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THE DETERMINATION OF THE SURFACE CONDUCTIVITY OF A PARTIALLY COATED DIELECTRIC*

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Abstract. A variational method is given for determining the essential supremum of the surface conductivity of a partially coated anisotropic dielectric medium from a knowledge of the far field pattern of the time-harmonic electric field at fixed frequency corresponding to an incident plane wave. It is assumed that the shape of the scatterer has been determined (e.g., by solving the far field equation and using the linear sampling method). Numerical examples are given for the scalar case with constant surface conductivity.

Key words. inverse scattering problem, interior transmission problem, electromagnetic waves, mixed boundary value problems

AMS subject classifications. 35P25, 35R30, 78A45

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1. Introduction.

In a previous paper in this journal [5], we considered the problem of determining the surface impedance of a perfect conductor that is partially coated with a dielectric from a knowledge of the far field pattern of the scattered electromagnetic wave corresponding to an incident time-harmonic plane wave at fixed frequency. Such problems are the simplest model for detecting hostile objects that have been partially coated with a dielectric in order to avoid detection by using such a coating to reduce the radar cross section of the scattered wave. In [5] it was shown that the solution of the far field equation that determines the shape of the scatterer by means of the linear sampling method [11] can also be used in conjunction with a variational method to determine the essential supremum of the surface impedance of the coated portion of the boundary, and numerical examples were given showing the viability of our method.

In this paper we consider the problem complementary to the one described above; i.e., we now wish to detect a benign object that has been partially coated by a thin conducting material in order to make it appear hostile [6]. An example of this is a wooden decoy in the shape of a tank that is partially coated by metallic paint. The problem is again to determine a coefficient (the surface conductivity) in the boundary condition from a knowledge of the far field pattern of the scattered electromagnetic wave corresponding to an incident time-harmonic plane wave. (The shape of the scatterer can again be determined by the linear sampling method.) However, the problem of determining the surface conductivity is considerably more complicated than the problem of determining the surface impedance of a coated perfect conductor since we now have a mixed boundary value problem for a penetrable obstacle. In particular, we now must consider an interior transmission problem with mixed boundary conditions, and the well-posedness of such problems is unknown.

The plan of our paper is as follows. After formulating the mathematical model for the scattering of time harmonic electromagnetic waves by an anisotropic medium

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that is partially coated by a thin conducting layer, we consider the scalar case corresponding to the scattering of electromagnetic waves by an infinite cylinder. We first show that in this case both the direct scattering problem and the interior transmission problem are well posed. We then use these results to derive a variational formula for the determination of the essential supremum of the surface conductivity of the coated portion of the boundary from a knowledge of the far field pattern of the scattered time-harmonic magnetic field. Finally, we derive an analogous formula for determining the surface conductivity in the case of Maxwell's equations in $\mathbb{R}^3$ under the assumption that the interior transmission problem in this case is well posed. We conclude by presenting some numerical examples for the scalar case with constant surface conductivity.

2. Formulation of the direct and inverse scattering problem. We consider the scattering of time-harmonic electromagnetic waves with frequency ω from an infinitely long cylindrical anisotropic dielectric partially coated with a very thin layer of a highly conductive material. We assume that the electric permittivity ϵ_0 and magnetic permeability μ_0 of the exterior dielectric background medium are positive constants, whereas the scatterer has the same magnetic permeability μ_0 as the exterior medium but the electric permittivity ϵ and the conductivity σ are real 3×3 matrix valued functions. After an appropriate scaling [12], the total electric and magnetic fields E, H satisfy the time-harmonic homogeneous Maxwell equations in the exterior of the scatterer,

$$\begin{cases} \nabla \times E - ikH = 0, \\ \nabla \times H + ikE = 0, \end{cases} \tag{2.1}$$

and the interior electric and magnetic fields E_0, H_0 solve the following equations in the interior of the scattering object:

$$\begin{cases} \nabla \times E_0 - ikH_0 = 0, \\ \nabla \times H_0 + ikN(x)E_0 = 0, \end{cases} \tag{2.2}$$

where $k^2 = \epsilon_0 \mu_0 \omega^2$ and the index of refraction is given by $N(x) = \frac{1}{\epsilon_0}\big(\epsilon(x) + i\frac{\sigma(x)}{\omega}\big)$.

Let the real valued function $\eta > 0$ defined on the coated portion of the boundary of the scatterer describe the physical properties of the highly conductive coating (see [1]). As shown in [8], the tangential component of the electric field is continuous across the boundary

$$\nu \times E - \nu \times E_0 = 0, \tag{2.3}$$

while the tangential component of the magnetic field is continuous only on the uncoated part of the boundary

$$\nu \times H - \nu \times H_0 = 0 \tag{2.4}$$

and satisfies the following relation on the coated part of the boundary:

$$\nu \times H - \nu \times H_0 = \eta(x)(\nu \times E) \times \nu. \tag{2.5}$$

The exterior field E, H is given by

$$E = E^i + E^s, \qquad H = H^i + H^s, \tag{2.6}$$

where E^s, H^s is the scattered field satisfying the Silver–Müller radiation condition at infinity [12] and E^i, H^i is the given incident field.

Now we assume that the scatterer is an infinitely long cylinder with axis in the z-direction and that the incident electromagnetic field is a plane wave propagating in the direction perpendicular to the cylinder. Let the bounded domain $D \subset \mathbb{R}^2$ with Lipschitz boundary Γ be the cross section of the cylinder such that the exterior domain $D_e := \mathbb{R}^2 \setminus \overline{D}$ is connected. We denote by ν the outward unit normal to Γ defined almost everywhere on Γ. The boundary $\Gamma = \Gamma_1 \cup \Pi \cup \Gamma_2$ is split into two open disjoint parts Γ_1 and Γ_2 having Π as their possible common boundary in Γ. Here Γ_1 corresponds to the uncoated part and Γ_2 corresponds to the coated part. We assume that the dielectric is orthotropic; i.e., the matrix N is of the form

$$N = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix},$$

and the functions N and η do not depend on z. If we consider incident waves such that the electric field is polarized perpendicular to the z-axis, then the magnetic fields have a component in only the z-direction, i.e.,

$$H^i = (0,\ 0,\ u^i), \quad H_0 = (0,\ 0,\ v), \quad H^s = (0,\ 0,\ u^s).$$

Assuming that N^{-1} exists and expressing the electric fields in terms of magnetic fields, (2.1)–(2.6) now lead to the following transmission problem for v and u:

$$\begin{array}{lll}
\text{(i)} & \nabla\cdot A\nabla v + k^2 v = 0 & \text{in } D, \\
\text{(ii)} & \Delta u + k^2 u = 0 & \text{in } D_e, \\
\text{(iii)} & v - u = 0 & \text{on } \Gamma_1, \\
\text{(iv)} & v - u = -i\eta \dfrac{\partial u}{\partial \nu} & \text{on } \Gamma_2, \\
\text{(v)} & \dfrac{\partial v}{\partial \nu_A} - \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma, \\
\text{(vi)} & u = u^s + u^i, & \\
\text{(vii)} & \lim\limits_{r\to\infty} \sqrt{r}\left(\dfrac{\partial u^s}{\partial r} - iku^s\right) = 0, &
\end{array} \tag{2.7}$$

$r = |x|$, where u^s is the scattered field and u^i is the given incident field. In the case of plane waves the incident field is given by $u^i := e^{ikx\cdot d}$, $d \in \Omega := \{x : |x| = 1\}$. Moreover,

$$\frac{\partial v}{\partial \nu_A}(x) := \nu(x) \cdot A(x)\nabla v(x), \qquad x \in \Gamma,$$

$$A = \frac{1}{n_{11}n_{22} - n_{12}n_{21}} \begin{pmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \end{pmatrix},$$

and the radiation condition (2.7)(vii) holds uniformly with respect to $\hat{x} = x/|x|$. Note that A is not the inverse of a 2×2 submatrix N but rather comes from substituting $H_0 = (0, 0, v)$ into (2.2).

In the following we assume that A is a 2×2 matrix valued function whose entries are continuously differentiable functions in $\overline{D}$ such that A is symmetric, $\mathcal{R}e(\,\bar{\xi} \cdot A\,\xi\,) \geq \gamma|\xi|^2$, and $\mathcal{I}m(\,\bar{\xi} \cdot A\,\xi\,) \leq 0$ for all $\xi \in \mathbb{C}^2$ and $x \in \overline{D}$, where γ is a positive constant. Note that, due to the symmetry of A, we have $\mathcal{R}e(\,\bar{\xi} \cdot A\,\xi\,) = \bar{\xi} \cdot \mathcal{R}e(A)\,\xi$ and $\mathcal{I}m(\,\bar{\xi} \cdot A\,\xi\,) = \bar{\xi} \cdot \mathcal{I}m(A)\,\xi$. Moreover, we require that $\eta \in L_\infty(\Gamma_2)$ and $\eta(x) \geq \eta_0 > 0$ for all $x \in \Gamma_2$.

Let $H^1(D)$ and $H^1_{\text{loc}}(D_e)$ denote the usual Sobolev spaces and $H^{\frac{1}{2}}(\Gamma)$ the corresponding trace space. For $\Gamma_2 \subset \Gamma$ we define

$$H^{\frac{1}{2}}(\Gamma_2) := \{u|_{\Gamma_2} : u \in H^{\frac{1}{2}}(\Gamma)\},$$

$$\tilde{H}^{\frac{1}{2}}(\Gamma_2) := \{u \in H^{\frac{1}{2}}(\Gamma_2) : \operatorname{supp} u \subseteq \overline{\Gamma}_2\},$$

and denote by $H^{-\frac{1}{2}}(\Gamma_2)$ and $\tilde{H}^{-\frac{1}{2}}(\Gamma_2)$ the dual spaces $(\tilde{H}^{\frac{1}{2}}(\Gamma_2))'$ and $(H^{\frac{1}{2}}(\Gamma_2))'$, respectively, with L^2 as a pivot space (for details, see [16]). We recall that a function in $\tilde{H}^{\frac{1}{2}}(\Gamma_2)$ and $\tilde{H}^{-\frac{1}{2}}(\Gamma_2)$ can be extended by zero to a function in $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$, respectively. Note that for $u \in H^1(D)$ with $\Delta u \in L^2(D)$ the trace $\frac{\partial u}{\partial \nu} \in H^{-\frac{1}{2}}(\Gamma)$ is well defined.

For later use we also define the Hilbert space

$$\mathbb{H}^1(D, \Gamma_2) := \left\{ u \in H^1(D) \quad \text{such that} \quad \frac{\partial u}{\partial \nu} \in L^2(\Gamma_2) \right\}$$

equipped with the usual graph norm

$$\|u\|^2_{\mathbb{H}^1(D,\Gamma_2)} := \|u\|^2_{H^1(D)} + \left\| \frac{\partial u}{\partial \nu} \right\|^2_{L^2(\Gamma_2)}.$$

The *forward scattering problem* reads: Given D, A, η as above and the incident field $u^i \in H^1_{\text{loc}}(\mathbb{R}^2)$, find $v \in H^1(D)$ and $u \in H^1_{\text{loc}}(D_e)$ that satisfy (2.7), where the boundary conditions are assumed in the sense of the trace operator. In what follows, we refer to this mixed transmission problem as (MTP).

It is known [12] that solutions of the Helmholtz equation that satisfy the Sommerfeld radiation condition (2.7)(vi) have the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}) + O(r^{-3/2}), \quad r \to \infty, \tag{2.8}$$

where $u_\infty(\hat{x})$ is the *far field pattern* of the radiating solution u^s. In the case of incident plane waves, $u_\infty(\hat{x})$ depends on the incident direction d, which we indicate by $u_\infty(\hat{x}, d)$. The *inverse scattering problem* we are concerned with is to determine D and η from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ of the scattered field u^s for $\hat{x}$, $-d \in \Omega_0$, where Ω_0 is a subset of the unit circle Ω. Note that no a priori knowledge of the amount of coating is required.

3. The direct scattering problem. First we want to show that the mixed transmission problem (2.7) is well posed.

LEMMA 3.1. *The problem* (MTP) *has at most one solution.*

Proof. Let $v \in H^1(D)$ and $u \in H^1_{\mathrm{loc}}(D_e)$ be the solution of (2.7) corresponding to the incident wave $u^i \equiv 0$. Applying Green's formula in D and $D_e \cap B_R$, where B_R is a disk of radius R containing D, and using the transmission conditions, we have

$$\begin{aligned}
&\int_D \left(\nabla \overline{v} \cdot A \nabla v - k^2 |v|^2\right) dy + \int_{D_e \cap B_R} \left(|\nabla u|^2 - k^2 |u|^2\right) dy \\
&= \int_\Gamma \overline{v} \cdot \frac{\partial v}{\partial \nu_A} \, ds - \int_\Gamma \overline{u} \cdot \frac{\partial u}{\partial \nu} \, ds + \int_{S_R} \overline{u} \cdot \frac{\partial u}{\partial \nu} \, ds \\
&= i \int_{\Gamma_2} \frac{1}{\eta} \, |v - u|^2 \, ds + \int_{S_R} \overline{u} \cdot \frac{\partial u}{\partial \nu} \, ds.
\end{aligned}$$

Now taking the imaginary part of both sides and using the fact that $\mathcal{I}m(A) \leq 0$ is a real valued matrix and $\eta \geq \eta_0 > 0$, we obtain

$$\mathcal{I}m \int_{S_R} u \cdot \frac{\partial \overline{u}}{\partial \nu} \, ds \geq 0.$$

Finally, an application of Rellich's lemma and the unique continuation principle yield $u = v = 0$. ∎

In order to give a variational formulation of the problem (MTP) we introduce the Dirichlet-to-Neumann map $\Lambda : H^{\frac{1}{2}}(S_R) \to H^{-\frac{1}{2}}(S_R)$, which maps $h \in H^{\frac{1}{2}}(S_R)$ to $\frac{\partial \tilde{u}}{\partial \nu}$, where $\tilde{u}$ solves the exterior Dirichlet problem for the Helmholtz equation in $\mathbb{R}^2 \backslash \overline{B}_R$ with Dirichlet boundary data h. The following result is known [12], [15].

LEMMA 3.2. *There exists an operator* $\Lambda_0 : H^{\frac{1}{2}}(S_R) \to H^{-\frac{1}{2}}(S_R)$ *such that*

$$\int_{S_R} \overline{\varphi} \, \Lambda_0 \varphi \, ds \leq 0 \tag{3.1}$$

and $\Lambda - \Lambda_0$ *is a compact operator from* $H^{\frac{1}{2}}(S_R)$ *to* $H^{-\frac{1}{2}}(S_R)$.

Integrating by parts the equations of (MTP) with a test function φ, we can put (MTP) into the following variational form: Find $w \in H^1(B_R \backslash \overline{\Gamma}_2)$ such that

$$\begin{aligned}
&\int_D \left(\nabla \overline{\varphi} \cdot A \nabla w - k^2 \overline{\varphi} \, w\right) dy + \int_{D_e \cap B_R} \left(\nabla \overline{\varphi} \cdot \nabla w - k^2 \overline{\varphi} \, w\right) dy \\
&\quad - \int_{\Gamma_2} \frac{i}{\eta} \, [\overline{\varphi}] \cdot [w] \, ds - \int_{S_R} \overline{\varphi} \, \Lambda w \, ds = - \int_{S_R} \overline{\varphi} \, \Lambda u^i \, ds + \int_{S_R} \overline{\varphi} \frac{\partial u^i}{\partial \nu} \, ds
\end{aligned} \tag{3.2}$$

for any function $\varphi \in H^1(B_R \backslash \overline{\Gamma}_2)$, where $[u] = u^+|_{\Gamma_2} - u^-|_{\Gamma_2}$ denotes the jump of u across Γ_2. Note that for $u \in H^1(B_R \backslash \overline{\Gamma}_2)$ the jump $[u] \in \tilde{H}^{\frac{1}{2}}(\Gamma_2)$. Let us denote by $\mathcal{A}_1$ and $\mathcal{A}_2$ the following sesquilinear forms:

$$\begin{aligned}
\mathcal{A}_1(w, \varphi) := &\int_D \left(\nabla \overline{\varphi} \cdot A \nabla w + \overline{\varphi} \, w\right) dy + \int_{D_e \cap B_R} \left(\nabla \overline{\varphi} \cdot \nabla w + \overline{\varphi} \, w\right) dy \\
&- \int_{\Gamma_2} \frac{i}{\eta} \, [\overline{\varphi}] \cdot [w] \, ds - \int_{S_R} \overline{\varphi} \, \Lambda_0 w \, ds
\end{aligned} \tag{3.3}$$

and

$$\mathcal{A}_2(w, \varphi) := - \int_{B_R} (k^2 + 1) \, \overline{\varphi} \, w \, dy - \int_{S_R} \overline{\varphi} \, (\Lambda_0 - \Lambda) w \, ds,$$

respectively. Then (3.2) becomes the following: Find $w \in H^1(B_R\setminus\overline{\Gamma}_2)$ such that

$$\mathcal{A}_1(w,\varphi)+\mathcal{A}_2(w,\varphi)=L(\varphi) \qquad \forall\varphi\in H^1(B_R\setminus\overline{\Gamma}_2), \tag{3.4}$$

where $L(\varphi)$ denotes the continuous antilinear form defined by the right-hand side of (3.2). Obviously if w is a solution of (3.4), then $v := w|_D$ and $u := w|_{B_R\cap D_e}$ satisfy the differential equations and the transmission conditions of (MTP). Then using Green's formula and the radiation condition, one can extend $w = u-u^i$ to a radiating solution of the Helmholtz equation in the exterior domain D_e (see, e.g., [15]).

Next we want to show that there exists a function $w \in H^1(B_R\setminus\overline{\Gamma}_2)$ that satisfies (3.4). The uniqueness of (3.4) is equivalent to the uniqueness of a solution to (MTP) (see Lemma 3.1). Note that, due to (2.7(iv)) and (2.7(v)), if $u \in H^1(D)$ and $v \in H^1_{\mathrm{loc}}(D^e)$ solve (2.7), then $w \in H^1(B_R\setminus\overline{\Gamma}_2)$. Using the classical trace theorems and Cauchy–Schwarz inequality, the chain of continuous imbeddings

$$\tilde{H}^{\frac12}(\Gamma_2)\subset H^{\frac12}(\Gamma_2)\subset L^2(\Gamma_2)\subset \tilde{H}^{-\frac12}(\Gamma_2)\subset H^{-\frac12}(\Gamma_2),$$

and the boundedness of A and η, we obtain

$$|\mathcal{A}_1(w,\varphi)|\le C_1\|w\|_{H^1(B_R\setminus\overline{\Gamma}_2)}\|\varphi\|_{H^1(B_R\setminus\overline{\Gamma}_2)}$$

with $C_1>0$ independent of w and φ. Hence $\mathcal{A}_1$ is bounded. Furthermore, from the fact that $\mathcal{R}e(A)$ is positive definite together with Lemma 3.2, we obtain the following coercivity result:

$$\mathcal{R}e\left(\mathcal{A}_1(w,w)\right)\ge C_2\|w\|^2_{H^1(B_R\setminus\overline{\Gamma}_2)},$$

where the constant $C_2>0$ does not depend on w.

Next, based on the Riesz representation theorem, we define an operator K : $H^1(B_R\setminus\overline{\Gamma}_2)\to H^1(B_R\setminus\overline{\Gamma}_2)$ by

$$(Kw,\ \varphi)=\mathcal{A}_2(w,\ \varphi) \qquad \forall w,\varphi\in H^1(B_R\setminus\overline{\Gamma}_2).$$

The compact embedding of $H^1(B_R\setminus\overline{\Gamma})$ into $L^2(B_R)$ and the compactness of the operator $\Lambda-\Lambda_0$ from Lemma 3.2 imply that the operator K is compact.

The above analysis shows that the Fredholm alternative can be applied to (3.2), which, together with the uniqueness of a solution to (3.2), implies the solvability of (3.2) and therefore the solvability of (2.7). Summarizing the above analysis, we have proved the following theorem.

THEOREM 3.3. *For any incident field $u^i\in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ there exists a unique solution $(v,u)\in H^1(D)\times H^1_{\mathrm{loc}}(D_e)$ of* (MTP) *which depends continuously on u^i.*

3.1. The interior transmission problem. As will be seen in what follows, an important role in solving the inverse problem of determining D and η is played by the *interior transmission problem*: Given $f\in H^{\frac12}(\Gamma)$, $h\in H^{-\frac12}(\Gamma)$, and $r\in L^2(\Gamma_2)$, find $v\in H^1(D)$ and $w\in\mathbb{H}^1(D,\Gamma_2)$ such that

$$\begin{aligned}
&\text{(i)} && \nabla\cdot A\nabla v+k^2\,v=0 && \text{in } D,\\
&\text{(ii)} && \Delta w+k^2\,w=0 && \text{in } D,\\
&\text{(iii)} && v-w=f|_{\Gamma_1} && \text{on } \Gamma_1,\\
&\text{(iv)} && v-w=-i\eta\,\frac{\partial w}{\partial\nu}+f|_{\Gamma_2}+r && \text{on } \Gamma_2,\\
&\text{(v)} && \frac{\partial v}{\partial\nu_A}-\frac{\partial w}{\partial\nu}=h && \text{on } \Gamma.
\end{aligned} \tag{3.5}$$

In the remainder of the paper we will refer to (3.5) as (IMTP). The well-posedness of the interior transmission problem in the case when $\eta \equiv 0$ and $r = 0$ is established in [7]. Here we will adapt the variational approach used in [7] to our mixed transmission case. In order to avoid repetition we will only sketch the proof, emphasizing the changes due to the boundary terms involving η. We first modify (IMTP) to

$$
\begin{array}{lll}
\text{(i)} & \nabla\cdot A\nabla v - m\,v = \ell_1 & \text{in } D,\\
\text{(ii)} & \Delta w - \,w = \ell_2 & \text{in } D,\\
\text{(iii)} & v - w = f|_{\Gamma_1} & \text{on } \Gamma_1,\\
\text{(iv)} & v - w = -i\eta\,\dfrac{\partial w}{\partial \nu} + f|_{\Gamma_2} + r & \text{on } \Gamma_2,\\
\text{(v)} & \dfrac{\partial v}{\partial \nu_A} - \dfrac{\partial w}{\partial \nu} = h & \text{on } \Gamma,
\end{array}
\tag{3.6}
$$

where $m > 0$, $\ell_1 \in L^2(D)$, and $\ell_2 \in L^2(D)$. We will now reformulate (3.6) as an equivalent variational problem. To this end let

$$
\mathbb{W}(D) = \{\mathbf{w} \in L^2(D)^2 \;:\; \nabla\cdot \mathbf{w} \in L^2(D), \;\; \text{and } \operatorname{curl}\mathbf{w} = 0 \text{ and } \nu\cdot\mathbf{w}|_{\Gamma_2} \in L^2(\Gamma_2)\}
$$

equipped with the natural norm

$$
\|\mathbf{w}\|^2_{\mathbb{W}} = \|\mathbf{w}\|^2_{L^2} + \|\nabla\cdot\mathbf{w}\|^2_{L^2} + \|\nu\cdot\mathbf{w}\|^2_{L^2},
$$

and denote by $\langle\cdot\,,\,\cdot\rangle$ the duality pairing between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$. We also introduce the duality identity

$$
\langle\varphi,\;\boldsymbol{\psi}\cdot\nu\rangle = \int_D \varphi\,\nabla\cdot\boldsymbol{\psi}\,dx + \int_D \nabla\varphi\cdot\boldsymbol{\psi}\,dx
\tag{3.7}
$$

for $(\varphi,\,\boldsymbol{\psi}) \in H^1(D)\times\mathbb{W}(D)$.

By doing exactly the same as in the proof of Theorem 3.3 in [7], one can show that the modified interior transmission problem (3.6) is equivalent to the following variational problem: Find $V = (v,\mathbf{w}) \in H^1(D)\times\mathbb{W}(D)$ such that

$$
\mathcal{A}(V,\Psi) = L(\Psi), \qquad \Psi \in H^1(D)\times\mathbb{W}(D),
\tag{3.8}
$$

where the sesquilinear form $\mathcal{A}$ defined in $(H^1(D)\times\mathbb{W}(D))^2$ is given by

$$
\begin{aligned}
\mathcal{A}(V,\Psi) = {}& \int_D A\nabla v\cdot\nabla\bar{\varphi}\,dx + \int_D m\,v\,\bar{\varphi}\,dx + \int_D \nabla\cdot\mathbf{w}\,\nabla\cdot\bar{\boldsymbol{\psi}}\,dx + \int_D \mathbf{w}\cdot\bar{\boldsymbol{\psi}}\,dx\\
& - i\int_{\Gamma_2}\eta\,(\mathbf{w}\cdot\nu)\,(\bar{\boldsymbol{\psi}}\cdot\nu),ds - \big\langle v,\,\bar{\boldsymbol{\psi}}\cdot\nu\big\rangle - \langle\bar{\varphi},\,\mathbf{w}\cdot\nu\rangle
\end{aligned}
\tag{3.9}
$$

and the antilinear form L is given by

$$
L(\Psi) = \int_D (\ell_1\,\bar{\varphi} + \ell_2\,\nabla\cdot\bar{\boldsymbol{\psi}})\,dx - i\int_{\Gamma_2}\eta\,r\,(\bar{\boldsymbol{\psi}}\cdot\nu) + \langle\bar{\varphi},\,h\rangle - \big\langle f,\,\bar{\boldsymbol{\psi}}\cdot\nu\big\rangle\,.
$$

The modified interior transmission problem (3.6) has a unique solution $(v,w) \in H^1(D)\times\mathbb{H}^1(D,\Gamma_2)$ if and only if the variational problem (3.8) has a unique solution $V \in H^1(D)\times W(D)$. If (v,w) is the unique solution (3.6), then $V = (v,\nabla w)$ is a unique solution to (3.8). Conversely if V is the unique solution to (3.8), then the unique solution (v, w) to (3.6) is such that $V = (v, \nabla w)$.

Now assume that there exists a constant $\gamma > 1$ such that $\bar{\xi} \cdot \mathcal{R}e(A)\, \xi \geq \gamma |\xi|^2$ and choose $m > 1$. Classical trace theorems and Schwarz's inequality ensure the continuity of the sesquilinear form $\mathcal{A}$ and the antilinear form L. On the other hand, by taking the real and the imaginary part of $\mathcal{A}(V, V)$, we have from the assumptions on $\mathcal{R}e(A)$, $\mathcal{I}m(A)$, and η that

$$|\mathcal{A}(V,V)| \geq \gamma \|v\|^2_{H^1(D)} + \|\mathbf{w}\|^2_{L^2(D)} + \|\nabla \cdot \mathbf{w}\|^2_{L^2(D)} - 2\mathcal{R}e(\langle \bar{v},\, \nu \cdot \mathbf{w} \rangle) + \eta_0 \|\nu \cdot \mathbf{w}\|^2_{L^2(\Gamma_2)}.$$

From the duality identity (3.7) and Schwarz's inequality we have

$$2\mathcal{R}e(\langle \bar{v},\, \nu \cdot \mathbf{w} \rangle) \leq |\, \langle \bar{v},\, \mathbf{w} \rangle \,| \leq \|v\|_{H^1(D)} (\|\mathbf{w}\|^2_{L^2(D)} + \|\nabla \cdot \mathbf{w}\|^2_{L^2(D)})^{\frac{1}{2}}.$$

Hence since $\gamma > 1$, we conclude that

$$|\mathcal{A}(V,V)| \geq \frac{\gamma - 1}{\gamma + 1} (\|v\|^2_{H^1(D)} + \|\mathbf{w}\|^2_{L^2(D)} + \|\nabla \cdot \mathbf{w}\|^2_{L^2(D)}) + \eta_0 \|\nu \cdot \mathbf{w}\|^2_{L^2(\Gamma_2)},$$

which means that $\mathcal{A}$ is coercive; i.e.,

$$|\mathcal{A}(V,V)| \geq C(\|v\|^2_{H^1(D)} + \|\mathbf{w}\|^2_{W(D)}),$$

where $C = \min((\gamma - 1)/(\gamma + 1),\, \eta_0)$. Therefore from the Lax–Milgram theorem we have that the variational problem (3.8) is uniquely solvable, whence the modified interior transmission problem has a unique solution (u, v) that satisfies

$$\|v\|_{H^1(D)} + \|w\|_{\mathbb{H}^1(D,\Gamma_2)} \leq C(\|f\|_{H^{\frac{1}{2}}(\Gamma)} + \|h\|_{H^{-\frac{1}{2}}(\Gamma)} + \|r\|_{L^2(\Gamma_2)}),$$

where $C > 0$ is independent on f, h, r.

THEOREM 3.4. *Assume that* $\bar{\xi} \cdot \mathcal{R}e(A)\, \xi \geq \gamma |\xi|^2$ *with* $\gamma > 1$ *and* $\eta(x) \geq \eta_0 > 0$. *Then the Fredholm alternative can be applied to the problem* (IMTP).

Proof. Let us define

$$\mathcal{Y}(D) := \big\{(v, w) \in H^1(D) \times \mathbb{H}^1(D, \Gamma_2) \;:\; \nabla \cdot A \nabla v \in L^2(D) \text{ and } \Delta w \in L^2(D)\big\}$$

and consider the operator $\mathcal{G}$ from $\mathcal{Y}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\Gamma_1) \times L^2(\Gamma_2) \times H^{-\frac{1}{2}}(\Gamma)$ defined by

$$\mathcal{G}(v, w) = \left\{ \nabla \cdot A \nabla v - mv,\, \Delta w - w,\, (v - w)|_{\Gamma_1}, \left(v - w + i\eta \frac{\partial w}{\partial \nu} \right)_{\Gamma_2}, \left(\frac{\partial v}{\partial \nu_A} - \frac{\partial w}{\partial \nu} \right)_{\Gamma} \right\},$$

where $m > 1$. We have shown that the inverse of $\mathcal{G}$ exists and is continuous. Since $\mathcal{G}$ is continuous, we deduce that $\mathcal{G}$ is a bijective operator. Now consider the operator $\mathcal{T}$ from $\mathcal{Y}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\Gamma_1) \times L^2(\Gamma_2) \times H^{-\frac{1}{2}}(\Gamma)$ defined by

$$\mathcal{T}(v, w) = \{(k^2 + m)v,\; (k^2 + 1)w,\; 0,\; 0,\; 0\}.$$

By the compact embedding of $H^1(D)$ into $L^2(D)$, the operator $\mathcal{T}$ is compact. Hence $\mathcal{G} + \mathcal{T}$ is a Fredholm operator of index one, which proves the theorem. ∎

By modifying the variational approach of [9] in a similar way, one can also prove the following result.

THEOREM 3.5. *Assume that* $\bar{\xi}\cdot\mathcal{R}e(A^{-1})\,\xi \geq \gamma|\xi|^2$ *with* $\gamma > 1$. *Then the Fredholm alternative can be applied to the problem* (IMTP).

LEMMA 3.6. *Assume that* $\bar{\xi}\cdot\mathcal{I}m(A)\,\xi < 0$ *at a point* $x_0 \in D$ *and* $\eta \geq \eta_0 > 0$ *on* Γ_2. *Then* (IMTP) *has at most one solution.*

Proof. Let us consider the homogeneous problem (i.e., $f = h = r = 0$). Applying the divergence theorem to $\overline{v}$ and $A\nabla v$, making use of the boundary conditions, and applying Green's theorem for $\overline{w}$ and w, we obtain

$$\int_D \nabla\overline{v}\cdot A\nabla v\,dy - \int_D k^2|v|^2\,dy = \int_D |\nabla w|^2\,dy - \int_D k^2|w|^2\,dy + \int_{\Gamma_2} i\eta\left|\frac{\partial w}{\partial\nu}\right|^2\,ds.$$

Hence

$$\mathcal{I}m\left(\int_D \nabla\overline{v}\cdot A\nabla v\,dy\right) = 0 \qquad \text{and} \qquad \int_{\Gamma_2}\eta\left|\frac{\partial w}{\partial\nu}\right|^2\,ds = 0.$$

Since $\bar{\xi}\cdot\mathcal{I}m(A)\xi < 0$ in a small ball $B_{x_0}\subset D$, from the first equality we obtain that $\nabla v = 0$ in B_{x_0}, whence $v \equiv 0$ in D since the unique continuation principle holds for (3.5)(i). From the boundary conditions and the integral representation, formula w also vanishes in D. ∎

We summarize the above analysis in the following theorem.

THEOREM 3.7. *Assume that* $\bar{\xi}\cdot\mathcal{I}m(A)\,\xi < 0$ *at a point* $x_0 \in D$ *and* $\eta \geq \eta_0 > 0$. *In addition, assume that there exists a constant* $\gamma > 1$ *such that*

$$\textit{either}\quad \bar{\xi}\cdot\mathcal{R}e(A)\,\xi \geq \gamma|\xi|^2 \quad \textit{or} \quad \bar{\xi}\cdot\mathcal{R}e(A^{-1})\,\xi \geq \gamma|\xi|^2 \qquad \forall\,\xi\in\mathbb{C}^2.$$

Then the interior transmission problem (IMTP) *has a unique solution* (v, w) *which satisfies*

$$\|v\|^2_{H^1(D)} + \|w\|^2_{\mathbb{H}^1(D,\Gamma_2)} \leq C\left(\|f\|_{H^{\frac{1}{2}}(\Gamma)} + \|h\|_{H^{-\frac{1}{2}}(\Gamma)} + \|r\|_{L^2(\Gamma_2)}\right). \tag{3.10}$$

The values of k for which (IMTP) is not uniquely solvable are called the *transmission eigenvalues*. The latter may occur, for example, if $\bar{\xi}\cdot\mathcal{I}m(A)\,\xi = 0$ in D. In this case, from the proof of Lemma 3.6 we obtain that $\frac{\partial w}{\partial\nu} = 0$ on Γ_2, whence the eigenvalues of (IMTP) form a subset of the transmission eigenvalues corresponding to the (usual) interior transmission problem discussed in [7]. Moreover, if $\Gamma_2 = \Gamma$, then the eigenvalues of (IMTP) form a subset of the Neumann eigenvalues of $-\nabla\cdot A\nabla$.

4. The inverse problem. The inverse problem that we consider here is to determine *both* the shape of the scattering object D and the surface conductivity η from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for all incident plane waves $u^i := e^{ikx\cdot d}$, $d \in \Omega$, and all observation directions $\hat{x}\in\Omega$. (Note that it suffices to know the far field pattern corresponding to all $d\in\Omega_1\subset\Omega$ and all $\hat{x}\in\Omega_2\subset\Omega$; of particular interest is the case $d = -\hat{x}\in\Omega_0\subset\Omega$.) We start the investigation of the inverse problem by stating a uniqueness theorem for determining the support D.

THEOREM 4.1. *Let the domains* D^1 *and* D^2 *with the boundaries* Γ^1 *and* Γ^2, *respectively; the matrix valued functions* A_1 *and* A_2; *and the functions* η_1 *and* η_2 *determined on the portions* $\Gamma_2^1 \subseteq \Gamma^1$ *and* $\Gamma_2^2 \subseteq \Gamma^2$, *respectively* (*either* Γ_2^1 *or* Γ_2^2 *or both can possibly be empty sets*), *satisfy the assumptions of* (MTP) *in section* 2.

Moreover, let us assume that either $\bar{\xi} \cdot \Re(A_1)\, \xi \geq \gamma|\xi|^2$ *or* $\bar{\xi} \cdot \Re(A_1^{-1})\, \xi \geq \gamma|\xi|^2$, *and either* $\bar{\xi} \cdot \Re(A_2)\, \xi \geq \gamma|\xi|^2$ *or* $\bar{\xi} \cdot \Re(A_2^{-1})\, \xi \geq \gamma|\xi|^2$ *for some* $\gamma > 1$. *If the far field patterns* $u^1_\infty(\hat{x}, d)$ *corresponding to the data* D^1, A_1, η_1 *and* $u^2_\infty(\hat{x}, d)$ *corresponding to the data* D^2, A_2, η_2 *coincide for all* $\hat{x}, d \in \Omega$, *then* $D^1 \equiv D^2$.

This theorem is proved in [8] for the case of Maxwell's equations in $\mathbb{R}^3$. In the scalar case under consideration, one can adapt the approach of Hähner in [15] to prove the above theorem. Note that the main ingredient of Hähner's approach is the well-posedness of the (modified) interior transmission problem investigated in section 3.1.

The next question to ask is the uniqueness of the surface conductivity η. From the above theorem we can now assume that D is known. Furthermore, we require that for an arbitrarily choice of Γ_2, A, and η there exist at least one incident plane wave such that the corresponding total field u satisfies $\frac{\partial u}{\partial \nu}\big|_{\Gamma_0} \neq 0$, where $\Gamma_0 \subset \Gamma$ is an arbitrary portion of Γ. In the context of our application this is a reasonable assumption since otherwise the portion of the boundary where $\frac{\partial u}{\partial \nu}\big|_{\Gamma_0} = 0$ for all incident plane waves will behave like a perfect conductor, contrary to the assumption that the metallic coating is thin enough for the incident field to penetrate into D. We can prove the following result.

THEOREM 4.2. *Assume that* $\eta \in C(\overline{\Gamma}_2)$ *and that* k *is not a Neumann eigenvalue for* $-\nabla \cdot A\nabla$. *Then, under the above assumption and for fixed* D *and* A, *the surface conductivity* η *is uniquely determined from the far field pattern* $u_\infty(\hat{x}, d)$ *for all* $\hat{x}, d \in \Omega$.

Proof. Let D and A be fixed, and suppose there exist $\eta_1 \in C(\overline{\Gamma}^1_2)$ and $\eta_2 \in C(\overline{\Gamma}^2_2)$ such that the corresponding scattered fields $u^{s,1}$ and $u^{s,2}$, respectively, have the same far field patterns $u^1_\infty(\hat{x}, d) = u^2_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \Omega$. Then from Rellich's lemma, $u^{s,1} = u^{s,2}$ in $\mathbb{R}^2 \backslash D$. Hence, from the transmission condition, the difference $V = v^1 - v^2$ satisfies

$$\nabla\cdot A\nabla V + k^2\, V = 0 \quad \text{in } D, \tag{4.1}$$

$$\frac{\partial V}{\partial \nu_A} = 0 \quad \text{on } \Gamma, \tag{4.2}$$

$$V = -i(\tilde{\eta}_1 - \tilde{\eta}_2)\frac{\partial u^1}{\partial \nu} \quad \text{on } \Gamma, \tag{4.3}$$

where $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are the extension by zero of η_1 and η_2, respectively, to the whole of Γ and $u^1 = u^{s,1} + u^i$. Assuming that k is not a Neumann eigenvalue for $-\nabla\cdot A\nabla$ (in particular, this is the case if $\mathcal{I}m(A) < 0$ at $x_0 \in D$, (4.1)), (4.2) implies that $V = 0$ in D, and hence (4.3) becomes

$$(\tilde{\eta}_1 - \tilde{\eta}_2)\frac{\partial u^1}{\partial \nu} = 0 \quad \text{on} \quad \Gamma$$

for all incident waves. Since for a given $\Gamma_0 \subset \Gamma$ there exists at least one incident plane wave such that $\frac{\partial u^1}{\partial \nu}\big|_{\Gamma_0} \neq 0$, the continuity of η_1 and η_2 in $\overline{\Gamma}^1_2$ and $\overline{\Gamma}^2_2$, respectively, implies that $\tilde{\eta}_1 = \tilde{\eta}_2$. ∎

We now define the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$ by

$$Fg(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d) \tag{4.4}$$

and introduce the far field equation

$$(Fg)(\hat{x}) = \gamma e^{-ik\hat{x}\cdot z}, \quad g \in L^2(\Omega), \quad z \in D, \tag{4.5}$$

where $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ and $\gamma e^{-ik\hat{x}\cdot z}$ is the far field pattern of the fundamental solution $\Phi(x,z) := \frac{i}{4}H_0^{(1)}(k|x-z|)$ to the Helmholtz equation in $\mathbb{R}^2$, with $H_0^{(1)}$ being a Hankel function of the first kind of order zero. A reconstruction of D can be obtained by using the linear sampling method which characterizes the support D from a solution of the far field equation (4.5) (see, e.g., [3], [7]). Assuming that D is known, our goal is to provide a formula for computing the L_∞ norm of η in terms of the solution of the far field equation (4.5).

To this end, assuming that k is not a transmission eigenvalue, for $z \in D$ we denote by v_z and w_z the unique solution of the interior transmission problem

$$\begin{aligned}
&\nabla\cdot A\nabla v_z + k^2\, v_z = 0 && \text{in } D,\\
&\Delta w_z + k^2\, w_z = 0 && \text{in } D,\\
&v_z - (w_z + \Phi(\cdot,z)) = 0 && \text{on } \Gamma_1,\\
&v_z - (w_z + \Phi(\cdot,z)) = -i\eta\,\frac{\partial}{\partial\nu}(w_z + \Phi(\cdot,z)) && \text{on } \Gamma_2,\\
&\frac{\partial v_z}{\partial \nu_A} - \frac{\partial}{\partial\nu}(w_z + \Phi(\cdot,z)) = 0 && \text{on } \Gamma.
\end{aligned} \tag{4.6}$$

We recall that a Herglotz wave function with kernel $g \in L^2(\Omega)$ is an entire solution of the Helmholtz equation defined by

$$v_g(x) = \int_\Omega e^{ikx\cdot d} g(d)ds(d), \qquad x \in \mathbb{R}^2. \tag{4.7}$$

The following theorem holds.

THEOREM 4.3. *Assume that k is not a transmission eigenvalue. Let $\epsilon > 0$, $z \in D$, and (w_z, v_z) be the unique solution of* (4.6)*. Then there exists a Herglotz wave function $v_{g_\epsilon^z}$ with kernel $g_\epsilon^z \in L^2(\Omega)$ such that*

$$\|w_z - v_{g_\epsilon^z}\|_{\mathbb{H}^1(D,\Gamma_2)} \le \epsilon. \tag{4.8}$$

Moreover, there exists a positive constant $c > 0$ independent of ϵ and z such that

$$\|(Fg_\epsilon^z)(\hat{x}) - \gamma e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} \le c\epsilon. \tag{4.9}$$

Proof. To prove the first part of the theorem we first show that the operator $\mathcal{H} : L^2(\Omega) \to H^{\frac{1}{2}}(\Gamma_1) \times L^2(\Gamma_2)$ defined by

$$(\mathcal{H}g)(x) := \begin{cases} \displaystyle\int_\Omega e^{-iky\cdot\hat{x}} g(\hat{x})ds(\hat{x}), & y \in \Gamma_1,\\ \displaystyle\frac{\partial}{\partial\nu}\int_\Omega e^{-iky\cdot\hat{x}} g(\hat{x})ds(\hat{x}) + i\int_\Omega e^{-iky\cdot\hat{x}} g(\hat{x})ds(\hat{x}), & y \in \Gamma_2, \end{cases} \tag{4.10}$$

has dense range. To this end it suffices to show that the corresponding dual operator $\mathcal{H}^* : \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \times L^2(\Gamma_2) \to L^2(\Omega)$ defined by

$$\langle \mathcal{H}g,\, \phi\rangle_{H^{\frac{1}{2}}(\Gamma_1),\tilde{H}^{-\frac{1}{2}}(\Gamma_1)} + \langle \mathcal{H}g,\, \psi\rangle_{L^2(\Gamma_2),L^2(\Gamma_2)} = \langle g,\, \mathcal{H}^*(\phi,\,\psi)\rangle_{L^2(\Omega),L^2(\Omega)}$$

for all $g \in L^2(\Omega)$, $\phi \in \tilde{H}^{-\frac{1}{2}}(\Gamma_1)$, $\psi \in L^2(\Gamma_2)$ is injective. By interchanging the order of integration, one can show that

$$\mathcal{H}^*(\phi,\,\psi)(\hat{x}) = \int_\Gamma e^{-iky\cdot\hat{x}}\tilde{\phi}(y)\,ds(y) + \int_\Gamma \frac{\partial e^{-iky\cdot\hat{x}}}{\partial\nu}\tilde{\psi}(y)\,ds(y) + i\int_\Gamma e^{-iky\cdot\hat{x}}\tilde{\psi}(y)\,ds(y),$$

where $\tilde{\phi} \in H^{-\frac{1}{2}}(\Gamma)$ and $\tilde{\psi} \in L^2(\Gamma)$ are the extension by zero to the whole boundary Γ of ϕ and ψ, respectively. Assume that $\mathcal{H}^*(\phi, \psi) = 0$. Since $\mathcal{H}^*(\phi, \psi)$ is, up to a constant, the far field pattern of the potential

$$P(x) = \int_\Gamma \Phi(x,y)\tilde{\phi}(y)\, ds(y) + \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu}\tilde{\psi}(y)\, ds(y) + i\int_\Gamma \Phi(x,y)\tilde{\psi}(y)\, ds(y),$$

which satisfies the Helmholtz equation in D_e, from Rellich's lemma we have that $P(x) = 0$ in D_e. As $x \to \Gamma$, the following jump relations (in the L^2 limit sense [12], [16]) hold:

$$P^+ - P^-|_{\Gamma_1} = 0, \qquad\qquad P^+ - P^-|_{\Gamma_2} = \psi,$$

$$\left.\frac{\partial P^+}{\partial \nu} - \frac{\partial P^-}{\partial \nu}\right|_{\Gamma_1} = -\phi, \qquad\qquad \left.\frac{\partial P^+}{\partial \nu} - \frac{\partial P^-}{\partial \nu}\right|_{\Gamma_2} = -i\psi,$$

where by the superscript + and − we distinguish the limits obtained by approaching the boundary Γ from D^e and D, respectively. Using the fact that $P^+ = \frac{\partial P^+}{\partial \nu} = 0$, we see that P satisfies the Helmholtz equation and

$$P^-|_{\Gamma_1} = 0, \qquad \left.\frac{\partial P^-}{\partial \nu} + iP^-\right|_{\Gamma_2} = 0,$$

where the equalities are understood in the L^2 limit sense. Using Green's theorem and a parallel surface argument, one can conclude as in Theorem 2.1 in [3] that $P = 0$ in D, whence from the above jump relations $\phi = \psi = 0$.

Now let $w \in \mathbb{H}^1(D, \Gamma_2)$ be a solution of the Helmholtz equation in D. From the above we can approximate $w|_{\Gamma_1} \in H^{\frac{1}{2}}(\Gamma_1)$ and $\frac{\partial w}{\partial \nu} + iw|_{\Gamma_2} \in L^2(\Gamma_2)$ by $\mathcal{H}g$. Hence using the a priori estimate for the solution of the mixed boundary value problem for the Helmholtz equation (see Theorem 2.3 in [3]),

$$\|w\|_{H^1(D)} + \left\|\frac{\partial w}{\partial \nu}\right\|_{L^2(\Gamma_2)} \le C\left(\|w\|_{H^{\frac{1}{2}}(\Gamma_1)} + \left\|\frac{\partial w}{\partial \nu} + iw\right\|_{L^2(\Gamma_2)}\right),$$

we obtain that w can be approximated by a Herglotz wave function v_g with respect to the $\mathbb{H}^1(D, \Gamma_2)$-norm, which proves the first part of the theorem. Note that, by a change of variable, v_g defined by (4.7) can be written as $\int_\Omega e^{-ikx\cdot d} g(d) ds(d)$.

Next let $z \in D$. Then $\gamma e^{-ik\hat{x}\cdot z}$ is the far field pattern of the radiating solution $\Phi(x, z)$. Let w_z and v_z be the unique solution of (4.6). Obviously v_z and $\Phi(x, z)$ satisfy (MTP) with incident field the $H^1_{\text{loc}}(\mathbb{R})$-extension of w_z. The well-posedness of (MTP) (section 3) together with the classical trace theorems and the approximation of w_z by a Herglotz wave function $v_{g^z_\epsilon}$ show that for every $\epsilon > 0$

$$\|(Fg^z_\epsilon)(\hat{x}) - \gamma e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} \le c_1\|u^s_{g^z_\epsilon} - \Phi(\cdot, z)\|_{H^1(D_e\cap B_R)} \le c\|v_{g^z_\epsilon} - w_z\|_{H^1(D)} \le c\epsilon$$

for $c_1, c > 0$, where $u^s_{g^z_\epsilon}$ is the scattered field corresponding to $v_{g^z_\epsilon}$ as the incident wave. (Note that by superposition Fg^z_ϵ coincides with $u^s_{g^z_\epsilon}$.) This ends the proof. □

Now let us define W_z by

$$W_z := w_z + \Phi(\cdot, z). \tag{4.11}$$

In particular, since $w_z \in \mathbb{H}^1(D, \Gamma_2)$, $\Delta w_z \in L^2(D)$, and $z \in D$, we have that $W_z|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$, $\frac{\partial W_z}{\partial \nu}|_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$, and $\frac{\partial W_z}{\partial \nu}|_{\Gamma_2} \in L^2(\Gamma_2)$.

LEMMA 4.4. *For every two points z_1 and z_2 in D we have that*

$$-2\int_D \nabla v_{z_1}\cdot \mathcal{I}m(A)\nabla \overline{v}_{z_2}\,dx + 2\int_{\Gamma_2}\eta(x)\frac{\partial W_{z_1}}{\partial\nu}\frac{\partial \overline{W}_{z_2}}{\partial\nu}\,ds = -4k\pi|\gamma|^2 J_0(k|z_1-z_2|) + i\left(w_{z_1}(z_2)-\overline{w}_{z_2}(z_1)\right), \tag{4.12}$$

where w_{z_1}, W_{z_1} and w_{z_2}, W_{z_2} are defined by (4.6) *and* (4.11), *respectively, and J_0 is a Bessel function of order zero.*

Proof. Let z_1 and z_2 be two points in D and v_{z_1}, w_{z_1}, W_{z_1} and v_{z_2}, w_{z_2}, W_{z_2} the corresponding functions defined by (4.6) and (4.11). Applying the divergence theorem to $v_{z_1}, \overline{v}_{z_2}$ and using (4.6) together with the fact that A is symmetric, we have

$$\int_\Gamma\left(v_{z_1}\frac{\partial\overline{v}_{z_2}}{\partial\nu_{\overline{A}}} - \overline{v}_{z_2}\frac{\partial v_{z_1}}{\partial\nu_A}\right)ds = \int_D\left(\nabla v_{z_1}\cdot\overline{A}\nabla\overline{v}_{z_2} - \nabla\overline{v}_{z_2}\cdot A\nabla v_{z_1}\right)dx$$
$$+\int_D\left(v_{z_1}\nabla\cdot\overline{A}\nabla\overline{v}_{z_2} - \overline{v}_{z_2}\nabla\cdot A\nabla v_{z_1}\right)dx = -2i\int_D\nabla v_{z_1}\cdot\mathcal{I}m(A)\nabla\overline{v}_{z_2}\,dx.$$

On the other hand, from the boundary conditions we have

$$\int_\Gamma\left(v_{z_1}\frac{\partial\overline{v}_{z_2}}{\partial\nu_{\overline{A}}} - \overline{v}_{z_2}\frac{\partial v_{z_1}}{\partial\nu_A}\right)ds$$
$$= \int_\Gamma\left(W_{z_1}\frac{\partial\overline{W}_{z_2}}{\partial\nu} - \overline{W}_{z_2}\frac{\partial W_{z_1}}{\partial\nu}\right)ds - 2i\int_{\Gamma_2}\eta(x)\frac{\partial W_{z_1}}{\partial\nu}\frac{\partial\overline{W}_{z_2}}{\partial\nu}\,ds.$$

Hence

$$-2i\int_D\nabla v_{z_1}\cdot\mathcal{I}m(A)\nabla\overline{v}_{z_2}\,dx + 2i\int_{\Gamma_2}\eta(x)\frac{\partial W_{z_1}}{\partial\nu}\frac{\partial\overline{W}_{z_2}}{\partial\nu}\,ds$$
$$= \int_\Gamma\left(W_{z_1}\frac{\partial\overline{W}_{z_2}}{\partial\nu} - \overline{W}_{z_2}\frac{\partial W_{z_1}}{\partial\nu}\right)ds = \int_\Gamma\left(\Phi(\cdot,z_1)\frac{\partial\overline{\Phi(\cdot,z_2)}}{\partial\nu} - \overline{\Phi(\cdot,z_2)}\frac{\partial\Phi(\cdot,z_1)}{\partial\nu}\right)ds$$
$$+\int_\Gamma\left(w_{z_1}\frac{\partial\overline{\Phi(\cdot,z_2)}}{\partial\nu} - \overline{\Phi(\cdot,z_2)}\frac{\partial w_{z_1}}{\partial\nu}\right)ds + \int_\Gamma\left(\Phi(\cdot,z_1)\frac{\partial\overline{w}_{z_2}}{\partial\nu} - \overline{w}_{z_2}\frac{\partial\Phi(\cdot,z_1)}{\partial\nu}\right)ds.$$

Green's theorem applied to the radiating solution $\Phi(\cdot,z)$ of the Helmholtz equation in D_e implies that [13]

$$\int_\Gamma\left(\Phi(\cdot,z_1)\frac{\partial\overline{\Phi(\cdot,z_2)}}{\partial\nu} - \overline{\Phi(\cdot,z_2)}\frac{\partial\Phi(\cdot,z_1)}{\partial\nu}\right)ds = -2ik\int_\Omega\Phi_\infty(\cdot,z_1)\overline{\Phi_\infty(\cdot,z_2)}ds$$
$$= -2ik\int_\Omega|\gamma|^2 e^{-ik\hat{x}\cdot z_1}e^{ik\hat{x}\cdot z_2}\,ds = -4ik\pi|\gamma|^2 J_0(k|z_1-z_2|),$$

and from the representation formula for w_{z_1} and w_{z_2} we now obtain

$$-2i\int_D\nabla v_{z_1}\cdot\mathcal{I}m(A)\nabla\overline{v}_{z_2}\,dx + 2i\int_{\Gamma_2}\eta(x)\frac{\partial W_{z_1}}{\partial\nu}\frac{\partial\overline{W}_{z_2}}{\partial\nu}\,ds$$
$$= -4ik\pi|\gamma|^2 J_0(k|z_1-z_2|) + \overline{w}_{z_2}(z_1) - w_{z_1}(z_2).$$

Dividing both sides of the above relation by i yields the result. $\square$

In the following we consider a ball $B_r \subset D$ of radius r contained in D and define a subset of $L^2(\Gamma_2)$ by

$$\mathcal{V} := \left\{ f \in L^2(\Gamma_2) : \quad \begin{array}{c} f = \left.\dfrac{\partial W_z}{\partial \nu}\right|_{\Gamma_2} \text{ with } W_z = w_z + \Phi(\cdot, z), \\ z \in B_r \text{ and } w_z, v_z \text{ the solution of (4.6).} \end{array} \right\}$$

LEMMA 4.5. *Assuming that k is neither a transmission eigenvalue nor a Neumann eigenvalue for $-\nabla \cdot A\nabla$, then $\mathcal{V}$ is complete in $L^2(\Gamma_2)$.*

Proof. Let φ be a function in $L^2(\Gamma_2)$ such that for every $z \in B_r$

$$\int_{\Gamma_2} \frac{\partial W_z}{\partial \nu} \varphi \, ds = 0.$$

Construct $v \in H^1(D)$ and $w \in \mathbb{H}^1(D, \Gamma_2)$ as the unique solution of the interior transmission problem

$$\begin{array}{rll} \text{(i)} & \nabla \cdot A\nabla v + k^2 v = 0 & \text{in } D, \\ \text{(ii)} & \Delta w + k^2 w = 0 & \text{in } D, \\ \text{(iii)} & v - w = 0 & \text{on } \Gamma_1, \\ \text{(iv)} & v - w = -i\eta \dfrac{\partial w}{\partial \nu} + \varphi & \text{on } \Gamma_2, \\ \text{(v)} & \dfrac{\partial v}{\partial \nu_A} - \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma. \end{array}$$

Then we have

$$\begin{aligned} 0 &= \int_{\Gamma_2} \frac{\partial W_z}{\partial \nu} \varphi \, ds = \int_{\Gamma} \frac{\partial W_z}{\partial \nu} (v - w) \, ds + i \int_{\Gamma_2} \eta \frac{\partial W_z}{\partial \nu} \frac{\partial w}{\partial \nu} \, ds \\ &= \int_{\Gamma} \frac{\partial W_z}{\partial \nu} v \, ds - \int_{\Gamma} \frac{\partial W_z}{\partial \nu} w \, ds + i \int_{\Gamma_2} \eta \frac{\partial W_z}{\partial \nu} \frac{\partial w}{\partial \nu} \, ds. \end{aligned} \tag{4.13}$$

Next from the equations for v_z and v, the divergence theorem, and the transmission conditions, we have

$$\begin{aligned} \int_{\Gamma} \frac{\partial W_z}{\partial \nu} v \, ds &= \int_{\Gamma} \frac{\partial v_z}{\partial \nu_A} v \, ds = \int_{\Gamma} \frac{\partial v}{\partial \nu_A} v_z \, ds \\ &= \int_{\Gamma} \frac{\partial w}{\partial \nu} W_z \, ds - i \int_{\Gamma_2} \eta \frac{\partial W_z}{\partial \nu} \frac{\partial w}{\partial \nu} \, ds. \end{aligned} \tag{4.14}$$

Finally, substituting (4.14) into (4.13) and using the integral representation formula yields

$$\begin{aligned} 0 &= \int_{\Gamma} \left(\frac{\partial w}{\partial \nu} W_z - \frac{\partial W_z}{\partial \nu} w \right) ds = \int_{\Gamma} \left(\frac{\partial w}{\partial \nu} w_z - \frac{\partial w_z}{\partial \nu} w \right) ds \\ &= \int_{\Gamma} \left(\frac{\partial w}{\partial \nu} \Phi(\cdot, z) - \frac{\partial \Phi(\cdot, z)}{\partial \nu} w \right) ds = w(z) \qquad \forall z \in B_r. \end{aligned} \tag{4.15}$$

The unique continuation principle for the Helmholtz equation now implies that $w = 0$ in D. Hence if k is not a Neumann eigenvalue corresponding to $-\nabla \cdot A\nabla$ (e.g., if

$\mathcal{I}m(A) < 0$ at a point $x_0 \in D$), then $v \equiv 0$ and therefore $\varphi = 0$, which proves the lemma. ∎

Now we are ready to prove the main result of this section.

THEOREM 4.6. *Let $\eta \in L_\infty(\Gamma_2)$ be the surface conductivity of* (MTP)*, and assume that $\mathcal{I}m(A) = 0$ in D and k is neither a transmission eigenvalue nor a Neumann eigenvalue for $-\nabla \cdot A\nabla$. Then*

$$\|\eta\|_{L_\infty(\Gamma_2)} = \sup_{\substack{z_i,\, z_j \in B_r \\ \alpha_i \in \mathbb{C}}} \frac{\sum_{i,j} \alpha_i \overline{\alpha_j} \left(-4\pi k |\gamma|^2 J_0(k|z_i - z_j|) + i w_{z_i}(z_j) - i\overline{w}_{z_j}(z_i) \right)}{2 \| \sum_i \alpha_i \frac{\partial}{\partial \nu} (w_{z_i} + \Phi(\cdot; z_i)) \|^2_{L^2(\Gamma_2)}}, \tag{4.16}$$

where w_z is such that (w_z, v_z) solves (4.6) *and the sums are arbitrary finite sums.*

Proof. We recall that

$$\|\eta\|_{L_\infty(\Gamma_2)} := \text{ess sup}\, \eta = \sup_{f \in L^2(\Gamma_2)} \frac{1}{\|f\|^2_{L^2(\Gamma_2)}} \int_{\Gamma_2} \eta(x) |f|^2 ds.$$

The theorem then follows from Lemmas 4.4 and 4.5 by fixing first z_2 and then z_1 and considering linear combinations of $\frac{\partial W_z}{\partial \nu}$ for different $z \in B_r$. ∎

Given that D is known, w_z in the right-hand side of (4.16) still cannot be computed, since it depends on the unknown functions η and A. However, from Theorem 4.3, we can use in (4.16) an approximation to w_z given by the Herglotz wave function v_{g^z} with kernel g^z being the (regularized) solutions of the far field equation (4.5).

In the particular case where the coating is homogeneous, i.e., the surface conductivity is a positive constant $\eta > 0$, we can further simplify (4.16). In particular, fix an arbitrary point $z_0 \in B_r$ and consider $z_1 = z_2 = z_0$. Then (4.12) simply becomes

$$\eta = \frac{-2k\pi|\gamma|^2 - \mathcal{I}m\,(w_{z_0}(z_0))}{\| \frac{\partial}{\partial \nu} (w_{z_0} + \Phi(\cdot; z_0)) \|^2_{L^2(\Gamma_2)}}. \tag{4.17}$$

A drawback of (4.16) and (4.17) is that the extent of the coating Γ_2 is not known. Thus, in practice these expressions provide only a lower bound for $\|\eta\|_{L_\infty(\Gamma_2)}$. However, if the object is fully coated, that is, $\Gamma_2 = \Gamma$, we can compute an approximation of $\|\eta\|_{L_\infty(\Gamma_2)}$ by (4.12) and (4.17), where Γ_2 is replaced by of Γ.

5. Remarks on Maxwell's equations in $\mathbb{R}^3$.

The analysis of the previous three sections for the case of scattering by an infinite cylinder can in principle be extended to the scattering of electromagnetic waves by a bounded dielectric in $\mathbb{R}^3$. In this case the direct scattering problem is given by (2.1)–(2.6), and the existence of a unique solution to this problem was established in [8]. The results for the inverse scattering problem for an infinite cylinder established in section 4 of this paper can in turn be extended to the case of Maxwell's equations in $\mathbb{R}^3$, *provided* that one can establish the existence of a unique solution to the interior transmission problem

$$\begin{aligned} \nabla \times E_z - ikH_z &= 0 \\ \nabla \times H_z + ikE_z &= 0 \end{aligned} \qquad \text{in } D, \tag{5.1}$$

$$\begin{aligned} \nabla \times E_z^{\text{int}} - ikH_z^{\text{int}} &= 0 \\ \nabla \times H_z^{\text{int}} + ikN(x)E_z^{\text{int}} &= 0 \end{aligned} \qquad \text{in } D, \tag{5.2}$$

together with the boundary relations

$$
\begin{aligned}
&\nu \times E_z^{\text{int}} - \nu \times E_z = \nu \times E_e(\cdot, z, q) && \text{on } \Gamma,\\
&\nu \times H_z^{\text{int}} - \nu \times H_z = \nu \times H_e(\cdot, z, q) && \text{on } \Gamma_1,\\
&\nu \times H_z^{\text{int}} - \nu \times H_z = -\eta\,[\nu \times (E_z + E_e(\cdot, z, q)) \times \nu] + \nu \times H_e(\cdot, z, q) && \text{on } \Gamma_2,
\end{aligned} \tag{5.3}
$$

where $E_e(\cdot, z, q)$, $H_e(\cdot, z, q)$ is the electric dipole defined by

$$
E_e(x, z, q) := \frac{i}{k}\nabla_x \times \nabla_x \times q\,\Phi(x, z), \qquad H_e(x, z, q) := \nabla_x \times q\,\Phi(x, z), \tag{5.4}
$$

where $q \in \mathbb{R}^3$ is a constant vector and

$$
\Phi(x, z) := \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x-z|}.
$$

Unfortunately this result remains an open problem. (For the existence of a unique solution to a modified version of (5.1)–(5.3), see [8].)

Assuming the existence of a unique solution of (5.1)–(5.3), one can now proceed to derive the three-dimensional analogue of Theorem 4.6; i.e., if $\mathcal{I}m(N) = 0$ and k is not a transmission eigenvalue, then

$$
\|\eta\|_{L_\infty(\Gamma_2)} = \sup_{\substack{z_i \in B_r,\, q \in \mathbb{R}^3 \\ \alpha_i \in \mathbb{C}}} \frac{\sum_{i,j} \alpha_i \overline{\alpha_j}\left[-\|q\|^2 A(z_i, z_j, k, q) + q \cdot E_{z_i}(z_j) + q \cdot \overline{E}_{z_j}(z_i)\right]}{2\|\sum_i \alpha_i\, \nu \times (E_{z_i} + E_e(\cdot, z_i, q))\|^2_{L_t^2(\Gamma_2)}}, \tag{5.5}
$$

where $B_r \subset D$ is a ball of radius r;

$$
A(z_i, z_j, k, q) = \frac{k^2}{6\pi}\left[2j_0(k|z_i - z_j|) + j_2(k|z_i - z_j|)(3\cos^2\phi - 1)\right], \tag{5.6}
$$

j_0 and j_2 being spherical Bessel functions of order 0 and 2, respectively; ϕ is the angle between $(z_i - z_j)$ and q; and E_z, E_z^{int} is the unique solution of the interior transmission problem (5.1)–(5.3). In particular, E_z can be approximated by

$$
E_{g_z}(x) := ik\int_\Omega e^{ikx\cdot d} g_z(d)\,ds(d), \tag{5.7}
$$

where $\Omega := \{x \in \mathbb{R}^3 : |x| = 1\}$ and g_z is the (regularized) solution of the *far field equation*

$$
\int_\Omega E_\infty(\hat{x}, d, g(d))\,ds(d) = E_{e,\infty}(\hat{x}, z, q). \tag{5.8}
$$

Here E_∞ is the electric far field pattern corresponding to the scattering problem (2.1)–(2.6), and $E_{e,\infty}$ is the electric far field pattern of the electric dipole (5.4). For details in the case of a perfect conductor coated by a dielectric, see section 3 of [5].

In the special case when η is a constant, (5.5) simplifies to

$$
\eta = \frac{-\frac{k^2}{6\pi}\|q\|^2 + \mathcal{R}e\,(q \cdot E_{z_0}(z_0))}{\|\nu \times (E_{z_0} + E_e(\cdot, z_0, q))\|^2_{L_t^2(\Gamma_2)}}, \tag{5.9}
$$

where z_0 is an arbitrary point in D.

6. Numerical examples. In this section we shall present some numerical tests of the inversion scheme using synthetic far field data for the Helmholtz equation. For a given scatterer, the far field data is computed by using a cubic finite element code to approximate the near field, and then employing a near to far field transformation [18]. The finite element computational domain is terminated by a rectilinear perfectly matched layer using a linear absorption function in the layer [2], [10].

Having computed approximate values of the far field pattern at N uniformly spaced points on the unit circle for N incoming waves, we have an $N \times N$ matrix $\mathcal{A}$ of approximate far field data

$$\mathcal{A}_{m,n} = u_{h,\infty}(d_m, d_n) \quad \text{where} \quad d_m = \left(\cos\left(\frac{2\pi(m-1)}{N}\right),\ \sin\left(\frac{2\pi(m-1)}{N}\right)\right)^T$$

for $1 \le m, n \le N$, where $u_{h,\infty}$ is the finite element far field pattern. To this we add further noise with parameter ϵ to obtain $\mathcal{A}_\epsilon$ using

$$(\mathcal{A}_\epsilon)_{m,n} = \mathcal{A}_{m,n}(1 + \epsilon(\xi_{1,m,n} + i\xi_{2,m,n})),$$

where $\xi_{1,m,n}$ and $\xi_{2,m,n}$ are given by a random number generator, uniformly distributed in the range $[-1, 1]$. Unless otherwise stated, $\epsilon = 0.01$ in these studies.

For a given sampling point z, the discrete far field equation is then to compute $\vec{g} = (g_1, \dots, g_N)$ such that $\mathcal{A}_\epsilon \vec{g} = \vec{b}$, where

$$b_m = N\gamma \exp\frac{(-ik(z \cdot d_m))}{(2\pi)}, \qquad 1 \le m \le N.$$

This ill-conditioned problem is solved approximately using the Tikhonov regularization and the Morozov discrepancy principle as described, for example, in [14].

6.1. Exact knowledge of the boundary. We start as in [5], assuming an exact knowledge of the boundary in order to assess the accuracy of (4.17) without the added error of computing an approximation to the boundary of the scatterer. In this case, for a given scatterer, we compute $\vec{g}$ for $z = z_0$ using the Morozov method outlined in the previous section, and then approximate (4.17) using the trapezoidal rule with 100 integration points. After limited experiments, we choose $z_0 = (0, 0)^T$ (both upcoming examples have this point as their centroid).

To simplify the presentation, we have limited our discussion to two scatterers: an ellipse given by $x = 0.5\cos(s)$ and $y = 0.2\sin(s)$, $s \in [0, 2\pi]$, and the rectangle $[-0.5, 0.5] \times [0.4, 0.4]$. In (2.7) we choose $A = (1/4)I$. In all cases $k = 5$.

For the ellipse we consider either a fully coated or partially coated object. The partially coated boundary is shown in Figure 6.1. In Figure 6.2 we show results of the reconstruction of a range of conductivities η for the fully coated ellipse, partially coated ellipse, and fully coated rectangle. For each exact η we compute the far field data, add noise, and compute an approximation to w_{z_0}, as discussed before. Despite the noise on the data, η is well approximated in the case of the fully coated scatterers, provided that the conductivity is not too large. In all cases the approximation of η deteriorates for large conductivities, and as expected, (4.17) leads to an underestimate of η when the boundary is partially coated. These limited examples suggest that (4.17) provides a viable method for reconstructing η, provided that η is small enough and the boundary of the scatterer is known sufficiently accurately.

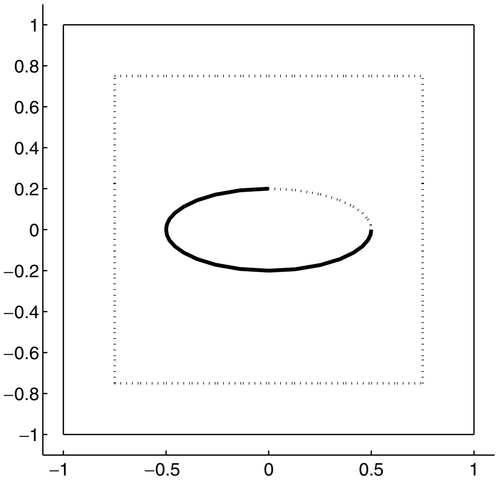

FIG. 6.1. *A diagram showing the coated portion of the partially coated ellipse as a thick line. The dotted square is the inner boundary of the PML, and the solid square is the boundary of the finite element computational domain.*

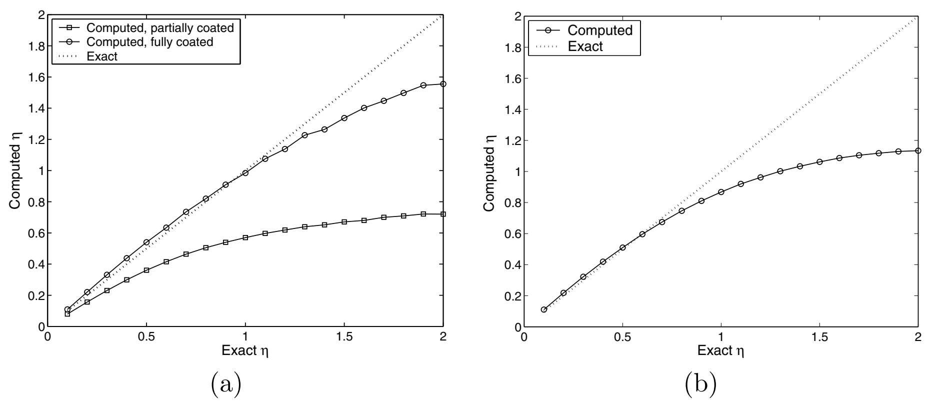

FIG. 6.2. *Computation of* η *using the exact boundary. Panel* (a) *shows results for the fully and the partially coated ellipse. Panel* (b) *shows the corresponding results for the fully coated rectangle. Clearly in all cases the approximation of* η *deteriorates for large conductivities.*

6.2. The ellipse. We now wish to investigate the solution of the full inverse problem. We start by using the standard linear sampling method to approximate the boundary of the scatterer. In particular we compute $1/\|\vec{g}\|$ for z on a uniform grid in the sampling domain. In the upcoming numerical results we have arbitrarily chosen $N = 61$, and we sample on a 101×101 grid on the square $[-1, 1] \times [-1, 1]$. This procedure takes around 10 seconds in MATLAB on an Apple G5 computer, so it is not time-consuming.

Having computed $\vec{g}$ for each sample point, we have a discrete level set function $1/\|\vec{g}\|$. Choosing a contour value C then provides a reconstruction of the support of the given scatterer. We extract the edge of the reconstruction and then fit this using a trigonometric polynomial of degree M, assuming that the reconstruction is star-like with respect to the origin. (For more advanced applications it would be necessary to

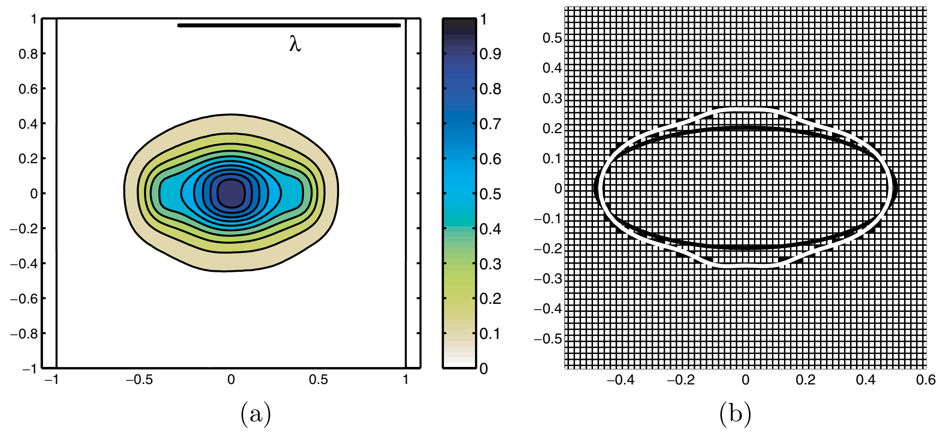

FIG. 6.3. *Illustration of the steps in the computation of η. First the standard linear sampling method is used to compute $1/\|\vec{g}\|$, as shown in panel* (a) (*the bar labeled λ shows the wavelength of the radiation*). *Choosing a cutoff C* (*in this case $C = 0.3$*), *the surface in panel* (a) *provides an approximation to the boundary of the scatterer shown as shaded blocks in panel* (b). *Each square in this figure contains one sampling point z at its center. We also show in panel* (b) *the outline of the true scatterer as a smooth solid line, and as a white line the fit of the trigonometric series to the reconstruction. In this case C is chosen too small and the computed boundary lies outside the true scatterer.*

employ a more elaborate smoothing procedure.) Thus for an angle θ the radius of the reconstruction is given by

$$r(\theta) = \Re\left(\sum_{n=-M}^{M} r_n \exp(in\theta)\right),$$

where r is measured from the origin (since in all the examples here the origin is within the scatterer). The coefficients r_n are found using a least squares fit to the boundary identified in the previous step of the algorithm. Once we have a parameterization of the reconstructed boundary, we can compute the normal to the boundary and evaluate (4.17) for some choice of z_0 (in the examples always $z_0 = (0,0)^T$) using the trapezoidal rule with 100 points. This provides our reconstruction of η.

Figure 6.3 shows the main steps for evaluating our prediction of η for the ellipse. Here we choose $\eta = 1$ on the entire ellipse (fully coated). In (a) we see a plot of $1/\|\vec{g}\|$ (normalized so that the maximum value is 1) as a function of position. In this case the choice $\epsilon = 0.01$ for the additional error in the far field pattern gives an error of 1.3% in the spectral norm for $\mathcal{A}$.

We then make the arbitrary choice $C = 0.3$ (i.e., due to the normalization, the value is 0.3 times the maximum of $1/\|\vec{g}\|$). Figure 6.3(b) shows a plot of the pixels separating regions where $1/\|\vec{g}\| > C$ and $1/\|\vec{g}\| < C$. For clarity, we have graphed only the region $[-0.6, 0.6] \times [-0.6, 0.6]$. The black pixels in Figure 6.3(b) are then fitted using $M = 8$ in the trigonometric polynomial for $r(\theta)$, and the resulting curve is shown as a light curve on the figure. We also indicate, using a thick black line, the true ellipse. We have deliberately chosen a contour value C that does not give the best reconstruction of the ellipse so that the different geometric features can be easily seen. Using this reconstruction results in a predicted value of $\eta = 0.8372$ (compared to the true value $\eta = 1$).

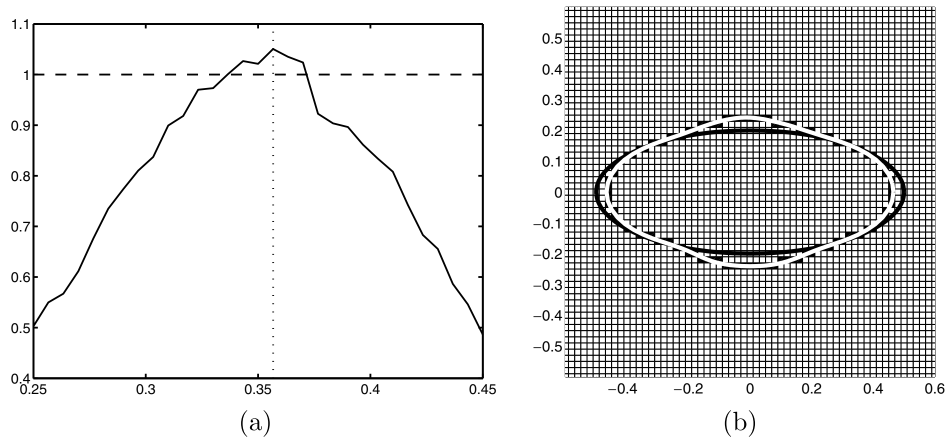

FIG. 6.4. *Panel* (a) *shows the computed value of* η *as a function of the cutoff* C. *The dashed line is the true value* $\eta = 1$, *and the dotted line marks the maximum predicted* η. *The corresponding reconstruction of the ellipse is shown in panel* (b) *using the same convention as in panel* (b) *of Figure* 6.3.

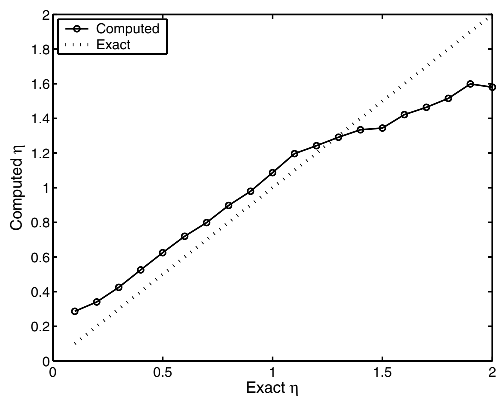

FIG. 6.5. *Reconstruction of a range of* η. *For each exact* η *we apply the reconstruction algorithm using a range of cutoffs and plot the corresponding reconstruction. An exact reconstruction would lie on the dotted line.*

With both scatterers in this study we have observed that a poor choice of the cutoff C tends to result in a predicted value of η that is too small. Therefore we now suggest sweeping through a range of values of C, and we find that the maximum value of η correlates with a good reconstruction of the scatterer and a better approximation of the true value of η. We show this in Figure 6.4 for the fully coated ellipse. The largest predicted value of η is $\eta = 1.05$ when $C = .3567$, and the reconstruction of the scatterer is better than choosing $C = 0.3$.

The reconstruction algorithm is next investigated for a range of values of η. For each exact η we apply the reconstruction algorithm using multiple cutoffs and plot the corresponding reconstruction of η. The results are shown in Figure 6.5 and should be compared to those in Figure 6.2(a). Given that the shape of the object and the parameter η are both being reconstructed, the results show reasonable agreement of the reconstruction up to approximately $\eta = 1.5$. For larger values of η the reconstruction deteriorates, perhaps because the field inside the scatterer diminishes as η

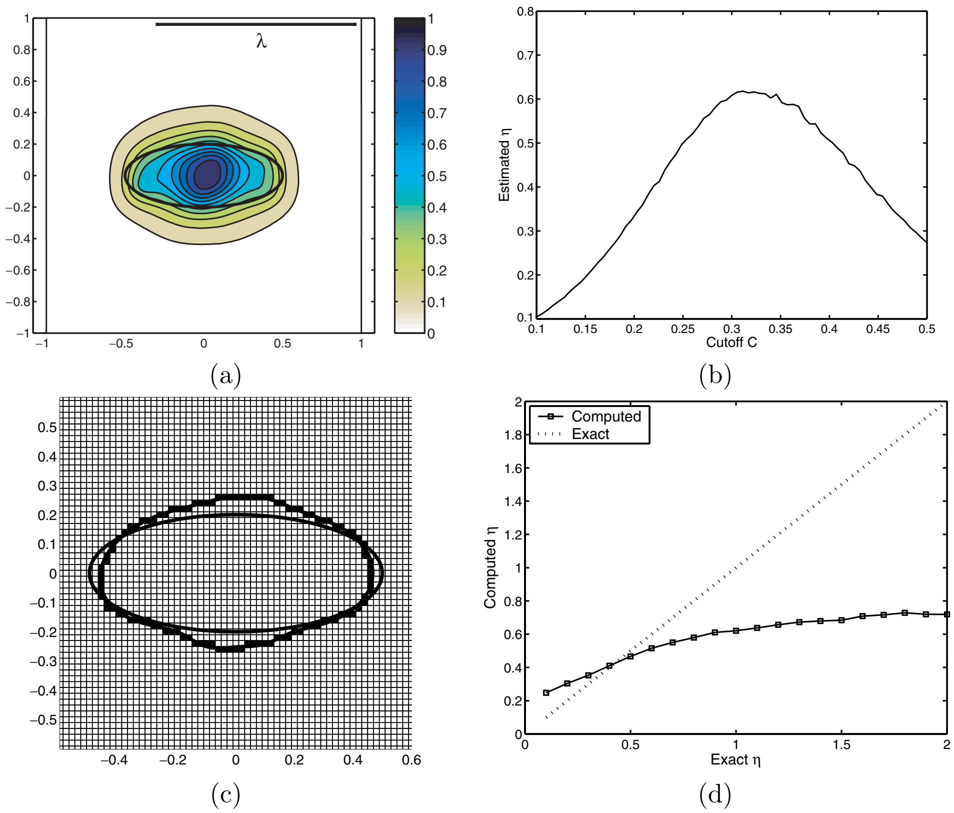

FIG. 6.6. *Reconstruction of the partially coated ellipse for* $\eta = 1$. (a) *The indicator function* $1/\|\vec{g}\|$ *resulting from the standard linear sampling method.* (b) *The computed value of* η *for a range of cutoffs* C. *The best reconstruction* (*maximum value of* η) *is* $\eta = 0.61793$ *when* $C = 0.3114$. (c) *The reconstruction of the ellipse using* $C = 0.3114$. (d) *The reconstruction of a range of* η*; this should be compared to Figure* 6.2(a).

increases. In this case the linear sampling method is able to provide a sufficiently accurate approximation of the ellipse so that the reconstruction of η in Figures 6.2 and 6.5 is of comparable accuracy.

Next we consider the partially coated ellipse (see Figure 6.1). The inversion algorithm is unchanged (both the boundary of the scatterer and η are reconstructed). The results are shown in Figure 6.6(a)–(c) when $\eta = 1$, and the results for a range of η are shown in Figure 6.6(d). The linear sampling method can still reconstruct the ellipse with reasonable fidelity despite the partial coating, and so the results in Figure 6.6(d) and Figure 6.2(a) are comparable. Recall that, for a partially coated obstacle, (4.17) provides only a lower bound for η.

6.3. Rectangular scatterer. Finally we show the reconstruction of the surface conductivity of the fully coated rectangular scatterer. Results for a range of η are shown in Figure 6.7. Comparing this to the reconstruction computed using the exact boundary (shown in Figure 6.2(b)), the results are much worse.

The deterioration of the results for the full inversion scheme can be explained by considering one choice of η in detail. In Figure 6.8 we show the full reconstruction procedure for $\eta = 1$. As in the case of the ellipse, we use the linear sampling method

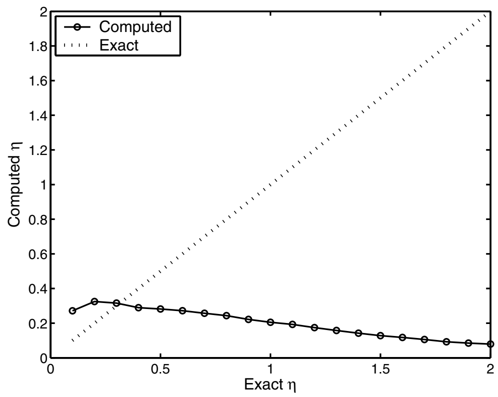

FIG. 6.7. *Results of reconstructing a range of conductivities for the fully coated rectangle. We plot the computed conductivity against its exact value. The results should be compared to Figure* 6.2(b), *and are seen to be substantially worse than that case.*

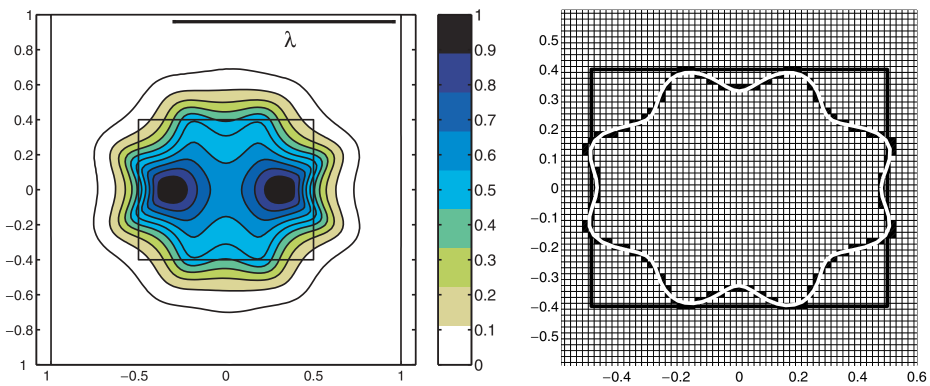

FIG. 6.8. *Details of the reconstruction of η when the exact value is $\eta = 0.5$.* (a) *The indicator function computed by the linear sampling method. Clearly, whatever the choice of C, the reconstruction of the scatterer will not provide an accurate normal.* (b) *The best reconstruction corresponding to $C = 0.52$, which yields the computed value of $\eta = 0.28$.*

to provide an indicator function for the boundary of the rectangle, but, compared to the ellipse, the reconstruction of the boundary shown in Figure 6.8(b) (at the best cutoff C) is now quite poor. From this reconstruction we need to compute the normal derivative of w_{z_0}. It is clear that this will be poorly approximated, and thus (4.17) will provide a poor approximation to η.

7. Conclusion. We have provided a method for estimating the surface conductivity of a scatterer from far field measurements. Numerical experiments show that this method can be combined with the linear sampling method to simultaneously identify the shape of the scatterer and the conductivity, provided that the shape of the scatterer can be computed with sufficient accuracy. Limitations include the fact that the method becomes inaccurate for large values of the surface conductivity, and the quality of the reconstruction of the conductivity can also be adversely influenced by the quality of the reconstruction of the scatterer. This may in part be due to the need to use the normal derivative of the Herglotz wave function in (4.17). We now plan to investigate the use of the electric far field pattern, which should allow us to avoid the normal derivative. However considerable mathematical difficulties need to be overcome, and in particular the existence of a solution of the interior transmission problem is not known in this case.

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