The Interior Transmission Eigenvalue Problem for Absorbing Media‡

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Abstract. In the recent years the transmission eigenvalue problem has been extensively studied for non-absorbing media. In this paper we initiate the study of this problem for absorbing media. In particular we show that, in the case of absorbing media, transmission eigenvalues form a discrete set, exist for sufficiently small absorption and for spherically stratified media exist without this assumption. For constant index of refraction we also obtain regions in the complex plane where the transmission eigenvalues cannot exist and obtain a priori estimate for real transmission eigenvalues.

1. Introduction

In the recent years transmission eigenvalues have become an important area of research in inverse scattering theory. This interest is motivated by the fact that transmission eigenvalues carry information about the material properties of the scattering object and that these eigenvalues can in principle be determined from the scattering data [4]. To see how transmission eigenvalues arise in scattering theory, consider the simplest scattering problem

$$\Delta_3 u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^3 \tag{1}$$

$$u = u^i + u^s \tag{2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - ik\sqrt{n_b}u^s\right) = 0 \tag{3}$$

where $k > 0$ is the wave number, $r = |x|$, u^i is the incident field, u^s is the scattered field, the Sommerfeld radiation condition (3) is assumed to hold uniformly in $\hat{x} = x/|x|$, and $n(x)$ is the index of refraction which is assumed to be bounded such that $\Re(n) > 0$,

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$\Im(n) \geq 0$. For the purpose of this motivation, we assume that there exists a bounded region D such that $n(x) = n_b$ for $x \in \mathbb{R}^3 \setminus \overline{D}$, where n_b is a constant i.e. the inhomogeneous medium with support D and index of refraction $n(x)$ is embedded in a homogeneous background with constant index of refraction n_b. The support D of $n(x) - n_b$ is bounded, connected and has a connected piecewise smooth boundary ∂D. In the case of plane incident waves $u^i(x) = e^{ik\sqrt{n_b}x\cdot d}$, $|d| = 1$, the solution $u \in H^1_{loc}(\mathbb{R}^3)$ of (1)-(3) satisfies the asymptotic behavior [11]

$$u^s(x) = \frac{e^{i\sqrt{n_b}kr}}{r}\left(u_\infty(\hat{x}, d, k) + O\left(\frac{1}{r}\right)\right) \tag{4}$$

where $u_\infty(\hat{x}, d, k)$ is the *far field pattern* of the scattered field. We now define the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$, $\Omega = \{x : |x| = 1\}$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d, k) g(d)\, ds_d. \tag{5}$$

Then it can be shown that F is injective with dense range if and only if there does not exist a nontrivial solution v, w of the *interior transmission problem*

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in } D \tag{6}$$

$$\Delta v + k^2 n_b v = 0 \qquad \text{in } D \tag{7}$$

$$v = w \qquad \text{on } \partial D \tag{8}$$

$$\frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \qquad \text{on } \partial D \tag{9}$$

where ν is the outward normal to D and v is a *Herglotz wave function*, i.e. a solution of the Helmholtz equation (7) of the form

$$v_g(x) := \int_\Omega g(d) e^{ik\sqrt{n_b}x\cdot d} ds_d \tag{10}$$

for $g \in L^2(\Omega)$ (see [11] Theorem 8.9, and [17] Theorem 4.4). Values of $k \in \mathbb{C}$ such that there exists a nontrivial solution of (6)-(9) are called *transmission eigenvalues*.

Until now all of the research on transmission eigenvalues (c.f. [5], [6], [14], [16], [19] and the references contained therein) has only considered the case when $\Im(n) = 0$ and $\Im(n_b) = 0$, i.e. the case when absorption is not present in either the background or inhomogeneity. This restriction was made in order to avoid certain mathematical difficulties in dealing with non-selfadjoint operators. Here we remove this restriction and initiate the study of the transmission eigenvalue problem for absorbing media. In particular, we will consider the case with absorption in both the inhomogeneity and background medium of the form $n(x) = \epsilon_1(x) + i\dfrac{\gamma_1(x)}{k}$, $x \in D$ and $n_b = \epsilon_0 + i\dfrac{\gamma_0}{k}$.

The plan of the paper is as follows. In the next section, based on the analytic Fredholm theory we will show that transmission eigenvalues form at most a discrete set. In addition, making use of the stability of eigenvalues for closed operators under small perturbations as described in Kato's book [18], we prove that (complex) transmission eigenvalues exist provided that the absorption in the media and background is small enough. We will then show in Section 3 that for the case of a spherically stratified medium that there exists infinitely many (complex) transmission eigenvalues for arbitrary absorption. In the final section of this paper we will establish eigenvalue-free zones in the complex plane for general absorbing media. In particular, we provide estimates for real transmission eigenvalues (if they exist) in terms of material properties of the media and show that they can be used to obtain information on the index of refraction $n(x)$. We remark that the results of Section 2 and Section 4 hold also in $\mathbb{R}^2$.

2. The Transmission Eigenvalue Problem

We start with the investigation of the interior transmission eigenvalue problem for the general case of absorbing media. In particular, let D denote a bounded connected region of $\mathbb{R}^3$ with piecewise smooth boundary ∂D and ν the outward normal vector to ∂D. Then the interior transmission eigenvalue problem reads:

$$\Delta w + k^2 \left(\epsilon_1(x) + i\frac{\gamma_1(x)}{k}\right) w = 0 \qquad \text{in } D \tag{11}$$

$$\Delta v + k^2 \left(\epsilon_0(x) + i\frac{\gamma_0(x)}{k}\right) v = 0 \qquad \text{in } D \tag{12}$$

$$v = w \qquad \text{on } \partial D \tag{13}$$

$$\frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \qquad \text{on } \partial D \tag{14}$$

where $w \in L^2(D)$ and $v \in L^2(D)$ such that $w - v \in H^2(D)$. In fact $u := w - v$ is in $H_0^2(D)$ which is the subspace of functions in $H^2(D)$ with zero traces of u and $\partial u/\partial\nu$ on the boundary ∂D. Here we assume that $\epsilon_1 \in L^\infty(D)$ and $\gamma_1 \in L^\infty(D)$ such that $\epsilon_1(x) \geq \eta_1 > 0$, $\gamma_1(x) \geq 0$ almost everywhere in D, and similarly $\epsilon_0 \in L^\infty(D)$ and $\gamma_0 \in L^\infty(D)$ such that $\epsilon_0(x) \geq \eta_0 > 0$, $\gamma_0(x) \geq 0$. It is possible to write (11)-(14) as an eigenvalue problem for the fourth order differential equation

$$\big(\Delta + k^2\epsilon_1(x) + ik\gamma_1(x)\big) \frac{1}{k\epsilon_c(x) + i\gamma_c(x)} \big(\Delta + k^2\epsilon_0(x) + ik\gamma_0(x)\big)\, u = 0 \tag{15}$$

for $u \in H_0^2(D)$, where we denote by $\epsilon_c := (\epsilon_1 - \epsilon_0)$ and $\gamma_c := (\gamma_1 - \gamma_0)$ the respective contrasts. The following equivalence result can be proven [6], [20]:

Lemma 2.1. *If $w \in L^2(D)$ and $v \in L^2(D)$ are such that $w - v \in H^2(D)$ and w, v satisfy (11)-(14) then $u := w - v \in H_0^2(D)$ satisfies (15). Conversely, if $u := w - v \in H_0^2(D)$ is a solution of (15) then $w := \dfrac{-1}{k^2\epsilon_c + ik\gamma_c}(\Delta + k^2\epsilon_0 + ik\gamma_0)u \in L^2(D)$ and $v = w - u \in L^2(D)$ satisfy (11)-(14).*

In variational form (15) is formulated as the problem of finding $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{k\epsilon_c + i\gamma_c} \left[\Delta u + (k^2\epsilon_0 + ik\gamma_0)u\right] \left[\Delta\overline{v} + (k^2\epsilon_1 + ik\gamma_1)\overline{v}\right] dx = 0 \tag{16}$$

for all $v \in H_0^2(D)$. It is easy to see that the interior transmission problem (11)-(14) does not have purely imaginary eigenvalues $k = i\tau$ as long as $\tau > 0$ is such that $\tau\epsilon_c + \gamma_c > 0$.

Indeed, after integrating by parts and using the zero boundary boundary conditions, we have that

$$0 = \int_D \frac{1}{\tau\epsilon_c + \gamma_c} \left[\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u\right] \left[\Delta \overline{u} - (\tau^2\epsilon_1 + \tau\gamma_1)\overline{u}\right] \, dx$$

$$= \int_D \frac{1}{\tau\epsilon_c + \gamma_c} \left|\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u\right|^2 \, dx - \tau \int_D \left[\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u\right] \overline{u} \, dx$$

$$= \int_D \frac{1}{\tau\epsilon_c + \gamma_c} \left|\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u\right|^2 \, dx + \tau \int_D |\nabla u|^2 \, dx + \tau^2 \int_D (\tau\epsilon_0 + \gamma_0)|u|^2 \, dx$$

which implies that $u = 0$ in D. In a similar way, by exchanging subindices $_1$ and $_0$ one can show the same result for $\tau\epsilon_c + \gamma_c < 0$. The situation is not clear for $k = i\tau$ for which $\tau\epsilon_c + \gamma_c$ changes sign. For example if $\epsilon_0 > 0$, $\epsilon_1 > 0$, $\gamma_0 > 0$ and $\gamma_1 > 0$ are all positive constants then $k = i\tau_0$ where $\tau_0 = \frac{\gamma_1 - \gamma_0}{\epsilon_1 - \epsilon_0}$ is an eigenvalue and the corresponding eigenspace is infinite dimensional since for any solution v to the Helmholtz equation $\Delta v - \tau_0(\tau_0\epsilon_0 + i\gamma_0)v = 0$, v and $w = v$ are eigenfunctions.

Remark 2.1. If both bounded contrasts ϵ_c and γ_c are positive, more specifically $\epsilon_c(x) \geq \theta > 0$ and $\gamma_c(x) \geq 0$ almost everywhere in D, then $k = i\tau$ where τ is such that $\tau \geq -\dfrac{\sup_D \gamma_c}{\inf_D \epsilon_c}$ or $\tau \leq -\dfrac{\inf_D \gamma_c}{\sup_D \epsilon_c}$ is not a transmission eigenvalue.

Remark 2.2. If $\epsilon_c(x) \geq \theta > 0$ and $|\gamma_c(x)| < M$ almost everywhere in D, then $k = i\tau$ where $\tau > 0$ is large enough such that $\tau \geq \dfrac{M}{\inf_D \epsilon_c}$ is not a transmission eigenvalue.

In the following we assume that the real part of $k \in \mathbb{C}$ is positive. Furthermore, we assume that the contrast ϵ_c is bounded and does not change sign, more specifically due to the symmetric role of ϵ_1 and ϵ_0 we require that $0 < \theta \leq \epsilon_c(x) < N$ almost everywhere in D, whereas the contrast γ_c is only bounded, i.e. $|\gamma_c(x)| < M$ almost everywhere in D.

Lemma 2.2. *Assume that $0 < \theta \leq \epsilon_c(x) < N$ and $|\gamma_c(x)| < M$ almost everywhere in D. Then the set of transmission eigenvalues in the region $G_\sigma := \{k = \kappa + i\tau : \kappa \geq \sigma > 0$ and $\tau \leq 2M/\theta\} \cup \{k = \kappa + i\tau : \kappa \in \mathbb{R}$ and $\tau \geq 2M/\theta\}$ is discrete.*

Proof. Let us define the following sesquilinear forms on $H_0^2(D)$:

$$\mathcal{A}_k(u, v) = \int_D \frac{1}{k\epsilon_c + i\gamma_c} \Delta u \, \Delta \overline{v} \, dx$$

$$\mathcal{B}_k(u, v) = \int_D \left[k\frac{k\epsilon_1 + i\gamma_1}{k\epsilon_c + i\gamma_c} \Delta u \, \overline{v} + k\frac{k\epsilon_0 + i\gamma_0}{k\epsilon_c + i\gamma_c} u \, \Delta \overline{v} + k^2 \frac{(k\epsilon_0 + i\gamma_0)(k\epsilon_1 + i\gamma_1)}{k\epsilon_c + i\gamma_c} u \, \overline{v} \right] \, dx.$$

From our assumption we have that $|k\epsilon_c + i\gamma_c| \geq \beta > 0$ almost everywhere in D and therefore by using the Riesz representation theorem the above bilinear forms define bounded linear operators $\mathbf{A}_k : H_0^2(D) \to H_0^2(D)$ and $\mathbf{B}_k : H_0^2(D) \to H_0^2(D)$ such that

$$(\mathbf{A}_k u, v)_{H^2(D)} := \mathcal{A}_k(u, v) \text{ and } (\mathbf{B}_k u, v)_{H^2(D)} := \mathcal{B}_k(u, v) \quad \text{for all } u, v \in H_0^2(D). \tag{17}$$

In terms of these operators the transmission eigenvalue problem takes the form

$$(\mathbf{A}_k + \mathbf{B}_k)\, u = 0, \qquad u \in H_0^2(D). \tag{18}$$

In particular, k is a transmission eigenvalue if and only if the kernel of the operator $\mathbf{A}_k + \mathbf{B}_k$ is non-trivial. Since the L^2-norm of the Laplacian $\|\Delta u\|_{L^2(D)}$ is equivalent to $\|u\|_{H_0^2(D)}$ for $u \in H_0^2(D)$, $\mathbf{A}_k$ is invertible for fixed $k \in G_\sigma \subset \mathbb{C}$. A standard argument making use of the compact embedding of $H_0^2(D)$ and $H_0^1(D)$ in $L^2(D)$, implies that the operator $\mathbf{B}_k$ is compact. Since (18) becomes $\left(\mathbf{I} + \mathbf{A}_k^{-1}\mathbf{B}_k\right) u = 0$, if k is a transmission eigenvalue -1 is an eigenvalue of the compact (non-selfadjoint) operator $\mathbf{A}_k^{-1}\mathbf{B}_k$ and hence transmission eigenvalues have finite multiplicity. Note that the eigenfunctions of $\mathbf{A}_k^{-1}\mathbf{B}_k$ are elements of the kernel of $\mathbf{A}_k + \mathbf{B}_k$ and vice versa. Next we show that the set of transmission eigenvalues is discrete and to this end we apply the analytic Fredholm theory. Obviously, the bilinear forms $\mathcal{A}_k(\cdot,\cdot)$ and $\mathcal{B}_k(\cdot,\cdot)$ depend analytically on $k \in G_\sigma \subset \mathbb{C}$, and thus the mapping $k \mapsto \mathbf{A}_k$ and $k \mapsto \mathbf{B}_k$ are weakly analytic in this region and hence strongly analytic. Therefore, $k \mapsto \mathbf{A}_k^{-1}$ is also strongly analytic and so is $k \mapsto \mathbf{A}_k^{-1}\mathbf{B}_k$. Furthermore, from Remark 2.2, $k_0 = i\tau$ for some $\tau > 2M/\theta$ is not a transmission eigenvalue, i.e. the kernel of $\mathbf{A}_{k_0} + \mathbf{B}_{k_0}$ and hence of $\mathbf{I} + \mathbf{A}_{k_0}^{-1}\mathbf{B}_{k_0}$, is nontrivial. Hence from the analytic Fredhom theory [11] we can conclude that the set of transmission eigenvalues in the region $G_\sigma \subset \mathbb{C}$ of the complex plane is discrete (possibly empty) with ∞ as the only possible accumulation point. $\square$

Now since the region $k \in \mathbb{C}$ such that $\Re(k) > 0$ is included in $\bigcup_{n=1}^{\infty} G_{1/n}$ we have proven the following theorem:

Theorem 2.3. *Assume that $0 < \theta \le \epsilon_c(x) < N$ and $|\gamma_c(x)| < M$ almost everywhere in D. Then the set of transmission eigenvalues $k \in \mathbb{C}$, $\Re(k) > 0$ is discrete (possibly empty).*

The existence of transmission eigenvalues for absorbing media is in general an open problem. In the next section we will show the existence of transmission eigenvalues in special cases for absorbing spherically stratified media. However, for small enough conductivities γ_0 and γ_1, using perturbation theory [18] it is possible to show the existence of transmission eigenvalues near the real axis. To this end we recall the following result from [6] on the existence of real transmission eigenvalues for the non-absorbing case.

Theorem 2.4. *Assume that both $\gamma_0 = 0$ and $\gamma_1 = 0$ almost everywhere in D and $\epsilon_0 \in L^\infty(D)$ and $\epsilon_1 \in L^\infty(D)$ are such that $\epsilon_0(x) \ge \theta_0 > 0$, $\epsilon_1(x) \ge \theta_1 > 0$ and $\epsilon_c := \epsilon_1 - \epsilon \ge \theta > 0$ almost everywhere in D. Then there exists an infinite set of positive real transmission eigenvalues that accumulate only at $+\infty$. Furthermore, the smallest real transmission eigenvalue $k_0 > 0$ satisfies $k_0 > \dfrac{\lambda(D)}{\sup_D \epsilon_c}$, where $\lambda(D) > 0$ is the first Dirichlet eigenvalue for $-\Delta$ in D.*

Our aim is to use the upper semicontinuity of the spectrum of linear operators. To this end we rewrite the eigenvalue problem (11)-(14) in a different equivalent form.

Note that we already know by Theorem 2.3 that in the right half plane (11)-(14) has a discrete point spectrum. Obviously, in terms of $u := w - v$ (11)-(14) can be written as

$$\Delta u + \left(k^2\epsilon_1 + ik\gamma_1\right) u + \left(k^2\epsilon_c + ik\gamma_c\right) v = 0 \qquad \text{in } D \tag{19}$$

$$\Delta v + \left(k^2\epsilon_0 + ik\gamma_0\right) v = 0 \qquad \text{in } D, \tag{20}$$

together with the boundary conditions

$$u = 0 \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D. \tag{21}$$

These equations make sense for $u = H_0^2(D)$ and $v \in L^2(D)$ such that $\Delta v \in L^2(D)$. Setting $X(D) := H_0^2(D) \times \{v \in L^2(D) : \ \Delta v \in L^2(D)\}$, we can define the linear operators $\mathbb{A}, \mathbb{B}, \mathbb{D} : L^2(D) \times L^2(D) \to L^2(D) \times L^2(D)$ by

$$\mathbb{A} = \begin{pmatrix} \Delta_{00} & 0 \\ 0 & \Delta \end{pmatrix}, \qquad \mathbb{B}_\gamma = \begin{pmatrix} i\gamma_1 & i\gamma_c \\ 0 & i\gamma_0 \end{pmatrix}, \qquad \mathbb{D}_\epsilon = \begin{pmatrix} \epsilon_1 & \epsilon_c \\ 0 & \epsilon_0 \end{pmatrix}$$

where Δ_{00} indicate that the Laplacian acts on a function in $H_0^2(D)$, i.e. with zero Cauchy data on ∂D. Let $\mathbf{p} := \begin{pmatrix} u \\ v \end{pmatrix}$ and note that the domain of definition of $\mathbb{A}$ is $X(D)$ and $\mathbb{A}$ is an unbounded densely defined operator in $L^2(D) \times L^2(D)$. Furthermore, $\mathbb{A}$ is a closed operator, i.e. for any sequence $\{\mathbf{p}_n\} \in X(D)$ such that $\mathbf{p}_n \to \mathbf{p}$ in $L^2(D) \times L^2(D)$ and $\mathbb{A}\mathbf{p}_n \to \mathbf{q}$, we have that $\mathbf{p} \in X(D)$ and $\mathbb{A}\mathbf{p} = \mathbf{q}$. Indeed, since $\|\Delta_{00} u\|_{L^2(D)}$ defines an equivalent norm in $H_0^2(D)$, if $u_n \to u$ in $L^2(D)$ and $\Delta_{00} u_n \to q_1$ in $L^2(D)$ then $u \in H_0^2(D)$ and $q_1 = \Delta_{00} u$. Similarly if $v_n \to v$ in $L^2(D)$ and $\Delta v_n \to q_2$ in $L^2(D)$ then $\Delta v = q_2$. The operators $\mathbb{B}_\gamma$ and $\mathbb{D}_\epsilon$ are bounded in $L^2(D) \times L^2(D)$ and $\mathbb{D}_\epsilon^{-1}$ exists in $L^2(D) \times L^2(D)$ and is given by

$$\mathbb{D}_\epsilon^{-1} = \frac{1}{\epsilon_0 \epsilon_1} \begin{pmatrix} \epsilon_0 & -\epsilon_c \\ 0 & \epsilon_1 \end{pmatrix}.$$

Thus the transmission eigenvalue problem is equivalent to the following quadratic eigenvalue problem

$$\mathbb{A}\mathbf{p} + k\mathbb{B}_\gamma \mathbf{p} + k^2 \mathbb{D}_\epsilon \mathbf{p} = \mathbf{0}, \qquad \mathbf{p} \in L^2(D) \times L^2(D). \tag{22}$$

Introducing $\mathbf{U} = \begin{pmatrix} \mathbf{p} \\ k\mathbb{D}_\epsilon \, \mathbf{p} \end{pmatrix}$ the eigenvalue problem (22) becomes

$$(\mathbb{K}\mathbf{U} - k\mathbb{I}_{\epsilon,\gamma})\mathbf{U} = \mathbf{0} \qquad \mathbf{U} \in (L^2(D) \times L^2(D))^2, \tag{23}$$

where the 4×4 matrix operators $\mathbb{K}$ and $\mathbb{I}_{\gamma,\epsilon}$ are given by

$$\mathbb{K} := \begin{pmatrix} \mathbb{A} & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad \mathbb{I}_{\epsilon,\gamma} := \begin{pmatrix} -\mathbb{B}_\gamma & -\mathbb{I} \\ \mathbb{D}_\epsilon & 0 \end{pmatrix}$$

where $\mathbb{I}$ is the identity operator in $L^2(D) \times L^2(D)$. By straightforward calculation we obtain $\mathbb{I}_{\epsilon,\gamma}^{-1} := \mathbb{D}_\epsilon^{-1} \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{D}_\epsilon & -\mathbb{B}_\gamma \end{pmatrix}$ which is a bounded operator in $L^2(D) \times L^2(D)$. Thus, we have that the original transmission eigenvalue problem (11)-(14) is equivalent to the eigenvalue problem for the closed (unbounded) operator $\mathbb{T}_{\epsilon,\gamma} := \mathbb{I}_{\epsilon,\gamma}^{-1}\mathbb{K}$ (note $\mathbb{T}_{\epsilon,\gamma}$ is closed since it is the product of a closed operator with bounded operator in $(L^2(D)\times L^2(D))^2$). Let us denote by $\mathbb{T}_{\epsilon,\gamma=0}$ the operator defined as above corresponding to the non-absorbing case, i.e. $\gamma_0 = 0$ and $\gamma_1 = 0$ almost everywhere in D ($\mathbb{B}_{\gamma=0}$ becomes the zero operator). Let $\Sigma(\mathbb{T}_{\epsilon,\gamma})$ be the spectrum of $\mathbb{T}_{\epsilon,\gamma}$ and $\mathcal{R}(k;\mathbb{T}_{\epsilon,\gamma})$ the resolvent of $\mathbb{T}_{\epsilon,\gamma}$. We have proven in Theorem 2.3 that $\mathcal{R}(k;\mathbb{T}_{\epsilon,\gamma}) = (\mathbb{T}_{\epsilon,\gamma} - k\mathbb{I})^{-1}$ is well defined for all $k\in\mathbb{C}$ such that $\Re(k) > 0$ except for a discrete set of k without any interior accumulation point (possibly empty). Furthermore, from Theorem 2.4 we already know that $\Sigma(\mathbb{T}_{\epsilon,\gamma=0})$ contains infinitely many isolated points lying on the positive real axis, which indeed are real transmission eigenvalues. Our aim is to use the stability of eigenvalues for closed operators under small perturbations as described in [18] (Chapter 4, Section 3). To this end we need to define what small perturbation means and prove that $\mathbb{T}_{\epsilon,\gamma}$ is a small perturbation of $\mathbb{T}_{\epsilon,\gamma=0}$ assuming that the absorptions γ_0 and γ_1 are small enough.

To do this we set $\mathbb{P} := \mathbb{T}_{\epsilon,\gamma} - \mathbb{T}_{\epsilon,\gamma=0}$ and by straightforward calculation we see that the perturbation $\mathbb{P}$ is a bounded operator in $(L^2(D)\times L^2(D))^2$ given by

$$\mathbb{P} = \begin{pmatrix} 0 & 0 \\ 0 & -\mathbb{D}_\epsilon^{-1}\mathbb{B}_\gamma \end{pmatrix}.$$

According to [18], the perturbation $\mathbb{P}$ is considered small if the so-called gap between the two closed operators $\mathbb{T}_{\epsilon,\gamma}, \mathbb{T}_{\epsilon,\gamma=0}$, denoted by $\hat{\delta}(\mathbb{T}_{\epsilon,\gamma}, \mathbb{T}_{\epsilon,\gamma=0})$ is small. For the sake of the reader's convenience we include here the definition of the gap $\hat{\delta}(T,S)$ between two closed operators T and S on a Banach space X. In particular

$$\hat{\delta}(T,S) = \max(\delta(T,S),\delta(S,T)), \quad \text{where} \quad \delta(T,S) = \sup_{u\in G(T),\|u\|=1} \operatorname{dist}(u, G(S))$$

where $G(T)$ and $G(S)$ are the graphs of T and S respectively, which are closed subsets of $X\times X$. In particular, if $S = T + A$ with A a bounded operator in X then (see [18], Chapter 4, Theorem 2.14)

$$\hat{\delta}(T+A,T) \le \|A\|.$$

In our case it is now easy to show that

$$\hat{\delta}(\mathbb{T}_{\epsilon,\gamma}, \mathbb{T}_{\epsilon,\gamma=0}) \le \|\mathbb{P}\| \le \|\mathbb{D}_\epsilon^{-1}\mathbb{B}_\gamma\| \le 4\frac{\sup_D(\epsilon_0)+\sup_D(\epsilon_1)}{\inf_D(\epsilon_0)\inf_D(\epsilon_1)}\left(\sup_D(\gamma_0)+\sup_D(\gamma_1)\right) \quad (24)$$

Now let k^* be a real transmission eigenvalue corresponding to the operator $\mathbb{T}_{\epsilon,\gamma=0}$, and consider a neighborhood $\mathcal{N}_\sigma(k^*)\subset\mathbb{C}$ of k^* of radius $\sigma>0$. Then there is a $\eta_{k^*}>0$ (of course depending on σ) such that this neighborhood contains at least one point in $\Sigma(\mathbb{T}_{\epsilon,\gamma})$ as long as $\hat{\delta}(\mathbb{T}_{\epsilon,\gamma}, \mathbb{T}_{\epsilon,\gamma=0}) < \eta_{k^*}$ since otherwise from [18] (Theorem 3.1, Chapter 4) $\mathcal{N}_\sigma(k^*)$ must be included in both resolvents, $\mathcal{R}(k;\mathbb{T}_{\epsilon,\gamma})$ and $\mathcal{R}(k;\mathbb{T}_{\epsilon,\gamma=0})$. Thus we have shown that for small absorption there is at least one transmission eigenvalue near k^*.

Theorem 2.5. *Let $\epsilon_0 \in L^\infty(D)$ and $\epsilon_1 \in L^\infty(D)$ satisfy $\epsilon_0(x) \geq \theta_0 > 0$, $\epsilon_1(x) \geq \theta_1 > 0$ and $\epsilon_c := \epsilon_1 - \epsilon \geq \theta > 0$ almost everywhere in D, and let $k_i > 0$, $i = 0, 1, \ldots, \ell$ be the first ℓ real transmission eigenvalues (multiple eigenvalues are counted once) corresponding to (11)-(14) for non-absorbing media, i.e. for $\gamma_0 = \gamma_1 = 0$ almost everywhere in D. Then for every $\sigma > 0$ there is a $\tilde{\eta} > 0$ (depending on σ) such that if the absorption in the media is such that $\sup_D \gamma_0 + \sup_D \gamma_1 < \tilde{\eta}$, there exist at least $\ell + 1$ transmission eigenvalues corresponding to (11)-(14) each in a σ-neighborhood of k_i, $i = 0, 1, \ldots, \ell$.*

Proof. To prove this theorem, from (24) it suffices to choose $\tilde{\eta} = \max(\tilde{\eta}_{k_1}, \tilde{\eta}_{k_2}, \cdots \tilde{\eta}_{k_\ell})$ where

$$\tilde{\eta}_{k_i} < \eta_{k_i} \frac{\inf_D(\epsilon_0) \inf_D(\epsilon_1)}{4 \sup_D(\epsilon_0) + 4 \sup_D(\epsilon_1)}$$

and η_{k_i} is the size of the perturbation corresponding to k_i, $i = 0, 1, \cdots \ell$, as discussed above. $\square$

Remark 2.3. Following [1], [9], and [21], it is possible to prove the discreteness of transmission eigenvalues if the condition $\epsilon_c := \epsilon_1 - \epsilon \geq \theta > 0$ is assumed to hold only on a neighborhood of ∂D but this is beyond the scope of this paper.

3. Spherically Stratified Media

In the above section the existence of transmission eigenvalues was shown under the assumption that the absorption in the medium and the background are sufficiently small (unfortunately it is not possible to quantify the magnitude of the absorption in order to guarantee the existence of transmission eigenvalues). For arbitrary absorption, the existence of transmission eigenvalues is still open. However, in this section we show that in the case of spherically stratified media there exist infinitely many complex transmission eigenvalues. To this end, let $B := \{x : |x| < a\}$ and consider the interior transmission eigenvalue problem

$$\Delta_3 w + k^2 \left(\epsilon_1(r) + i \frac{\gamma_1(r)}{k} \right) w = 0 \qquad \text{in } B \tag{25}$$

$$\Delta_3 v + k^2 \left(\epsilon_0 + i \frac{\gamma_0}{k} \right) v = 0 \qquad \text{in } B \tag{26}$$

$$v = w \qquad \text{on } \partial B \tag{27}$$

$$\frac{\partial v}{\partial r} = \frac{\partial w}{\partial r} \qquad \text{on } \partial B \tag{28}$$

where $\epsilon(r)$ and $\gamma_1(r)$ are continuous functions of r in $\overline{B}$ such that $\epsilon_1(a) = \epsilon_0$ and and ϵ_0 and γ_0 are positive constants. We look for a solution of (25)-(28) in the form

$$\begin{aligned} v(r) &= c_1 j_0(k \tilde{n}_0 r) \\ w(r) &= c_2 \frac{y(r)}{r} \end{aligned} \tag{29}$$

where $\tilde{n}_0 := \left(\epsilon_0 + i\frac{\gamma_0}{k}\right)^{1/2}$ (where the branch cut is chosen such that $\tilde{n}_0$ has positive real part), j_0 is a spherical Bessel function of order zero, $y(r)$ is a solution of

$$y'' + k^2\left(\epsilon_1(r) + i\frac{\gamma_1(r)}{k}\right) y = 0 \tag{30}$$

$$y(0) = 0, \quad y'(0) = 1 \tag{31}$$

for $0 < r < a$ and c_1 and c_2 are constants. Then there exist constants c_1 and c_2, not both zero, such that (29) will be a nontrivial solution of (25)-(28) provided

$$d := \text{Det}\begin{pmatrix} \dfrac{y(a)}{a} & -j_0(k\tilde{n}_0 a) \\ \\ \dfrac{d}{dr}\left[\dfrac{y(r)}{r}\right]_{r=a} & -\dfrac{d}{dr}\left[j_0(k\tilde{n}_0 r)\right]_{r=a} \end{pmatrix} = 0. \tag{32}$$

We will derive an asymptotic expansion for $y(r)$ for large k to show that there exist an infinite set of complex values of k such that (32) is true.

Following [13] (p. 84 - see also page 89), we see that (30) has a fundamental set of solutions $y_1(r)$ and $y_2(r)$ defined for $r \in [a,\, b]$ such that

$$y_j(r) = Y_j(r)\left[1 + O\left(\frac{1}{k}\right)\right] \tag{33}$$

as $k \to \infty$, uniformly for $0 \le r \le a$ where

$$Y_j(r) = \exp\left[\beta_{0j}k + \beta_{1j}\right]$$

$$(\beta'_{0j})^2 + \epsilon_1(r) = 0 \tag{34}$$

$$2\beta'_{0j}\beta_{1j} + i\gamma_1(r) + \beta''_{0j} = 0. \tag{35}$$

From (34) we see that, modulo arbitrary constants,

$$\begin{aligned} \beta_{0j} &= \pm\int_0^r \sqrt{\epsilon_1(\rho)}\, d\rho \\ \beta_{ij} &= \mp\frac{1}{2}\int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(r)}} d\rho + \log\left[\epsilon_1(r)\right]^{-1/4} \end{aligned} \tag{36}$$

where $j = 1$ corresponds to the upper sign and $j = 2$ corresponds to the lower sign. Substituting back into (33) and using the initial condition (31) we see that

$$y(r) = \frac{1}{ik\left[\epsilon_1(0)\epsilon_1(r)\right]^{1/4}} \sinh\left[ik\int_0^r \sqrt{\epsilon_1(\rho)}\, d\rho - \frac{1}{2}\int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}} d\rho\right] + O\left(\frac{1}{k^2}\right) \tag{37}$$

as $k \to \infty$. Similarly,

$$j_0(k\tilde{n}_0 r) = \frac{1}{ik\sqrt{\epsilon_0}r} \sinh\left[ik\sqrt{\epsilon_0}r - \frac{1}{2}\frac{\gamma_0}{\sqrt{\epsilon_0}}r\right] + O\left(\frac{1}{k^2}\right) \tag{38}$$

as $k \to \infty$. Using (37), (38), and the fact that these expressions can be differentiated with respect to r, implies that

$$d = \frac{1}{ika^2\left[\epsilon_1(0)\epsilon_0\right]^{1/4}} \sinh\left[ik\sqrt{\epsilon_0}a - ik\int_0^a \sqrt{\epsilon_1(\rho)}\,d\rho \right.$$
$$\left. - \frac{1}{2}\frac{\gamma_0 a}{\sqrt{\epsilon_0}} + \frac{1}{2}\int_0^a \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}}d\rho\right] + O\left(\frac{1}{k^2}\right) \tag{39}$$

as $k \to \infty$.

We now want to use (39) to deduce the existence of transmission eigenvalues. In the case when there is no absorption ($\gamma_0 = \gamma_1 = 0$) this is a simple consequence of Bolzano's theorem (c.f. Section 8.4 of [11]). However this argument is no longer applicable in the present case and we must use more sophisticated arguments. We first note that since j_0 is an even function of its argument, $j_0(k\tilde{n}_0 r)$ is an entire function of k of order one and finite type. By representing $y(r)$ in terms of j_0 via a transformation operator (c.f. [10], p. 47-49) it is seen that $y(r)$ also has this property and hence so does d. Furthermore, d is bounded as $k \to \infty$. For $k < 0$, d has the asymptotic behavior (39) with γ_0 replaced by $-\gamma_0$ and γ_1 replaced by $-\gamma_1$ and hence d is also bounded as $k \to -\infty$. By analyticity k is bounded on any compact subset of the real axis and therefore $d(k)$ is bounded on the real axis. Now assume that there are not an infinite number of (complex) zeros of $d(k)$. Then by Hadamard's factorization theorem $d(k)$ is of the form

$$d(k) = k^m e^{ak+b} \prod_{\ell=1}^{n} \left(1 - \frac{k}{k_\ell}\right) e^{k/k_\ell}$$

for integers m and n and constants a and b. But this contradicts the asymptotic behavior of $d(k)$. Hence $d(k)$ has an infinite number of (complex) zeros, i.e. there exist an infinite number of transmission eigenvalues.

4. Transmission Eigenvalues Free Zones

A natural question to ask next is where do complex transmission eigenvalues lie (if they exist)? The goal of this section is to establish eigenvalue-free zones in the complex plane which will in turn provide bounds for transmission eigenvalues in terms of the material properties of the medium. Of particular interest from a practical point of view are the estimates for real transmission eigenvalues (if they exist) since they can be measured from the scattering data [4]. For this purpose we limit ourselves to the simple case where the refraction indexes are constant. Let $k = \kappa + i\tau$ be a complex wave number. We recall that $k^2 n_j = k^2\varepsilon_j + ik\gamma_j$, $j = 0, 1$. Therefore

$$k^2 n_j = \alpha_j + i\beta_j \quad j = 0, 1$$

with

$$\alpha_j = (\kappa^2 - \tau^2)\varepsilon_j - \tau\gamma_j \quad \text{and} \quad \beta_j = \kappa(2\tau\varepsilon_j + \gamma_j).$$

Complex transmission eigenvalues are those for which there exists a non-trivial solution $u \in H_0^2(D)$ to

$$\mathcal{P}u := (\Delta + \alpha_0 + i\beta_0)(\Delta + \alpha_1 + i\beta_1)u = 0 \quad \text{in } D.$$

In the following we will denote by $(\cdot,\, \cdot)$ the $L^2(D)$ scalar product and $\|\cdot\|$ the associated norm. The symmetry of the operator $\mathcal{P}$ implies that

$$(a) \quad (\Re(\mathcal{P})u,\, u) = 0 \quad \text{and} \quad (b) \quad (\Im(\mathcal{P})u,\, u) = 0 \tag{40}$$

where

$$\Re(\mathcal{P}) := (\Delta + \alpha_0)(\Delta + \alpha_1) - \beta_0\beta_1$$
$$\Im(\mathcal{P}) := \beta_1(\Delta + \alpha_0) + \beta_0(\Delta + \alpha_1)$$

4.1. The case when $\beta_0 + \beta_1 \neq 0$.

We first observe that

$$(\Re(\mathcal{P})u,\, u) \;=\; \|(\Delta + \alpha_0)u\|^2 + (\alpha_1 - \alpha_0)\left((\Delta + \alpha_0)u,\, u\right) - \beta_0\beta_1 \|u\|^2 .$$

Equation (40b) implies that

$$(\Delta u,\, u) = -\frac{(\beta_0\alpha_1 + \beta_1\alpha_0)}{\beta_0 + \beta_1} \|u\|^2 . \tag{41}$$

Consequently

$$\begin{aligned}(\Re(\mathcal{P})u,\, u) &= \|(\Delta + \alpha_0)u\|^2 + \left((\alpha_1 - \alpha_0)\left(\alpha_0 - \frac{(\beta_0\alpha_1 + \beta_1\alpha_0)}{\beta_0 + \beta_1}\right) - \beta_0\beta_1\right)\|u\|^2 \\ &= \|(\Delta + \alpha_0)u\|^2 - \beta_0\left(\frac{(\alpha_1 - \alpha_0)^2}{\beta_0 + \beta_1} + \beta_1\right)\|u\|^2\end{aligned}$$

Therefore if

$$\beta_0\left(\frac{(\alpha_1 - \alpha_0)^2}{\beta_0 + \beta_1} + \beta_1\right) \le 0. \tag{42}$$

then k is not a transmission eigenvalue. This extends the known result of non-existence of real transmission eigenvalues when one medium is absorbing and the other one is not. The latter case corresponds for instance to $\gamma_0 = 0$ and $\gamma_1 \neq 0$, i.e. since k is real, $\tau = 0$ and therefore $\beta_0 = 0$ which is a case included in (42). By symmetry, transmission eigenvalues also do not exist if

$$\beta_1\left(\frac{(\alpha_1 - \alpha_0)^2}{\beta_0 + \beta_1} + \beta_0\right) \le 0.$$

Let us introduce

$$\mu_0(D) := \min_{u \in H_0^2(D), u \neq 0} \|\Delta u\|^2 / \|u\|^2$$

which is the first clamped plate eigenvalue for the biharmonic operator $-\Delta^2$ in D. Regrouping differently the terms in $(\Re(\mathcal{P})u,\, u)$ we also observe that

$$\begin{aligned}(\Re(\mathcal{P})u,\, u) &= \|\Delta u\|^2 + (\alpha_0 + \alpha_1)(\Delta u,\, u) + (\alpha_0\alpha_1 - \beta_0\beta_1)\|u\|^2 \\ &= \|\Delta u\|^2 - \frac{\alpha_0 + \alpha_1}{\beta_0 + \beta_1}(\beta_0\alpha_1 + \beta_1\alpha_0)\|u\|^2 + (\alpha_0\alpha_1 - \beta_0\beta_1)\|u\|^2 .\end{aligned} \tag{43}$$

One therefore deduces that no transmission eigenvalues exist if

$$\mu_0(D) > \frac{\alpha_0+\alpha_1}{\beta_0+\beta_1}(\beta_0\alpha_1+\beta_1\alpha_0)+\beta_0\beta_1-\alpha_0\alpha_1.$$

which is equivalent to

$$\frac{\alpha_0^2\beta_1+\alpha_1^2\beta_0}{\beta_0+\beta_1}+\beta_0\beta_1 < \mu_0(D). \tag{44}$$

A similar type of condition can be obtained from (41) since if we set

$$\lambda_0(D) = \min_{u\in H_0^1(D), u\neq 0} \|\nabla u\|^2 / \|u\|^2$$

which is the first Dirichlet eigenvalue for $-\Delta$ in D, then we easily see that no transmission eigenvalues exist if

$$\frac{\beta_0\alpha_1+\beta_1\alpha_0}{\beta_0+\beta_1} < \lambda_0(D). \tag{45}$$

One consequence of (44) and (45) is that for real transmission eigenvalues we have the lower bounds

$$k^2 \geq \lambda_0(D)\frac{\gamma_0+\gamma_1}{\gamma_0\varepsilon_1+\gamma_1\varepsilon_0}$$

and

$$\mu_0(D) \leq k^4\frac{\gamma_0\varepsilon_1^2+\gamma_1\varepsilon_0^2}{\gamma_0+\gamma_1}+k^2\gamma_0\gamma_1$$

For an illustration of the region excluded by inequalities (44) and (45) we refer to Figures 1 and 2.

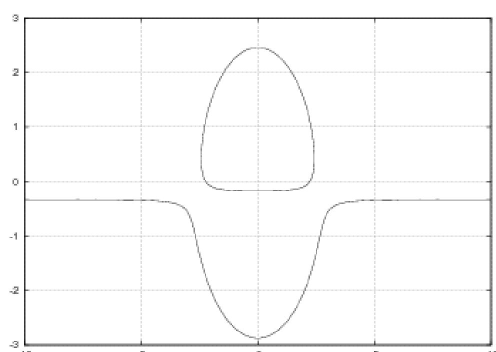


Figure 1: Parameters: $\varepsilon_0 = 1, \varepsilon_1 = 2, \gamma_0 = 1, \gamma_1 = 0, \mu_0(D) = 100$. The free zone defined by (44) corresponds to the complement of the connected region containing the point $(0, -1)$.

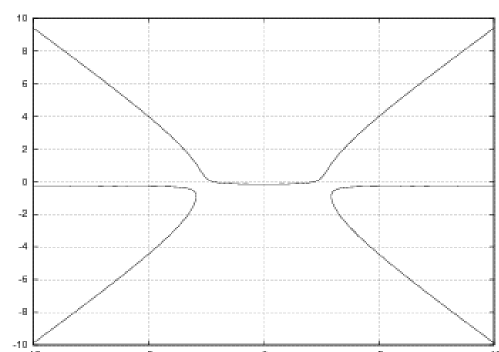


Figure 2: Parameters: $\varepsilon_0 = 1, \varepsilon_1 = 2, \gamma_0 = 1, \gamma_1 = 0, \lambda_0(D) = 10$. The free zone defined by (45) corresponds to the complement of the connected region containing the point $(0, -2)$.

4.2. The general case.

In the general case, i.e. including the case when $\beta_0 + \beta_1 = 0$, one can obtain a priori estimates on the transmission eigenvalues by testing for the positivity of

$$(\Re(\mathcal{P})u,\, u) \;=\; \|\Delta u\|^2 - (\alpha_0+\alpha_1)\,\|\nabla u\|^2 + (\alpha_0\alpha_1 - \beta_0\beta_1)\,\|u\|^2 . \quad (46)$$

To this end, we consider the *subregion of the complex plane defined by* $(\alpha_0+\alpha_1) \le 0$. From the definition of α_0 and α_1 we notice that this subregion does not cover real transmission eigenvalues. Now, from (46) one obviously sees that in the case when $(\alpha_0+\alpha_1) \le 0$, no transmission eigenvalues exist if

$$\mu_0(D) - (\alpha_0+\alpha_1)\lambda_0(D) + (\alpha_0\alpha_1 - \beta_0\beta_1) > 0. \quad (47)$$

In particular no transmission eigenvalues are present in the region defined by

$$(\alpha_0+\alpha_1) \le 0 \;\text{ and }\; \beta_0\beta_1 < \alpha_0\alpha_1$$

Figure 3 represents an illustrative example of the transmission eigenvalues free zone defined by (47).

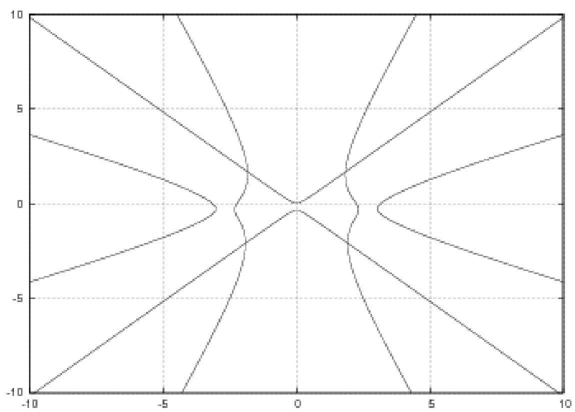


Figure 3: Parameters: $\varepsilon_0 = 2, \varepsilon_1 = 1, \gamma_0 = 0, \gamma_1 = 1, \mu_0(D) = 100$. The eigenvalue free zone defined by (44) corresponds to the connected regions containing the points $(5,7), (5,-7), (-5,7), (-5,-7)$.

4.3. An estimate for small values of $|k|$.

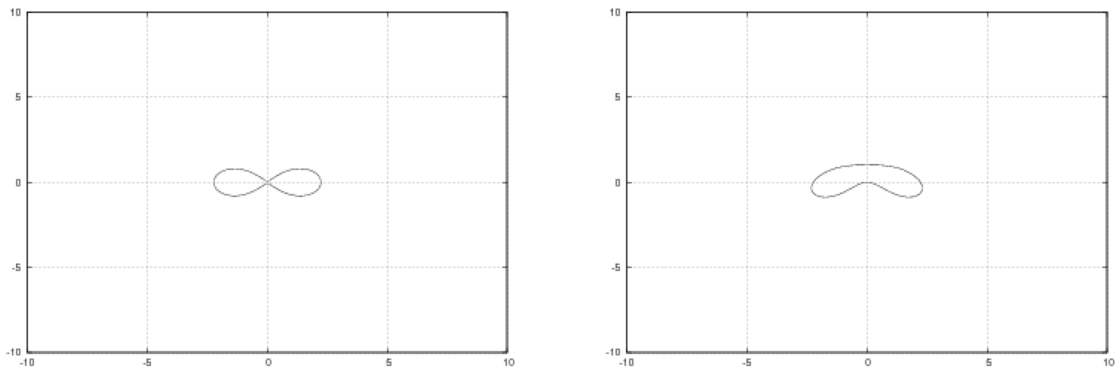
In the general case we can reproduce the approach in [2] [3], [6], [12] on obtaining bounds for real transmission eigenvalues. To this end let as assume that $\alpha_0 > \alpha_1$. Then

$$(\Re(\mathcal{P})u,\, u) \;=\; \|\Delta u + \alpha_0 u\|^2 + (\alpha_0 - \alpha_1)\left(\|\nabla u\|^2 - \alpha_0\,\|u\|^2\right) - \beta_0\beta_1\,\|u\|^2 . \quad (48)$$

Using the fact that $\lambda_0(D)\,\|u\|^2 \le \|\nabla u\|^2$ one deduces that no transmission eigenvalues exist in the region

$$\begin{cases} \alpha_0 > \alpha_1 \\ \alpha_0 + \frac{\beta_0\beta_1}{\alpha_0-\alpha_1} \le \lambda_0(D). \end{cases} \quad (49)$$

By symmetry, no transmission eigenvalues are present in the region defined by interchanging the role of the indices 0 and 1 in (49). The following figures (Figure 4) illustrate the domains defined by (49) in both cases.



Parameters: $\varepsilon_0 = 1, \varepsilon_1 = 2,$ $\gamma_0 = 0, \gamma_1 = 0, \lambda_0(D) = 10$

Parameters: $\varepsilon_0 = 1, \varepsilon_1 = 2,$ $\gamma_0 = 0, \gamma_1 = 1, \lambda_0(D) = 10$

Figure 4: The free zone defined by (49) corresponds to the interior of the indicated closed curves.

4.4. An a priori estimate for the first real transmission eigenvalue.

Using the known estimates for real transmission eigenvalues for the non-absorbing case we can obtain a finer estimate for real transmission eigenvalues for the absorbing case (if they exist). To this end, let us denote by k_0 the first real transmission eigenvalue corresponding to the non-absorbing case (i.e. $\gamma_0 = 0$ and $\gamma_1 = 0$) and $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ which are assumed to be different, for instance $0 < \varepsilon_0 < \varepsilon_1$. The existence of k_0 is proven in [6]. This transmission eigenvalue is defined as the smallest positive real k_0 such that there exists a non trivial solution $u_0 \in H_0^2(D)$ of

$$(\Delta + k_0^2\varepsilon_0)(\Delta + k_0^2\varepsilon_1)u_0 = 0 \quad \text{in } D.$$

It is then known [6], [7] that

$$k_0^2\varepsilon_0 \left\|\nabla u\right\|^2 \leq \frac{\varepsilon_0}{\varepsilon_1 - \varepsilon_0} \left\|\Delta u + k_0^2\varepsilon_0 u\right\|^2 + (k_0^2\varepsilon_0)^2 \left\|u\right\|^2 \quad \forall u \in H_0^2(D).$$

This is equivalent to

$$\left\|\nabla u\right\|^2 \leq \frac{1}{k_0^2(\varepsilon_1 + \varepsilon_0)} \left\|\Delta u\right\|^2 + \frac{k_0^2\varepsilon_1\varepsilon_0}{(\varepsilon_1 + \varepsilon_0)} \left\|u\right\|^2 \quad \forall u \in H_0^2(D). \tag{50}$$

Using this estimate one obtains from (46) that

$$(\Re(\mathcal{P})u,\, u) \;\; \geq \;\; \left\|\Delta u\right\|^2 - \tfrac{(\alpha_0+\alpha_1)}{k_0^2(\varepsilon_1+\varepsilon_0)} \left(\left\|\Delta u\right\|^2 + k_0^4\varepsilon_1\varepsilon_0 \left\|u\right\|^2\right) + (\alpha_0\alpha_1 - \beta_0\beta_1) \left\|u\right\|^2.$$

Next we obtain an a priori estimate for the real part κ of a complex transmission eigenvalue k. We first observe that

$$\alpha_j \leq \kappa^2\varepsilon_j \quad j = 0, 1.$$

Consequently

$$(\Re(\mathcal{P})u,\, u) \;\geq\; \|\Delta u\|^2 \left(1 - \tfrac{\kappa^2}{k_0^2}\right) + (\alpha_0\alpha_1 - \beta_0\beta_1 - \kappa^2 k_0^2 \varepsilon_1\varepsilon_0)\,\|u\|^2 . \quad (51)$$

We also observe that

$$\alpha_0\alpha_1 - \beta_0\beta_1 = \kappa^4\varepsilon_1\varepsilon_0 - \kappa^2\theta(\kappa,\tau)$$

where

$$\theta(\kappa,\tau) := (1-(y/x)^2)(\varepsilon_0 y + \gamma_0)(\varepsilon_1 y + \gamma_1) + y\,(2\varepsilon_0(\varepsilon_1 y + \gamma_1) + 2\varepsilon_1(\varepsilon_0 y + \gamma_0) + y\varepsilon_0\varepsilon_1)\,.$$

In particular $\theta(\kappa,\tau) > 0$ for positive κ and $-\delta < \tau < \kappa$ with small enough $\delta > 0$. Assume that $\kappa^2 \leq k_0^2$ then, from $\|\Delta u\|^2 \geq \mu_0 \|u\|^2$ and (51), we infer that

$$(\Re(\mathcal{P})u,\, u) \;\geq\; \left(\left(1 - \tfrac{x^2}{k_0^2}\right)(\mu_0 - x^2 k_0^2\varepsilon_1\varepsilon_0) - x^2\theta(x,y)\right)\|u\|^2 .$$

Consequently, no transmission eigenvalues exist if

$$\left(1 - \frac{\kappa^2}{k_0^2}\right)(\mu_0 - \kappa^2 k_0^2 \varepsilon_1\varepsilon_0) > \kappa^2\theta(\kappa,\tau),$$

from which we can write a weaker condition, i.e. no transmission eigenvalues exist if

$$\left(1 - \frac{\kappa^2}{k_0^2}\right)(\mu_0 - k_0^4 \varepsilon_1\varepsilon_0) > \kappa^2\theta(\kappa,\tau).$$

In case that $k_0^2 \leq \sqrt{\mu_0}/\sqrt{\varepsilon_1\varepsilon_0}$ (which is possible for example for large ϵ_1 [19]) we can get a simpler estimate, e.g. the previous inequality implies that there are no complex transmission eigenvalue with real part κ that satisfies

$$\kappa^2 < k_0^2\left(1 + \frac{\theta(\kappa,\tau)}{\mu_0 - k_0^4\varepsilon_1\varepsilon_0}\right)^{-1}. \quad (52)$$

In particular, the latter condition implies that there are no real transmission eigenvalues k, i.e. for $\tau = 0$, that satisfy

$$k^2 = \kappa^2 < k_0^2\left(1 + \frac{\gamma_0\gamma_1}{\mu_0 - k_0^4\varepsilon_1\varepsilon_0}\right)^{-1} \quad (53)$$

since now $\theta(\kappa, 0) = \gamma_0\gamma_1$.

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