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THE DETERMINATION OF THE SURFACE IMPEDANCE OF A PARTIALLY COATED OBSTACLE FROM FAR FIELD DATA*

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Abstract. A variational method is given for determining the essential supremum of the surface impedance of a partially coated perfect conductor from a knowledge of the far field pattern of the time-harmonic electric field at fixed frequency. It is assumed that the shape of the scatterer has been determined (e.g., by solving the far field equation and using the linear sampling method). Numerical examples are given for the scalar case with constant surface impedance.

Key words. inverse scattering problem, impedance boundary condition, electromagnetic waves

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1. Introduction.

In order to avoid detection by radar, hostile objects are often partially coated by a material designed to reduce the radar cross section of the scattered wave. From the point of view of target identification a key question to answer is, given the shape of a scattering obstacle (which can be determined, for example, by the linear sampling method [2], [3]), is the obstacle coated or not and if so what are the electrical properties of the coating? The simplest example of such a problem is the case of a perfect conductor that is partially coated by a dielectric. In this case the direct scattering problem is a mixed boundary value problem for Maxwell's equations where on the coated part of the boundary the electromagnetic field satisfies an impedance boundary condition [9], [12] and on the remaining part of the boundary the tangential component of the total electric field vanishes. The inverse problem of determining whether or not the obstacle is coated, and, if so, what the values of the surface impedance are, is complicated by the fact that the extent of the coating (if indeed the object is coated at all!) is not known a priori.

In this paper we will provide a variational method for determining the essential supremum of the surface impedance (which may be zero if the scatterer is not coated!) from a knowledge of the far field pattern of the scattered electric field corresponding to a time-harmonic incident plane wave at fixed frequency. In the special case where the surface impedance is a constant, this of course yields this constant. However, in neither case does our method provide information on how much of the scattering obstacle is coated. (In particular, there could be no coating at all or the entire obstacle could be coated!) Our analysis is based on our recent investigations of the inverse scattering problem for partially coated obstacles where the aim was to determine the shape of the scattering obstacle with unknown boundary condition from a knowledge of the electric far field pattern [2], [3]. As we show in this paper, the far field equation that was used in [2] and [3] to determine the shape can also be used in conjunction with a variational method to determine the essential supremum of the surface impedance on the coated portion of the boundary. Although for the sake of exposition we assume in

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this paper that we have full-aperture far field data, we point out at the end of section 3 how all of our results remain valid for the practical case of limited-aperture data.

Given the shape of the scattering obstacle, the problem of determining lower bounds for the surface impedance in the scalar case when the obstacle is completely coated has previously been considered by Colton and Kress [6] (full-aperture scattering data) and Colton and Piana [8] (limited-aperture scattering data). In particular, the paper of Colton and Piana has had a strong influence on the approach used in the present paper. We also draw the reader's attention to a recent paper of Akduman and Kress [1], where a potential theoretic method is given for determining the surface impedance in the case when the shape of the scatterer is known and the obstacle is completely coated.

The plan of our paper is as follows. We first consider the scattering of time-harmonic plane waves by a partially coated infinite cylinder (which in fact can be totally coated, partially coated or not coated at all). This leads to the investigation of a mixed boundary value problem for the two-dimensional Helmholtz equation in the exterior of a bounded domain D with Lipschitz boundary Γ. Assuming the surface impedance $\lambda = \lambda(x)$ on the coated portion Γ_I of Γ is in $L_\infty(\Gamma_I)$, we derive a variational method for determining ess sup $\lambda(x)$ from a knowledge of the far field pattern of the scattered wave. We then extend this result to the case of Maxwell's equations in $\mathbb{R}^3$. In the final section of our paper we consider several numerical examples in the scalar case when the surface impedance is a constant.

2. The scalar case. We consider the scattering of an electromagnetic time harmonic plane wave by a perfectly conducting infinite cylinder that is (partially) coated by an inhomogeneous dielectric material. This leads to a mixed boundary value problem for the Helmholtz equation [2]. In particular let $D \subset \mathbb{R}^2$ be an open bounded region with Lipschitz boundary Γ such that $\mathbb{R}^2 \setminus \overline{D}$ is connected. We assume that the boundary Γ has a Lipschitz dissection $\Gamma = \Gamma_D \cup \Pi \cup \Gamma_I$, where Γ_D and Γ_I are disjoint, relatively open subsets of Γ, having Π as their common boundary in Γ (see e.g., [10]). Furthermore, boundary conditions of Dirichlet and impedance type with the surface impedance a bounded measurable function $\lambda \in L_\infty(\Gamma_I)$ are specified on Γ_D and Γ_I, respectively. We assume that the surface impedance is positive and uniformly bounded, i.e., $\lambda(x) \geq \lambda_0 > 0$ for $x \in \Gamma_I$. Let ν denote the unit outward normal vector defined almost everywhere on $\Gamma_D \cup \Gamma_I$. The total field $u = u^s + e^{ikx\cdot d}$ given as the sum of the unknown scattered wave and incident plane wave satisfies

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D}, \tag{2.1a}$$

$$u = 0 \quad \text{on} \quad \Gamma_D, \tag{2.1b}$$

$$\frac{\partial u}{\partial \nu} + i\lambda(x) u = 0 \quad \text{on} \quad \Gamma_I, \tag{2.1c}$$

where $k > 0$ is the wave number and d is a unit vector describing the incident direction. Moreover, the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{2.2}$$

uniformly in $\hat{x} = x/|x|$ with $r = |x|$.

The well-posedness of the exterior mixed boundary value problem is established in [2] (in [2] λ was assumed to be constant, but all the results remain valid if $\lambda = \lambda(x) \in L_\infty(\Gamma_I)$). In particular it is shown that the direct scattering problem (2.1a)–(2.2) has a unique solution $u \in H_{loc}(D_e)$.

It is easy to see [5] that the scattered field has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O(r^{-3/2}), \tag{2.3}$$

where u_∞ is the *far field pattern* of the scattered wave. The far field pattern defines the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d) ds(d), \qquad g \in L^2(\Omega). \tag{2.4}$$

The corresponding interior mixed boundary value problem is also studied in [2]. In particular we consider the following problem: find $u_z \in H^1(D)$ that satisfies

$$\Delta u_z + k^2 u_z = 0 \quad \text{in} \quad D, \tag{2.5a}$$

$$u_z = -\Phi(\cdot, z) \quad \text{on} \quad \Gamma_D, \tag{2.5b}$$

$$\frac{\partial u_z}{\partial \nu} + i\lambda(x) u_z = -\frac{\partial \Phi(\cdot, z)}{\partial \nu} - i\lambda(x)\Phi(\cdot, z) \quad \text{on} \quad \Gamma_I \tag{2.5c}$$

for a fixed $z \in D$, where Φ is the fundamental solution to the Helmholtz equation defined by

$$\Phi(x, z) := \frac{i}{4} H_0^{(1)}(k|x - z|) \tag{2.6}$$

with $H_0^{(1)}$ being a Hankel function of the first kind of order zero. Then in [2] it is shown that (2.5a)–(2.5c) has a unique solution $u_z \in H^1(D)$ provided $\Gamma_I \neq \emptyset$ and $\lambda \neq 0$.

Next we introduce the far field equation

$$(Fg)(\hat{x}) = \gamma e^{-ik\hat{x}\cdot z}, \qquad g \in L^2(\Omega), \quad z \in D, \tag{2.7}$$

where $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ and $\gamma e^{-ik\hat{x}\cdot z}$ is the far field pattern of $\Phi(x, z)$.

A Herglotz wave function with kernel $g \in L^2(\Omega)$ is an entire solution of the Helmholtz equation defined by

$$v_g(x) = \int_\Omega e^{ikx\cdot d} g(d) ds(d), \qquad x \in \mathbb{R}^2.$$

The following theorem is proved in [2].

THEOREM 2.1. *Let $\epsilon > 0$, $z \in D$, and u_z be the unique solution of* (2.5a)–(2.5c). *Then there exists a Herglotz wave function $v_{g_\epsilon^z}$ with kernel $g_\epsilon^z \in L^2(\Omega)$ such that*

$$\|u_z - v_{g_\epsilon^z}\|_{H^1(D)} \leq \epsilon. \tag{2.8}$$

Moreover, there exists a positive constant $c > 0$ independent of ϵ such that

$$\|(Fg_\epsilon^z)(\hat{x}) - \gamma e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} \leq c\epsilon. \tag{2.9}$$

Now let us define w_z by

$$w_z := u_z + \Phi(\cdot, z). \tag{2.10}$$

In particular, since $u_z \in H^1(D)$ and $z \in D$, we have that $w_z|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$, $\frac{\partial w_z}{\partial \nu}|_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$, and

$$w_z|_{\Gamma_D} = 0 \qquad \text{and} \qquad \left(\frac{\partial w_z}{\partial \nu} + i\lambda w_z\right)|_{\Gamma_I} = 0 \tag{2.11}$$

interpreted in the sense of the trace theorem.

LEMMA 2.2. *For every two points z_1 and z_2 in D we have that*

$$2\int_{\Gamma_I} w_{z_1}\lambda(x)\overline{w}_{z_2}\,ds = -4k\pi|\gamma|^2 J_0(k|z_1-z_2|) + i\left(u_{z_1}(z_2) - \overline{u}_{z_2}(z_1)\right), \tag{2.12}$$

where u_{z_1}, w_{z_1} and u_{z_2}, w_{z_2} are defined by (2.5a)–(2.5c) *and* (2.10)*, respectively, and J_0 is a Bessel function of order zero.*

Proof. Let z_1 and z_2 be two points in D and u_{z_1}, w_{z_1} and u_{z_2}, w_{z_2} the corresponding functions defined by (2.5a)–(2.5c) and (2.10). From (2.11) we have that

$$\begin{aligned}
2i\int_{\Gamma_I} w_{z_1}\lambda(x)\overline{w}_{z_2}\,ds &= \int_\Gamma \left(w_{z_1}\frac{\partial \overline{w}_{z_2}}{\partial\nu} - \overline{w}_{z_2}\frac{\partial w_{z_1}}{\partial \nu}\right) ds \\
&= \int_\Gamma \left(\Phi(\cdot,z_1)\frac{\partial\overline{\Phi(\cdot,z_2)}}{\partial\nu} - \overline{\Phi(\cdot,z_2)}\frac{\partial\Phi(\cdot,z_1)}{\partial\nu}\right) ds \\
&\quad + \int_\Gamma \left(u_{z_1}\frac{\partial\overline{\Phi(\cdot,z_2)}}{\partial\nu} - \overline{\Phi(\cdot,z_2)}\frac{\partial u_{z_1}}{\partial\nu}\right) ds \\
&\quad + \int_\Gamma \left(\Phi(\cdot,z_1)\frac{\partial\overline{u}_{z_2}}{\partial\nu} - \overline{u}_{z_2}\frac{\partial\Phi(\cdot,z_1)}{\partial\nu}\right) ds.
\end{aligned}$$

From Green's theorem applied to the radiating solution $\Phi(\cdot, z)$ of the Helmholtz equation in D_e and the uniformity of the asymptotic relation (2.3) we have (see [7])

$$\begin{aligned}
&\int_\Gamma \left(\Phi(\cdot,z_1)\frac{\partial\overline{\Phi(\cdot,z_2)}}{\partial\nu} - \overline{\Phi(\cdot,z_2)}\frac{\partial\Phi(\cdot,z_1)}{\partial\nu}\right) ds = -2ik\int_\Omega \Phi_\infty(\cdot,z_1)\overline{\Phi_\infty(\cdot,z_2)}ds \\
&= -2ik\int_\Omega |\gamma|^2 e^{-ik\hat{x}\cdot z_1}e^{ik\hat{x}\cdot z_2}\,ds = -4ik\pi|\gamma|^2 J_0(k|z_1-z_2|).
\end{aligned}$$

Now from the representation formula for u_{z_1} and u_{z_2} we obtain

$$2i\int_{\Gamma_I} w_{z_1}\lambda(x)\overline{w}_{z_2}\,ds = -4ik\pi|\gamma|^2 J_0(k|z_1-z_2|) + \overline{u}_{z_2}(z_1) - u_{z_1}(z_2).$$

Finally, dividing both sides of the above relation by i yields the result. ∎

In the following let us consider a ball $B_r \subset D$ of radius r contained in D and denoted by

$$\mathcal{W} := \left\{ f \in L^2(\Gamma_I) : \begin{array}{c} f = w_z|_{\Gamma_I} \text{ with } w_z = u_z + \Phi(\cdot,z), \\ z \in B_r \text{ and } u_z \text{ the solution of (2.5a)–(2.5c)} \end{array} \right\}.$$

Now we are ready to prove the main result of this section.

THEOREM 2.3. *Let $\lambda \in L_\infty(\Gamma_I)$ be the surface impedance of the scattering problem* (2.1a)–(2.2)*. Then*

$$\|\lambda\|_{L_\infty(\Gamma_I)} = \sup_{\substack{z_i \in B_r \\ \alpha_i \in \mathbb{C}}} \frac{\sum_{i,j} \alpha_i \overline{\alpha_j} \left[-4\pi k|\gamma|^2 J_0(k|z_i - z_j|) + i\left(u_{z_i}(z_j) - \overline{u}_{z_j}(z_i)\right)\right]}{2\| \sum_i \alpha_i \left(u_{z_i} + \Phi(\cdot; z_i)\right)\|^2_{L^2(\Gamma)}}, \tag{2.13}$$

where u_z *is the solution to* (2.5a)–(2.5c) *and the sums are arbitrary finite sums.*

Proof. First we show that $\mathcal{W}$ is complete in $L^2(\Gamma_I)$. To this end let φ be a function in $L^2(\Gamma_I)$ such that for every $z \in B_r$

$$\int_{\Gamma_I} w_z \varphi \, ds = 0.$$

Construct $v \in H^1(D)$ as the unique solution of the interior mixed boundary value problem [2]

$$\begin{aligned} \Delta v + k^2 v = 0 \qquad & \text{in} \qquad D, \\ v = 0 \qquad & \text{on} \qquad \Gamma_D, \\ \frac{\partial v}{\partial \nu} + i\lambda(x) v = \varphi \qquad & \text{on} \qquad \Gamma_I. \end{aligned}$$

Then for every $z \in B_r$, using the boundary conditions and the integral representation formula, we have that

$$\begin{aligned} 0 = \int_{\Gamma_I} w_z \varphi \, ds &= \int_{\Gamma_I} w_z \left(\frac{\partial v}{\partial \nu} + i\lambda v\right) ds = \int_{\Gamma} w_z \left(\frac{\partial v}{\partial \nu} + i\lambda v\right) ds \\ &= \int_{\Gamma} \left(u_z \frac{\partial v}{\partial \nu} + i\lambda u_z v + \Phi(\cdot, z)\frac{\partial v}{\partial \nu} + i\lambda \Phi(\cdot, z) v\right) ds \\ &= \int_{\Gamma} \left[u_z \frac{\partial v}{\partial \nu} + v\left(-\frac{\partial u_z}{\partial \nu} - \frac{\partial \Phi(\cdot, z)}{\partial \nu} - i\lambda \Phi(\cdot, z)\right)\right] ds \\ &+ \int_{\Gamma} \left(\Phi(\cdot, z)\frac{\partial v}{\partial \nu} + i\lambda v \Phi(\cdot, z)\right) ds = v(z). \end{aligned}$$

Now the unique continuation principle implies that $v(z) = 0$ for all $z \in D$, whence from the trace theorem $\varphi = 0$.

We now show that

$$\|\lambda\|_{L_\infty(\Gamma_I)} := \text{ess sup}\, \lambda = \sup_{f \in L^2(\Gamma_I)} \frac{1}{\|f\|^2_{L^2(\Gamma_I)}} \int_{\Gamma_I} \lambda(x)|f|^2 ds.$$

The theorem then follows from Lemma 2.2 and the denseness of $\mathcal{W}$ in $L^2(\Gamma_I)$ by fixing first z_2 and then z_1 and considering linear combinations of w_z for different $z \in B_r$ together with the fact that $\|w_z\|_{L^2(\Gamma)} = \|w_z\|_{L^2(\Gamma_I)}$. (Note that w_{z_1} and w_{z_2} are not orthogonal with respect to $\lambda(x)$ and hence two different points are needed.) To prove the above identity, let $C = \text{ess sup}\, \lambda > 0$. Obviously,

$$\frac{1}{\|f\|^2_{L^2(\Gamma_I)}} \int_{\Gamma_I} \lambda(x)|f|^2 ds \le C \qquad \forall f \in L^2(\Gamma_I).$$

Now for every $0 < \epsilon < C$ the set $M_\epsilon = \{x \in \Gamma_I : |\lambda(x)| \geq C - \epsilon\}$ has a positive measure and for an $f_\epsilon \in L^2(\Gamma_I)$ supported in M_ϵ we have

$$\frac{1}{\|f_\epsilon\|^2_{L^2(\Gamma_I)}} \int_{\Gamma_I} \lambda(x)|f_\epsilon|^2 ds \geq (C - \epsilon),$$

which ends the proof. $\square$

Given that D is known (for example, by using the far field equation and the linear sampling method as discussed in [2]), u_z in the right-hand side of (2.13) still cannot be computed since it depends on the unknown function λ. However, from Theorem 2.1, we can use in (2.13) an approximation to u_z given by the Herglotz wave function v_{g^z} with kernel g^z being the (regularized) solutions of the far field equation (2.7).

In the particular case where the surface impedance is a positive constant $\lambda > 0$ we can further simplify the formula (2.13). In particular, fix an arbitrary point $z_0 \in B_r$ and consider $z_1 = z_2 = z_0$. Then (2.12) simply becomes

$$\lambda = \frac{-2k\pi|\gamma|^2 - \operatorname{Im}(u_{z_0}(z_0))}{\|u_{z_0} + \Phi(\cdot; z_0)\|^2_{L^2(\Gamma)}}. \tag{2.14}$$

Note that the expressions on the right-hand sides of (2.13) and (2.14) can be used as a target signature to detect if an obstacle is coated or not. In particular an object is coated if and only if the numerator is nonzero.

3. The vector case. We now turn our attention to the electromagnetic scattering problem for a (partially) coated perfect conductor in $\mathbb{R}^3$. In particular let $D \subset \mathbb{R}^3$ be a bounded region with boundary Γ such that $D_e := \mathbb{R}^3 \setminus \overline{D}$ is connected. Each simply connected piece of D is assumed to be a Lipschitz curvilinear polyhedron. Moreover, we assume that the boundary $\Gamma = \Gamma_D \cup \Pi \cup \Gamma_I$ is split into two disjoint parts Γ_D and Γ_I having Π as their possible common boundary in Γ and that each part Γ_D and Γ_I can be written as the union of a finite number of open smooth faces $(\Gamma_D^j)_{j=1,\dots,N_D}$ and $(\Gamma_I^j)_{j=1,\dots,N_I}$, respectively, where e_{ij} denotes the common edge of two adjacent faces Γ^i and Γ^j. Let ν denote the unit outward normal defined almost everywhere on Γ.

The direct scattering problem for the scattering of a time-harmonic electromagnetic plane wave by a partially coated obstacle D is to find an electric field E and a magnetic field $H := \frac{1}{ik}\operatorname{curl} E$ such that

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \tag{3.1a}$$

$$\nu \times E = 0 \quad \text{on} \quad \Gamma_D, \tag{3.1b}$$

$$\nu \times \operatorname{curl} E - i\lambda(x)(\nu \times E) \times \nu = 0 \quad \text{on} \quad \Gamma_I, \tag{3.1c}$$

where the surface impedance $\lambda \in L_\infty(\Gamma_I)$ satisfies $\lambda(x) \geq \lambda_0 > 0$. The total electric field E is given by

$$E = E^i + E^s, \tag{3.2}$$

where E^s is the scattered field satisfying the Silver–Müller radiation condition

$$\lim_{r\to\infty} (\operatorname{curl} E^s \times x - ikrE^s) = 0 \tag{3.3}$$

uniformly in $\hat{x} = x/|x|$, where $r = |x|$ and the incident field E^i is given by

$$E^i(x) := \frac{i}{k}\operatorname{curl}\operatorname{curl} p e^{ikx\cdot d} = ik(d\times p)\times d e^{ikx\cdot d}, \tag{3.4}$$

where $k > 0$ is the wave number, d is a unit vector giving the direction of propagation, and p is the polarization vector. The well-posedness of the direct problem is established in [3] (in [3] λ was assumed to be constant, but all the results remain valid if $\lambda = \lambda(x) \in L_\infty(\Gamma_I)$). In particular it is shown that there exists a unique solution E, and $H = \frac{1}{ik}\operatorname{curl} E$ of (3.1a)–(3.4), and, moreover, $E \in X(D_e \cap B_R, \Gamma_I)$ for every ball of radius R containing D, where $X(D_e \cap B_R, \Gamma_I)$ is the Sobolev space defined by

$$X(D_e \cap B_R, \Gamma_I) := \{u \in H(\operatorname{curl}, D_e \cap B_R) \,:\, \nu \times u|_{\Gamma_I} \in L^2_t(\Gamma_I)\}$$

with

$$H(\operatorname{curl}, D_e \cap B_R) := \{u \in (L^2(D_e \cap B_R))^3 \,:\, \operatorname{curl} u \in (L^2(D_e \cap B_R))^3\},$$

$$L^2_t(\Gamma_I) := \{u \in (L^2(\Gamma_I))^3 \,:\, \nu \cdot u = 0 \quad \text{on} \quad \Gamma_I\}.$$

The scattered electric field E^s has the asymptotic behavior [5]

$$E^s(x) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right)\right\}$$

as $|x| \to \infty$, where E_∞ is a tangential vector field defined on the unit sphere Ω and known as the *electric far field pattern*. The electric far field operator $F : L^2_t(\Omega) \to L^2_t(\Omega)$ is then defined by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))ds(d), \qquad \hat{x} \in \Omega, \tag{3.5}$$

for $g \in L^2_t(\Omega)$. Note that by superposition Fg is the electric far field pattern of the exterior mixed boundary value problem corresponding to the electromagnetic Herglotz pair with kernel ikg as incident field. An *electromagnetic Herglotz pair* is defined to be a pair of vector fields of the form

$$E_g(x) = \int_\Omega e^{ikx\cdot d} g(d)ds(d), \qquad H_g(x) = \frac{1}{ik}\operatorname{curl} E_g(x), \tag{3.6}$$

where $g \in L^2_t(\Omega)$. It is easily seen that E_g, H_g is a solution of Maxwell's equations $\operatorname{curl} E - ikH = 0$, $\operatorname{curl} H + ikE = 0$ in $\mathbb{R}^3$. Now let us consider the electric dipole with polarization q defined by

$$E_e(x, z, q) := \frac{i}{k}\operatorname{curl}_x \operatorname{curl}_x q\,\Phi(x, z), \qquad H_e(x, z, q) := \operatorname{curl}_x q\,\Phi(x, z), \tag{3.7}$$

where Φ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$ defined by

$$\Phi(x, z) := \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x-z|}, \quad x \neq z \quad \text{and} \quad x, z \in \mathbb{R}^3.$$

If $z \in D$, then $E_e(x,z,q)$ and $H_e(x,z,q)$ satisfy Maxwell's equations in $\mathbb{R}^3 \setminus \overline{D}$, and the corresponding electric far field pattern $E_{e,\infty}(\hat{x},z,q)$ is given by

$$E_{e,\infty}(\hat{x},z,q) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x}\, e^{-ik\hat{x}\cdot z}. \tag{3.8}$$

As in the scalar case, we also need the interior mixed boundary value problem corresponding to the scattering problem which is studied in detail in [3]. (For the case when either $\Gamma_I = \emptyset$ or $\Gamma_D = \emptyset$, see [11].) In particular, let $E_z \in X(D,\Gamma_I)$ be the unique solution of

$$\operatorname{curl}\operatorname{curl} E_z - k^2 E_z = 0 \quad \text{in} \quad D, \tag{3.9a}$$

$$\nu \times [E_z + E_e(\cdot,z,q)] = 0 \quad \text{on} \quad \Gamma_D, \tag{3.9b}$$

$$\nu \times \operatorname{curl}(E_z + E_e(\cdot,z,q)) - i\lambda[\nu \times (E_z + E_e(\cdot,z,q))] \times \nu = 0 \quad \text{on } \Gamma_I \tag{3.9c}$$

for a fixed but arbitrary $z \in D$. Define

$$W_z := E_z + E_e(\cdot,z,q) \tag{3.10}$$

and let $u_T := (\nu \times u) \times \nu$ be the tangential component of a function $u \in H(\operatorname{curl}, D)$. Note that $(W_z)_T|_{\Gamma_I} \in L^2_t(\Gamma_I)$ and that W_z depends on the artificial polarization q as well. We now look for a solution to the far field equation

$$Fg(\hat{x}) = E_{e,\infty}(\hat{x},z,q), \qquad z \in D, \tag{3.11}$$

where F is given by (3.5). We have the following result (see [3, Thm. 3.2]).

THEOREM 3.1. *For every $\epsilon > 0$ and $z \in D$ there exists an electric Herglotz wave function $E_{g^z_\epsilon}$ with kernel $g^z_\epsilon \in L^2_t(\Omega)$ such that*

$$\|E_z - ikE_{g^z_\epsilon}\|_{X(D,\Gamma_I)} \le \epsilon, \tag{3.12}$$

where E_z is the solution of (3.9a)–(3.9c). *Moreover, there exists a positive constant $c > 0$ independent of ϵ such that*

$$\|(Fg^z_\epsilon)(\hat{x}) - E_{e,\infty}(\hat{x},z,q)\|_{L^2_t(\Omega)} \le c\epsilon. \tag{3.13}$$

Our next aim is to find a relation that connects the surface impedance λ with E_z.

LEMMA 3.2. *For every two points z_1 and z_2 in D and polarization $q \in \mathbb{R}^3$ we have that*

$$2\int_{\Gamma_I} (W_{z_1})_T \cdot \lambda(\overline{W}_{z_2})_T\, ds = -\|q\|^2 A(z_1,z_2,k,q) + k\,\left(q \cdot E_{z_1}(z_2) + q \cdot \overline{E}_{z_2}(z_1)\right),$$

where E_{z_1}, E_{z_2} and W_{z_1}, W_{z_2} are defined by (3.9a)–(3.9c) *and* (3.10), *respectively, and $A(z_1,z_2,k,q)$ is a computable number depending only on $z_1, z_2, k,$ and q.*

Proof. By applying the second vector Green's formula and using the boundary conditions for E_{z_1} and E_{z_2} on Γ we obtain

$$\begin{aligned}
2i\int_{\Gamma_I} (W_{z_1})_T \cdot \lambda(\overline{W}_{z_2})_T\, ds &= \int_\Gamma \left(\nu \times W_{z_1} \cdot \operatorname{curl}\overline{W}_{z_2} - \nu \times \overline{W}_{z_2} \cdot \operatorname{curl} W_{z_1}\right) ds \\
&= \int_\Gamma \Big(\nu \times E_e(\cdot,z_1,q) \cdot \operatorname{curl}\overline{E_e(\cdot,z_2,q)} - \nu \times \overline{E_e(\cdot,z_2,q)} \cdot \operatorname{curl} E_e(\cdot,z_1,q)\Big)\, ds \\
&\quad + \int_\Gamma \Big(\nu \times E_{z_1} \cdot \operatorname{curl}\overline{E_e(\cdot,z_2,q)} - \nu \times \overline{E_e(\cdot,z_2,q)} \cdot \operatorname{curl} E_{z_1}\Big)\, ds \\
&\quad + \int_\Gamma \left(\nu \times E_e(\cdot,z_1,q) \cdot \operatorname{curl}\overline{E}_{z_2} - \nu \times \overline{E}_{z_2} \cdot \operatorname{curl} E_e(\cdot,z_1,q)\right) ds.
\end{aligned} \tag{3.14}$$

One can easily see that if $E \in H(\mathrm{curl}, D)$ and $H = \frac{1}{ik}\mathrm{curl}\, E$ is a solution of Maxwell's equations and $z \in D$, we have

$$\begin{aligned}&\nu \times E_e(y, z, q) \cdot \mathrm{curl}_y\, \overline{E}(y) = -\frac{i}{k}(-ik)\mathrm{curl}_z\, \mathrm{curl}_z\, q\Phi(y, z) \cdot (\nu \times \overline{H}(y))\\ &= -q \cdot \mathrm{curl}_z\, \mathrm{curl}_z\, \Phi(y, z)(\nu \times \overline{H}(y))\end{aligned}$$

and

$$\nu \times \overline{E}(y) \cdot \mathrm{curl}_y\, E_e(y, z, q) = ik\nu \times \overline{E}(y) \cdot H_e(y, z, q) = ikq \cdot \mathrm{curl}_z\Phi(y, z)(\nu \times \overline{E}(y)),$$

and therefore from the Stratton–Chu formula

$$\int_\Gamma \big(\nu \times E_e(y, z, q) \cdot \mathrm{curl}_y\, \overline{E}(y) - \nu \times \overline{E}(y) \cdot \mathrm{curl}_y\, E_e(y, z, q)\big) = ikq \cdot \overline{E}(z). \tag{3.15}$$

Moreover (see [7]),

$$\begin{aligned}&\int_\Gamma \Big(\nu \times E_e(\cdot, z_1, q) \cdot \mathrm{curl}\, \overline{E_e(\cdot, z_2, q)} - \nu \times \overline{E_e(\cdot, z_2, q)} \cdot \mathrm{curl}\, E_e(\cdot, z_1, q)\Big)\, ds\\ &\quad = -2ik \int_\Omega E_{e,\infty}(\cdot, z_1, q) \cdot \overline{E_{e,\infty}(\cdot, z_2, q)} ds\\ &\quad = -\frac{ik^3}{8\pi^2} \int_\Omega ((\hat{x} \times q) \times \hat{x}) \cdot ((\hat{x} \times q) \times \hat{x})\, e^{-ik\hat{x}\cdot(z_1 - z_2)} ds\\ &\quad = -\frac{ik^3}{8\pi^2} \int_\Omega (\|q\|^2 - (\hat{x} \cdot q)^2)\, e^{-ik\hat{x}\cdot(z_1 - z_2)} ds := -i\|q\|^2 A(z_1, z_2, k, q),\end{aligned} \tag{3.16}$$

where by straightforward calculations

$$A(z_1, z_2, k, q) = \frac{k^3}{6\pi} \left[2j_0(k|z_1 - z_2|) + j_2(k|z_1 - z_2|)(3\cos^2\phi - 1)\right] \tag{3.17}$$

with j_0 and j_2 being spherical Bessel functions of order 0 and 2, respectively, and ϕ is the angle between $(z_1 - z_2)$ and q. Hence using (3.15) and (3.16) in (3.14) and dividing both sides of (3.14) by i yield the result. $\square$

Next we consider a subset $\mathcal{E}$ of $L^2_t(\Gamma_I)$ defined by

$$\mathcal{E} := \left\{ f \in L^2_t(\Gamma_I) : \begin{array}{c} f = (W_z)_T|_{\Gamma_I} \text{ with } W_z = E_z + E_e(\cdot, z, q), \\ z \in B_r,\ E_z \text{ the solution of (3.9a)–(3.9c) and } q \in \mathbb{R}^3 \end{array} \right\},$$

where B_r is a ball of radius r contained in D.

LEMMA 3.3. *$\mathcal{E}$ is complete in $L^2_t(\Gamma_I)$.*

Proof. Let $\varphi \in L^2_t(\Gamma_I)$ such that for every $z \in B_r$

$$\int_{\Gamma_I} (W_z)_T \cdot \varphi\, ds = 0.$$

Let $E \in X(D, \Gamma_I)$ be the solution of the interior mixed boundary value problem [3]

$$\begin{aligned}&\mathrm{curl}\, \mathrm{curl}\, E - k^2 E = 0 \quad \text{in} \quad D,\\ &\nu \times E = 0 \quad \text{on} \quad \Gamma_D,\\ &\nu \times \mathrm{curl}\, E - i\lambda E_T = \varphi \quad \text{on} \quad \Gamma_I.\end{aligned}$$

Then for $z \in B_r$ and $q \in \mathbb{R}^3$, using the fact that $(W_z)_T = E_T = 0$ on Γ_D, the second vector Green's formula, and (3.15), we have that

$$
\begin{aligned}
0 &= \int_{\Gamma_I} (W_z)_T \cdot \varphi \, ds = \int_{\Gamma} W_z \cdot (\nu \times \operatorname{curl} E - i\lambda E_T) \, ds \\
&= \int_{\Gamma} [E_z \cdot (\nu \times \operatorname{curl} E) - i\lambda E_z \cdot E_T + E_e(\cdot, z, q) \cdot (\nu \times \operatorname{curl} E) - i\lambda E_e(\cdot, z, q) \cdot E_T] \, ds \\
&= \int_{\Gamma} [E_z \cdot (\nu \times \operatorname{curl} E) - E \cdot (\nu \times \operatorname{curl} E_z)] \, ds \\
&+ \int_{\Gamma} [-E \cdot (\nu \times \operatorname{curl} E_e(\cdot, z, q)) + i\lambda E_T \cdot E_e(\cdot, z, q)] \, ds \\
&+ \int_{\Gamma} [E_e(\cdot, z, q) \cdot (\nu \times \operatorname{curl} E) - i\lambda E_e(\cdot, z, q) \cdot E_T] \, ds \\
&= \int_{\Gamma} [E_e(\cdot, z, q) \cdot (\nu \times \operatorname{curl} E) - E \cdot (\nu \times \operatorname{curl} E_e(\cdot, z, q))] \, ds \\
&= -\int_{\Gamma} [(\nu \times E_e(\cdot, z, q)) \cdot \operatorname{curl} E - (\nu \times E) \cdot \operatorname{curl} E_e(\cdot, z, q)] \, ds = ikq \cdot E(z).
\end{aligned}
$$

Thus $q \cdot E(z) = 0$ holds for all polarizations $q \in \mathbb{R}^3$ and $z \in B_r$, and hence $E(z) = 0$ for $z \in B_r$. By the unique continuation principle for the solution of Maxwell's equations in D we now see that $E \equiv 0$ in D, whence, by the trace theorem, $\varphi \equiv 0$, which proves the lemma. $\square$

Combining Lemmas 3.2 and 3.3, we can prove in the same way as in the last part of the proof of Theorem 2.3 the main result of this section.

THEOREM 3.4. *Let $\lambda \in L_\infty(\Gamma_I)$ be the surface impedance of the scattering problem* (3.1a)–(3.4). *Then*

$$
\begin{aligned}
(3.18)\quad & \|\lambda\|_{L_\infty(\Gamma_I)} \\
&= \sup_{\substack{z_i \in B_r,\, q \in \mathbb{R}^3 \\ \alpha_i \in \mathbb{C}}} \frac{\sum_{i,j} \alpha_i \overline{\alpha_j} \left[-\|q\|^2 A(z_i, z_j, k, q) + k \left(q \cdot E_{z_i}(z_j) + q \cdot \overline{E}_{z_j}(z_i) \right) \right]}{2 \| \sum_i \alpha_i (W_{z_i})_T \|^2_{L^2_t(\Gamma)}},
\end{aligned}
$$

where $W_z = E_z + E_e(\cdot, z, q)$ with E_z being the solution to (3.9a)–(3.9c)*, $A(z_i, z_j, k, q)$ is given by* (3.17)*, and the sums are arbitrary finite sums.*

In the particular case where λ is a positive constant and setting $z_1 = z_2 = z_0 \in B_r$, we obtain the following formula for constant surface impedance:

$$
\lambda = \frac{-\frac{k^2}{6\pi}\|q\|^2 + k \operatorname{Re}(q \cdot E_{z_0})}{\|(W_{z_0})_T\|^2_{L^2_t(\Gamma)}}, \tag{3.19}
$$

where $W_{z_0} = E_{z_0} + E_e(\cdot, z_0, q)$ with E_{z_0} being the solution of (3.9a)–(3.9c) corresponding to $z_0 \in B_r$.

In both cases (3.18) and (3.19) E_z cannot be computed since λ appears in the boundary conditions. However, from Theorem 3.1 we can approximate E_z by the electric field ikE_{g^z} of the Herglotz electromagnetic pair with kernel ikg^z, where g^z is a (regularized) solution of the far field equation (3.11) for $z \in B_r \subset D$ and E_∞ is the measured far field data (we again assume that D is known by using the far field equation (3.11) and the linear sampling method as discussed in [3]). We note that, as in the scalar case, the numerator on the right-hand side of (3.18) and (3.19) can be used as a target signature to detect whether or not a object is coated.

We conclude this section by remarking that, in both scalar and vector cases, it suffices to know only the far field data for a limited-aperture $\Omega_0 \subset \Omega$. In particular, in sections 2.3 and 3.2 of [4] it is proved that a Herglotz wave function and an electromagnetic Herglotz pair and their first derivatives can be approximated uniformly on a compact subset of a disk B_R of radius R by a Herglotz wave function and an electromagnetic Herglotz pair, respectively, with kernel supported in a subset $\Omega_0 \subset \Omega$. The kernel of this new Herglotz wave function can now be used in place of g_ϵ^z in Theorems 2.1 and 3.1, and therefore the corresponding $v_{g_\epsilon^z}$ and $E_{g_\epsilon^z}$ can be used as approximations of u_z and E_z, respectively, in the above formulas.

4. Numerical examples. In this section we give some results of numerical experiments performed in the scalar case when the surface impedance λ is a constant. As shown in section 2, an approximation for λ is given by

$$\frac{-2k\pi|\gamma|^2 - \operatorname{Im}(v_{g^z}(z))}{\|v_{g^z} + \Phi(\cdot; z)\|^2_{L^2(\Gamma)}}, \qquad z = (x, y) \in D, \tag{4.1}$$

where $v_{g^z} = \int_0^{2\pi} g^z(d) e^{ik(x\cos\theta + y\sin\theta)} d\theta$, $d = (\cos\theta, \sin\theta)$, and the kernel g^z is the solution of the far field equation

$$\int_0^{2\pi} u_\infty(d, \hat{x}) g^z(d) d\theta = \gamma e^{-ik\hat{x}\cdot z}, \qquad z \in B_r \subset D.$$

The far field data is generated by the method of integral equations and is corrupted by random noise. We fix $k = 3$, select a domain D, boundaries Γ_D and Γ_I (in most of our examples $\Gamma_D = \emptyset$), and a constant λ and then solve the corresponding forward problem. We compute the far field pattern for 100 incident directions and observation directions equally distributed on the unit circle and add random noise of 1% or 10% to the Fourier coefficients of the far field pattern. Tikhonov regularization and the Morozov discrepancy principle are then used to solve the ill-posed discrete far field equation (see section 4 of [2] for details). We choose the sampling points z on a uniform grid of 101×101 points in the square region $[-5,\ 5]^2$ and compute the corresponding g^z. To visualize the obstacle we plot the level curves of the inverse of the discrete ℓ_2 norm of g (note that by the linear sampling method the boundary of the obstacle is characterized as the set of points where the L^2-norm of g starts to become large; see [2]). Then we compute (4.1) at the sampling points in the disk centered at the origin with radius 0.5 (in our examples this circle is always inside D). Although (4.1) is theoretically a constant, because of the ill-posed nature of the far field equation we evaluated (4.1) at all the grid points z in the disk and exhibit the maximum, the average, and the median of the computed values of (4.1). In all tested cases there are some outliers for the minimum value but this is not the case for the maximum. The average and median of the numbers obtained by evaluating (4.1) at the sampling points show that these numbers accumulate near the maximum value and that the average, median, and maximum each provides a reasonable approximation to the true impedance.

For our examples we select two scatterers shown in Figure 4.1 (the kite and the peanut).

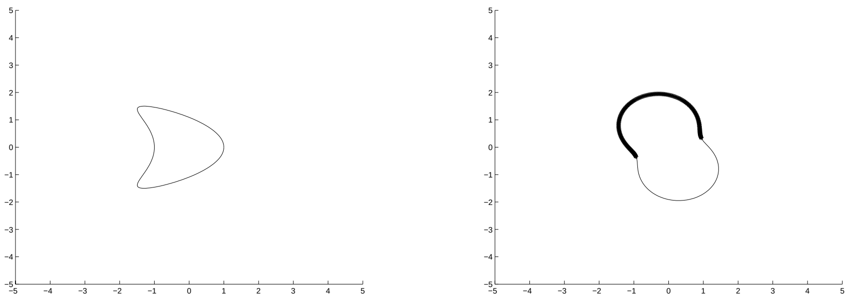

FIG. 4.1. *The boundary of the scatterers used in this study: kite/peanut. When a mixed condition is used for the peanut, the thicker portion of the boundary is* Γ_D.

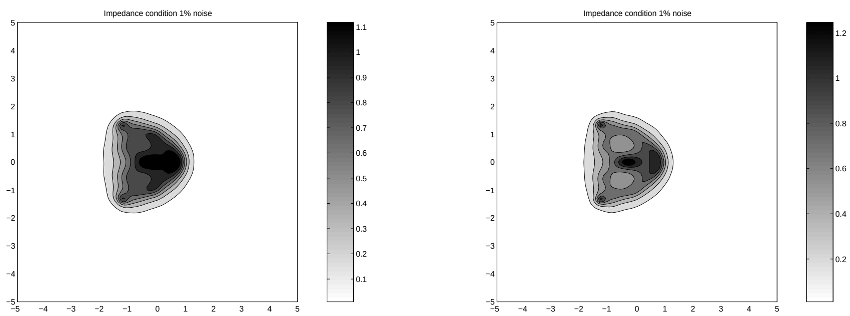

FIG. 4.2. *These figures show the reconstruction of a kite with impedance boundary condition with* 1% *noise: on the left with* $\lambda = 5$ *and on the right with* $\lambda = 9$.

We have obviously left open a number of interesting numerical questions, e.g., what is observed when $\lambda = 0$, what is the dependence of the algorithm on the wave number k, etc. In particular, the examples given here are preliminary in nature. Note that only in the example of the peanut do we consider an object that is really partially coated.

4.1. The kite. We consider the impedance boundary value problem for the kite described by the equation (the left curve in Figure 4.1)

$$x(t) = (1.5\sin(t), \cos(t) + 0.65\cos(2t) - 0.65)\,, \qquad 0 \le t \le 2\pi,$$

with impedance $\lambda = 2$, $\lambda = 5$, and $\lambda = 9$. In Figure 4.2 we show two examples of the reconstructed kite (the reconstructions for the other tested cases look similar). Note that the reconstruction of the boundary is quite accurate so one obtains a good guess for the equation of the boundary Γ of the scatterer. In the numerical results for the reconstructed λ shown in Tables 4.1 and 4.2 we use the exact boundary Γ when we compute the $L^2(\Gamma)$-norm that appears in the denominator of (4.1).

4.2. The peanut. Next we consider a peanut described by the equation (the right curve in Figure 4.1)

$$x(t) = \left(\sqrt{\cos^2(t) + 4\sin^2(t)}\,\cos(t), \quad \sqrt{\cos^2(t) + 4\sin^2(t)}\,\sin(t), \qquad 0 \le t \le 2\pi\right)$$

rotated by $\pi/9$. Here we choose the surface impedance $\lambda = 2$ and $\lambda = 5$ and consider the case of a totally coated peanut (i.e., impedance boundary value problem) as well as

TABLE 4.1
The reconstruction of the surface impedance λ for the kite with 1% noise.

	Maximum	Average	Median
λ=2	2.050	1.975	1.982
λ=5	4.976	4.679	4.787
λ=9	8.883	8.342	8.403

TABLE 4.2
The reconstruction of the surface impedance λ for the kite with 10% noise.

	Maximum	Average	Median
λ=2	2.043	1.960	1.957
λ=5	4.858	4.513	4.524
λ=9	9.0328	8.013	7.992

of a partially coated peanut (i.e., mixed Dirichlet-impedance boundary value problem with Γ_I being the lower half of the peanut as shown in Figure 4.1). Two examples of the reconstructed peanut are presented in Figure 4.3 where, as expected, one notices that for the mixed case the Dirichlet portion of the boundary is more visible. In practice the exact boundary is not available to compute the $L^2(\Gamma)$-norm in (4.1). As suggested by the reconstruction of the peanut, the natural guess for the boundary of the scatterer is the ellipse shown by dashed line in Figure 4.4. So we also examine the sensitivity of our formula on the approximation of the boundary by using this ellipse for computing $\|v_{g^z} + \Phi(\cdot; z)\|_{L^2(\Gamma)}$ in (4.1). The recovered values of λ for our experiments are shown in Tables 4.3 and 4.4.

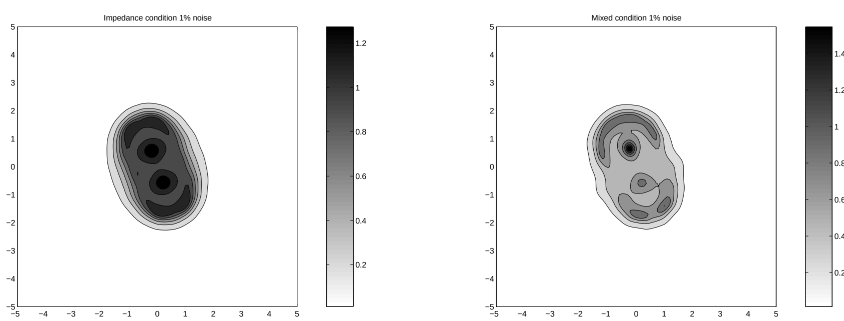

FIG. 4.3. *The figure on the left shows the reconstruction of a peanut with impedance boundary condition with $\lambda = 5$. The figure on the right shows the reconstruction of a peanut with mixed condition with $\lambda = 5$ on the impedance part. Both examples are for $k = 3$ with 1% noise.*

4.3. Conclusions. We have presented the results of some numerical experiments for the scalar case with constant surface impedance. The only a priori information we use is that the coating is homogeneous. Our results suggest that the maximum, median, and average values obtained by evaluating (4.1) at a set of sampling points in a disk closely approximate the true value of λ. We have further shown that even if the boundary of the scatterer is not known exactly, reasonable approximations to the impedance can still be obtained. Numerical experiments need to be done in $\mathbb{R}^3$ and for the nonhomogeneous coating where the scheme is a variational problem. This will be the subject of a forthcoming work.

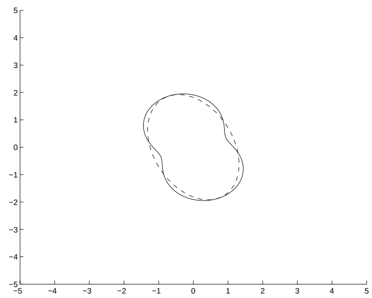

FIG. 4.4. *The dashed line is the approximated boundary we use for computing* $\|v_{g^z} + \Phi(\cdot; z)\|_{L^2(\Gamma)}$ *in* (4.1) *in the case of a peanut with impedance boundary condition.*

TABLE 4.3
Reconstruction of λ *for the peanut with* 1% *noise.*

	Maximum	Average	Median
λ=2 impedance	2.192	1.992	1.979
λ=2 imped., approx. bound.	2.395	1.823	1.886
λ=2 mixed conditions	2.595	2.207	2.257
λ=5 impedance	5.689	4.950	5.181
λ=5 imped., approx. bound.	5.534	4.412	4.501
λ=5 mixed conditions	5.689	4.950	5.180

TABLE 4.4
Reconstruction of λ *for the peanut with* 10% *noise.*

	Maximum	Average	Median
λ=2 impedance	2.297	1.985	1.978
λ=2 imped., approx. bound.	2.301	1.828	1.853
λ=2 mixed conditions	2.681	2.335	2.374
λ=5 impedance	5.335	4.691	4.731
λ=5 imped., approx. bound.	5.806	4.231	4.313
λ=5 mixed conditions	5.893	4.649	4.951

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