ISSN 0002-9920 (print)
ISSN 1088-9477 (online)

Notices
of the American Mathematical Society

October 2021 Volume 68, Number 9

American Mathematical Society
AMS
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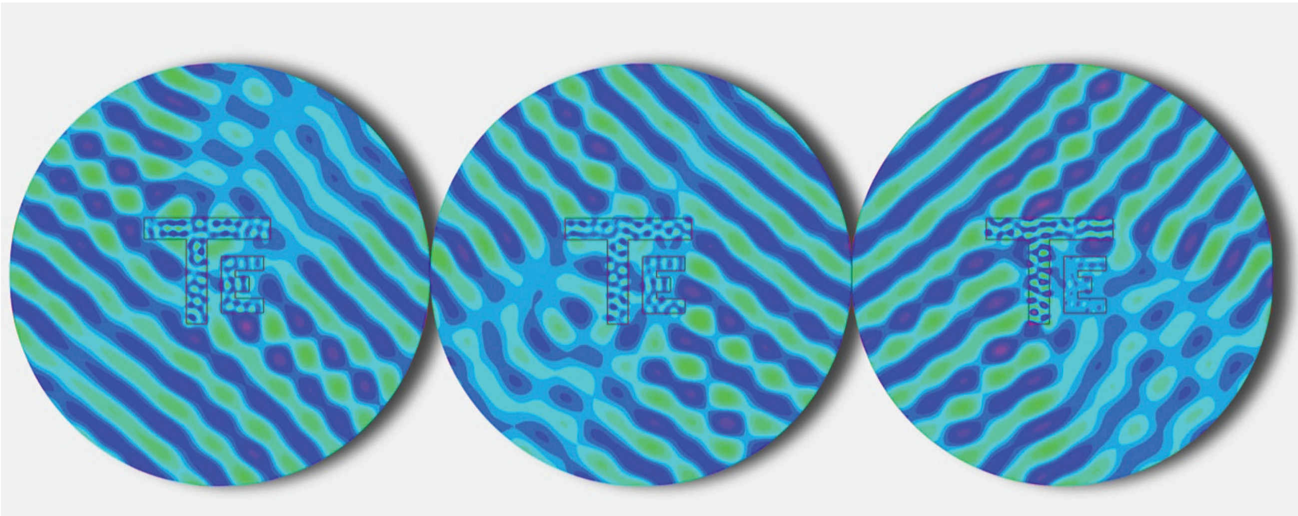
See the JMM 2022 Section on page 1639.

Transmission Eigenvalues



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Introduction

The study of eigenvalue problems for partial differential equations has a long history during which a variety of themes has emerged. Although historically such efforts have focused on eigenvalue problems defined on bounded domains, the importance of scattering theory in modern mathematical physics has led to an intensive study of eigenvalue problems in unbounded domains connected with the Schrödinger equation and the wave equation for propagation in an inhomogeneous medium. A particularly noteworthy development in this latter direction has been the theory of scattering resonances which now play a central role in mathematical scattering theory. For a magisterial presentation of this theory we refer the reader to the monograph by Dyatlov and Zworski [DZ19]. More recently, a new eigenvalue problem in scattering theory has attracted increased attention both inside and outside the scattering community. This new problem is called the *transmission eigenvalue problem* and in a certain sense exhibits a duality relation to the theory of scattering resonances. The purpose of this short survey is to introduce this class of nonselfadjoint eigenvalue problems to the wider mathematical community. For further information on transmission eigenvalues, including applications to inverse scattering theory, we refer the reader to the monograph [CCH16] and Chapter 10 of [CK19].

The transmission eigenvalue problem arises in the study of wave propagation in an inhomogeneous medium. Hence, for the benefit of the reader who is not an expert in scattering theory, we begin by describing the basic elements of acoustic scattering theory (see [CK19]). Broadly speaking, acoustic scattering theory is concerned with the effect an inhomogeneous medium has on an incident wave. In particular, if the total field u is viewed as the sum of an incident field u^i and a scattered field u^s, then the scattering problem is to determine u^s from a knowledge of u^i and the differential equation governing the wave motion. More specifically, assume that the incident field is given by the time-harmonic acoustic plane wave

$$u^i(x,t) = e^{i(kx\cdot\hat{y}-\omega t)},$$

where $x \in \mathbb{R}^3$, t denotes time, $k = \omega/c_0$ is the wave number, ω is the frequency, c_0 is the speed of sound, and the unit

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Communicated by Notices *Associate Editor Reza Malek-Madani.*

DOI: https://doi.org/10.1090/noti2350

vector $\hat{y}$ is the direction of propagation. Then factoring out the term $e^{-i\omega t}$, the simplest acoustic scattering problem for the case of an inhomogeneous medium is to find the total field u such that

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^3, \tag{1}$$

$$u(x) = u^i(x) + u^s(x), \tag{2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{3}$$

where $u^i(x) = e^{ikx\cdot\hat{y}}$, $r = |x|$, $n = c_0^2/c^2$ is the refractive index where $c = c(x)$ is the speed of sound in the inhomogeneous medium, and (3) is the *Sommerfeld radiation condition* which holds uniformly with respect to $\hat{x} := x/|x|$ and guarantees that the scattered field is outgoing. It is assumed that $1-n$ has compact support in a bounded region D having piecewise smooth boundary ∂D such that $\mathbb{R}^3 \setminus D$ is connected and that $n \in L^\infty(D)$ is such that $n(x) > 0$ for $x \in D$. It can be shown that there is a unique solution $u^s \in H^2_{loc}(\mathbb{R}^3)$ to the scattering problem (1)–(3) and that $u^s(x) := u^s(x, \hat{y}, k)$ (for fixed $\hat{y}$ and k) has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r} u^\infty(\hat{x}) + O\left(\frac{1}{r^2}\right), \quad r \to \infty, \tag{4}$$

where the function $u^\infty(\hat{x}) := u^\infty(\hat{x}, \hat{y}, k)$ is called the *far field pattern* and is an infinitely differentiable function of $\hat{x}$ on the unit sphere S^2. *Rellich's lemma* says that $u^\infty(\hat{x})$ for $\hat{x} \in S^2$ determines $u^s(x)$ for $x \in \mathbb{R}^3 \setminus \overline{D}$. Note that if we vary the incident direction $\hat{y} \in S^2$, $u^\infty(\hat{x}, \hat{y}, k)$ is also infinitely differentiable with respect to $\hat{y}$, and in fact it can be shown that $u^\infty(\hat{x}, \hat{y}, k) = u^\infty(-\hat{y}, -\hat{x}, k)$.

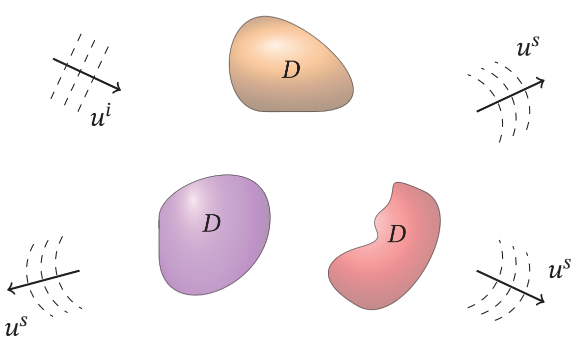

Figure 1. Sketch of the scattering problem.

Associated with the scattering problem (1)–(3) are two basic nonselfadjoint eigenvalue problems. The first of these, and by far the most studied, is the theory of scattering resonances which seeks values of the wave number k such that for $u^i = 0$ there exists a nontrivial solution u of (1)–(2) with an appropriate modification of the radiation condition (3) accounting for $\Im(k) < 0$. Such values of k are called *scattering resonances* and can be shown to form a discrete set lying in the lower half of the complex k-plane.

The second class of eigenvalue problems associated with the scattering problem (1)–(3), and one of more recent origin, is the theory of transmission eigenvalues and this is the topic of our survey article. Now, instead of asking for a nontrivial solution u of (1)–(3) for which $u^i = 0$, we ask if there is a nontrivial solution u of (1)–(3) for which $u^s = 0$. In other words, we ask the question of whether we are able to construct an incident field which does not scatter. Values of k for which this is possible will lead to the theory of *transmission eigenvalues*. Of particular interest to us in the sequel will be incident fields $u^i = v_g$, where v_g is defined by

$$v_g(x) = \int_{S^2} g(\hat{y}) e^{ikx\cdot\hat{y}}\, ds \tag{5}$$

and $g \in L^2(S^2)$ is referred to as the kernel of v_g. Solutions of the Helmholtz equation of the form (5) are called *Herglotz wave functions* and are extensively discussed in [CK19].

We now proceed to outline the basic theory of transmission eigenvalues and in particular their dual relationship to scattering resonances. We begin by noting that a solution v of the Helmholtz equation

$$\Delta v + k^2 v = 0 \qquad \text{in } \mathbb{R}^3 \tag{6}$$

is of the form v_g in (5) if and only if

$$\sup_{R>0} \frac{1}{\sqrt{R}} \|v\|_{L^2(B_R)} < \infty,$$

where B_R is the ball of radius R centered at the origin. Every v_g in the space of Herglotz wave functions can be uniquely decomposed as $v_g := u_g - u_g^s$, where the total field u_g and the scattered field u_g^s satisfy (1)–(3) with $u^i := v_g$. The *scattering operator (matrix)* as defined by Lax and Phillips in [LP89] maps $v_g \mapsto u_g$ and for k such that $\Im(k) \geq 0$ is an isomorphism in appropriate Banach spaces. A heuristic argument for the latter can be given using the Lipmann-Schwinger equation for the solution of (1)–(3) with $u^i := v_g$ in terms of the compact k-analytic integral operator $T(k) : L^2(B_R) \to L^2(B_R)$,

$$(I - T(k))u = v_g, \tag{7}$$

where $D \subset B_R$ and

$$T(k)u := k^2 \int_{\mathbb{R}^3} \frac{e^{ik|x-y|}}{4\pi|x-y|}(1 - n(y))u(y)\, dy.$$

A fixed point argument implies that for $|k|$ small enough $I - T(k)$ is invertible, and hence by the Analytic Fredholm Theorem we have that $u_g := (I - T(k))^{-1} v_g$ is meromorphic for $k \in \mathbb{C}$. Furthermore, for k such that $\Im(k) \geq 0$, uniqueness of the scattering problem implies that u_g is analytic and thus its poles are in the lower-half complex plane.

Of particular interest to us in the sequel will be the "incoming-to-outgoing" mapping $v_g \mapsto u_g^s := u_g - v_g$. We shall characterize this in terms of the far field pattern defined in (4). To this end let u_g^∞ denote the far field pattern of the scattered field u_g^s corresponding to the incident field v_g. The compact linear operator $F(k) : L^2(S^2) \to L^2(S^2)$ defined by

$$F(k) : g \mapsto u_g^\infty \tag{8}$$

is called the *far field operator* (otherwise referred to as the *relative scattering operator*). Clearly from (5) and by linearity

$$u_g^\infty(\hat{x}) = (F(k)g)(\hat{x}) = \int_{S^2} u^\infty(\hat{x}, \hat{y}, k) g(\hat{y})\, ds,$$

where $u^\infty(\hat{x}, \hat{y}, k)$ is the far field pattern of the scattered field due to an incident plane wave $e^{ikx\cdot\hat{y}}$ (4). The scattering operator $\mathcal{S}(k) : L^2(S^2) \to L^2(S^2)$ can then be expressed as [LP89]

$$\mathcal{S}(k) := I + \frac{ik}{2\pi} F(k).$$

If $\Im(n) = 0$, then $F(k)$ is normal and $\mathcal{S}(k)$ is unitary for real $k > 0$, which is not the case if $\Im(n) > 0$ on a subset of D of nonzero measure. Both are analytic operator-valued functions of k in the upper-half complex plane. The scattering poles are the poles of the meromorphic extension of $\mathcal{S}(k)$ in the lower-half complex plane.

Now we are ready to formally introduce the transmission eigenvalue problem. An application of Rellich's lemma implies that the incident field v_g with $g \in \operatorname{Kern} F(k)$ does not scatter. Straightforward calculation reveals that the kernel of $F(k)$ consists of all $g \in L^2(S^2)$ such that, if v_g is the corresponding Herglotz wave function, $v := v_g|_D$ and u satisfy the *transmission eigenvalue problem*

$$\begin{cases} \Delta u + k^2 n(x) u = 0 & \text{in} \quad D, \\ \Delta v + k^2 v = 0 & \text{in} \quad D, \\ u = v & \text{on} \quad \partial D, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} & \text{on} \quad \partial D. \end{cases} \tag{9}$$

A value of $k \in \mathbb{C}$ is said to be a *transmission eigenvalue* if (9) has nontrivial solutions $u \in L^2(D)$, $v \in L^2(D)$, such that $u - v \in H_0^2(D)$. We call the pair (u, v) the corresponding eigenfunction. In general, at a transmission eigenvalue the part v of the corresponding eigenfunction does not take the form of a Herglotz wave function and hence the kernel of the far field operator is in general empty. Thus the set of *nonscattering wave numbers* k for which $\operatorname{Kern} F(k) \neq \{0\}$ is a subset (possibly empty) of the transmission eigenvalues. There is a special configuration for which transmission eigenvalues and nonscattering frequencies coincide, namely the case of spherically stratified media which is discussed in the next section. Allowing u^i to be any function $v \in H_{inc}(D)$ with

$$v \in H_{inc}(D) := \{v \in L^2(D) : \ \Delta v + k^2 v = 0\},$$

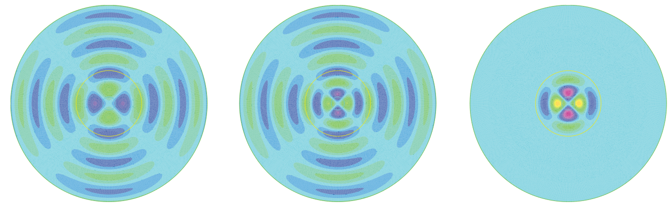

Figure 2. Illustration of a transmission eigenvalue for a dielectric medium formed by two concentric circles with outer radius = 1 and inner radius = 0.3. The refractive index $n = 2$ in the annulus and $n = 3$ in the inner circle. The wave number is $k = 7.223$. Left: Incident field; a Herglotz wave function with kernel $g(\theta) = exp(-2i\theta)$. Middle: Total field. Right: Scattered field (=0 outside the circle of radius 1).

the scattered field u^s corresponding to v can be defined as $u^s \in H^2_{loc}(\mathbb{R}^3)$ satisfying

$$\Delta u^s + k^2 n u^s = k^2(1-n)v \quad \text{in} \quad \mathbb{R}^3 \tag{10}$$

together with the Sommerfeld radiation condition (3). Note that $H_{inc}(D)$ is a Hilbert space which densely contains the Herglotz wave functions v_g. Defining the operator $\mathcal{G}(k) : H_{inc}(D) \to L^2(S^2)$ mapping $v \mapsto u^\infty$, where u^∞ is the far field associated with the solution of (10), which is a compact linear operator, then we can equivalently define transmission eigenvalues as the values of k for which $\operatorname{Kern} \mathcal{G}(k)$ is nontrivial (in fact the part v of the corresponding eigenfunction belongs to $\operatorname{Kern} \mathcal{G}$). In addition, the relation

$$F(k)g = \mathcal{G}(k)\mathcal{H}(k)g$$

holds, where

$$\mathcal{H}(k) : g \mapsto v_g|_D \tag{11}$$

and we already observed that $\overline{\mathcal{H}(L^2(S^2))} = H_{inc}(D)$. Hence, in general, at a transmission eigenvalue one can construct a Herglotz wave function v_g of unit $L^2(D)$-norm, that produces an arbitrarily small scattered field u_g^s (see Figures 2 and 3).

Spherically Stratified Media

As mentioned above, the scattering problem for spherically symmetric media is of great importance since it provides an example where the set of nonscattering frequencies and transmission eigenvalues are the same. More precisely, when D is a ball of radius a centered at the origin and $n := n(r)$, $r = |x|$, is a *radial real-valued function*, the part v of a transmission eigenfunction is indeed a Herglotz wave function and hence transmission eigenvalues coincide with the values of $k \in \mathbb{C}$ for which $\operatorname{Kern} F(k) \neq \{0\}$. Figure 2 gives an example in two dimensions. To see explicitly what the transmission eigenvalues are in this case, we use as incident field the Herglotz wave function $v = j_\ell(k|x|)Y_\ell^m(\hat{x})$, where j_ℓ is a spherical Bessel function and Y_ℓ^m is a spherical harmonic of order $\ell \in \mathbb{N}_0$, $m = -\ell \cdots \ell$. Straightforward calculations by separation of variables lead to the

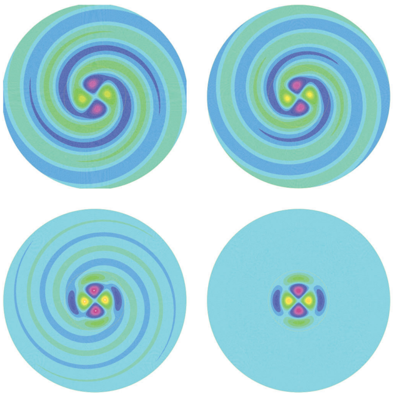

Figure 3. We represent the scattered fields for the same scattering experiment as in Figure 2 with an incident field being a Herglotz wave function with kernel $g(\theta) = exp(-2i\theta)$ but for different wave numbers. From top to bottom and left to right $k = 6$, $k = 6.5$, $k = 7$, $k = 7.223$ (a transmission eigenvalue).

following expression for the scattered field for $r > a$ and the corresponding far field, respectively,

$$u^s(x) := \frac{C_\ell(k;n)}{W_\ell(k;n)} h_\ell^{(1)}(k|x|) Y_\ell^m(\hat{x}),$$
$$u^\infty(\hat{x}) := \frac{C_\ell(k;n)}{W_\ell(k;n)} \frac{1}{k} Y_\ell^m(\hat{x}),$$

where $h_\ell^{(1)}(r)$ is the Hankel function of the first kind of order ℓ and

$$C_\ell(k;n) = \det\begin{pmatrix} y_\ell(a) & -j_\ell(ka) \\ y_\ell'(a) & -kj_\ell'(ka) \end{pmatrix},$$
$$W_\ell(k;n) = \det\begin{pmatrix} y_\ell(a) & -h_\ell^{(1)}(ka) \\ y_\ell'(a) & -kh_\ell^{(1)'}(ka) \end{pmatrix}$$

with y_ℓ the solution to

$$y'' + \frac{2}{r} + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) y = 0$$

which as $r \to 0$ behaves like $j_\ell(kr)$, i.e., $\lim_{r\to 0} r^{-\ell} y_\ell(r) = \frac{\sqrt{\pi}k^\ell}{2^{\ell+1}\Gamma(\ell+3/2)}$. Thus transmission eigenvalues are those values of $k \in \mathbb{C}$ such that $C_\ell(k;n) = 0$, whereas the scattering poles are $k \in \mathbb{C}$ for which $W_\ell(k;n) = 0$. The latter set lies in the lower half of the complex plane. If k is a zero of $C_\ell(k;n)$ (i.e., a transmission eigenvalue), then the part v of the corresponding eigenfunction is $v = j_\ell(k|x|)Y_\ell^m(\hat{x})$ which is an entire solution of the Helmholtz equation in $\mathbb{R}^3$. All transmission eigenvalues for a spherically stratified medium are obtained in this way by choosing $\ell \in \mathbb{N}$. Hence in this case, at any real transmission eigenvalue, there is at least one incident wave $v = j_\ell(k|x|)Y_\ell(\hat{x})$ for some $\ell \in \mathbb{N}$ that does not scatter (in fact there are at least $2\ell+1$ linearly independent nonscattering incident waves).

It is possible to give more details on the structure of transmission eigenvalues for spherically stratified media if we focus on the case of spherically symmetric incident fields ($\ell = 0$ in the above discussion), in other words when the transmission eigenfunction is radially symmetric. Hence in the rest of this section we consider only transmission eigenvalues with spherically symmetric eigenfunctions. These are the zeros of $C_0(k;n)$, and at such a zero $v := j_0(kr)$ and $u := y_0(r)$ satisfy the transmission eigenvalue problem:

$$\begin{cases} \Delta u + k^2 n(r) u = 0 & \text{for} \quad |r| < a, \\ \Delta v + k^2 v = 0 & \text{for} \quad |r| < a, \\ v(a) - u(a) = 0 & \text{and} \quad v'(a) - u'(a) = 0. \end{cases}$$

Letting $y_0(r) := y(r)/r$, where now $y(r)$ satisfies $y'' + k^2 n(r) y = 0$, and noting that $j_0(kr) = \sin(kr)/kr$, we obtain that k is a transmission eigenvalue, i.e., $C_0(k;n) = 0$, if and only if

$$d(k) := \det\begin{pmatrix} y(a) & -\sin(ka)/k \\ y'(a) & -\cos ka \end{pmatrix} = 0.$$

We now note that $d(k)$ is an entire function of k that is real for real k and is bounded on the real axis. Hence, by Hadamard's factorization theorem, if $d(k)$ is not identically zero, then there exists a countable set of transmission eigenvalues (cf. [CCH20] for this conclusion in a similar case). It can be shown that if $d(k)$ is identically equal to zero, then $n(r)$ is identically equal to one [CCH16], Section 5.1. From now on we assume that

$$\delta := \int_0^a \sqrt{n(\rho)}\, d\rho \neq a. \tag{12}$$

An asymptotic analysis shows that

$$\begin{aligned} d(k) &= \frac{1}{ka^2}\left[\frac{1}{[n(0)n(a)]^{1/4}} \sin(k\delta)\cos(ka) \right. \\ &\quad \left. - \left[\frac{n(a)}{n(0)}\right]^{1/4} \cos(k\delta)\sin(ka)\right] + O\left(\frac{1}{k^2}\right) \end{aligned}$$

as $k \to \infty$ and hence there exist an infinite number of positive transmission eigenvalues.

Example. Let $n(r) = n_0^2$ with n_0 a positive constant. When $n_0 = 1/2$ we have that

$$d(k) = \frac{2}{k}\sin^3\left(\frac{ka}{2}\right)$$

and hence $d(k)$ has a set of real zeros and no complex zeros. When $n_0 = 2/3$ we have that

$$d(k) = \frac{1}{k}\sin^3\left(\frac{ka}{3}\right)\left[3 + 2\cos\left(\frac{2ka}{3}\right)\right]$$

and hence $d(k)$ has an infinite set of real and complex zeros.

This example demonstrates the perplexing structure of transmission eigenvalues and in particular their dependence on the properties of the contrast of the medium.

Now let $n \in C^2[0, a]$ and assume (12) holds. Then one can prove the following results [CK19].

Theorem 1. *Assume that either* $1 < \sqrt{n(r)} < \delta/a$ *or* $\delta/a < \sqrt{n(r)} < 1$ *for* $0 \leq r \leq a$*. Then there exist infinitely many real and infinitely many complex transmission eigenvalues.*

Theorem 2. *Assume that* $n(a) \neq 1$*. Then, if complex eigenvalues exist, they all lie in a strip parallel to the real axis.*

The above theorems are quite different if we relax the assumption that $n(a) \neq 1$. In particular if $n(a) = 1$ and $n'(a) = 0$, then there exist infinitely many real and infinitely many complex transmission eigenvalues only under extra assumptions, e.g., $n''(a) \neq 0$. Furthermore, if $n(a) = 1$ and either $n'(a)$ or $n''(a)$ is nonzero, then the set of transmission eigenvalues does not lie inside any fixed strip parallel to the real axis. The extension of these results to the general media case will be discussed later in this paper.

Given the existence of both real and complex transmission eigenvalues for a spherically stratified medium with spherically symmetric eigenfunctions, it is a natural question to ask if whether or not a knowledge of all eigenvalues, both real and complex including multiplicity, uniquely determine $n(r)$. Recent progress in this direction has been obtained by Aktosun, Gintides, and Papanicolaou and by Colton and Leung (see [CCH16], Section 5.1 and references therein). Both of these papers assumed that $0 < n(r) \leq 1$ for $r < a$ as well as the fact that $n(a) = 1$ and $n'(a) = 0$. The case when $n(r) > 1$ is still open. However it can be shown that all transmission eigenvalues (not only those with spherically symmetric eigenfunctions), real and complex including multiplicity, uniquely determine $n(r)$ in both the cases when $0 < n(r) \leq 1$ and $n(r) \geq 1$ for $r \leq a$ provided that such information is known along with the value of $n(0)$ [CCH16], Section 5.2.

Note that a similar asymptotic analysis used above for $C_0(k; n)$ can be employed to prove that $W_0(k; n)$ has infinitely many complex zeros, thus proving the existence of an infinite set of scattering poles for spherically symmetric media (see [CCH20] for more details).

Nonscattering Frequencies and Transmission Eigenvalues

The basic question that led to the transmission eigenvalue problem is the injectivity of the far field operator $F(k)$. In particular, if the far field operator $F(k)$ is not injective, then there is a Herglotz wave function v_g which is not scattered by the homogeneity. We can more generally define *nonscattering wave numbers* as the values of $k \in \mathbb{R}$ for which there exists an incident wave v, a solution of the Helmholtz equation $\Delta v + k^2 v = 0$ in $\mathbb{R}^3$, which doesn't scatter by the inhomogeneity. A necessary condition for k to be a nonscattering wave number is that k is a transmission eigenvalue with $u := (u^s + v)|_D$ and $v := v|_D$ as corresponding eigenfunction. On the other hand, a real transmission eigenvalue k is a nonscattering wave number if the part v of the corresponding transmission eigenfunction is also defined as a solution of the Helmholtz equation outside D. Whether at a transmission eigenvalue such an extension is possible remains largely open. In the case of a spherically stratified media, we already saw that there exist infinitely many nonscattering wave numbers and furthermore the set of nonscattering wave numbers coincides with the set of real transmission eigenvalues. The spherically symmetric configuration is unstable with respect to nonscattering. Recently in [VX] it is shown in $\mathbb{R}^2$ and for constant refractive index $n \neq 1$ that if the disk is perturbed even slightly to an ellipse with arbitrarily small eccentricity, then there exist at most finitely many positive wave numbers for which a Herglotz wave function with a fixed, smooth nontrivial density can be nonscattering. When the boundary of the inhomogeneity D contains a corner, at a transmission eigenvalue it is impossible to extend the part v of the transmission eigenfunction as a solution to the Helmholtz equation in the vicinity of the corner outside D. Thus, for such inhomogeneities, nonscattering wave numbers do not exist. This result was first proved in [BPS14] for D having a right angle and nonvanishing $n - 1$ at the corner, followed by [EH18] for inhomogeneities containing arbitrary corners and edges. The recent paper [CV21] contributes to filling the gap between spherically symmetric media and inhomogeneities containing a corner in relation to nonscattering waves (with the exception of a few earlier partial results for analytic boundaries; cf. references in [CV21]). Roughly speaking, employing techniques on free boundary regularity, the authors show that if there is a point $x_0 \in \partial D$, such that n is analytic in a neighborhood of x_0, but the boundary is not analytic in any neighborhood of x_0, then every incident field v is scattered, provided $(n(x_0) - 1)v(x_0) \neq 0$. A similar result is proven for n that are less regular locally near x_0, in which case it is proven that if the boundary is not sufficiently smooth locally (related to the order of smoothness of n), then every incident wave is scattered, again provided $(n(x_0) - 1)v(x_0) \neq 0$. The study of the behavior of transmission eigenfunctions, in particular of the part v satisfying the Helmholtz equation, is important since one can obtain information about v from scattering data and this information can be used in inverse scattering theory. The results in [CV21] also shed light into boundary regularity of transmission eigenfunctions.

Inside-outside Duality

The fact that the injectivity of the far field operator can be related to eigenvalues of an interior problem is referred to in the literature as inside-outside duality [KL13]. It leads in particular to a numerical algorithm for determining real transmission eigenvalues from the far field operator. The algorithm described in [CCH16], Section 4.4, is based on a study of the phase of the eigenvalues of the far field operator. For media with real-valued index of refraction n, the compact far field operator $F(k)$ is normal and therefore admits a sequence of eigenvalues $\lambda_j(k)$ that accumulates at the origin. One then can prove the following theorem, where we have set $\hat{\lambda}_j(k) := \lambda_j(k)/|\lambda_j(k)|$ and $v_j(k) := \mathcal{H}(k)g_j/\|\mathcal{H}(k)g_j\|_{L^2(D)}$, where g_j is an eigenfunction associated with $\lambda_j(k)$ and $\mathcal{H}(k)$ is defined by (11). This theorem states how to determine real transmission eigenvalues along with an approximation of the v part of the corresponding eigenfunction from the far field operator. We refer the reader to [ACH18] for details.

Theorem 3. *Assume that $n-1 \geq \alpha > 0$ (respectively, $1-n \geq \alpha > 0$) in D for some constant α. Let $k_0 > 0$, and let $\{k_\ell\}$ be a sequence of positive numbers converging to k_0 as $\ell \to \infty$. If there exists a sequence $\{\hat{\lambda}^\ell\}$ such that $\hat{\lambda}^\ell = \hat{\lambda}_{j_\ell}(k_\ell)$ for some j_ℓ and $\hat{\lambda}^\ell \to -1$ (respectively, $\hat{\lambda}^\ell \to +1$) as $\ell \to \infty$, then k_0 is a transmission eigenvalue. Moreover, the sequence $\{v_{j_\ell}(k_\ell)\}$ admits a subsequence that converges strongly in $L^2(D)$ to a nontrivial v solving (9) with $k = k_0$.*

The converse of the statement in this theorem has been proved only for cases where n is a sufficiently small perturbation of a constant [KL13].

The inside-outside duality also leads to a duality between the set of transmission eigenvalues and the set of scattering poles. In particular, these two sets are interchangeable if instead of the relative scattering operator for the scattering problem (1)–(3) we consider the relative scattering operator for an appropriate interior scattering problem. Such a duality was first established in [CCH20] and we sketch the approach given there for the special case when the interior relative scattering operator is defined on a sphere inside D centered at the origin. This allows us to give an equivalent definition of the involved operators in terms of Fourier series representation which better reveals the duality between the exterior and interior operators (since there is no natural dual definition of the far field for the interior problem). To explain this, we first represent the far field operator defined by (8) as a Fourier series in terms of spherical harmonics $\{Y_\ell^m(\hat{x})\}$. For $g \in L^2(S^2)$ we set

$$g(\hat{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell,m} Y_\ell^m(\hat{x})$$

and define an isometry $\mathcal{J}$ between $L^2(S^2)$ and $l^2(\mathbb{Z})$ by the mapping $g \mapsto \tilde{g} := \{g_{\ell,m}\}$. Thus we now have a new representation of the far field operator, denoted by $\tilde{F}(k) : l^2(\mathbb{Z}) \mapsto l^2(\mathbb{Z})$:

$$\tilde{F}(k) = \mathcal{J}^{-1^*} F(k) \mathcal{J}^{-1},$$

where $\mathcal{J}^{-1^*}$ denotes the L^2-adjoint of $\mathcal{J}^{-1}$. The discussion on transmission eigenvalues in connection to the kernel of the operator $F(k)$ can now be carried over in exactly the same way as before if we replace $F(k)$ by $\tilde{F}(k)$. To introduce the duality, we first observe that the operator $\tilde{F}(k)$ can be equivalently defined using scattered waves associated with incident spherical waves. More precisely, let $u^s_{\ell,m}(x)$ be the scattered field corresponding to the incident wave (which is a Herglotz wave function)

$$v(x) := j_\ell(k|x|)Y_\ell^m(\hat{x}).$$

Outside a ball B_R of radius R containing D, the scattered field can be expanded as

$$u^s_{\ell,m}(x) = \sum_{p=0}^{\infty} \sum_{q=-p}^{p} a^{p,q}_{\ell,m} h^{(1)}_p(k|x|)\overline{Y_p^q}(\hat{x})$$

(note $\overline{Y_p^q} = Y_p^{-q}$). Then, up to a multiplicative constant the Fourier coefficients of $\tilde{F}(k)\tilde{g}$ are

$$(\tilde{F}(k)\tilde{g})_{p,q} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell,m} a^{p,q}_{\ell,m}. \tag{13}$$

Now reversing the role of the incident and scattered waves, we can define similarly to (13) an interior far field operator to characterize the scattering poles where in the following we assume that k is complex with $\Im(k) < 0$. To this end, let $B_\delta \subset D$ be a ball centered at the origin, and for an outgoing solution to the Helmholtz equation

$$w_{\ell,m}(x) = j_\ell(k\delta) h^{(1)}_\ell(k|x|)Y_\ell^m(\hat{x})$$

we denote by $(u_{\ell,m}, v_{\ell,m}) \in L^2(D) \times L^2(D)$ the solution of the interior transmission problem

$$\begin{cases} \Delta u_{\ell,m} + k^2 n(x) u_{\ell,m} = 0 & \text{in} \quad D, \\ \Delta v_{\ell,m} + k^2 v_{\ell,m} = 0 & \text{in} \quad D, \\ u_{\ell,m} - v_{\ell,m} = w_{\ell,m} & \text{on} \quad \partial D, \\ \frac{\partial u_{\ell,m}}{\partial \nu} - \frac{\partial v_{\ell,m}}{\partial \nu} = \frac{\partial w_{\ell,m}}{\partial \nu} & \text{on} \quad \partial D. \end{cases}$$

Inside B_δ, the field $v_{\ell,m}$ can be expanded as

$$v_{\ell,m}(x) = \sum_{p=0}^{\infty} \sum_{q=-p}^{p} b^{p,q}_{\ell,m} \frac{j_p(k|x|)}{j_p(k\delta)} \overline{Y_p^q}(\hat{x}).$$

Thus, we can define the interior far field operator $\tilde{F}_{int} : l^2(\mathbb{Z}) \mapsto l^2(\mathbb{Z})$ by its Fourier coefficients

$$(\tilde{F}_{int}(k)\tilde{g})_{p,q} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell,m} b^{p,q}_{\ell,m}. \tag{14}$$

Observe that this operator can be defined for any $k \in \mathbb{C}$ with $\Im(k) < 0$ that does not coincide with a transmission eigenvalue. Now, let k be such that there exists a $\tilde{g} \neq 0$ with $\tilde{F}_{int}(k)\tilde{g} = 0$. Then by unique continuation we have that

$$v_{\tilde{g}} := \sum_{n=0}^{\infty} \sum_{m=-n}^{n} g_{n,m} v_{n,m} = 0 \qquad \text{in} \;\; D,$$

and consequently one can show that $w \in H^2_{loc}(\mathbb{R}^3)$ defined by

$$w = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell,m} u_{\ell,m} \qquad \text{in } D$$

and

$$w = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell,m} w_{\ell,m} \qquad \text{in } \mathbb{R}^3 \setminus D$$

is a nontrivial solution of

$$\Delta w + k^2 n w = 0 \quad \text{in } \mathbb{R}^3$$

and

$$w = \int_{\partial D} \Phi(\cdot, y) \frac{\partial w(y)}{\partial \nu} - w(y) \frac{\partial \Phi(\cdot, y)}{\partial \nu} \, dy \ \text{ in } \mathbb{R}^3 \setminus D.$$

This means that such a value of k is a scattering pole. For the given inhomogeneity (D, n), there is a duality between $\tilde{F}(k)$ whose kernel is related to the transmission eigenvalues and $\tilde{F}_{int}(k)$ whose kernel is related to the scattering poles. Both are defined by similar expressions, but $\tilde{F}(k)$ corresponds to the exterior scattering problem due to an incident Herglotz wave function, whereas $\tilde{F}_{int}(k)$ corresponds to the interior scattering problem due to an incident outgoing spherical wave. In [CCH20] this duality is expressed in terms of a near field operator for the exterior scattering problem and an interior operator associated with w being a single layer potential associated with a boundary strictly included in D. The interior operator $\tilde{F}_{int}(k)$ coincides (up to an isometry) with the interior operator considered in [CCH20] when the single layer potential is supported by ∂B_δ.

Transmission Eigenvalues for General Media

We now turn our attention to the study of the spectral properties of the transmission eigenvalue problem for general media. Despite its inherent connection with scattering theory, the transmission eigenvalue problem is a new spectral problem leading to many interesting and challenging mathematical questions. The transmission eigenvalue problem has a deceptively simple formulation, namely two elliptic PDEs in a bounded domain that share the same Cauchy data on the boundary, but presents a perplexing mathematical structure. In particular, it is a nonselfadjoint eigenvalue problem for a nonstrongly elliptic operator, and hence the investigation of its spectral properties becomes challenging. Roughly speaking, the spectral properties depend on the assumptions on the contrast $n-1$ of the media. Questions central to the inverse scattering theory include: the discreteness of the spectrum, the location of transmission eigenvalues in the complex plane, the existence of transmission eigenvalues, and the determination of real transmission eigenvalues from scattering data, which is important since real transmission eigenvalues can be used to obtain information about the material properties of the scattering media. For a more comprehensive and detailed discussion on the transmission eigenvalue problem and its application in inverse scattering we refer the reader to the monograph [CCH16].

Setting $k^2 := \tau$, the transmission eigenvalue problem (9) written in terms of $w = u - v$ and replacing $k^2 v$ by v becomes: find $w \in H^2_0(D)$ and $v \in L^2(D)$ such that

$$\begin{cases} \Delta w + \tau n w = (1-n) v & \text{in} \quad D, \\ \Delta v + \tau v = 0 & \text{in} \quad D. \end{cases} \tag{15}$$

We have already noticed in the spherically stratified case that this eigenvalue problem is not selfadjoint (since complex eigenvalues can exist). Obviously this is a nonstandard system of two elliptic partial differential equations since one unknown has two boundary conditions while the other has none. A first natural idea is to eliminate v from the system which obviously can be done if $1/(n-1)$ is in $L^\infty(D)$. Under this assumption, $w \in H^2_0(D)$ satisfies

$$(\Delta + \tau) \frac{1}{n-1} (\Delta + \tau n) w = 0 \quad \text{in } D. \tag{16}$$

We therefore obtain a quadratic eigenvalue problem. Note that although the transmission eigenvalue problem (16) has the structure of a quadratic pencil of operators, it appears that available results on these types of operators are not applicable to our problem due to the incorrect signs of the involved operators. This formulation can be used however to prove discreteness of transmission eigenvalues provided that the contrast function $n-1$ keeps a definite sign in the whole domain D. Since $\Delta \frac{1}{(n-1)} \Delta w$ is an equivalent norm in $H^2_0(D)$, from an application of the Analytic Fredholm Theorem one easily sees that the set of transmission eigenvalues is at most discrete with infinity as the only accumulation point and that all transmission eigenvalues have finite multiplicity. Furthermore, by rewriting (16) one can also prove the existence of an infinite set of real transmission eigenvalues. For instance, if the contrast $n-1$ is positive definite in D, then the main idea is to consider for fixed τ the generalized selfadjoint eigenvalue problem

$$(\Delta + \tau) \frac{1}{n-1} (\Delta + \tau) w + \tau^2 w = -\lambda(\tau) \Delta w \text{ in } D \tag{17}$$

and notice that transmission eigenvalues of (16) are solutions to $\lambda(\tau) = \tau$. Proving existence of zeros to this equation exploits a min-max principle for $\lambda(\tau)$ and the construction of special subspaces using known results on transmission eigenvalues for spherically symmetric problems. This reasoning leads to the following theorem, where $n \in L^\infty(D)$ and we set

$$0 < n_* = \inf_D(n) \ \text{ and } \ n^* := \sup_D(n) < +\infty.$$

Theorem 4. *If either $1 < n_*$ or $n^* < 1$, then there exists an infinite sequence of real transmission eigenvalues $\tau_j > 0$ for $j \in \mathbb{N}$ accumulating only at $+\infty$.*

The proof of the above theorem also provides monotonicity results for real transmission eigenvalues with respect to n. More specifically, let $\tau_j := \tau_j(n, D) > 0$ for $j \in \mathbb{N}$ be the increasing sequence of the transmission eigenvalues for the media with support D and refractive index $n(x)$ such that τ_j is the smallest zero of $\lambda_j(\tau) = \tau$ for some eigenvalue λ_j of (17). More specifically, let D_1 and D_2 be two domains such that $D_1 \subset D \subset D_2$. Then for $1 < n_*$ the monotonicity property

$$\begin{aligned}\tau_j(n^*, D_2) \le \tau_j(n^*, D) \le \tau_j(n, D)\\ \le \tau_j(n_*, D) \le \tau_j(n_*, D_1)\end{aligned}$$

holds, whereas for $n^* < 1$

$$\begin{aligned}\tau_j(n_*, D_2) \le \tau_j(n_*, D) \le \tau_j(n, D)\\ \le \tau_j(n^*, D) \le \tau_j(n^*, D_1).\end{aligned}$$

In particular, the above inequalities are valid for the smallest real transmission eigenvalue $\tau_1(n, D) > 0$. This fact can be used to show that a constant refractive index $n \neq 1$ is uniquely determined from $\tau_1(n, D)$. Furthermore, the smallest real transmission eigenvalue satisfies $\tau_1(n, D) > \frac{\lambda(D)}{n^*}$ if $1 < n_*$ and $\tau_1(n, D) > \lambda(D)$ if $n^* < 1$, where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D.

In the cases when n does not keep the same sign in D, the approach described above does not work anymore. In particular, the existence of real transmission eigenvalues is still an open problem for this case and all the known spectral results are obtained under the standard assumption that the contrast $n - 1$ has a definite sign in a boundary neighborhood $D_\delta := \{x \in D;\ \mathrm{dist}(x, \partial D) < \delta\}$. Then, if we assume that either $1 < \inf_{D_\delta}(n)$ or $\sup_{D_\delta}(n) < 1$, under the same L^∞-regularity assumption on n as above it is proven that the set of transmission eigenvalues is at most discrete with infinity as the only possible accumulation point (see [CCH16], Section 3.1, and references therein). The proof of discreteness is based on first showing that the resolvent of (15) is Fredholm of index zero. The uniqueness of the solution is then proved for $\tau = i\lambda$ with λ sufficiently large, exploiting the exponential decay inside D of solutions to the Helmholtz equation for such wave numbers. The proof is now completed by appealing to the Analytic Fredholm Theorem. However, the real interest related to the discreteness of transmission eigenvalues is in the case when the contrast $n - 1$ is allowed to change sign in $\overline{D}$, and this case is still not understood. The discreteness of transmission eigenvalues in this case, if it holds, would imply the uniqueness of the sound speed for the wave equation with arbitrary source which is a question that arises in thermo-acoustic imaging.

In the case when both the domain D and the real-valued refractive index n are C^∞-smooth, with the additional assumption that $n \neq 1$ on ∂D, a complete characterization of the spectrum of the transmission eigenvalue problem is presented in [Rob13]. This study is done in the framework of semiclassical analysis, relating the transmission eigenvalue problem to the spectrum of a Hilbert-Schmidt operator whose resolvent exhibits the desired growth properties following the approach of Agmon in [Agm10]. This reasoning leads to the following spectral theorem.

Theorem 5. *Assume that $n \in C^\infty(\overline{D})$, ∂D is of class C^∞, $n(x) \ge n_0 > 0$ for $x \in D$, and $n \neq 1$ on ∂D. Then there exist an infinite number of transmission eigenvalues $k \in \mathbb{C}$ and the space spanned by the generalized eigenfunctions (w, v) is dense in $H_0^2(D) \times \{L^2(D),\ \Delta u \in L^2(D)\}$.*

Another important question is the location of the transmission eigenvalues in the complex plane $\mathbb{C}$. In particular, it is desirable to know if there exists a half plane in $\mathbb{C}$ free of transmission eigenvalues. This is an important question for analyzing the time-domain interior transmission problem which is the main building block for the time-domain linear sampling method for inhomogeneous media [CMS21]. It was the paper by Vodev [Vod18] which shed light onto this issue. To explain the result in this paper, we first remark that the transmission eigenvalue problem (9) can be recast in terms of the difference of two Dirichlet-to-Neumann operators. More precisely, let us define $\mathcal{N}_q(k) : \varphi \mapsto \frac{\partial u}{\partial \nu}$, where u solves

$$\begin{cases} \Delta u + k^2 q u = 0 & \text{in } D,\\ u = \varphi & \text{on } \partial D \end{cases}$$

(provided k^2 is not a Dirichlet eigenvalue). Then, the transmission eigenvalue problem can be viewed as finding $k \in \mathbb{C}$ for which there exists a nontrivial u such that

$$\mathcal{T}(k)u := \mathcal{N}_n(k)u - \mathcal{N}_1(k)u = 0.$$

The operator $\mathcal{T}(k) : H^{-1/2+s}(\partial D) \to H^{1/2+s}(\partial D)$, $0 \le s \le 1$, is one order smoothing and is Fredholm with index zero. The eigenvalue free zone in $\mathbb{C}$ corresponds to $k \in \mathbb{C}$ for which $\mathcal{T}(k)^{-1}$ exists. In [Vod18] it is proven that all transmission eigenvalues k lie in a horizontal strip about the real axis. In addition this paper provides k-explicit bounds for the norm of the inverse of $\mathcal{T}(k)$ as well as Weyl's asymptotic estimates for the transmission eigenvalues. The main tool in obtaining these results is the derivation of refined high frequency estimates for the Dirichlet-to-Neumann operator in the framework of semiclassical analysis. Therefore these results require C^∞ regularity for both D and n and are summarized in the following theorem.

Theorem 6. *Assume that* $n \in C^\infty(\overline{D})$*,* ∂D *is of class* C^∞*,* $n(x) \geq n_0 > 0$ *for* $x \in D$*, and* $n \neq 1$ *on* ∂D*. The following hold:*

i) *There are no transmission eigenvalues in the region* $\{k \in \mathbb{C} : |\Im(k)| > \gamma\}$ *for some constant* $\gamma > 0$*. In this region,* $\mathcal{T}(k)^{-1} : H^1(\partial D) \to L^2(\partial D)$ *is bounded, and if in addition* $\Re(k) > 1$*,*

$$\|\mathcal{T}(k)^{-1}\| \leq c|k|^{-1} \quad \text{for some } c > 0.$$

ii) *Let* $N(r) := \#\{k \text{ trans. eigen. } |k| \leq r\}$*. Then*

$$N(r) = \frac{r^3}{6\pi^2}\int_D \left(1 + n(x)^{3/2}\right) dx + O_\epsilon\left(r^{2+\epsilon}\right)$$

for all $0 < \epsilon << 1$*, where the order term depends on* ϵ*.*

We conclude this section by listing a few important open questions. Although for the spherically symmetric media it is proven that complex transmission eigenvalues exist, for general media (D, n) it is not known if this is the case. Also in the case when the contrast $n - 1$ is of one sign in a neighborhood of ∂D but otherwise it changes sign inside D, it is not known if *real* transmission eigenvalues exist, a question which is important in the use of transmission eigenvalues to obtain information on the refractive index n. Nothing is known about the spectral properties of the transmission eigenvalue problem in the case when there is a point P on the boundary ∂D for which $n - 1$ changes sign in every neighborhood of P inside D. Finally the case of absorbing media, i.e., media with refractive index $n(x) := n_1(x) + \frac{i}{k}n_2(x)$ with n_1 and n_2 positive definite real-valued functions, is not fully understood. Although the discreteness of transmission eigenvalues is proven under some assumptions on $n_1 - 1$, nothing is known about the existence of the eigenvalues and their location in the complex plane.

Generalizations

Transmission eigenvalues in scattering by anisotropic media. Of great interest in many applications is the scattering problem for anisotropic media having support D where the wave propagation of the total field u inside D is governed by

$$\nabla \cdot A(x)\nabla u + k^2 n(x) u = 0,$$

where A is a positive definite 3×3 symmetric matrix-valued function. The same analysis of the injectivity of the far field operator as previously discussed leads to the following transmission eigenvalue problem: find $u \in H^1(D)$ and $v \in H^1(D)$ such that

$$\left\{\begin{array}{lll} \nabla \cdot A\nabla u + k^2 n u = 0 & \text{in} & D, \\ \Delta v + k^2 v = 0 & \text{in} & D, \\ u = v & \text{on} & \partial D, \\ \nu \cdot A\nabla u = \nu \cdot \nabla v & \text{on} & \partial D. \end{array}\right. \tag{18}$$

Results on discreteness, existence of real transmission eigenvalues, and their monotonicity properties in terms of A and n can be found in [CCH16], Sections 3.2 and 4.3, under the assumptions that $A - I$ and $n - 1$ have one sign in D. In the same way as for the case of $A = I$ one can show that real transmission eigenvalues can be determined from the scattering data. In inverse scattering theory, it is known that in general matrix-valued coefficients cannot be uniquely determined from the scattering data. Hence, using real transmission eigenvalues to detect changes in anisotropic media becomes quite important (see [CCH16] and the references therein). More generally, completeness results on transmission eigenfunctions and Weyl's estimates on the counting function are obtained in [NN21] for A and n continuous in a neighborhood of ∂D satisfying the following two conditions:

$$\langle A(x)\nu, \nu\rangle \langle A(x)\xi, \xi\rangle - \langle A(x)\nu, \xi\rangle \neq 1 \;\; \forall\, x \in \partial D$$

and for every $\xi \in \mathbb{R}^3 \setminus \{0\}$ orthogonal to the outward normal vector ν at any $x \in \partial D$, and

$$\langle A(x)\nu, \nu\rangle\, n(x) \neq 1 \qquad \text{for all } x \in \partial D$$

(the first condition is known as the complementing condition due to Agmon, Douglis, and Nirenberg). Under C^∞ regularity for both the boundary and the coefficients and assuming that $A(x) = a(x)I$, where a is a scalar function, the location in the complex plane of the transmission eigenvalues is studied in [Vod18]. Unfortunately, in this case (as opposed to the media with $A = I$) it is not possible to prove that there is a half plane of $\mathbb{C}$ free of transmission eigenvalues. For example, if

$$(a(x) - 1)(a(x)n(x) - 1) < 1 \qquad \text{for all } x \in \partial D$$

it is proven that there are infinitely many transmission eigenvalues arbitrarily close to the imaginary axis.

Transmission eigenvalues in hyperbolic geometry and the Riemann zeta function. The concept of nonscattering modes and transmission eigenvalues appears in other scattering configurations. As an example, [BBDCP18] discusses such a concept for the scattering by objects inside an infinite waveguide, i.e., an infinite tube with one direction of propagation. In this section we describe it in connection with scattering theory for automorphic solutions to the wave equation in the hyperbolic plane with isometries corresponding to a specific group. We refer the reader to the monograph by Lax and Phillips [LP76] for a discussion on the mathematical foundation of wave propagation in this framework and scattering theory for hyperbolic surfaces. Limited to automorphic forms with respect to Fuchsian groups of the first kind that has only cusps at infinity, scattering theory has a profound connection to fundamental results from analytic number theory. In scattering theory of automorphic forms the concern has always been with the poles of the scattering matrix. In [CC19] the first effort is made to understand the counterpart of transmission eigenvalues in this framework, and in the following we sketch some of its results.

To explain the main ideas we start by recalling some basic concepts of the hyperbolic surface generated by specific groups. The hyperbolic plane

$$\mathbb{H} = \{z = x + iy : \quad x \in \mathbb{R},\ y \in \mathbb{R}^+\}$$

is a Riemannian manifold with the complete metric

$$ds^2 = y^{-2}(dx^2 + dy^2)$$

with geodesics the Euclidean semicircles and lines perpendicular to the x-axis. This differential form on $\mathbb{H}$ is invariant with respect to the group of Möbius transformations acting on the whole compactified complex plane $\hat{\mathbb{C}}$, which are fractional linear transformations $\frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$. This group is isomorphic with $SL_2(\mathbb{R}) \setminus \pm I$. Derived for the hyperbolic theory of wave propagation in the hyperbolic plane [LP76], the Helmholtz equation in the case of $\mathbb{R}^3$ is now replaced by

$$y^2\Delta u + s(1-s)u = 0. \tag{19}$$

Given a large isometry group of $\mathbb{H}$, a natural way to obtain a hyperbolic surface is by a quotient $\Gamma \setminus \mathbb{H}$ (the sets of orbits) for some subgroup Γ of $SL_2(\mathbb{R})$. The fundamental domain $F := \Gamma \setminus \mathbb{H}$ corresponding to such a subgroup is a region in $\mathbb{H}$ whose distinct points are not equivalent (different modulo Γ) and such that any orbit of Γ contains points in the closure of F in the $\hat{\mathbb{C}}$ topology. There are various ways to visualize a fundamental domain (see Figure 5), and we refer the reader to the monograph by Iwaniec [Iwa02] for details and references on this matter. A function $f : \mathbb{H} \to \mathbb{C}$ is called automorphic with respect to Γ if

$$f(\gamma z) = f(z) \ \text{ for all } \gamma \in \Gamma, \tag{20}$$

i.e., f lives on $F := \Gamma \setminus \mathbb{H}$. Because of (20), an automorphic function is completely determined by its values on $\overline{F}$. Now we are ready to formulate the scattering by a hyperbolic surface $\Gamma \setminus \mathbb{H}$. Let u^i be a solution of (19). The nonidentically zero total field $u := u^i + u^s$ satisfies the equation

$$y^2\Delta u + s(1-s)u = 0, \qquad z = (x, y) \in F.$$

On the boundary of F periodic boundary conditions are imposed:

$$\begin{aligned} u(\gamma z) &= u(z), & z \in \partial F,\ \gamma \in \Gamma, \\ \frac{\partial u}{\partial \nu}(\gamma z) &= \frac{\partial u}{\partial \nu}(z), & z \in \partial F,\ \gamma \in \Gamma. \end{aligned}$$

Scattering happens because the incoming free wave u^i needs to be periodified on F. In connection with this scattering problem, we consider the concept of transmission eigenvalues for fundamental domains generated by some particular discrete groups, more specifically Fuchsian groups of the first kind. A Fuchsian group Γ of the first kind is a discrete subgroup of $SL_2(\mathbb{R})$ that has a finite number of generators and a fundamental domain of finite volume. We further consider Fuchsian groups of the first kind that are noncocompact, which means that the closure in $\hat{\mathbb{C}}$ of the fundamental domain is not compact. Such groups have only cusps (see Figure 4). The cusps are formed by the two sides of F meeting orthogonally at a vertex in $\partial\mathbb{H}$.

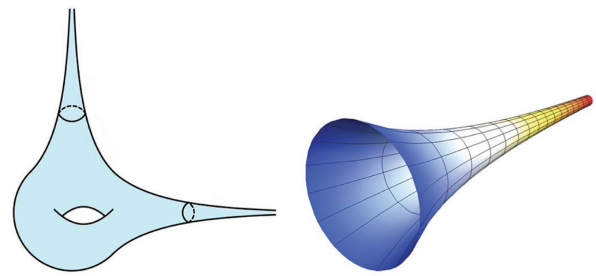

Figure 4. Sketch of a hyperbolic surface with two cusps (left) and part of a hyperbolic cusp (right) isometrically embedded into $\mathbb{R}^3$.

Now, let y^s and y^{1-s} be solutions to (19) which are invariant under $z \mapsto z+1$. Similarly to the Sommerfeld radiation condition in the Euclidean case (3) we have that $u := y^s$ satisfies

$$\left.\frac{\partial u}{\partial \nu}\right|_{\mathbb{H}} - su = y\frac{\partial u}{\partial y} - su = 0.$$

So if $\Im(s) > 0$, $\Im(1-s) < 0$, then y^s is incoming away from the cusp (traveling toward the scattering medium like $e^{ikx\cdot d}$ in $\mathbb{R}^3$) and y^{1-s} is outgoing toward the cusp (traveling away from the scattering medium like $e^{-ikx\cdot d}$ in $\mathbb{R}^3$). For $\Re(s) > 1$ and incident field $u^i := y^s$ the solution $u(z)$ of the above scattering problem is given by an Eisenstein series (which roughly plays the same role as Fourier series in the Euclidean case) which as $y \to \infty$ within cusps behaves as

$$\delta_{\mathbf{ab}}y^s + \varphi_{\mathbf{ab}}(s)y^{1-s} + O((1 + y^{-\Re(s)})e^{-2\pi y})$$

uniformly in $z \in \mathbb{H}$, where $\delta_{\mathbf{ab}}$ is the Kronecker delta vanishing when $\mathbf{a}, \mathbf{b}$ are inequivalent cusps. The corresponding scattering matrix is

$$\Phi(s) := (\varphi_{\mathbf{ab}}(s)), \text{ where } \mathbf{a} \text{ and } \mathbf{b} \text{ run over all cusps,}$$

and it describes "incoming-to-outgoing" far field behavior in a similar manner as the far field operator in $\mathbb{R}^3$ introduced earlier. The scattering matrix has a meromorphic continuation to $s \in \mathbb{C}$, and the corresponding scattering poles are the poles of this meromorphic extension of $\varphi_{\mathbf{ab}}(s)$. In [CC19] a *transmission eigenvalue* associated with a cusp $\mathbf{a}$ is defined as the value $\lambda := s(1-s)$ that corresponds to s being a zero of the entry $\varphi_{\mathbf{aa}}(s)$ of the scattering matrix $\Phi(s)$. In particular, if s is a zero of $\varphi_{\mathbf{aa}}(s)$, then the incident wave y^s propagates through without seeing any boundaries of the fundamental domain. In this sense, transmission eigenvalues correspond to nonscattering energies. One could also consider the zeros of off-diagonal entries $\varphi_{\mathbf{ab}}(s)$ for two inequivalent cusps $\mathbf{a}$ and $\mathbf{b}$. This case would mean that an observer sitting at the cusp $\mathbf{b}$ sees no effects of an incident field y^s coming into the media from the cusp $\mathbf{a}$, which corresponds to nontransmitted energies in the language of waveguides [BBDCP18]. It is known that the poles of main diagonal terms of the scattering matrix are also poles for the off-diagonal terms [Iwa02]; however, such a connection between the zeros of the main diagonal and off-diagonal terms remains open. Note that in the scattering by hyperbolic surfaces generated by Fuchsian groups of the first kind a Rellich's type lemma does not hold and thus nonscattering is meant in an asymptotic sense.

We conclude this section by giving an explicit expression for the scattering matrix for two particular examples of Fuchsian groups of the first kind. In these examples the transmission eigenvalues (nonscattering energies) are related to the zeros of the Riemann zeta function, whereas nontransmitted energies are absent.

The *modular group* $SL_2(\mathbb{Z})$ is the subgroup of $SL_2(\mathbb{R})$ containing 2×2 matrices with integer entries. The fundamental domain of $SL_2(\mathbb{Z})$ contains only one inequivalent cusp (see Figure 5). For the modular group the scattering matrix contains only one element and is given by

$$\varphi_{\infty\infty} = \pi^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}, \tag{21}$$

where $\zeta(s)$ is the Riemann zeta function.

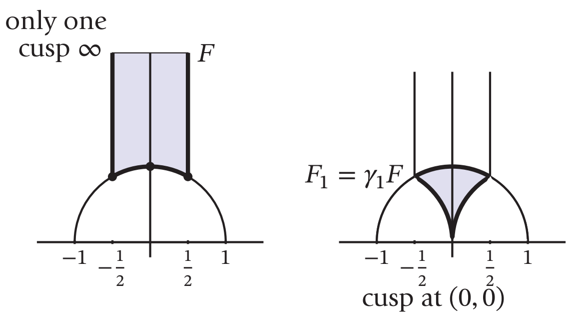

Figure 5. Left: Shaded region depicts the Ford fundamental domain F for $\Gamma = SL_2(\mathbb{Z})$. Right: Shaded region depicts an equivalent fundamental domain $F_1 = \gamma_1 F$, where $\gamma_1 : z \to -1/z$. The image of F under Γ tesselates $\mathbb{H}$.

The *Hecke congruence subgroup* $\Gamma_0(N)$ of level N for N an integer is defined as

$$\Gamma_0(N) = \left\{\gamma \in SL_2(\mathbb{Z}),\ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\text{mod } N)\right\}.$$

When $N := p$ is prime the fundamental domain $\Gamma_0(p) \setminus \mathbb{H}$ contains only two inequivalent cusps, namely at zero and infinity, and the scattering matrix is given by

$$\Phi(s) = \begin{pmatrix} \varphi_{\infty\infty} & \varphi_{\infty 0} \\ \varphi_{0\infty} & \varphi_{00} \end{pmatrix} = \psi(s)N_p(s),$$

where

$$\psi(s) = \pi^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}$$

and $N_p(s)$ is a 2×2 matrix with nonvanishing entries.

In these two examples we see that, except for $\lambda = 1$ and $\lambda = 1/4$ corresponding to trivial zeros $s = 0$ and $s = 1/2$, i.e., the poles of $\zeta(2s)$, all the transmission eigenvalues $\lambda = s(1-s)$ correspond to s being the zeros of $\zeta(2s-1)$ such that $\Im(2s-1) \neq 0$. Thus, the *Riemann hypothesis* is equivalent to the statement that *all* transmission eigenvalues lie on the parabola $x = \frac{3}{16} + 4y^2$ except for finitely many trivial ones (see Figure 6).

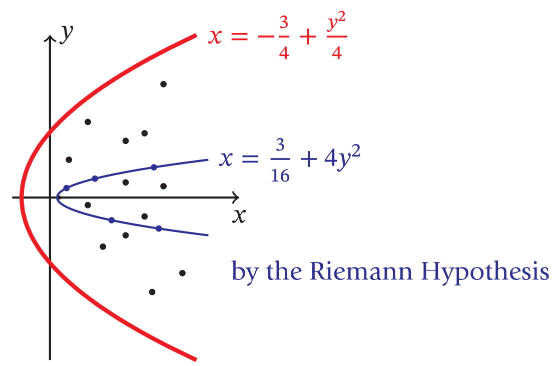

Figure 6. Black dots indicate possible location of transmission eigenvalues. There are infinitely many transmission eigenvalues on the inner parabola and all eigenvalues lie inside the outer parabola. If the Riemann hypothesis is true, all transmission eigenvalues lie on the inner parabola. There are no real transmission eigenvalues except for the trivial ones, at 0 and 1/4.

In [CC19] the reader can find a few more explicit examples. However in general for discrete Fuchsian subgroups it is not possible to derive an explicit expression for the scattering matrix. We also mention that [CC19] provides estimates for the counting function of transmission eigenvalues. As this paper remarks, the density of transmission eigenvalues appears to be related to the work of Phillips and Sarnack on the existence of cusp forms.

ACKNOWLEDGMENTS. The research of F. Cakoni was supported in part by AFOSR Grant FA9550-20-1-0024 and NSF Grant DMS-1602802. The research of D. Colton was partially supported by the AFOSR Grant FA9550-20-1-0024.

References

[Agm10] Shmuel Agmon, *Lectures on elliptic boundary value problems*, AMS Chelsea Publishing, Providence, RI, 2010. Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr.; Revised edition of the 1965 original, DOI 10.1090/chel/369. MR2589244

[ACH18] Lorenzo Audibert, Lucas Chesnel, and Houssem Haddar, *Transmission eigenvalues with artificial background for explicit material index identification* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **356** (2018), no. 6, 626–631, DOI 10.1016/j.crma.2018.04.015. MR3806890

[BPS14] Eemeli Blåsten, Lassi Päivärinta, and John Sylvester, *Corners always scatter*, Comm. Math. Phys. **331** (2014), no. 2, 725–753, DOI 10.1007/s00220-014-2030-0. MR3238529

[BBDCP18] Anne-Sophie Bonnet-Ben Dhia, Lucas Chesnel, and Vincent Pagneux, *Trapped modes and reflectionless modes as eigenfunctions of the same spectral problem*, Proc. A. **474** (2018), no. 2213, 20180050, 14, DOI 10.1098/rspa.2018.0050. MR3832481

[CC19] Fioralba Cakoni and Sagun Chanillo, *Transmission eigenvalues and the Riemann zeta function in scattering theory for automorphic forms on Fuchsian groups of type I*, Acta Math. Sin. (Engl. Ser.) **35** (2019), no. 6, 987–1010, DOI 10.1007/s10114-019-8128-8. MR3952699

[CCH16] Fioralba Cakoni, David Colton, and Houssem Haddar, *Inverse scattering theory and transmission eigenvalues*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 88, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2016, DOI 10.1137/1.9781611974461.ch1. MR3601119

[CCH20] Fioralba Cakoni, David Colton, and Houssem Haddar, *A duality between scattering poles and transmission eigenvalues in scattering theory*, Proc. A. **476** (2020), no. 2244, 20200612, 19. MR4203099

[CMS21] Fioralba Cakoni, Peter Monk, and Virginia Selgas, *Analysis of the linear sampling method for imaging penetrable obstacles in the time domain*, Anal. PDE **14** (2021), no. 3, 667–688, DOI 10.2140/apde.2021.14.667. MR4259870

[CV21] Fioralba Cakoni and Michael Vogelius, *Singularities almost always scatter: Regularity results for non-scattering inhomogeneities*, arXiv:2104.05058 (2021), no. 11.

[CK19] David Colton and Rainer Kress, *Inverse acoustic and electromagnetic scattering theory*, Applied Mathematical Sciences, vol. 93, Springer, Cham, 2019. Fourth edition of [MR1183732], DOI 10.1007/978-3-030-30351-8. MR3971246

[DZ19] Semyon Dyatlov and Maciej Zworski, *Mathematical theory of scattering resonances*, Graduate Studies in Mathematics, vol. 200, American Mathematical Society, Providence, RI, 2019, DOI 10.1090/gsm/200. MR3969938

[EH18] Johannes Elschner and Guanghui Hu, *Acoustic scattering from corners, edges and circular cones*, Arch. Ration. Mech. Anal. **228** (2018), no. 2, 653–690, DOI 10.1007/s00205-017-1202-4. MR3766986

[Iwa02] Henryk Iwaniec, *Spectral methods of automorphic forms*, 2nd ed., Graduate Studies in Mathematics, vol. 53, American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, 2002, DOI 10.1090/gsm/053. MR1942691

[KL13] Andreas Kirsch and Armin Lechleiter, *The inside-outside duality for scattering problems by inhomogeneous media*, Inverse Problems **29** (2013), no. 10, 104011, 21, DOI 10.1088/0266-5611/29/10/104011. MR3116206

[LP76] Peter D. Lax and Ralph S. Phillips, *Scattering theory for automorphic functions*, Annals of Mathematics Studies, No. 87, Princeton Univ. Press, Princeton, N.J., 1976. MR0562288

[LP89] Peter D. Lax and Ralph S. Phillips, *Scattering theory*, 2nd ed., Pure and Applied Mathematics, vol. 26, Academic Press, Inc., Boston, MA, 1989. With appendices by Cathleen S. Morawetz and Georg Schmidt. MR1037774

[NN21] Hoai-Minh Nguyen and Quoc-Hung Nguyen, *The Weyl law of transmission eigenvalues and the completeness of generalized transmission eigenfunctions*, J. Funct. Anal. **281** (2021), no. 8, 109146, DOI 10.1016/j.jfa.2021.109146. MR4280271

[Rob13] Luc Robbiano, *Spectral analysis of the interior transmission eigenvalue problem*, Inverse Problems **29** (2013), no. 10, 104001, 28, DOI 10.1088/0266-5611/29/10/104001. MR3116196

[Vod18] Georgi Vodev, *High-frequency approximation of the interior Dirichlet-to-Neumann map and applications to the transmission eigenvalues*, Anal. PDE **11** (2018), no. 1, 213–236, DOI 10.2140/apde.2018.11.213. MR3707296

[VX] Michael Vogelius and Jingni Xiao, *Finiteness results concerning non-scattering wave numbers for incident plane- and Herglotz waves*, SIAM J. Math. Anal. (to appear).

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