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A boundary integral equation method for the transmission eigenvalue problem

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ABSTRACT
We propose a new integral equation formulation to characterize and compute transmission eigenvalues for constant refractive index that play an important role in inverse scattering problems for penetrable media. As opposed to the recently developed approach by Cossonnière and Haddar [1,2] which relies on a two by two system of boundary integral equations our analysis is based on only one integral equation in terms of Dirichlet-to-Neumann or Robin-to-Dirichlet operators which results in a noticeable reduction of computational costs. We establish Fredholm properties of the integral operators and their analytic dependence on the wave number. Further we employ the numerical algorithm for analytic non-linear eigenvalue problems that was recently proposed by Beyn [3] for the numerical computation of transmission eigenvalues via this new integral equation.

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1. Introduction

The transmission eigenvalue problem arises in scattering theory for inhomogeneous media. If n denotes the refractive index of an inhomogeneous medium with support $D \in \mathbb{R}^m$, $m = 2, 3$, in acoustic or specially polarized electromagnetic scattering, the transmission eigenvalue problem is formulated as finding $k \in \mathbb{C}$ for which the following homogeneous problems

$$\Delta w + k^2 n w = 0 \quad \text{in } D, \tag{1.1}$$

$$\Delta v + k^2 v = 0 \quad \text{in } D, \tag{1.2}$$

$$w = v \quad \text{on } \partial D, \tag{1.3}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D, \tag{1.4}$$

have non-trivial solutions. Here we assume that D is bounded and has a connected complement $\mathbb{R}^m \setminus D$ with sufficiently smooth boundary ∂D, and ν denotes the outward unit normal vector. Transmission eigenvalues can be seen as the extension of the concept of resonant frequencies for impenetrable objects to the case of penetrable media. They are related to non-scattering frequencies. As explained in [4,5], if k is a real transmission eigenvalue and v can be extended outside D as a solution to the Helmholtz equation, then the extended field does not scatter at this wave number k. The transmission eigenvalue problem is a non-linear and non-selfadjoint eigenvalue problem that

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is not covered by the standard theory of eigenvalue problems for elliptic equations, and as such in recent years its analysis has been an attractive subject of investigation. Questions such as discreteness of transmission eigenvalues, existence of real transmission eigenvalues, location of the transmission eigenvalues in the complex plane, and in general spectral properties of the transmission eigenvalue problem under various assumptions on the contrast $n-1$, have been the central concerns in various studies (see e.g. [6–12]). Various mathematical methods have been developed to study this eigenvalue problem, including variational methods,[6,10] boundary integral equation methods,[2,13] and semi-classical analysis.[9,11] On the other hand, it is shown in [14,15] that transmission eigenvalues can be determined from scattering data which together with the establishment of monotonicity properties of transmission eigenvalues in terms of the refractive index [6,16] opens the possibility to use transmission eigenvalues as target signature in the inverse medium problem. These developments drove the need to derive numerical approaches for computing transmission eigenvalues based on finite elements or boundary elements methods.[1,17–20] Since the transmission eigenvalue problem is non-selfadjoint there is the possibility of complex transmission eigenvalues and, indeed, in [21] it was shown that this is the case for spherically stratified media. Whether or not there exist complex transmission eigenvalues for general media, remains an open problem.

In this paper we develop a boundary integral equation approach for the transmission eigenvalue problem in the case of a constant refractive index. Boundary integral equation methods were first used in the context of the transmission eigenvalue problem by Cossonnière and Haddar [1,2], who used Green's representation formula for v and w to derive a system of two linear boundary integral equations that are equivalent to the transmission eigenvalue problem. For the corresponding two by two matrix valued operator which depends non-linearly on the eigenvalue parameter k, Cossonnière and Haddar analyzed the kernel and also used it to numerically compute transmission eigenvalues. Here, in this paper we propose a new formulation which leads to only one linear boundary integral equation that also depends non-linearly on the eigenvalue parameter k. Our main idea is to derive the integral equation from a characterization of the transmission eigenvalues in terms of the Dirichlet-to-Neumann operator as defined by

$$N_{k,n} : \varphi \mapsto \frac{\partial u}{\partial \nu}\,, \tag{1.5}$$

where u is the unique solution to

$$\Delta u + k^2 n u = 0 \quad \text{in } D, \tag{1.6}$$

$$u = \varphi \quad \text{on } \partial D, \tag{1.7}$$

assuming that k^2 is not an eigenvalue for this problem. If we further assume that k^2 is not an eigenvalue for the case when $n = 1$, then k is a transmission eigenvalue if and only if the kernel of the operator

$$M(k) := N_{k,n} - N_{k,1} \tag{1.8}$$

is non-trivial. Furthermore, if φ is such that $(N_{k,n} - N_{k,1})\varphi = 0$ then the transmission eigenfunctions w and v, i.e. the non-trivial solutions of (1.1)–(1.4), are the solutions of (1.6)–(1.7) with n and $n = 1$, respectively.

The plan of the paper is as follows. In the following Section 2 we will analyze the operator M given by (1.8) for the case of a constant refractive index n. We will establish that in an appropriate Sobolev space setting M is Fredholm and analytic in k in all of $\mathbb{C}$ except the Dirichlet eigenvalues that we needed to exclude in the definition of M. To get rid of the latter restriction, in the subsequent Section 3 we extend our analysis by replacing the Dirichlet-to-Neumann operator by the Robin-to-Dirichlet operator. Then, in particular, as a main result we can use our simplified integral equation approach to reestablish that the set of transmission eigenvalues is discrete. In the final Section 4 we conclude with numerical examples using our integral equation to compute real transmission eigenvalues by an efficient algorithm that was recently developed by Beyn [3] for analytic non-linear eigenvalue problems. Besides the reduction in the complexity of the integral equations with a reduction of the computational costs by about 50 %, an additional advantage of our approach lies in the fact that the integral equation based on the Dirichlet-to-Neumann operator (or on the Robin-to-Dirichlet operator) characterizes the interior transmission eigenvalues, i.e. the transmission eigenvalues for D, without any additional conditions whereas the integral equations from [1,2] simultaneously characterize both the interior and also the exterior transmission eigenvalues, i.e. the transmission eigenvalues for the complement of D. Hence, the latter integral equations require additional conditions to distinguish the interior and exterior eigenvalues within the numerical computation.

2. Integral equations based on the Dirichlet-to-Neumann operator

In this section we assume that $0 < n \neq 1$ is constant and let $k_n := k\sqrt{n}$. From now on we will write N_k and N_{k_n} for $N_{k,1}$ and $N_{k,n}$, respectively. We consider the problem of finding $k \in \mathbb{C}$ for which there exists non-zero $v \in L^2(D)$ and $w \in L^2(D)$ with $w - v \in H^2(D)$ satisfying

$$\Delta w + k^2 n w = 0 \qquad \text{in } D, \tag{2.1}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } D, \tag{2.2}$$

$$w = v \qquad \text{on } \partial D, \tag{2.3}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D. \tag{2.4}$$

As mentioned above we are concerned with analyzing the kernel of $N_{k_n} - N_k$ and for the time being we need to assume that k^2 and k_n^2 are not Dirichlet eigenvalues for the negative Laplacian in D. In Section 3 we will discuss how to remedy this restriction. In order to represent the Dirichlet-to-Neumann operator we use boundary integral operators and to this end we need to introduce the single-layer potential $\mathcal{S}_k$ defined by

$$\left(\mathcal{S}_k \psi\right)(x) := 2 \int_{\partial D} \psi(y) \Phi_k(x,y)\, \mathrm{d}s_y, \quad x \in \mathbb{R}^m \setminus \partial D,$$

where

$$\Phi(x,y) = \begin{cases} \dfrac{i}{4} H_0^{(1)}(k|x-y|) & \text{in } \mathbb{R}^2, \\[2ex] \dfrac{1}{4\pi} \dfrac{e^{ik|x-y|}}{|x-y|} & \text{in } \mathbb{R}^3. \end{cases} \tag{2.5}$$

(The factor 2 in the definition of $\mathcal{S}_k$ later on avoids the occurrence of a factor $1/2$ in our representations of the Dirichlet-to-Neumann and the Robin-to-Dirichlet operator.) It is known (see e.g. [22, Theorem 7.2]) that if ∂D is $C^{1,1}$-smooth the linear mapping $\mathcal{S}_k : H^{s-\frac{1}{2}}(\partial D) \to H^{s+1}(D)$ is continuous for $-1 \leq s \leq 1$. We define the restriction of $\mathcal{S}_k$ and of its normal derivative to the boundary ∂D by

$$(S_k \psi)(x) := 2 \int_{\partial D} \psi(y) \Phi(x,y)\, \mathrm{d}s_y, \quad x \in \partial D, \tag{2.6}$$

$$(K_k' \psi)(x) := 2 \int_{\partial D} \psi(y) \frac{\partial}{\partial \nu_x} \Phi(x,y)\, \mathrm{d}s_y, \quad x \in \partial D. \tag{2.7}$$

Hence (see e.g. [22,23])

$$S_k : H^{-\frac{1}{2}+s}(\partial D) \longrightarrow H^{\frac{1}{2}+s}(\partial D) \tag{2.8}$$

$$K'_k : H^{-\frac{1}{2}+s}(\partial D) \longrightarrow H^{-\frac{1}{2}+s}(\partial D) \tag{2.9}$$

are continuous for $-1 \leq s \leq 1$.

As mentioned above the solution v and w of the transmission problem eigenvalue problem are in $L^2_\Delta(D)$, where

$$L^2_\Delta(D) := \left\{ u \in L^2(D) : \Delta u \in L^2(D) \right\}.$$

Therefore their trace and their normal derivative on the boundary live in $H^{-\frac{1}{2}}(\partial D)$ and $H^{-\frac{3}{2}}(\partial D)$, respectively. Indeed if $u \in L^2_\Delta(D)$ then its trace $u \in H^{-\frac{1}{2}}(\partial D)$ is defined by duality using the identity

$$\langle u, \tau \rangle_{H^{-\frac{1}{2}}(\partial D), H^{\frac{1}{2}}(\partial D)} = \int_D \big(u\Delta w - w\Delta u\big)\, \mathrm{d}x$$

where $w \in H^2(D)$ is such that $w = 0$ and $\partial w/\partial \nu = \tau$. Similarly, the trace of $\partial u/\partial \nu \in H^{-\frac{3}{2}}(\partial D)$ is defined by duality using the identity

$$\left\langle \frac{\partial u}{\partial \nu}, \tau \right\rangle_{H^{-\frac{3}{2}}(\partial D), H^{\frac{3}{2}}(\partial D)} = -\int_D \big(u\Delta w - w\Delta u\big)\, \mathrm{d}x$$

where $w \in H^2(D)$ is such that $w = \tau$ and $\partial w/\partial \nu = 0$. Hence to represent v and w by a single-layer potential approach we must work with densities in $H^{-\frac{3}{2}}(\partial D)$ (i.e. for $s = -1$ in the above). Obviously, $\mathcal{S}_k\psi$ satisfies the Helmholtz equation, hence we can conclude that $\mathcal{S}_k : H^{-\frac{3}{2}}(\partial D) \to L^2_\Delta(D)$ is continuous. More importantly, by a duality argument it is possible to extend the jump relations for single-layer potentials across ∂D to the case of potentials with weaker densities. More specifically, the following lemma is proven in Theorem 3.1 in [2] (see also [22]).

Lemma 2.1: *The single-layer potential $\mathcal{S}_k : H^{-\frac{3}{2}}(\partial D) \to L^2_\Delta(D)$ satisfies*

$$(\mathcal{S}_k\psi)^-_{\partial D} = S_k\psi \qquad \text{in } H^{-\frac{1}{2}}(\partial D),$$

$$\frac{\partial(\mathcal{S}_k\psi)^-}{\partial \nu}\Big|_{\partial D} = K'_k\psi + \psi \qquad \text{in } H^{-\frac{3}{2}}(\partial D),$$

where the bounded linear operators

$$S_k : H^{-\frac{3}{2}}(\partial D) \longrightarrow H^{-\frac{1}{2}}(\partial D) \qquad K'_k : H^{-\frac{3}{2}}(\partial D) \longrightarrow H^{-\frac{3}{2}}(\partial D)$$

are given by (2.6) *and* (2.7)*, respectively, and the superscript indicates that the boundary ∂D is approached from inside D.*

The following lemma is a particular case of Theorem 7.17 in [22].

Lemma 2.2: *Assume that ∂D is of class $C^{2,1}$. Then $S_k : H^{s-\frac{1}{2}}(\partial D) \to H^{s+\frac{1}{2}}(\partial D)$ is Fredholm with index zero for $-1 \leq s \leq 1$. In addition its kernel does not depend on s in this range.*

As a simple consequence of Lemma 2.2 we have the following result.

Corollary 2.3: *Assume that ∂D is of class $C^{2,1}$ and k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D. Then $S_k^{-1} : H^{s+\frac{1}{2}}(\partial D) \to H^{s-\frac{1}{2}}(\partial D)$ exist and is bounded for $-1 \leq s \leq 1$.*

Proof: The standard theory of single-layer potentials implies that the kernel of $S_k : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is non-trivial because of our assumption on k^2 not to be a Dirichlet eigenvalue, whence Lemma 2.2 implies the result. □

From our discussion we can now conclude that the Dirichlet-to-Neumann operator $N_k : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{3}{2}}(\partial D)$ for the $L^2(D)$ solutions of (1.2) is represented by

$$N_k = (I + K_k')S_k^{-1},$$

provided that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D.

Now we consider the difference of the Dirichlet-to-Neumann operators corresponding to k and k_n given by

$$M(k) := (I + K_k')S_k^{-1} - (I + K_{k_n}')S_{k_n}^{-1} \tag{2.10}$$

and have the following regularity result.

Lemma 2.4: *The linear operators*

$$\varphi \mapsto \mathcal{S}_k S_k^{-1}\varphi - \mathcal{S}_{k_n} S_{k_n}^{-1}\varphi \tag{2.11}$$

from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$ and $M(k) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ are bounded.

Proof: By definition $M(k)\varphi$ is the normal derivative on the boundary ∂D of

$$u := \mathcal{S}_k S_k^{-1}\varphi - \mathcal{S}_{k_n} S_{k_n}^{-1}\varphi, \quad \varphi \in H^{-\frac{1}{2}}(\partial D).$$

We can rewrite u as

$$u = \mathcal{S}_k(S_k^{-1} - S_{k_n}^{-1})\varphi - (\mathcal{S}_{k_n} - \mathcal{S}_k)S_{k_n}^{-1}\varphi. \tag{2.12}$$

From Theorem 3.2 in [2] we have that $(\mathcal{S}_{k_n} - \mathcal{S}_k)$ is a pseudo-differential operator of order $-7/2$ which implies that the operator

$$(\mathcal{S}_{k_n} - \mathcal{S}_k) : H^{-\frac{3}{2}}(\partial D) \to H^2(D) \tag{2.13}$$

is bounded. By the trace theorem this implies that

$$(S_k - S_{k_n}) : H^{-\frac{3}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$$

is also bounded. Now writing

$$S_{k_n}^{-1} - S_k^{-1} = S_k^{-1}(S_k - S_{k_n})S_{k_n}^{-1}$$

and using from Corollary 2.3 that both $S_{k_n}^{-1} : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{3}{2}}(\partial D)$ and $S_k^{-1} : H^{\frac{3}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ are bounded, we find that

$$S_k^{-1} - S_{k_n}^{-1} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D) \tag{2.14}$$

is bounded. Now putting (2.13) and (2.14) into (2.12) together with the boundedness of $\mathcal{S}_k : H^{\frac{1}{2}}(\partial D) \to H^2(D)$ and $S_{k_n}^{-1} : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{3}{2}}(\partial D)$ shows that the operator (2.11) indeed is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$. From this the statement on $M(k)$ follows by taking the normal trace. □

Combining all the above results, we can conclude that k is a transmission eigenvalue if and only if the kernel of $M(k) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ given by (2.10) is non-trivial. To analyze the kernel of $M(k)$ we want to show that $M(k) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is a Fredholm operator of index zero. To this end we show that $M(i\kappa)$ is coercive for $\kappa > 0$.

Theorem 2.5: *Let $\kappa > 0$ and $\kappa_n := \kappa\sqrt{n}$. Then*

$$(\kappa^2 - \kappa_n^2)M(i\kappa) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$$

is coercive, i.e.

$$(\kappa^2 - \kappa_n^2)\,\langle M(i\kappa)\varphi, \varphi\rangle_{H^{\frac{1}{2}}(\partial D), H^{-\frac{1}{2}}(\partial D)} \geq C\|\varphi\|^2_{H^{-\frac{1}{2}}(\partial D)}$$

for all $\varphi \in H^{-\frac{1}{2}}(\partial D)$ and some $C > 0$. (Here, $\langle\cdot,\cdot\rangle$ denotes the sesquilinear duality pairing.)

Proof: For $u, v \in H^2(D)$, we transform

$$\begin{aligned}
&\int_D v(\Delta - \kappa^2)\,(\Delta - \kappa_n^2)u\,\mathrm{d}x \\
&\qquad - \int_D \left[\Delta u \Delta v + \left(\kappa^2 + \kappa_n^2\right) \operatorname{grad} u \cdot \operatorname{grad} v + \kappa^2\kappa_n^2 uv\right] \mathrm{d}x \\
&\quad = \int_D (v\Delta\Delta u - \Delta v \Delta u)\,\mathrm{d}x - (\kappa^2 + \kappa_n^2)\int_D (v\Delta u + \operatorname{grad} u \cdot \operatorname{grad} v)\,\mathrm{d}x.
\end{aligned}$$

From this, by Green's theorem we obtain

$$\begin{aligned}
&\int_D v(\Delta - \kappa^2)\,(\Delta - \kappa_n^2)u\,\mathrm{d}x \\
&\qquad - \int_D \left[\Delta u \Delta v + \left(\kappa^2 + \kappa_n^2\right) \operatorname{grad} u \cdot \operatorname{grad} v + \kappa^2\kappa_n^2 uv\right] \mathrm{d}x \\
&\quad = \int_{\partial D} \left(v\,\frac{\partial \Delta u}{\partial \nu} - \Delta u\,\frac{\partial v}{\partial \nu}\right) \mathrm{d}s - \left(\kappa^2 + \kappa_n^2\right)\int_{\partial D} v\,\frac{\partial u}{\partial \nu}\,\mathrm{d}s.
\end{aligned} \tag{2.15}$$

For $v = \bar{u}$ the second domain integral is equivalent to the $\|\cdot\|_{H^2}$ norm as can be seen with the aid of Green's representation formula, that is,

$$\int_D \left(|\Delta u|^2 + (\kappa^2 + \kappa_n^2)|\operatorname{grad} u|^2 + \kappa^2\kappa_n^2|u|^2\right)\,\mathrm{d}x \geq c\,\|u\|^2_{H^2(D)} \tag{2.16}$$

for all $u \in H^2(D)$ and some constant $c > 0$.

Now, for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ we define

$$u := \mathcal{S}_{i\kappa} S_{i\kappa}^{-1}\varphi - \mathcal{S}_{i\kappa_n} S_{i\kappa_n}^{-1}\varphi$$

which belongs to $H^2(D)$ by Lemma 2.4. Then

$$(\Delta - \kappa^2)(\Delta - \kappa_n^2)u = 0 \tag{2.17}$$

and

$$\Delta u = \kappa^2\,\mathcal{S}_{i\kappa} S_{i\kappa}^{-1}\varphi - \kappa_n^2\,\mathcal{S}_{i\kappa_n} S_{i\kappa_n}^{-1}\varphi.$$

Therefore, we have the boundary conditions

$$u = 0 \quad \text{and} \quad \Delta u = (\kappa^2 - \kappa_n^2)\varphi \quad \text{on } \partial D. \tag{2.18}$$

We set $v = \bar{u}$ in (2.15) and use (2.18) and $\dfrac{\partial u}{\partial \nu} = M(i\kappa)$ to obtain

$$\int_D \left[|\Delta u|^2 + (\kappa^2 + \kappa_n^2)|\operatorname{grad} u|^2 + \kappa^2\kappa_n^2|u|^2\right] \mathrm{d}x$$

$$= \int_{\partial D} \Delta u \frac{\partial \bar{u}}{\partial \nu} \,\mathrm{d}s = (\kappa^2 - \kappa_n^2) \int_{\partial D} \varphi \overline{M(i\kappa)\varphi} \,\mathrm{d}s.$$

Hence, together with (2.16) we get the coercivity estimate

$$(\kappa^2 - \kappa_n^2) \int_{\partial D} \varphi \overline{M(i\kappa)\varphi} \,\mathrm{d}s \geq \tilde{C} \, \|u\|^2_{H^2(D)} \geq C\|\varphi\|^2_{H^{-\frac{1}{2}}(\partial D)} \tag{2.19}$$

for all $\varphi \in H^{-\frac{1}{2}}(\partial D)$ and some constants $\tilde{C}, C > 0$, where for the latter inequality we have used (2.18) and the definition of the trace of φ by duality as in the beginning of this section. □

Theorem 2.6: *The operator*

$$M(k) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} M(i|k|) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$$

is compact.

Proof: For $\varphi \in H^{-\frac{1}{2}}(\partial D)$ we define

$$u := \mathcal{S}_k S_k^{-1}\varphi - \mathcal{S}_{k_n} S_{k_n}^{-1}\varphi \quad \text{and} \quad u_i := \mathcal{S}_{i|k|} S_{i|k|}^{-1}\varphi - \mathcal{S}_{i|k_n|} S_{i|k_n|}^{-1}\varphi$$

and let

$$U := u + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} \, u_i.$$

Then $U \in H^2(D)$ by Lemma 2.4 and it satisfies the boundary conditions (see (2.18))

$$U = 0 \quad \text{and} \quad \Delta U = 0 \quad \text{on} \quad \partial D. \tag{2.20}$$

Furthermore, it is straightforward to check that

$$\Delta\Delta U = F(u, u_i) \tag{2.21}$$

where

$$\begin{aligned} F(u, u_i) := &-k^2 k_n^2 u - (k^2 + k_n^2)\Delta u \\ &- \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} \left[|k|^2|k_n|^2 u_i - (|k|^2 + |k_n|^2)\Delta u_i\right] \end{aligned} \tag{2.22}$$

belongs to $L^2(D)$. Now, we can apply a regularity theorem on the Poisson equation which guarantees that the unique solution $v \in H_0^1(D)$ of $\Delta v = f$ for $f \in H^m(D)$ belongs to $H^{m+2}(D)$ and that the linear mapping taking f into v is bounded from $H^m(D)$ into $H^{m+2}(D)$ for $m = 0, 1, \ldots$ (see Theorem 1.3 in [24, p.305]). First we show that this property can be extended to solutions $v \in L^2(D)$ that vanish on ∂D in the sense of the $H^{-\frac{1}{2}}(\partial D)$ trace. For this we observe from the definition of the $H^{-\frac{1}{2}}(\partial D)$ trace as discussed at the beginning of this section that for any harmonic function $v \in L^2(D)$ vanishing on the boundary ∂D we have that $\int_D v \Delta w \mathrm{d}x = 0$ for all $w \in H^2(D)$ with $w = 0$ on ∂D. Inserting the solution $w \in H_0^1(D)$ of $\Delta w = \overline{v}$ which automatically belongs to $H^2(D)$ by the above theorem yields $v = 0$ in D. For a solution $v \in L^2(D)$ of $\Delta v = f$ for $f \in L^2(D)$ with vanishing $H^{-\frac{1}{2}}(\partial D)$ trace on ∂D

we denote by $\tilde{v}$ the solution of $\Delta\tilde{v} = f$ in $H_0^1(D)$ and apply the uniqueness result for the difference $v - \tilde{v}$ to obtain that $v = \tilde{v} \in H_0^1(D)$.

Applying this regularity lifting first for ΔU we obtain that $\Delta U \in H^2(D)$ with the mapping $F \mapsto \Delta U$ bounded from $L^2(D)$ into $H^2(D)$. Applying the regularity property again for U then in turn shows $U \in H^4(D)$ with the mapping $\Delta U \mapsto U$ bounded from $H^2(D)$ into $H^4(D)$. Since by Lemma 2.4 the mappings $\varphi \mapsto u$ and $\varphi \mapsto u_i$ are bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$, summarizing we have that $\varphi \mapsto U$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^4(D)$. Therefore, the mapping $\varphi \mapsto \partial_\nu U$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{3}{2}}(\partial D)$. Now, noting that

$$\frac{\partial U}{\partial \nu} = M(k) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} M(i|k|)$$

the statement of the theorem follows from the compact embedding of $H^{\frac{3}{2}}(\partial D)$ into $H^{\frac{1}{2}}(\partial D)$. $\square$

At this point we would like to remark that the regularity result used in the proof of Theorem 2.6 also gives rise to a different and somewhat shorter proof of Lemma 2.4. However, since in the subsequent section the corresponding regularity arguments from the proof of Theorem 3.3 are not sufficient to prove the corresponding Lemma 3.2 we kept the proof of Lemma 2.4 in the same spirit as that of Lemma 3.2.

Let us denote by $\mathbb{E}$ the set of all positive k such that k^2 or k_n^2 is a Dirichlet eigenvalue for $-\Delta$ in D. Then $M(k)$ is defined for $k \in \mathbb{C} \setminus \mathbb{E}$ and analytic since the kernels of the integral operators are analytic in k. Then Theorems 2.5 and 2.6 imply the following result.

Theorem 2.7: *$M(k) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is a Fredholm operator with index zero and analytic in $\mathbb{C} \setminus \mathbb{E}$*

The drawback of this approach is that we have to exclude the Dirichlet eigenvalues, and as a remedy to this restriction, in the next section we develop similar ideas for a modified approach.

3. Integral equations based on the Robin-to-Dirichlet operator

Instead of using the Dirichlet-to-Neumann operator, we now propose to express the transmission eigenvalues in terms of the Robin-to-Dirichlet operator. To this end, we define the Robin-to-Dirichlet operator $R_{k,n} : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ by

$$R_{k,n} : \varphi \mapsto u|_{\partial D} \tag{3.1}$$

where $u \in L^2_\Delta(D)$ solves

$$\Delta u + k^2 n u = 0 \quad \text{in } D, \tag{3.2}$$

$$\frac{\partial u}{\partial \nu} - i\eta u = \varphi \quad \text{on } \partial D \tag{3.3}$$

where $\eta > 0$. Note that an application of Green's integral theorem shows that $R_{k,n}$ is well defined for $k \in \mathbb{C}$ such that Re $(k) > 0$ and Im $(k) \geq 0$ (one can use $\eta < 0$ in the impedance condition instead if Im $(k) < 0$). Now k is a transmission eigenvalue corresponding to (2.1)–(2.4) if and only if the kernel of the operator

$$P(k;\eta) := R_{k,1} - R_{k,n}$$

is non-trivial. Formally, the Robin-to-Dirichlet operator can be written in terms of boundary integral operators as

$$R_{k,1} = S_k(I + K'_k - i\eta S_k)^{-1} \quad \text{and} \quad R_{k,n} = S_{k_n}(I + K'_{k_n} - i\eta S_{k_n})^{-1}$$

where from now on we again use the notation $k_n := k\sqrt{n}$.

Lemma 3.1: *Assume that ∂D is of class $C^{2,1}$. Then for $-1 \le s \le 1$ the operator $(I + K'_k - i\eta S_k)^{-1} : H^{s-\frac{1}{2}}(\partial D) \to H^{s-\frac{1}{2}}(\partial D)$ exists and is bounded.*

Proof: The operators S_k and K'_k are compact on $H^{s-\frac{1}{2}}(\partial D)$ since S_k is a pseudo-differential operator of order -1 as noted above, and K'_k is also a pseudo-differential operator of order (at least) -1 (in fact in $\mathbb{R}^2$ the order is -3, since the continuously differentiable kernel of K'_k has a second derivative with a logarithmic singularity). Hence obviously $(I + K'_k - i\eta S_k) : H^{s-\frac{1}{2}}(\partial D) \to H^{s-\frac{1}{2}}(\partial D)$ is Fredholm with index zero. Thus, to prove the result we need to show that the kernel of $(I + K'_k - i\eta S_k)$ in $H^{s-\frac{1}{2}}(\partial D)$ is trivial. We first consider $s = 0$ and assume that $\varphi \in H^{-\frac{1}{2}}(\partial D)$ is in the kernel of $(I + K'_k - i\eta S_k)$. Then $u := \mathcal{S}_k\varphi$ is an $H^1(D)$ solution of the Helmholtz equation satisfying (in the sense of the trace approaching ∂D from inside D)

$$\frac{\partial u}{\partial \nu} - i\eta u = 0 \quad \text{on } \partial D.$$

An application of Green's first integral theorem implies $u = 0$ in D provided Re $(k) > 0$ and Im $(k) \ge 0$. On the other hand $u := \mathcal{S}_k\varphi$ is an $H^1_{\text{loc}}(\mathbb{R}^m \setminus \overline{D})$ radiating solution to the Helmholtz equation satisfying $u = 0$ on ∂D because of the jump relations for the single-layer potential across ∂D. The well-known uniqueness for the exterior Dirichlet problem (see e.g. [25]) implies that $u = 0$ in $\mathbb{R}^m \setminus \overline{D}$. The jump property of the normal derivative of $\mathcal{S}_k\varphi$ across ∂D gives $\varphi = 0$, i.e. the kernel of $(I + K'_k - i\eta S_k)$ is trivial in $H^{-\frac{1}{2}}(\partial D)$. From this, using the Fredholm alternative in dual systems (see [26, Theorem 4.20]) the kernel is also trivial in $H^{s-\frac{1}{2}}(\partial D)$ for $-1 \le s \le 1$. □

We note that the statement of Lemma 3.1 is also true in the case where $k = i$ and $\eta = i$ because of the uniqueness for the Robin problem $\Delta u - u = 0$ in D with $\partial_\nu u + u = 0$ on ∂D.

We need to analyze the kernel of the operator

$$P(k;\eta) := S_k(I + K'_k - i\eta S_k)^{-1} - S_{k_n}(I + K'_{k_n} - i\eta S_{k_n})^{-1} \tag{3.4}$$

which, analogous to the case of the difference of the Dirichlet-to-Neumann operators in Section 2, is more smoothing than the Robin-to-Dirichlet operator itself. More specifically we have the following result.

Lemma 3.2: *The linear operators*

$$\varphi \mapsto \mathcal{S}_k(I + K'_k - i\eta S_k)^{-1}\varphi - \mathcal{S}_{k_n}(I + K'_{k_n} - i\eta S_{k_n})^{-1}\varphi \tag{3.5}$$

from $H^{-\frac{3}{2}}(\partial D)$ into $H^2(D)$ and $P(k;\eta) : H^{-\frac{3}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ are bounded.

Proof: For $\varphi \in H^{-\frac{3}{2}}(\partial D)$ we define

$$u := \mathcal{S}_k(I + K'_k - i\eta S_k)^{-1}\varphi - \mathcal{S}_{k_n}(I + K'_{k_n} - i\eta S_{k_n})^{-1}\varphi, \tag{3.6}$$

which can be rewritten as

$$\begin{aligned} u = {} & \mathcal{S}_k\left((I + K'_k - i\eta S_k)^{-1} - (I + K'_{k_n} - i\eta S_{k_n})^{-1}\right)\varphi \\ & -(\mathcal{S}_{k_n} - \mathcal{S}_k)(I + K'_{k_n} - i\eta S_{k_n})^{-1}\varphi. \end{aligned} \tag{3.7}$$

Theorem 3.2 in [2] implies that $(\mathcal{S}_{k_n} - \mathcal{S}_k) : H^{-\frac{3}{2}}(\partial D) \to H^2(D)$ is bounded and so is the operator $(\mathcal{S}_{k_n} - \mathcal{S}_k)(I + K'_{k_n} - i\eta S_{k_n})^{-1} : H^{-\frac{3}{2}}(\partial D) \to H^2(D)$ by Lemma 3.1. Next we can write

$$\begin{aligned}&\left((I + K'_k - i\eta S_k)^{-1} - (I + K'_{k_n} - i\eta S_{k_n})^{-1}\right)\\ &\quad = (I + K'_k - i\eta S_k)^{-1}\left((K'_k - K'_{k_n}) - i\eta(S_k - S_{k_n})\right)(I + K'_{k_n} - i\eta S_{k_n})^{-1}\end{aligned}$$

Again from Theorem 3.2 in [2] we have that $(S_{k_n} - S_k) : H^{-\frac{3}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ and consequently $(K'_k - K'_{k_n}) : H^{-\frac{3}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ are bounded. Hence combining these mapping properties with Lemma 3.1 and the fact that $\mathcal{S}_k : H^{\frac{1}{2}}(\partial D) \to H^2(D)$ is continuous shows that the first term of (3.8) also maps $H^{-\frac{3}{2}}(\partial D)$ continuously into $H^2(D)$.

Finally, by definition $P(k;\eta)\varphi$ is the trace of u on ∂D and the result of the lemma follows from the above. □

In view of the note after Lemma 3.1 obviously the statement of Lemma 3.2 is also true in the case where k is purely imaginary and $\eta = i$.

Summarizing up to this point we have that $k \in \mathbb{C}$ with Re $(k) > 0$ and Im $(k) \geq 0$ is a transmission eigenvalue if and only if the kernel of $P(k,\eta) : H^{-\frac{3}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ as given by (3.4) is non-trivial.

Theorem 3.3: *For $\kappa > 0$, $\kappa_n := \kappa\sqrt{n}$ and $\eta = i$, the operator*

$$(\kappa_n^2 - \kappa^2)P(i\kappa; i) : H^{-\frac{3}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$$

is coercive, i.e.

$$(\kappa_n^2 - \kappa^2)\,\langle P(i\kappa; i)\varphi, \varphi\rangle_{H^{\frac{3}{2}}(\partial D), H^{-\frac{3}{2}}(\partial D)} \geq C\|\varphi\|^2_{H^{-\frac{3}{2}}(\partial D)}$$

for all $\varphi \in H^{-\frac{3}{2}}(\partial D)$ and some $C > 0$.

Proof: Analogous to the proof of Theorem 2.5, for $\varphi \in H^{\frac{3}{2}}(\partial D)$ we consider

$$u := \mathcal{S}_{i\kappa}(I + K'_{i\kappa} + S_{i\kappa})^{-1}\varphi - \mathcal{S}_{i\kappa_n}(I + K'_{i\kappa_n} + S_{i\kappa_n})^{-1}\varphi.$$

Then again (2.17) is valid and

$$\Delta u = \kappa^2\,\mathcal{S}_{i\kappa}(I + K'_{i\kappa} + S_{i\kappa})^{-1}\varphi - \kappa_n^2\mathcal{S}_{i\kappa_n}(I + K'_{i\kappa_n} + S_{i\kappa_n})^{-1}\varphi.$$

The boundary conditions now are

$$u = P(i\kappa; i)\varphi, \quad \frac{\partial u}{\partial\nu} + u = 0, \quad \frac{\partial\Delta u}{\partial\nu} + \Delta u = (\kappa^2 - \kappa_n^2)\varphi \quad \text{on } \partial D. \tag{3.8}$$

We set $v = \bar{u}$ in (2.15) and use (2.17) and (3.8) to obtain

$$\begin{aligned}&-\int_D \left[|\Delta u|^2 + (\kappa^2 + \kappa_n^2)|\operatorname{grad} u|^2 + \kappa^2\kappa_n^2|u|^2)\right] dx\\ &\quad = (\kappa^2 - \kappa_n^2)\int_{\partial D} \varphi\overline{P(i\kappa)\varphi}\, ds + (\kappa^2 + \kappa_n^2)\int_{\partial D} |u|^2\, ds.\end{aligned}$$

Hence, together with (2.16) and the trace theorem we get the coercivity estimate

$$(\kappa_n^2 - \kappa^2) \int_{\partial D} \varphi \overline{P(i\kappa; i)\varphi}\, ds \geq \tilde{C}\, \|u\|^2_{H^2(D)} \geq C\|\varphi\|^2_{H^{-\frac{3}{2}}(\partial D)} \tag{3.9}$$

for all $\varphi \in H^{\frac{3}{2}}(\partial D)$ and some constants $\tilde{C}, C > 0$. □

Theorem 3.4: *The operator*

$$P(k;\eta) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} P(i|k|; i) : H^{-\frac{3}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$$

is compact.

Proof: Following the lines of the proof of Theorem 2.6, for $\varphi \in H^{-\frac{3}{2}}(\partial D)$ we define

$$\begin{aligned} u &:= \mathcal{S}_k(I + K_k' - i\eta S_k)^{-1}\varphi - \mathcal{S}_{k_n}(I + K_{k_n}' - i\eta S_{k_n})^{-1}\varphi, \\ u_i &:= \mathcal{S}_{i|k|}(I + K_{i|k|}' + S_{i|k|})^{-1}\varphi - \mathcal{S}_{i|k_n|}(I + K_{i|k_n|}' + S_{i|k_n|})^{-1}\varphi, \end{aligned}$$

and set

$$U := u + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2}\, u_i.$$

Then $U \in H^2(D)$ again satisfies

$$\Delta\Delta U = F(u, u_i) \tag{3.10}$$

where $F(u, u_i) \in L^2(D)$ is given by (2.22). On the boundary ∂D we have that

$$\frac{\partial U}{\partial \nu} + U = (1 + i\eta)u \tag{3.11}$$

and

$$\frac{\partial \Delta U}{\partial \nu} + \Delta U = (1 + i\eta)\Delta u. \tag{3.12}$$

and note that by Lemma 3.2 the trace of u on ∂D is in $H^{\frac{3}{2}}(\partial D)$ and that of Δu in $H^{-\frac{1}{2}}(\partial D)$ with the corresponding maps from $\varphi \in H^{-\frac{3}{2}}(\partial D)$ into the traces being bounded. We know from standard elliptic theory, i.e. from the Lax–Milgram theorem, that for $F \in L^2(D)$ and $g := (1 + i\eta)\Delta u|_{\partial D} \in H^{-\frac{1}{2}}(\partial D)$ there exists a unique solution $V \in H^1(D)$ to

$$\Delta V = F \quad \text{in } D, \quad \frac{\partial V}{\partial \nu} + V = g \quad \text{on } \partial D,$$

and that the mapping $(F, g) \mapsto V$ is continuous from $L^2(D) \times H^{-1/2}(\partial D)$ into $H^1(D)$ (see [24]). Then the difference $w := V - \Delta U \in L^2_\Delta(D)$ is harmonic and satisfies $\partial_\nu w + w = 0$ on ∂D. Hence, by Green's integral formula, the trace $w|_{\partial D} \in H^{-\frac{3}{2}}(\partial D)$ belongs to the kernel of $(I + K_0' + S_0)$ in $H^{-\frac{3}{2}}(\partial D)$ where K_0' and S_0 are the boundary integral operators given by (2.6) and (2.7) with $k = 0$. By the same reasoning as in Lemma 3.1 we conclude that the kernel of this operator is trivial, whence $\Delta U = V$ follows. Therefore $\Delta U \in H^1(D)$ and the linear operator taking $\varphi \in H^{-\frac{3}{2}}(\partial D)$ into $\Delta U \in H^1(D)$ is bounded.

Now we are in the position to apply a regularity result which implies that the solution $v \in H^1(D)$ of $\Delta v - v = G$ in D with Neumann boundary condition $\partial_\nu v = g$ on ∂D for $G \in H^1(D)$ and $g \in H^{\frac{3}{2}}(\partial D)$ belongs to $H^3(D)$ and the mapping taking (G, g) from $H^1(D) \times H^{\frac{3}{2}}(\partial D)$ into $H^3(D)$ is continuous (see Proposition 7.5 in [24, p.350]). In view of the boundedness of the mapping $\varphi \mapsto u|_{\partial D}$ from

$H^{-\frac{3}{2}}(\partial D)$ into $H^{\frac{3}{2}}(\partial D)$, applying this regulartity property to $v = U$ shows that $U \in H^3(D)$ with the map $\varphi \to U$ bounded from $H^{-\frac{3}{2}}(\partial D)$ into $H^3(D)$. Hence the mapping $\varphi \mapsto U|_{\partial D}$ is bounded from $H^{-\frac{3}{2}}(\partial D)$ into $H^{\frac{5}{2}}(\partial D)$ and noting that

$$U|_{\partial D} = P(k;\eta) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} P(i|k|; i)$$

the statement of the theorem follows from the compact embedding of $H^{\frac{5}{2}}(\partial D)$ into $H^{\frac{3}{2}}(\partial D)$. □

In summary, Theorems 3.3 and 3.4 imply the following result, from which in particular we can reestablish the well-known discreteness of the transmission eigenvalues for the special case of a constant refractive index.

Theorem 3.5: *$P(k;\eta) : H^{-\frac{3}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ is a Fredholm operator with index zero and analytic in $\{k \in \mathbb{C} : \mathrm{Re}\,(k) > 0 \text{ and } \mathrm{Im}\,(k) \geq 0\}$.*

Remark 3.6: Our integral equation approach can be used to study the transmission eigenvalue problem for a more general refractive index $n(x)$ in the same way as in Section 4 in [2] or Section 3.2 in [13]. Fur such a generalization the only assumption needed in addition to the standard assumptions on the refractive index is that there exists a neighborhood of the boundary ∂D where $n(x)$ is constant. We leave out the details in this regard, since the emphasis of this paper is not to reproduce known results on the discreteness of transmission eigenvalues but rather to provide an alternative approach for computing transmission eigenvalues.

4. Numerical computation

In this final section we will illustrate the use of our boundary integral formulation for the actual computation of interior transmission eigenvalues. So far, in the literature, the majority of numerical methods are based on finite element methods applied after a transformation of the homogeneous interior transmission problem to a generalized eigenvalue problem for a fourth-order partial differential equation (see [17–19,27], among others). To our knowledge, boundary integral equations have been employed for the computation of transmission eigenvalues only by Cossonnière [1] and Kleefeld [20] using the two by two system of boundary integral equations proposed by Cossonnière and Haddar [2].

Noting that computing transmission eigenvalues is equivalent to finding wave numbers k such that $M(k)\varphi = 0$ (or $P(k)\varphi = 0$) has a non-trivial solution following Cossonnière's idea [1] one would compute the eigenvalues of the linear operator $M(k)$, that is, the eigenvalues of matrix approximations of $M(k)$ and look for those values of k where the smallest eigenvalue is close to zero. However, since the operator $M(k)$ is compact, let say, from $L^2(\partial D)$ to $L^2(\partial D)$ its eigenvalues will accumulate at zero and consequently, due to numerical errors, it is impossible to distinguish an eigenvalue zero from the surrounding eigenvalues close to zero. Instead of trying to remedy this difficulty by considering a generalized eigenvalue problem with a preconditioner as suggested by Cossonnière [1], we find it more attractive to use a new algorithm for solving non-linear eigenvalue problems for large-sized matrices A that are analytic with respect to the eigenvalue parameter proposed by Beyn [3]. This algorithm has already been applied by Kleefeld [20] for the computation of interior transmission eigenvalues in three dimensions. However, Kleefeld's work is based on the two by two system of Haddar and Cossonnière's boundary integral equations. In our case, the matrix A will be given by an approximation of the operator M or P via numerical quadratures.

To describe Beyn's algorithm, let $\Omega_0 \subset \mathbb{C}$ be a simply connected domain and $A : \Omega_0 \to \mathbb{C}^{m\times m}$ an analytic function with values in the space of complex $m \times m$ matrices. Consider the non-linear eigenvalue problem $A(k)v = 0$ and assume that A has only a finite number $\ell \ll m$ of eigenvalues $k_1, \ldots k_\ell$ in Ω_0 (counted according to their multiplicity). Beyn's approach uses Keldysh's formula for

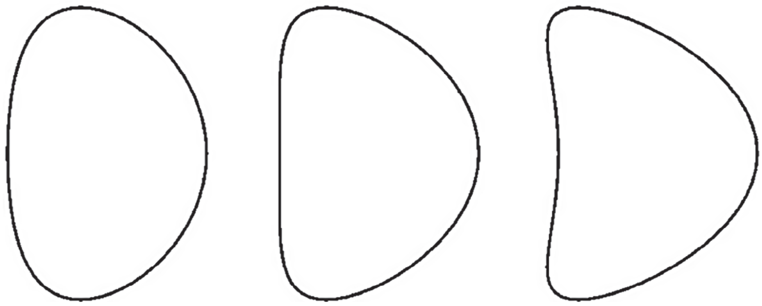

Figure 1. Shape of boundary (4.3) for $\varepsilon = 0.1$ (left), $\varepsilon = 0.2$ (middle), and $\varepsilon = 0.3$ (right).

the principle part of the resolvent $A^{-1}(k)$ and Cauchy's integral theorem to reduce the non-linear eigenvalue problem for the matrix A of size m to a linear eigenvalue problem of size ℓ.

For this let Ω be a simply connected domain with $\overline{\Omega} \subset \Omega_0$ with analytic boundary curve containing the eigenvalues $k_1, \ldots k_\ell$ of A. Then Beyn's algorithm is divided into three steps as follows:

Step 1. Choose a number $p \in \mathbb{N}$ and matrices $B \in \mathbb{C}^{m\times m}$ and $C \in \mathbb{C}^{m\times p}$ at random. Then compute matrices $A_0, A_1 \in \mathbb{C}^{m\times p}$ by evaluating the complex integrals

$$A_0 := \frac{1}{2\pi i}\int_{\partial\Omega} BA^{-1}(k)C\,\mathrm{d}k \quad \text{and} \quad A_1 := \frac{1}{2\pi i}\int_{\partial\Omega} k\,BA^{-1}(k)C\,\mathrm{d}k \tag{4.1}$$

numerically by the composite trapezoidal rule (after parameterizing the curve $\partial\Omega$). For the integrals in (4.1) we assume counterclockwise orientation of $\partial\Omega$ (although the orientation does not matter for the algorithm). Note that for the analytic periodic integrands the composite trapezoidal rule has an exponential convergence rate (see [26, Section 12.1]).

Step 2. Now perform a singular value decomposition

$$A_0 = U\Sigma V^*$$

with orthogonal matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{p\times p}$ and a diagonal matrix $\Sigma \in \mathbb{C}^{m\times p}$ with non-negative entries. With a sufficiently small number tol, in a rank test determine $1 \le \ell \le p$ such that

$$\sigma_1 \ge \cdots \ge \sigma_\ell > \text{tol} > \sigma_{\ell+1} \approx 0 \approx \cdots \approx \sigma_p$$

for the diagonal entries of Σ. If $\ell = p$, then increase p and go back to Step 1. Otherwise continue with Step 3.

Step 3. Compute the matrix $\widetilde{A} \in \mathbb{C}^{\ell\times\ell}$ by

$$\widetilde{A} := U_0^* A_1 V_0 \Sigma_0^{-1}$$

where the matrices $U_0 \in \mathbb{C}^{m\times\ell}$ and $V_0 \in \mathbb{C}^{p\times\ell}$ are obtained from U and V by deleting the the last $p-\ell$ columns, respectively, and $\Sigma_0 = \operatorname{diag}(\sigma_1, \ldots, \sigma_\ell)$. Finally, the eigenvalues $k_1, \ldots, k_\ell$ of A are obtained as the eigenvalues of the $\ell \times \ell$ matrix $\widetilde{A}$.

Summarizing, the non-linear eigenvalue problem $A(k)v = 0$ for the $m \times m$ matrix $A(k)$ has been reduced to a linear eigenvalue problem for the $\ell \times \ell$ matrix $\widetilde{A}$ with ℓ considerably smaller than m. The main computational cost is setting up the matrix $A(k)$ and inverting it for N values $k \in \partial\Omega$ where N is the number of quadrature points used in the trapezoidal rule for the integrals in (4.1).

For the method to work, it is essential that the matrix A_0 has rank ℓ. If the ℓ eigenvectors of A and the corresponding ℓ eigenvectors of the transposed matrix A' are linearly independent, it can be

Table 1. Interior transmission eigenvalues for ellipse with major axis $a = 1$.

$n = 2$				$n = 4$			
$b = 1$	$b = 0.8$	$b = 0.5$	$b = 0.3$	$b = 1$	$b = 0.8$	$b = 0.5$	$b = 0.3$
7.37512	7.63521	8.99951	12.32237	2.90260	3.13534	4.33068	6.55275
7.39666	8.13114	9.08377	12.34655	2.90260	3.48518	4.36895	6.56055
7.39666	8.43084	10.97143	16.48683	3.38419	3.54733	5.40918	8.09495
7.98435	8.51387	11.03990	16.49174	3.41205	3.88430	5.60124	8.15743
7.98435	8.95319	12.30361	17.01368	3.41205	4.14574	6.09152	8.88607
8.02926	9.00645	12.31572	17.14467	3.97647	4.49414	6.16797	8.93590
8.02926	9.24582	13.40175	19.44708	3.97647	4.55719	6.29163	9.52869
8.21647	9.34204	13.41828	19.44871	4.54698	5.05324	6.59713	9.75653
8.21647	9.47369	14.21896	19.92760	4.54698	5.06377	6.67731	9.94522
8.67540	9.76915	14.47108	19.94432	5.11604	5.65205	6.98627	10.09228

shown (see [3]) that the possibility of a rank defect in A_0 may be considered as non-generic because of the random choice of the matrices B and C. In [3] it also shown how the degenerate case where the eigenvectors of A and A' are not linearly independent can be dealt with by using higher order moments $\int_{\partial\Omega} k^s\, BA^{-1}(k)C\,\mathrm{d}k$, $s = 0, 1, \dots, s_0$, for some $s_0 \in \mathbb{N}$.

Before we conclude with two numerical examples, we note that as an immediate consequence of their definition the transmission eigenvalues $k(n)$ and $k(1/n)$ for constant refractive index n and $1/n$, respectively, are related by

$$k(1/n) = \sqrt{n}\, k(n). \tag{4.2}$$

Therefore we only present examples with $n > 1$.

For our first example we choose as boundary of the domain D an ellipse with major axis $a = 1$ and various choices for the minor axis b. For the second example the boundary curve is given by the parametric representation

$$z(t) = (0.75 \cos t + \varepsilon \cos 2t, \sin t), \quad 0 \le t \le 2\pi, \tag{4.3}$$

with various choices for the parameter ε. The shape of these curves is presented in Figure 1. For the contour integrals (4.1) we used ellipses

$$\partial\Omega = \{\gamma + \alpha \cos t + i\beta \sin t : t \in [0, 2\pi]\}$$

with positive parameters α, β and γ. Throughout our two examples we used 64 quadrature points for the composite trapezoidal rule for the two integrals in (4.1) and 128 quadrature points for the approximation of the boundary integral operators. For the latter we employed the approximations described in [25, Section 3.5] that are exponentially convergent for analytic boundaries.

The numerical results for the transmission eigenvalues for the refractive indexes $n = 2$ and $n = 4$ are shown in Tables 1 and 2. As an indication for the accuracy of the algorithm we observed that the imaginary part of the computed approximations for the transmission eigenvalues was always less than 10^{-14} and the condition number of the operator A at these values was always larger than 10^{13}. We also observed that the peaks of these condition numbers in the neighborhood of the transmission eigenvalues are extremely narrow. The accuracy, in higher decimals than given in the tables, can be improved by subdividing the range of transmission eigenvalues to be computed into smaller subintervals.

Our results for the ellipse with minor axis $b = 0.5$ and refractive index $n = 4$ concur with Cossonnière's results in [1, Figure 6.9] as obtained via looking for wave numbers where the linear sampling method fails, that is, where the norm of the regularized solution of the so-called far field equation peaks. However, as indicated in Table 1 our approach delivers more accurate numerical values for the transmission eigenvalues.

Table 2. Interior transmission eigenvalues for boundary curve (4.3).

$n = 2$			$n = 4$		
$\varepsilon = 0.1$	$\varepsilon = 0.2$	$\varepsilon = 0.3$	$\varepsilon = 0.1$	$\varepsilon = 0.2$	$\varepsilon = 0.3$
7.77900	7.93886	8.18170	3.27634	3.38239	3.51642
8.19492	8.18432	8.21173	3.59456	3.61769	3.69403
8.70986	8.69102	8.67801	3.73942	3.80771	3.87530
8.78226	8.75290	8.74716	4.07208	4.12712	4.21836
9.23137	9.36038	9.39387	4.37891	4.37938	4.43201
9.34688	9.38534	9.53699	4.74031	4.84546	4.96368
9.59654	9.59959	9.92163	4.79468	4.90266	5.07212
9.72272	9.86253	10.08375	5.27044	5.43618	5.64175
10.06567	10.18017	10.30841	5.30539	5.49621	5.70722
10.22065	10.35121	10.51804	5.84346	5.99979	6.21484

Despite the deficiency of the operator $M(k)$ at the Dirichlet eigenvalues, we found that the numerical results using $M(k)$ and $P(k; 0.5k)$ were identical. A heuristic explanation for this effect is the comparatively low probability for k and $k\sqrt{n}$ simultaneously being Dirichlet eigenvalues with the same eigenfunctions.

Comparing the computational costs for Beyn's algorithm as applied to Cossonnière and Haddar's two by two system on the one hand and our approach on the other hand we recall that the main numerical effort is setting up the matrix A and computing its inverse. Using m quadrature points on the boundary in our approach we have to deal with $m \times m$ matrices whereas in Cossonnière and Haddar's approach the same numerical accuracy requires $2m \times 2m$ matrices. Doubling the size of the matrix increases the computational costs for the inverse by a factor $8 = 2^3$. Taking into account that in setting up the matrices in our approach we also have to invert two matrices altogether we have a reduction of the computational costs for computing inverses by a factor $3/8$. Concerning the computation of the matrices our approach requires the single-layer operators and the operators for the normal derivative of the single-layer potential for the two different wave numbers k and $k\sqrt{n}$ whereas for Cossonnière and Haddar's integral equations also the corresponding operators for the double-layer potential are required, that is, in our approach we have a reduction of the computation costs for setting up the matrices by a factor $1/2$. Summarizing, using our integral equation approach reduces the computational costs in the application of Beyn's algorithm by a little more than 50 % as compared with the approach of Cossonnière and Haddar.

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