then use the linear sampling method to determine both the shape and the surface impedance of a partially coated perfect conductor without knowing a priori whether or not the obstacle is coated. We end our paper with an investigation of the inverse electromagnetic scattering problem for objects buried in a known inhomogeneous medium. In particular, we present a new sampling method which in certain cases avoids the need to compute the Green's function of the background media.

2. The scattering problem for a perfect conductor

We first consider the case when the scattering object D is a perfect conductor. In particular, let $D \subset \mathbb{R}^3$ be a bounded region such that $\mathbb{R}^3 \setminus \bar{D}$ is connected. The boundary Γ of D is assumed to be a Lipshitz curvilinear polyhedron and ν denotes the unit outward normal defined almost everywhere on Γ. After factoring out a term of the form $\mathrm{e}^{-\mathrm{i}\omega t}$ where ω is the frequency and expressing the magnetic field in terms of the electric field, we are led to the following boundary value problem for the scattered electric field E^{s}:

$$\operatorname{curl}\operatorname{curl} E^{\mathrm{s}} - k^2 E^{\mathrm{s}} = 0 \quad \text{in } D_{\mathrm{e}}, \tag{1}$$

$$\nu \times E^{\mathrm{s}} = f \quad \text{on } \Gamma, \tag{2}$$

where k is the wave number, $f = -\nu \times E^{\mathrm{i}}$ where E^{i} is the incident field given by

$$E^{\mathrm{i}}(x) = \frac{\mathrm{i}}{k} \operatorname{curl}\operatorname{curl} p\mathrm{e}^{\mathrm{i}kx\cdot d}. \tag{3}$$

$p \in \mathbb{R}^3$ is a polarization vector and $d \in \Omega := \{x \in \mathbb{R}^3;\ |x| = 1\}$ is the direction of propagation. Finally E^{s} is required to satisfy the Silver–Müller radiation condition

$$\lim_{|x|\to\infty} (\operatorname{curl} E^{\mathrm{s}} \times x - \mathrm{i}k|x|E^{\mathrm{s}}) = 0 \tag{4}$$

uniformly in $\hat{x} = x/|x|$.

To analyze the scattering problem we consider the following Hilbert spaces:

$$H(\operatorname{curl}, D) := \{u \in (L^2(D))^3 : \operatorname{curl} u \in (L^2(D))^3\},$$

$$L_t(\Gamma) := \{u \in (L^2(\Gamma))^3 : \nu \cdot u = 0 \quad \text{on } \Gamma\},$$

$$H^{-1/2}_{\mathrm{div}}(\Gamma) := \{u \in H^{-1/2}(\Gamma), \quad \operatorname{div}_\Gamma u \in H^{-1/2}(\Gamma)\},$$

$$H^{-1/2}_{\mathrm{curl}}(\Gamma) := \{u \in H^{-1/2}(\Gamma), \quad \operatorname{curl}_\Gamma u \in H^{-1/2}(\Gamma)\}.$$

Furthermore, $u \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ if $u \in H(\mathrm{curl}, B_R \setminus \bar{D})$ for every ball of radius R containing D. It is known that for u in $H(\mathrm{curl}, D)$ or in $H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$, $\nu \times u \in H_{\mathrm{div}}^{-1/2}(\Gamma)$ and $\nu \times (u \times \nu) \in H_{\mathrm{curl}}^{-1/2}(\Gamma)$, and a duality pairing is defined between $H_{\mathrm{div}}^{-1/2}(\Gamma)$ and $H_{\mathrm{curl}}^{-1/2}(\Gamma)$ through an integration by parts formula (see [2] for the interpretation of the above spaces in the case of nonsmooth boundaries). The proof of the following theorem can be found in [27].

Theorem 2.1. *There exists a unique solution* $E^{\mathrm{s}} \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ *to* (1)–(4) *which satisfies*

$$\|E^{\mathrm{s}}\|_{H(\mathrm{curl}, B_R \setminus \bar{D})} \leqslant C \|f\|_{H_{\mathrm{div}}^{-1/2}(\Gamma)} \tag{5}$$

for every ball B_R of radius R containing D where C is a positive constant independent of f.

The radiating electric field E^{s} has the asymptotic behavior [15]

$$E(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|} \left\{ E_\infty(\hat{x}) + \mathrm{O}\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty, \tag{6}$$

where E_∞ is an infinitely smooth tangential field defined on the unit sphere Ω and is called the *electric far field pattern*. In particular if $E^{\mathrm{s}}(x) := E^{\mathrm{s}}(x, d, p)$ corresponds to an incident plane wave given by (3), then $E_\infty(\hat{x}) := E_\infty(\hat{x}, d, p)$ depends on d and (linearly) on p.

Let us now recall two family of solutions to the (normalized) Maxwell equations. A (normalized) *electromagnetic Herglotz wave pair* is defined to be a pair of vector fields of the form [15]

$$E_g(x) = \mathrm{i}k \int_\Omega \mathrm{e}^{\mathrm{i}kx \cdot d} g(d) \, \mathrm{d}s(d), \quad H_g(x) = \frac{1}{\mathrm{i}k} \mathrm{curl}\, E_g(x), \tag{7}$$

where the kernel g is a tangential vector field in $L_t^2(\Omega)$. We call E_g an electric Herglotz function. In particular, E_g is an entire solution to (1). Furthermore, a (normalized) electric dipole with polarization q is defined by

$$E_{\mathrm{e}}(x, z, q) := \frac{\mathrm{i}}{k} \mathrm{curl}_x \, \mathrm{curl}_x \, q \, \Phi(x, z), \quad H_{\mathrm{e}}(x, z, q) := \mathrm{curl}_x \, q \, \Phi(x, z), \tag{8}$$

where $\Phi(x, z) := \mathrm{e}^{\mathrm{i}k|x-z|}/4\pi|x-z|$. In particular, E_{e} is a radiating solution to (1) outside a neighborhood of z and the corresponding far field pattern $E_{\mathrm{e},\infty}(\hat{x}, z, q)$ is given by

$$E_{\mathrm{e},\infty}(\hat{x}, z, q) = \frac{\mathrm{i}k}{4\pi} (\hat{x} \times q) \times \hat{x} \mathrm{e}^{-\mathrm{i}k\hat{x} \cdot z}. \tag{9}$$

The *inverse scattering problem* for a perfect conductor is to determine D from a knowledge of the electric far field pattern $E_\infty(\hat{x}, d, p)$ for $d \in \Omega_1 \subset \Omega$, $\hat{x} \in \Omega_2 \subset \Omega$ and three linearly independent polarizations p, where Ω_1 and Ω_2 are two open subsets of the unit sphere Ω (possibly $\Omega_1 = \Omega_2$).

Theorem 2.2. *D is uniquely determined by $E_\infty(\hat{x}, d, p)$, $d \in \Omega_1 \subset \Omega$, $\hat{x} \in \Omega_2 \subset \Omega$ and three linearly independent polarizations p.*

The proof of this uniqueness result is based on the ideas of Kirsch and Kress [24] (see [25, 15, Theorem 7.1]) which essentially make use of the well-posedness of the direct scattering problem and a clever use of the singular behavior of electric dipoles. We note that these ideas are closely related to the linear sampling method for reconstructing D which we discuss next.

We first assume that we know $E_\infty(\hat{x}, d, p)$ for all $d, \hat{x} \in \Omega$ (for limited aperture data see the next section). Then we can define the *far field operator* $F : L_t^2(\Omega) \to L_t^2(\Omega)$ by

$$(Fg)(\hat{x}) := \mathrm{i}k \int_\Omega E_\infty(\hat{x}, d, g(d)) \, \mathrm{d}s(d), \quad \hat{x} \in \Omega, \tag{10}$$

for $g \in L_t^2(\Omega)$. Note that by superposition Fg is the electric far field pattern of the scattered field E^{s} corresponding to the electric Herglotz function with kernel g as incident field i.e., E^{s} is the solution of (1)–(4) with $f := -\nu \times E_g$.

Next, we investigate the linear first kind integral equation (which we will call the *far field equation*):

$$Fg(\hat{x}) = E_{\mathrm{e},\infty}(\hat{x}, z, q). \tag{11}$$

If $z \in D$, it is seen that if $g = g_z$ is a solution to the far field equation then from Rellich's lemma (see [15]) the scattered field E^{s} due to the incident field E_g and the electric dipole $E_{\mathrm{e}}(\cdot, z, q)$ coincide in D_{e}. Hence, by the trace theorem, their tangential traces $\nu \times E^{\mathrm{s}} = -\nu \times E_g$ and $\nu \times E_{\mathrm{e}}(\cdot, z, q)$ coincide on Γ. Now letting $z \in D \to \Gamma$, since $\|\nu \times E_{\mathrm{e}}(\cdot, z, q)\|_{H_{\mathrm{div}}^{-1/2}(\Gamma)} \to \infty$, we conclude that $\|\nu \times E_g\|_{H_{\mathrm{div}}^{-1/2}(\Gamma)} \to \infty$, whence $\|g\|_{L_t^2(\Omega)} \to \infty$. This determines the boundary Γ of D. The above is only a heuristic argument since it is based on the assumption that g solves the far field equation. Unfortunately the far field equation has no solution for $z \in D$ (in fact as will be seen later, the far field equation is not solvable for any $z \in \mathbb{R}^3$). This follows from the fact that if g solves the far field equation then the electric Herglotz function E_g is the solution of the interior boundary value problem

$$\operatorname{curl}\operatorname{curl} E_z - k^2 E_z = 0 \quad \text{in } D, \tag{12}$$

$$\nu \times [E_z + E_{\mathrm{e}}(\cdot, z, q)] = 0 \quad \text{on } \Gamma \tag{13}$$

which is in general not possible. However, it can be shown that the unique solution $E_z \in H(\mathrm{curl}, D)$ of (12)–(13) (which exists provided k is not a Maxwell eigenvalue for D) can be approximated arbitrarily closely by an electric Herglotz function E_g [16]:

Theorem 2.3. *Assume that $\mathbb{R}^3 \setminus \bar{D}$ is connected. Then the set of electric Herglotz functions E_g with $g \in L_t^2(\Omega)$ is dense in the space*

$$M(D) := \left\{ u \in H(\mathrm{curl}, D) : \operatorname{curl}\operatorname{curl} u - k^2 u = 0 \right\}$$

with respect to the $H(\mathrm{curl}, D)$ norm. In particular, provided that k is not a Maxwell eigenvalue for D, the well-posedness of the interior problem (12)–(13) with $-\nu \times E_{\mathrm{e}}(\cdot, z, q)$ replaced by an arbitrary function $f \in H_{\mathrm{div}}^{-1/2}(\Gamma)$ implies that for every $\varepsilon > 0$ there exists a $g_\varepsilon \in L_t^2(\Omega)$ such that

$$\|\nu \times E_{g_\varepsilon} - f\|_{H_{\mathrm{div}}^{-1/2}(\Gamma)} < \varepsilon.$$

In order to understand better the far field equation, we introduce a bounded linear operator $B : H_{\mathrm{div}}^{-1/2}(\Gamma) \to L_t^2(\Omega)$ which maps the boundary data $f \in H_{\mathrm{div}}^{-1/2}(\Gamma)$ to the far field pattern E_∞ of the radiating solution E^{s} of (1)–(2). Then if F is the far field operator we have that

$$F = -B(\nu \times E_g).$$

B is a compact operator since it can be seen as a composition of a bounded operator mapping $f \in H_{\mathrm{div}}^{-1/2}(\Gamma)$ into $(\nu \times E^{\mathrm{s}}, \nu \times H^{\mathrm{s}}) \in [H_{\mathrm{div}}^{-1/2}(\partial B_R)]^2$, where B_R is a ball of radius R containing D, and the compact operator mapping this Cauchy data to the far field pattern given by

$$E_\infty(\hat{x}) = \frac{\mathrm{i}k}{4\pi} \hat{x} \times \int_{\partial B_R} \left\{ (\nu_y \times E^{\mathrm{s}}(y)) + (\nu_y \times H^{\mathrm{s}}(y)) \times \hat{x} \right\} \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} \, \mathrm{d}s_y.$$

The following lemmas play an important role in our analysis.

Lemma 2.4. *The operator $B : H_{\mathrm{div}}^{-1/2}(\Gamma) \to L_t^2(\Omega)$ is injective and has dense range.*

Proof. Injectivity is a consequence of Rellich's lemma and the uniqueness of the direct scattering problem. To prove that B has dense range, we consider the dual operator $B^\top : L_t^2(\Omega) \to H_{\mathrm{curl}}^{-1/2}(\Gamma)$ given by

$$\langle Bf, g \rangle_{L^2, L^2} = \langle f, B^\top g \rangle_{H_{\mathrm{div}}^{-1/2}, H_{\mathrm{curl}}^{-1/2}},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the denoted spaces. By changing the order of integration and using integration by parts we see that

$$\langle Bf, g\rangle = \frac{1}{4\pi}\int_{\Gamma}[f\cdot(\operatorname{curl} E_g - \operatorname{curl}\tilde{E})]\,\mathrm{d}s,$$

where $\tilde{E} \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3\backslash\bar{D})$ is the solution of the exterior problem (1)–(2) with data $f = v \times E_g$ and the electric Herglotz wave function E_g is written in the form

$$E_g(y) := \int_{\Omega} g(d)\mathrm{e}^{-\mathrm{i}k\mathrm{d}\cdot y}\,\mathrm{d}s(d).$$

Hence, noting that the above integral is interpreted in the sense of duality between $H_{\mathrm{div}}^{-1/2}(\Gamma)$ and $H_{\mathrm{curl}}^{-1/2}(\Gamma)$, we have that

$$(B^{\top}g)(x) = v \times (\operatorname{curl} E_g(x) - \operatorname{curl}\tilde{E}(x)) \times v, \quad x \in \Gamma.$$

Next we show that $B^{\top}$ is injective. To this end, $B^{\top}g = 0$ implies that $v \times \operatorname{curl} E_g = v \times \operatorname{curl}\tilde{E}$ and by definition we also have that $v \times E_g = v \times \tilde{E}$. Using the Stratton–Chu formula, this implies that $\tilde{E} = E_g = 0$, since $\tilde{E}$ is a radiating solution while E_g is an entire solution to (1). Hence $g = 0$. Recalling that injectivity of $B^{\top}$ implies that B has dense range this ends the proof. □

Lemma 2.5. *$E_{\mathrm{e},\infty}(\hat{x}, z, q)$ is in the range of B if and only if $z \in D$.*

Proof. If $z \in D$ then from the above we have that $B(-v \times E_{\mathrm{e}}(\cdot, z, q)) = E_{\mathrm{e},\infty}(\hat{x}, z, q)$. Now let $z \in \mathbb{R}^3\backslash D$ and assume that there is $f \in H_{\mathrm{div}}^{-1/2}(\Gamma)$ such that $Bf = E_{\mathrm{e},\infty}(\cdot, z, q)$. Then by Rellich's lemma the scattered field E^{s} corresponding to the boundary data f and the electric dipole $E_{\infty}(\cdot, z, q)$ coincide outside a ball that contains D and z. Applying the unique continuation principle we arrive at a contradiction since $E^{\mathrm{s}} \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3\backslash\bar{D})$ but $E_{\infty}(\cdot, z, q)$ is not.

Next we consider the ill-posed equation

$$Bf = E_{\mathrm{e},\infty}(\cdot, z, q), \quad z \in \mathbb{R}^3. \tag{14}$$

As noted in the proof of Lemma 2.5 for $z \in D$, $f_z := -v \times E_{\mathrm{e}}(\cdot, z, q)$ is the solution to (14). In particular, as $z \to \Gamma$, we have that $\|f_z\|_{H_{\mathrm{div}}^{-1/2}(\Gamma)} \to \infty$. If $z \in \mathbb{R}^3\backslash\bar{D}$, from Lemmas 2.4 and 2.5, by using Tikhonov regularization we can construct a regularized solution to (14). In particular, there exists $f_z := f_z^{\alpha}$ corresponding to a parameter $\alpha = \alpha(\delta)$ chosen by a regular regularization strategy (e.g., the Morozov discrepancy principle) such that

$$\|Bf_z - E_{\mathrm{e},\infty}(\hat{x}, z, q)\|_{L_t^2(\Omega)} < \gamma\delta$$

for an arbitrary noise level δ and a constant $\gamma \geqslant 1$, and

$$\lim_{\alpha\to 0}\|f_z\|_{H_{\mathrm{div}}^{-1/2}(\Gamma)} \to \infty.$$

Note that $\alpha \to 0$ as $\delta \to 0$. Finally approximating $-f_z$ arbitrarily close by $v \times E_g$ (see Theorem 2.3) yields the following result.

Theorem 2.6. *Assume that k is not a Maxwell eigenvalue for D and F is the far field operator corresponding to the scattering problem* (1)–(4). *Then*

(1) *For $z \in D$ and a given $\varepsilon > 0$ there exists a $g_z^{\varepsilon} \in L_t^2(\Omega)$ such that*

$$\|Fg_z^{\varepsilon} - E_{\mathrm{e},\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \varepsilon$$

and the corresponding Herglotz function $E_{g_z^{\varepsilon}}$ converges to the solution of (12)–(13) *in $H(\operatorname{curl}, D)$ as $\varepsilon \to 0$.*

(2) *For a fixed* $\varepsilon > 0$, *we have that*

$$\lim_{z\to\Gamma} \|E_{g_z^\varepsilon}\|_{H(\mathrm{curl},D)} = \infty \quad \textit{and} \quad \lim_{z\to\Gamma} \|g_z^\varepsilon\|_{L_t^2(\Omega)} \to \infty.$$

(3) *For* $z \in \mathbb{R}^3\setminus\bar{D}$ *and a given* $\varepsilon > 0$, *every* $g_z^\varepsilon \in L_t^2(\Omega)$ *that satisfies*

$$\|Fg_z^\varepsilon - E_{\mathrm{e},\infty}(\hat{x}, z, q)\|_{L_t^2(\Omega)} < \varepsilon$$

is such that

$$\lim_{\varepsilon\to 0} \|E_{g_z^\varepsilon}\|_{H(\mathrm{curl},D)} = \infty \quad \textit{and} \quad \lim_{\varepsilon\to 0} \|g_z^\varepsilon\|_{L_t^2(\Omega)} \to \infty.$$

The above result provides a characterization for the boundary Γ of the scattering object D. Unfortunately, since the behavior of $E_{g_z^\varepsilon}$ is described in terms of a norm depending on the unknown region D, $E_{g_z^\varepsilon}$ cannot be used to characterize D. Instead the linear sampling method characterizes the obstacle by the behavior of g_z^ε. In particular, given a discrepancy $\varepsilon > 0$ and g_z^ε the ε-approximate solution of the far field equation, the boundary of the scatterer is reconstructed as the set of points z where the $L_t^2(\Omega)$ norm of g_z^ε becomes large. An open question is how to obtain numerically the ε-approximate solution of the far field equation given by Theorem 2.6. In all numerical experiments implemented up to date, the Tikhonov regularization combined with the Morozov discrepancy principle is used to solve the far field equation [18] (see [10,13] for a detailed numerical study of the linear sampling method). Although all these experiments indicate that this regularized solution behaves in the same way as g_z^ε given by Theorem 2.6, in general there is no mathematical justification of this fact. However, for the case of the Helmholtz equation, Arens has shown in [1] that in certain cases applying a regular regularization technique to the far field equation leads to a solution g that exhibits the desired behavior.

Obviously, in the context of the above discussion, it would be desirable to modify the far field equation in a way that it has a solution if and only if $z \in D$. This desire motivated Kirsch to introduce the *factorization method* for solving the inverse scattering problem of shape reconstruction [21–23]. The applicability of the factorization method is still limited to a restricted class of scattering problems. In particular, to date the method has not been established for the case of Maxwell's equation for a perfect conductor, for partially coated obstacles and limited aperture scattering data. On the other hand, when applicable, the factorization method provides a mathematical justification for using the regularized solution of an appropriate far field equation to determine D. In the case of a perfect conductor, the factorization method has been shown to be valid for the case of a spherical scatterer in [10].

3. Limited aperture data

In many cases of practical interest, the far field data $E_\infty(\hat{x}, d, p)$ is restricted to the case when d and $\hat{x}$ are on a subset Ω_1 and Ω_2, respectively, of the unit sphere Ω (possibly $\Omega_1 = \Omega_2$). In the case of limited aperture data the far field (11) takes the form

$$\int_{\Omega_1} E_\infty(\hat{x}, d, g(d))\,\mathrm{d}s(d) = E_{\mathrm{e},\infty}(\hat{x}, z, q), \quad \hat{x} \in \Omega_2. \tag{15}$$

In order to handle this case, we note that from the proof of Theorem 2.6 the function $g_\varepsilon \in L_t^2(\Omega)$ is the kernel of a Herglotz wave function which approximates a solution to (1) in D with respect to the $H(\mathrm{curl}, D)$ norm (see Theorem 2.3). Therefore, as it is discussed in [3], to treat the limited aperture case it is enough to show that a Herglotz wave function and its first derivative can be approximated uniformly on compact subsets of a ball B_R of radius R by a Herglotz wave function with kernel supported in a subset of Ω. This new Herglotz wave function and the kernel can now be used in place of $E_{g_z^\varepsilon}$ and g_z^ε in Theorem 2.6. To this end, assuming that k is not a Maxwell eigenvalue for the ball B_R (this is not a restriction since we can always find a ball containing D and having this property), it suffices to show that the set of functions

$$E_g(x) := v \times \int_\Omega g(d)\mathrm{e}^{\mathrm{i}kx\cdot d}\,\mathrm{d}s(d), \quad g \in L^2_t(\Omega) \text{ with support in } \Omega_0 \subseteq \Omega$$

for some subset $\Omega_0 \subseteq \Omega$ is complete in $H^{-1/2}_{\mathrm{div}}(\partial B_R)$.

To this end, let $\varphi \in H^{-1/2}_{\mathrm{curl}}(\partial B_R)$ and assume that for a fixed $\Omega_0 \subset \Omega$ we have that

$$\int_{\partial B_R} \varphi(x)\left[\int_{\Omega_0} \overline{g}(d)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(d)\right]\mathrm{d}s(x) = 0 \tag{16}$$

for every $g \in L^2_t(\Omega_0)$, where the first integral is interpreted in the sense of duality pairing. We want to show that $\varphi = 0$. By interchanging the order of integration we arrive at

$$\int_{\Omega_0} \overline{g}(d)\left[\int_{\partial B_R} \varphi(x)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(x)\right]\mathrm{d}s(d) = 0$$

for every $g \in L^2_t(\Omega_0)$, which implies that

$$d \times \int_{\partial B_R} \varphi(x)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(x) \times d \equiv 0, \quad d \in \Omega_0. \tag{17}$$

The left-hand side of (17) coincides with the far field pattern $(A\varphi)_\infty$ of the surface potential defined by

$$(A\varphi)(y) := \frac{1}{k^2}\nabla_y \times \nabla_y \times \int_{\partial B_R} a(x)\Phi(x,y)\,\mathrm{d}s(x), \quad y \in \mathbb{R}^3\backslash\partial B_R, \quad \varphi \in H^{-1/2}_{\mathrm{curl}}(\partial B_R).$$

It is known [15] that $v \times A\varphi$ is continuous across the boundary ∂B_R and A maps $H^{-1/2}_{\mathrm{curl}}(\partial B_R)$ to $H_{\mathrm{loc}}(\mathbb{R}^3)$. Since $A\varphi$ is a radiating solution to (1) in $\mathbb{R}^3\backslash\bar{B}_R$, from Rellich's lemma, (17) implies that $A\varphi = 0$ outside B_R. In particular $v \times A\varphi = 0$. From the continuity of $v \times A\varphi$ we then have that $A\varphi$ satisfies (1) inside B_R and the zero boundary condition $v \times A\varphi = 0$, whence $A\varphi = 0$ in B_R. Finally, applying the jump relation for $v \times \nabla \times (A\varphi)$ across ∂B_R [15] we obtain that $\varphi \equiv 0$. This ends the proof. □

Examples of reconstruction with limited aperture data can be found in [10,13,17].

4. Partially coated objects

To fix our ideas, we discuss here only the scattering problem for a partially coated perfect conductor. Other scattering problems for partially coated obstacles are discussed in [17] and in the references therein. Let $D \subset \mathbb{R}^3$ be a bounded region with Lipshitz boundary Γ such that $D_{\mathrm{e}} := \mathbb{R}^3\backslash\overline{D}$ is connected. We assume that the boundary $\Gamma = \Gamma_D \cup \Pi \cup \Gamma_I$ is split into two disjoint parts Γ_D and Γ_I having Π as their possible common boundary in Γ. Again v denotes the unit outward normal defined almost everywhere on Γ. We assume that D is a perfectly conducting object that is coated on the part Γ_I of its boundary by a very thin layer of a dielectric material. Under appropriate assumptions [20], the first-order approximation of the scattering problem is described by the following mixed boundary value problem for the electric scattered field E^{s}:

$$\operatorname{curl}\operatorname{curl} E^{\mathrm{s}} - k^2E^{\mathrm{s}} = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{D}, \tag{18}$$

$$v \times E^{\mathrm{s}} = f \quad \text{on } \Gamma_D, \tag{19}$$

$$v \times \operatorname{curl} E^{\mathrm{s}} - \mathrm{i}\lambda(x)(v \times E^{\mathrm{s}}) \times v = h \quad \text{on } \Gamma_I, \tag{20}$$

$$\lim_{r\to\infty}(\operatorname{curl} E^{\mathrm{s}} \times x - \mathrm{i}krE^{\mathrm{s}}) = 0, \tag{21}$$

where the surface impedance $\lambda \in L_\infty(\Gamma_I)$ describing the physical properties of the coating satisfies $\lambda(x) \geqslant \lambda_0 > 0$. If the scattering is due to an incident electromagnetic plane wave then $f := -\nu \times E^{\mathrm{i}}$ and $h := -\nu \times \operatorname{curl} E^{\mathrm{i}} + \mathrm{i}\lambda(\nu \times E^{\mathrm{i}}) \times \nu$ where E^{i} is given by (3).

The well-posedness of the direct problem is established in [7]. In particular it is shown that there exist a unique solution $E^{\mathrm{s}} \in X(B_R \backslash \bar{D}, \Gamma_I)$ of (18)–(21) where

$$X(B_R \backslash \bar{D}, \Gamma_I) := \{u \in H(\mathrm{curl}, B_R \backslash \bar{D}) : \nu \times u|_{\Gamma_I} \in L^2_t(\Gamma_I)\}$$

for every ball of radius R containing D. The *inverse scattering problem* we are interested in is to determine *both* D and λ from a knowledge of the electric far field pattern $E_\infty(\hat{x}, d, p)$ for $d \in \Omega_1 \subset \Omega$, $\hat{x} \in \Omega_2 \subset \Omega$ and three linearly independent polarizations p.

Theorem 4.1. *Assume that D_1 and D_2 are two partially coated scattering obstacles with corresponding surface impedances λ_1 and λ_2 such that for a fixed wave number the electric far field patterns coincide for $d \in \Omega_1 \subset \Omega$, $\hat{x} \in \Omega_2 \subset \Omega$ and three linearly independent polarizations p. Then $D_1 = D_2$ and $\lambda_1 = \lambda_2$.*

Proof. First, following [25], one shows that $D_1 = D_2 = D$. As a part of this proof, one also obtains that the scattered field E^{s}_1 and E^{s}_2 corresponding to λ_1 and λ_2 coincide in $\mathbb{R}^3 \backslash \bar{D}$, whence $\nu \times E^{\mathrm{s}}_1 = \nu \times E^{\mathrm{s}}_2$ and $\nu \times \operatorname{curl} E^{\mathrm{s}}_1 - \mathrm{i}\lambda_1(\nu \times E^{\mathrm{s}}_1) \times \nu = \nu \times \operatorname{curl} E^{\mathrm{s}}_2 - \mathrm{i}\lambda_1(\nu \times E^{\mathrm{s}}_2) \times \nu$ on Γ. From the boundary condition we have

$$\nu \times (E^{\mathrm{s}}_j + E^{\mathrm{i}}) = 0 \quad \text{on } \Gamma_{D_j}, \qquad \nu \times \mathrm{curl}(E^{\mathrm{s}}_j + E^{\mathrm{i}}) - \mathrm{i}\lambda_1(\nu \times (E^{\mathrm{s}}_j + E^{\mathrm{i}})) \times \nu = 0 \quad \text{on } \Gamma_{I_j}$$

for $j = 1, 2$. First we observe that $\Gamma_{D_1} \cap \Gamma_{D_2} = \emptyset$, because otherwise both Cauchy data of the total field will be zero on a part of the boundary and from the Stratton–Chu formula one concludes that the total field is zero which is not the case. Hence, we have that $\Gamma_{I_1} = \Gamma_{I_2} = \Gamma_I$. Next

$$(\lambda_1 - \lambda_2)(\nu \times E^{\mathrm{s}}_1) \times \nu = 0 \quad \text{on } \Gamma_I,$$

and again we can conclude that $\lambda_1 = \lambda_2$ since otherwise both $\nu \times (E^{\mathrm{s}}_1 + E^{\mathrm{i}}) = 0$ and $\nu \times \mathrm{curl}(E^{\mathrm{s}}_1 + E^{\mathrm{i}}) = 0$ on a part of Γ, implying that $E^{\mathrm{s}}_1 + E^{\mathrm{i}} = 0$ in $\mathbb{R}^3 \backslash \bar{D}$ which is a contradiction. □

The shape of the scattering object D can be reconstructed without any a priori knowledge of Γ_D, Γ_I or λ by using the linear sampling method. In particular, similar results to the ones stated in Theorem 2.3, Lemmas 2.4 and 2.5 can be proven for the boundary data f, h and the boundary operator B which in this case takes (f, h) to E_∞ of the radiating solution E^{s} of (18)–(21). We note that complications arise in the proof of these results due to the nonstandard space for the tangential component $\nu \times u|_{\Gamma_D}$ of functions $u \in X(B_R \backslash \bar{D}, \Gamma_I)$. We also remark that if $\Gamma_I \neq \emptyset$ then the linear sampling method is valid for all (fixed) wave numbers k. We refer the reader to [7] for the complete proof of the linear sampling method for partially coated perfect conductors.

Assuming now that D is known, we want to determine the surface impedance λ by making use of the approximate solution g to the far field equation

$$(Fg)(\hat{x}) := \mathrm{i}k \int_\Omega E_\infty(\hat{x}, d, g(d))\, \mathrm{d}s(d) = E_{\mathrm{e},\infty}(\hat{x}, z, q) \tag{22}$$

that was used to determine D. Note that now $E_\infty(\hat{x}, d, p)$ is the far field pattern that correspond to (18)–(21). To this end let $E_z \in X(D, \Gamma_I)$ be the unique solution (see [7]) of

$$\operatorname{curl}\operatorname{curl} E_z - k^2 E_z = 0 \quad \text{in } D, \tag{23}$$

$$\nu \times [E_z + E_{\mathrm{e}}(\cdot, z, q)] = 0 \quad \text{on } \Gamma_D, \tag{24}$$

$$\nu \times \mathrm{curl}(E_z + E_{\mathrm{e}}(\cdot, z, q)) - \mathrm{i}\lambda[\nu \times (E_z + E_{\mathrm{e}}(\cdot, z, q))] \times \nu = 0 \quad \text{on } \Gamma_I \tag{25}$$

for a fixed but arbitrary $z \in D$ and define

$$W_z := E_z + E_{\mathrm{e}}(\cdot, z, q). \tag{26}$$

Denoting $u_\top := (\nu \times u) \times \nu$ we recall that $(W_z)_T \in H_{\mathrm{curl}}^{-1/2}(\Gamma)$ and $(W_z)_T|_{\Gamma_I} \in L_t^2(\Gamma_I)$. The following result holds (see [7, Theorem 3.2]):

Theorem 4.2. *For every $\varepsilon > 0$ and $z \in D$ there exists an electric Herglotz function $E_{g_z^\varepsilon}$ with kernel $g_z^\varepsilon \in L_t^2(\Omega)$ such that*

$$\|E_z - \mathrm{i}kE_{g_z^\varepsilon}\|_{X(D,\Gamma_I)} \leqslant \varepsilon, \tag{27}$$

where E_z is the solution of (23)–(25). *Moreover this g_z^ε is an approximate solution of the far field equation*

$$\|(Fg_z^\varepsilon)(\hat{x}) - E_{\mathrm{e},\infty}(\hat{x}, z, q)\|_{L_t^2(\Omega)} \leqslant \varepsilon. \tag{28}$$

Our next aim is to find a relation that connects the surface impedance λ with E_z. We need the following technical lemma:

Lemma 4.3. *For every two points z_1 and z_2 in D and polarization $q \in \mathbb{R}^3$ we have that*

$$2\int_{\Gamma_I} (\overline{W}_{z_1})_T \cdot \lambda (W_{z_2})_T \, \mathrm{d}s = -\|q\|^2 A(z_1, z_2, k) + k\,\mathrm{Re}(q \cdot E_{z_1}(z_2) + q \cdot \overline{E}_{z_2}(z_1)),$$

where E_{z_1}, E_{z_1} and W_{z_2}, W_{z_2} are defined by (23)–(25) *and* (26), *respectively, and $A(z_1, z_2, k)$ is a computable number depending only on z_1, z_2 and k.*

Proof. See [4]. □

Next, assuming that D is connected, we consider a subset $\mathscr{E}$ of $L_t^2(\Gamma_I)$ defined by

$$\mathscr{E} := \left\{ f \in L_t^2(\Gamma_I) : \begin{array}{c} f = (W_z)_T|_{\Gamma_I} \text{ with } W_z = E_z + E_{\mathrm{e}}(\cdot, z, q), \\ z \in B_r, \;\; E_z \text{ the solution of (12)–(13) and } q \in \mathbb{R}^3 \end{array} \right\},$$

where B_r is a ball of radius r contained in D.

Remark 4.1. If D is not connected, the lemma remains true if we replace B_r by a union of discs where each component contains one disc from the union.

Lemma 4.4. *$\mathscr{E}$ is complete in $L_t^2(\Gamma_I)$.*

Proof. Let $\varphi \in L_t^2(\Gamma_I)$. Let $E \in X(D, \Gamma_I)$ be the solution of the interior mixed boundary value problem

$$\mathrm{curl}\,\mathrm{curl}\, E - k^2 E = 0 \quad \text{in } D,$$

$$\nu \times E = 0 \quad \text{on } \Gamma_D,$$

$$\nu \times \mathrm{curl}\, E - \mathrm{i}\lambda E_T = \varphi \quad \text{on } \Gamma_I.$$

Then for $z \in B_r$ and $q \in \mathbb{R}^3$, using the fact that $(W_z)_T = E_T = 0$ on Γ_D, simple calculations show that

$$\begin{aligned}
0 &= \int_{\Gamma_I} (W_z)_T \cdot E_T \, ds = \int_{\Gamma} W_z \cdot (\nu \times \operatorname{curl} E - i\lambda E_T) \, ds \\
&= \int_{\Gamma} [E_z \cdot (\nu \times \operatorname{curl} E) - i\lambda E_z \cdot E_T] \, ds \\
&\quad + \int_{\Gamma} [E_e(\cdot, z, q) \cdot (\nu \times \operatorname{curl} E) - i\lambda E_e(\cdot, z, q) \cdot E_T] \, ds \\
&= \int_{\Gamma} [E_z \cdot (\nu \times \operatorname{curl} E) - E \cdot (\nu \times \operatorname{curl} E_z)] \, ds \\
&\quad - \int_{\Gamma} [E \cdot (\nu \times \operatorname{curl} E_e(\cdot, z, q)) - i\lambda E_T \cdot E_e(\cdot, z, q)] \, ds \\
&\quad + \int_{\Gamma} [E_e(\cdot, z, q) \cdot (\nu \times \operatorname{curl} E) - i\lambda E_e(\cdot, z, q) \cdot E_T] \, ds \\
&= \int_{\Gamma} [E_e(\cdot, z, q) \cdot (\nu \times \operatorname{curl} E) - E \cdot (\nu \times \operatorname{curl} E_e(\cdot, z, q))] \, ds \\
&= -\int_{\Gamma} [(\nu \times E_e(\cdot, z, q)) \cdot \operatorname{curl} E - (\nu \times E) \cdot \operatorname{curl} E_e(\cdot, z, q)] \, ds = ikq \cdot E(z).
\end{aligned}$$

Thus $q \cdot E(z) = 0$ holds for all polarizations $q \in \mathbb{R}^3$ and $z \in B_r$ and hence $E(z) = 0$ for $z \in B_r$. By the unique continuation principle for the solution of Maxwell's equations in D we now see that $E \equiv 0$ in D, whence by the trace theorem $\varphi \equiv 0$ which proves the lemma. □

Using Lemmas 4.3 and 4.4 we can prove the following result [4].

Theorem 4.5. *Let $\lambda \in L_\infty(\Gamma_I)$ be the surface impedance of the scattering problem* (18)–(21). *Then*

$$\|\lambda(x)\|_{L_\infty(\Gamma_I)} = \sup_{z_1, z_2 \in B_r, q \in \mathbb{R}^3} \frac{-\|q\|^2 A(z_1, z_2, k) + k \operatorname{Re}(q \cdot E_{z_1}(z_2) + q \cdot \overline{E}_{z_2}(z_1))}{2\|(W_{z_1})_T\|_{L_t^2(\Gamma)} \|(W_{z_2})_T\|_{L^2(\Gamma)}}, \tag{29}$$

where $W_{z_1} = E_{z_1} + E_e(\cdot, z_1, q)$ and $W_{z_2} = E_{z_2} + E_e(\cdot, z_2, q)$ with E_{z_1} and E_{z_2} being the solutions to (23)–(25) *corresponding to z_1 and z_2, respectively, and*

$$A(z_1, z_2, k) = \frac{k^3}{6\pi} [2 j_0(k|z_1 - z_2|) + j_2(k|z_1 - z_2|)(3\cos^2\phi - 1)].$$

In the particular case where λ is a positive constant and setting $z_1 = z_2 = z_0 \in B_r$, we obtain the following formula for constant surface impedance:

$$\lambda = \frac{-(k^2/6\pi)\|q\|^2 + k \operatorname{Re}(q \cdot E_{z_0})}{\|(W_{z_0})_T\|^2_{L_t^2(\Gamma)}}, \tag{30}$$

where $W_{z_0} = E_{z_0} + E_e(\cdot, z_0, q)$ with E_{z_0} being the solution of (23)–(25) corresponding to $z_0 \in B_r$.

In both cases (29) and (30) E_z cannot be computed since λ appears in the boundary conditions. However from Theorem 4.2 we can approximate E_z by the electric field E_{g^z} of the Herglotz electromagnetic pair with kernel g^z where g^z is a (regularized) solution of the far field equation for $z \in B_r \subset D$ and E_∞ is the measured far field data (recall that D is reconstructed by using the linear sampling method!). We end this section by remarking that from the discussion in Section 3 the results of this section are valid for limited aperture data. Numerical examples of reconstructions of D

for partially coated perfect conductor can be found in [7,10], while examples of reconstructions for both D and λ are given in [17].

Remark 4.2. If the thickness of the scattering object is much smaller than the other dimensions and the wave number, we can model the object by an open surface in $\mathbb{R}^3$ which we refer to as a screen. If one side of the screen is a perfect conductor and the other side a dielectric, the scattering problem becomes a mixed boundary value problem which can again be treated by using the linear sampling method [9].

5. Inhomogeneous background

In many practical situations the scattering object is imbedded in an inhomogeneous background. Furthermore, in applications such as mine detection or medical imagining, it is more suitable to use point sources as incident waves and to measure near field data i.e., the scattered electromagnetic field measured on a given surface. As a model problem we discuss here the case of a perfect conductor embedded in a known piecewise homogeneous background. All the ideas can also be generalized to the case of partially coated obstacles, penetrable scatterers, etc.

We assume that the magnetic permeability $\mu_0 > 0$ of the background medium is a positive constant whereas the electric permitivity $\varepsilon(x)$ and conductivity $\sigma(x)$ are piecewise constant. Moreover we assume that for $|x| > R$ for R sufficiently large, $\sigma = 0$ and $\varepsilon(x) = \varepsilon_0$. Then, after an appropriate scaling [15] and eliminating the magnetic field, we have that the electric field E in the background medium satisfies

$$\operatorname{curl}\operatorname{curl} E - k^2 n(x) E = 0,$$

where $k = \varepsilon_0\mu_0\omega^2$ and $n(x) = (1/\varepsilon_0)(\varepsilon(x) + \mathrm{i}(\sigma(x)/\omega))$ (ω is the frequency). Note that the piecewise constant function $n(x)$ satisfies $n(x) = 1$ for $|x| > R$, $\Re(n) > 0$ and $\Im(n) \geqslant 0$. The surface across which the refractive index of the background media $n(x)$ is discontinuous are assumed to be piecewise smooth. We assume that the scattering object D embedded in this medium is a perfect conductor such that $\mathbb{R}^3\setminus\bar{D}$ is connected and that the boundary Γ of D is piecewise smooth. Furthermore, we suppose that the incident field is an electric dipole located at $x_0 \in \Lambda$ with polarization $p \in \mathbb{R}^3$, where Λ is a smooth open surface situated in a layer with constant index of refraction n_s. Recall that in this case the point source is given by

$$E_{\mathrm{e}}(x, x_0, p, k_s) := \frac{\mathrm{i}}{k_s}\operatorname{curl}_x \operatorname{curl}_x p \frac{\mathrm{e}^{\mathrm{i}k_s|x-x_0|}}{4\pi|x - x_0|} \tag{31}$$

where $k_s^2 = k^2 n_s$. We denote by $\mathbb{G}(x, x_0)$ the free space Green's tensor of the background medium (which is the total field in the absence of the scatterer) and define $E^{\mathrm{i}}(x) := E^{\mathrm{i}}(x, x_0, p) = \mathbb{G}(x, x_0)p$ which satisfies

$$\operatorname{curl}\operatorname{curl} E^{\mathrm{i}}(x) - k^2 n(x) E^{\mathrm{i}}(x) = p\,\delta(x - x_0) \quad \text{in } \mathbb{R}^3. \tag{32}$$

Note that E^{i} can be written as

$$E^{\mathrm{i}}(x) = E_{\mathrm{e}}(x, x_0, p, k_s) + E_b^{\mathrm{s}}(x), \tag{33}$$

where $E_b^{\mathrm{s}} = E_b^{\mathrm{s}}(\cdot, x_0, p)$ is the electric scattered field due to the background medium. Then the scattering problem is given $E^{\mathrm{i}} = \mathbb{G}(\cdot, x_0)p$, find a solution $E \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3\setminus\overline{D} \cup \{x_0\})$ of

$$\operatorname{curl}\operatorname{curl} E - k^2 n(x) E = 0 \quad \text{in } \mathbb{R}^3\setminus\overline{D} \cup \{x_0\}, \tag{34}$$

$$\nu \times E = 0 \quad \text{on } \partial D, \tag{35}$$

$$E = E^{\mathrm{s}} + E^{\mathrm{i}}, \tag{36}$$

$$\lim_{|x|\to\infty} (\operatorname{curl} E^{\mathrm{s}} \times x - \mathrm{i}k|x|E^{\mathrm{s}}) = 0. \tag{37}$$

It is known that (34)–(37) is well-posed [27]. We remark that it is also possible to consider the problem of objects buried in unbounded multi-layer medium. In this case, the radiation condition and mathematical analysis become more complicated (see [19] for the case of two layered medium).

The *inverse scattering problem* is to determine D from a knowledge of the tangential component $\nu \times E$ of the total field measured on Λ for all incident point sources $E_{\mathrm{e}}(x, x_0, p, k_s)$, $x_0 \in \Lambda$ and three linearly independent polarization p, where ν is the unit normal to Λ.

We use the linear sampling method to solve the inverse problem which is based on finding a solution $\varphi_z \in (L^2(\Lambda))^3$ of the following integral equation of the first kind called the *near field equation*:

$$\int_\Lambda \nu(x) \times E^{\mathrm{s}}(x, y, \varphi_z(y))\, \mathrm{d}s(y) = \nu(x) \times \mathbb{G}(x, z)\, q, \quad x \in \Lambda, \ \ z \in \mathbb{R}^3. \tag{38}$$

Here ν is the unit outward normal to Λ and $q \in \mathbb{R}^3$ is an artificial polarization. Note that since E^{s} depends linearly on the polarization p, the *near field equation* is linear. We remind the reader that in our formulation E^{s} is in fact the scattered field due to the incident wave being $\mathbb{G}(\cdot, x_0)\, p$. Alternatively, one could replace E^{s} by the difference of the scattered field due to the point source and the scattered field due to the background.

In the same way as in Section 2 one can prove a similar result to the one stated in Theorem 2.6 for the near field equation, where the electric Herglotz function E_g and its kernel g are replaced by the single layer potential

$$(S\varphi)(x) := \int_\Lambda \varphi(y)\mathbb{G}(x, y)\, \mathrm{d}s(y)$$

with density φ. In particular, the boundary Γ of D is characterized as the set of points where the L^2-norm of the approximate (regularized) solution φ_z to the near field equation becomes large. The discussion at the end of Section 2 is also applicable to the solution of the near field equation. The reader can find in [11] some numerical experiments using the linear sampling method for buried perfect conductors in unbounded two layered medium.

A drawback of the linear sampling method is that the Green's function of the background medium appears in the equations to be solved. As reported in [11], the computation of the background Green's function is a difficult and very expensive task. We now show that under additional assumptions it is possible to avoid the need to compute the background Green's function. This is done by using an alternative linear sampling method first introduced in [12] for the scalar case and further developed in [11]. We end the paper by discussing the main ideas of this new sampling method.

We assume that there is a bounded region B containing D such that inside B the medium is homogeneous with constant index of refraction n_b. Define $k_b^2 := k^2 n_b$. Furthermore, assume that it is possible to measure *both* $\nu \times E$ and $\nu \times H$ on the boundary ∂B of B, where ν is the outward normal vector to ∂B and E and H are the total electric and magnetic fields, respectively. Note that $\nu \times H|_{\partial B} = (1/\mathrm{i}k_b)\nu \times \operatorname{curl} E$. The *inverse scattering problem* is to determine D from a knowledge of *both* $\nu \times E$ and $\nu \times H$ on ∂B for all incident point sources $E_{\mathrm{e}}(x, x_0, p, k_s)$, $x_0 \in \Lambda$ and three linearly independent polarization p.

For any function $W \in H(\mathrm{curl}, B)$, we define the *gap reciprocity function* by

$$\mathscr{R}(E, W) = \int_{\partial B} (\nu \times E) \cdot \operatorname{curl} W - (\nu \times W) \cdot \operatorname{curl} E\, \mathrm{d}s, \tag{39}$$

where the integral is interpreted in the sense of the duality between $H_{\mathrm{div}}^{-1/2}(\partial B)$ and $H_{\mathrm{curl}}^{-1/2}(\partial B)$. Note that E depends on $x_0 \in \Lambda$ and hence so does $\mathscr{R}$. In particular $\mathscr{R}(E, W) \in L^2(\Lambda)$. Next we consider a dense subset $\{A\phi : \phi \in X\}$ of

$$M(B) := \{u \in H(\mathrm{curl}, B) : \operatorname{curl}\operatorname{curl} u - k_b^2 u = 0\},$$

where X is a normed space. An example of such dense sets of solutions is the set of electric Herglotz functions

$$E_g(x) = \int_\Omega g(d)\mathrm{e}^{\mathrm{i}kd\cdot x}\, \mathrm{d}s(d), \quad g \in L_t^2(\Omega),$$

where Ω is the unit sphere, or the set of single layer potentials

$$(S\varphi)(x) := \int_{\partial B} \varphi(y)\mathbb{G}(x, y)\, \mathrm{d}s(y), \quad \varphi \in L_{\mathrm{div}}^2(\partial B).$$

Now we look for a solution $\phi \in X$ to the following integral equation

$$\mathscr{R}(E, A\phi) = \mathscr{R}(E, E_{\mathrm{e}}(\cdot, z, q, k_s)), \tag{40}$$

where $A\phi$ is one of the above choices (there are more possibilities of choosing $A\phi$). To fix our ideas we take $\{A\phi\}$ to be Herglotz wave functions E_g, $g \in L^2_t(\Omega)$. The following result is proven in [11]:

Theorem 5.1. *Provided that k is not a Maxwell eigenvalue for D, we have that*

(1) *If $z \in D$, for every $\varepsilon > 0$ there exists a $g_z^\varepsilon \in L^2_t(\Omega)$ such that*

$$\|\mathscr{R}(E, E_{g_z^\varepsilon}) - \mathscr{R}(E, E_{\mathrm{e}}(\cdot, z, q, k_s))\|_{L^2(\Lambda)} < \varepsilon$$

and the corresponding Herglotz function $E_{g_z^\varepsilon}$ converges to the solution E_z of

$$\operatorname{curl}\operatorname{curl} E_z - k_b^2 E_z = 0 \quad \text{in } D,$$

$$\nu \times [E_z + E_{\mathrm{e}}(\cdot, z, q, k_s)] = 0 \quad \text{on } \Gamma$$

in $H(\mathrm{curl}, B)$ as $\varepsilon \to 0$.

(2) *For a fixed $\varepsilon > 0$, we have that*

$$\lim_{z \to \Gamma} \|E_{g_z^\varepsilon}\|_{H(\mathrm{curl}, B)} = \infty \quad \text{and} \quad \lim_{z \to \Gamma} \|g_z^\varepsilon\|_{L^2_t(\Omega)} \to \infty.$$

(3) *If $z \in \mathbb{R}^3 \setminus \bar{D}$, for every $g_z^\varepsilon \in L^2_t(\Omega)$ satisfying*

$$\|\mathscr{R}(E, E_{g_z^\varepsilon}) - \mathscr{R}(E, E_{\mathrm{e}}(\cdot, z, q, k_s))\|_{L^2(\Lambda)} < \varepsilon$$

for an arbitrary $\varepsilon > 0$ we have that

$$\lim_{\varepsilon \to 0} \|E_{g_z^\varepsilon}\|_{H(\mathrm{curl}, B)} = \infty \quad \text{and} \quad \lim_{\varepsilon \to 0} \|g_z^\varepsilon\|_{L^2_t(\Omega)} \to \infty.$$

The approximate solution g_z^ε provided by this theorem can now be used in the same way as in the linear sampling method to characterize D (see the discussion at the end of Section 2). We remark that the reciprocity gap (40) with $A\phi := E_g$ can be transformed to the far field equation if the background medium is homogeneous and the incident field is a plane wave. It is an open problem how to use the reciprocity gap method with limited aperture data. Numerical examples using the gap reciprocity method for solving the inverse problem for buried objects can be found in [12] in the scalar case and in [11] for Maxwell's equations.

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