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THE PERTURBATION OF TRANSMISSION EIGENVALUES FOR INHOMOGENEOUS MEDIA IN THE PRESENCE OF SMALL PENETRABLE INCLUSIONS

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Abstract. This paper concerns the transmission eigenvalue problem for an inhomogeneous media of compact support containing small penetrable homogeneous inclusions. Assuming that the inhomogeneous background media is known and smooth, we investigate how these small volume inclusions affect the real transmission eigenvalues. Note that for practical applications the real transmission eigenvalues are important since they can be measured from the scattering data. In particular, in addition to proving the convergence rate for the eigenvalues corresponding to the perturbed media as inclusions' volume goes to zero, we also provide the explicit first correction term in the asymptotic expansion for simple eigenvalues. The correction terms involves the eigenvalues and eigenvectors of the unperturbed known background as well as information about the location, size and refractive index of small inhomogeneities. Thus, our asymptotic formula has the potential to be used to recover information about small inclusions from a knowledge of real transmission eigenvalues.

1. Introduction.

The transmission eigenvalue problem, which is a non-selfadjoint and non-linear eigenvalue problem, appears in the study of the scattering problem for inhomogeneous media [7]. The corresponding transmission eigenvalues can be determined from the scattering data [3], [13] and provide information about material properties of the scattering media [4], [5]. Hence they can play an important role in a variety of inverse problems related to target identification and non-destructive testing [11]. The transmission eigenvalue problem is a non-selfadjoint and nonlinear problem that is not covered by the standard theory of eigenvalue problems for elliptic operators. In the past few years transmission eigenvalues have become an important area of research in inverse scattering theory. Since the first proof of existence of transmission eigenvalues in [5] and [16], the interest in the transmission eigenvalue problem has increased, resulting in a number of important advancements in this

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area. For an updated survey on the topic we refer the reader to [7] (see also the special issue of Inverse Problems Vol. 29, No 10).

This paper is a continuation of the study started in [8]. In particular we consider the transmission eigenvalue problem corresponding to an inhomogeneous media perturbed by small volume penetrable inclusions, and investigate the behavior of the corresponding transmission eigenvalues as the volume of perturbation goes to zero. In the current paper the transmission eigenvalue problem is written as a non-linear eigenvalue problem for a compact operator following the fourth order formulation approach developed in [6] and [5]. This approach restricts us to study the perturbation of only real eigenvalues, which from practical application point of view is sufficient since the real eigenvalues are measurable from the scattering data. Our discussion allows for the unperturbed media to be inhomogeneous as opposed to [8] where the unperturbed media is assumed homogeneous. On the other hand, in [8], the convergence of the eigenvalues and eigenvectors was obtained for the entire spectrum (complex eigenvalues included). Unfortunately, there is a term missing in the expression for the first order correction in the asymptotic expansion of the eigenvalues calculated in [8]. In the current paper, for a simple transmission eigenvalue, we also obtain explicit formulas for the first correction term, hence correcting the formula obtained in [8]. Our calculations are based on a nonlinear version [14] of Osborn's theorem [15] which is valid only for simple eigenvalues. For the nonlinear eigenvalue problem, we are finding an eigenvalue λ of $T(\lambda)$, that is, the operator depends on λ. Simplicity is required when applying Osborn's theorem because the multiplicity introduces other eigenvalues λ_i, which are not equal to λ, and hence not *nonlinear* eigenvalues. Hence the case of multiple eigenvalues is still not completed and of interest since it is known that transmission eigenvalues, in some cases even the first eigenvalue, can have multiplicity greater than one. Of course other approaches that use some existing symmetry, such as J-selfadjointness [12], can be considered to study the perturbation of transmission eigenvalues. Note however that the transmission eigenvalue problem is inherently non-selfadjoint.

The paper is structured as follows. In the next section we formulate the transmission eigenvalue problem as a nonlinear eigenvalue problem for a compact operator and introduce the analytical framework. In Section 3 we prove the main convergence results needed to later obtain asymptotic expressions for the transmission eigenvalues. For the convenience of the reader in Section 4 we recall the nonlinear perturbation formula for the general case proven in [14] and use it to provide an explicit expression with a correction term for the perturbation of real simple transmission eigenvalues. We present the proof of a few technical lemmas in the Appendix.

2. Problem statement.

Let our domain $D \subset \mathbb{R}^d$ $(d = 2, 3)$ be bounded with C^2 boundary, and let $n_0 \in C^\infty(D)$ be the given smooth background coefficient which becomes constant near the boundary of D. That is, we assume that

$$n_0 - \hat{n}_0 \text{ has compact support in } D \text{ for some constant } \hat{n}_0. \tag{1}$$

This background n_0 will be perturbed by small volume inhomogeneities of arbitrary smooth shape. For $i = 1, \ldots, m$, we define the bounded shapes B_i to be smooth deformations of a ball centered at the origin, so that $z_i + \epsilon B_i$ is a small inhomogeneity centered at z_i. We also assume that ϵ is small enough so that each scaled ball is separated from the others and the boundary, in particular $(z_i + \epsilon\overline{B_i}) \cap (z_j + \epsilon\overline{B_j}) = \emptyset$

for $i \neq j$. We let W_ϵ be the union of these inhomogeneities, that is

$$W_\epsilon := \bigcup_{i=1}^{m} (z_i + \epsilon B_i),$$

and we define the perturbed squared index of refraction n_ϵ:

$$n_\epsilon(x) = \begin{cases} n_i & x \in z_i + \epsilon B_i, i = 1, \dots, m \\ n_0 & x \in D \setminus W_\epsilon \end{cases} \tag{2}$$

where the $n_i \in \mathbb{R}$ are constants. Let $H_0^2(D)$ denote the Sobolev space given by

$$H_0^2(D) := \left\{ u \in H^2(D) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D \right\}.$$

or, equivalently, the H^2 closure of $C_0^\infty(D)$ functions. Consider now the interior transmission eigenvalue problem corresponding to a scalar isotropic media with these small inhomogeneities. We wish to find nontrivial $w, v \in L^2(D)$ with $w - v \in H_0^2(D)$ satisfying

$$\Delta w + \tau n_\epsilon w = 0 \qquad \text{in } D \tag{3}$$

$$\Delta v + \tau v = 0 \qquad \text{in } D \tag{4}$$

$$w = v \qquad \text{on } \partial D \tag{5}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D. \tag{6}$$

Note that the boundary conditions are equivalently stated as $w - v \in H_0^2(D)$. In relation to the scattering problem, the eigenvalue parameter $\tau := k^2$ where k is the wave number proportional to the frequency. The transmission eigenvalue problem is non-selfadjoint and in special cases is known to have complex eigenvalues. However here we limit ourselves to the study of real eigenvalues, which are the only ones that can be measured from scattering data and used to obtain information about the media.

Definition 1. Values of $\tau \in \mathbb{R}_+$ for which the eigenvalue problem (3)-(6) has a nontrivial solution are called transmission eigenvalues.

In [6], the transmission eigenvalue problem was shown to posses an infinite set of real transmission eigenvalues assuming that $n_\epsilon - 1 \geq \alpha > 0$ or $0 \leq \beta \leq 1 - n_\epsilon(x) < 1$ for α and β independent of ϵ. Here we assume that $n_\epsilon - 1 \geq \alpha > 0$, and note that all of the following could just as likely be done in the second case. We also note that the existence of real or complex eigenvalues is proven under weaker assumptions in [17]. However, our approach makes fundamental use of the framework developed in [6], hence we need the above assumptions. Under our assumption, it is proven in [4], [9], that all transmission eigenvalues τ_ϵ are uniformly bounded below by $\tau_0 := \lambda_1(D)/\alpha + 1$ where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D. Therefore we may assume that all transmission eigenvalues lie in $[\tau_0, +\infty)$, and this assumption plays an important role in our proofs later on.

The goal of this paper is to find an asymptotic expansion with respect to ϵ for transmission eigenvalues of the perturbed problem. Since everything in our problem is real valued, throughout the paper, for sake of simplicity, we assume that the involved Sobolev spaces contain real valued functions over real numbers field.

Given our assumptions, the transmission eigenvalue problem (3) is equivalent to the fourth order nonlinear eigenvalue problem [5] for $u = w - v \in H_0^2(D)$

$$(\Delta + \tau n_\epsilon)\frac{1}{n_\epsilon - 1}(\Delta + \tau)u = 0, \tag{7}$$

which in variational form is stated as follows: find $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n_\epsilon - 1}(\Delta u + \tau u)(\Delta \phi + \tau n_\epsilon \phi)\,\mathrm{d}x = 0 \quad \text{for all } \phi \in H_0^2(D). \tag{8}$$

Following the definitions in [5], we rephrase this in terms of variationaly defined operators. Let us define the bilinear forms on $H_0^2(D) \times H_0^2(D)$,

$$\mathcal{A}_{\tau,\epsilon}(u,\phi) = \left(\frac{1}{n_\epsilon - 1}(\Delta u + \tau u), (\Delta \phi + \tau \phi)\right)_{L^2(D)} + \tau^2 (u,\phi)_{L^2(D)} \tag{9}$$

for $\epsilon \geq 0$ and

$$\mathcal{B}(u,\phi) = (\nabla u, \nabla \phi)_{L^2(D)}. \tag{10}$$

Note that due to the restrictions on n_ϵ, the bilinear forms will be bounded [6], [7]. Hence by the Riesz Representation Theorem, we may define operators $\mathbb{A}_{\tau,\epsilon}, \mathbb{B} : H_0^2(D) \to H_0^2(D)$ such that

$$\mathcal{A}_{\tau,\epsilon}(u,\phi) = (\mathbb{A}_{\tau,\epsilon} u, \phi)_{H_0^2(D)} \quad \text{and} \quad \mathcal{B}(u,\phi) = (\mathbb{B}u, \phi)_{H_0^2(D)} \tag{11}$$

for all $u, \phi \in H_0^2(D)$. Notice that $\mathbb{B}$ is compact due to the compact embedding of $H^2(D)$ into $H^1(D)$, and $\mathbb{A}_{\tau,\epsilon}$ is invertible since $\mathcal{A}_{\tau,\epsilon}$ is coercive for all $\tau > 0$ [6]. It is convenient to use the inner product and norm on $H_0^2(D)$ induced by the bilinear form $\mathcal{A}_{\tau,0}$:

$$(u,\phi)_A := \mathcal{A}_{\tau,0}(u,\phi) = (\mathbb{A}_{\tau,0} u, \phi)_{H_0^2(D)} \quad \text{and} \quad \|u\|_A := \sqrt{(u,u)_A} \tag{12}$$

where we note that these norms depend on τ. Here $\epsilon = 0$ corresponds to the unperturbed case, i.e. the media with refractive index n_0. We also denote the adjoint with respect to this inner product by *, that is for an operator $T : H_0^2(D) \to H_0^2(D)$,

$$(Tu, \phi)_A = (u, T^* \phi)_A \quad \text{for } u, v \in H_0^2(D). \tag{13}$$

We may now rewrite the variational form (8) of the transmission eigenvalue problem as

$$(\mathbb{A}_{\tau,\epsilon} u - \tau \mathbb{B} u, \phi)_{H_0^2(D)} = 0 \tag{14}$$

for $u, \phi \in H_0^2(D)$, or equivalently, finding $u \in H_0^2(D)$ such that

$$(I - \tau \mathbb{A}_{\tau,\epsilon}^{-1}\mathbb{B})u = 0. \tag{15}$$

Define the linear operator $T_\epsilon(\tau) : H_0^2(D) \to H_0^2(D)$ for $\epsilon \geq 0$ and $\tau \in \mathbb{C}$ such that

$$T_\epsilon(\tau) := \mathbb{A}_{\tau,\epsilon}^{-1}\mathbb{B}, \tag{16}$$

so we can write (15) as

$$\tau T_\epsilon(\tau) u = u. \tag{17}$$

We have now rephrased the problem as a nonlinear eigenvalue perturbation problem. That is, would like to find τ such that there exists a nontrivial $u \in H_0^2(D)$ satisfying

$$\tau T_\epsilon(\tau) u = u$$

for $\epsilon > 0$. If $\epsilon = 0$, then $n_\epsilon = n_0$ in the definition of $T_0(\tau)$, and this corresponds to our background problem. Our goal is to find a correction formula for the eigenvalues of the perturbed problem in terms of the eigenvalues and eigenvectors of the background problem. To this end, we will apply Theorem of [14], an application of Osborn's theorem for approximating the eigenvalues of compact operators [15]. We can apply the theorem if $T_\epsilon(\tau)$ converges to $T_0(\tau)$ in norm, and this convergence restricted to the eigenspace of $T_0(\tau)$ dictates the speed of which τ_ϵ approaches τ.

For sake of the reader convenience, we state here the main result of this paper which will be proven in Section 4.

Theorem 2.1. *Let $d = 2, 3$, u be a solution to (8) for $\epsilon = 0$ (i.e. $n_\epsilon := n_0$), and τ and τ_ϵ be transmission eigenvalues corresponding to n_0 and n_ϵ respectively, such that τ is simple. Then for u chosen such that $\|u\|_A = 1$,*

$$\tau_\epsilon - \tau = \epsilon^d \tau \sum_{i=1}^{m} |B_i| \frac{n_0(z_i) - n_i}{(n_0(z_i) - 1)^2} \frac{(\Delta u(z_i) + \tau u(z_i))^2}{1 + \tau^2 (DT_0(\tau)u, u)_A} + o(\epsilon^d).$$

where

$$(DT_0(\tau)u, u)_A = -2\frac{1}{\tau}\left(\frac{1}{n_0 - 1}u, \Delta u + \tau u\right)_{L^2(D)} - 2\,(u, u)_{L^2(D)}\,.$$

The proof of this results requires many technical and auxiliary results that will follow. The next section will contain some estimates which will give us these convergence rates and will show that the hypotheses of the theorem are satisfied. Section 4 contains the application of this eigenvalue correction theorem; and in the appendix we prove several technical lemmas used throughout the the paper.

3. **Operator convergence.** In this section we prove some lemmas which will allow us to successfully apply the nonlinear eigenvalue correction theorem [14]. First, we prove convergence in the H_0^2 norm for $\mathbb{A}_{\tau,\epsilon}^{-1} f$ assuming smoothness on f. Next, we introduce a correction term which we will use to improve the convergence rate. We then prove that we have operator norm convergence of $\mathbb{A}_{\tau,\epsilon}^{-1}\mathbb{B}$, and, finally, we derive an asymptotic formula for $\mathbb{A}_{\tau,\epsilon}^{-1} f$ in the inner product of $H_0^2(D)$.

To simplify the analysis, we will assume a single inhomogeneity $W_\epsilon = \epsilon B$ centered at the origin. The arguments carry over easily to the more general case.

3.1. **Strong convergence of $\mathbb{A}_{\tau,\epsilon}^{-1}$.** For a fixed $f \in H_0^2(D)$, define u_ϵ for $\epsilon > 0$ as

$$u_\epsilon = \mathbb{A}_{\tau,\epsilon}^{-1} f \quad \text{and} \quad u_0 = \mathbb{A}_{\tau,0}^{-1} f. \tag{18}$$

Lemma 3.1. *Let $\mathbb{A}_{\tau,\epsilon}$ be defined as in (11) and $f \in H_0^2(D)$. If $\mathbb{A}_{\tau,0}^{-1} f \in C^{2,\alpha}(D)$ for some $\alpha > 0$, then*

$$\|\mathbb{A}_{\tau,\epsilon}^{-1} f - \mathbb{A}_{\tau,0}^{-1} f\|_{H_0^2(D)} \le C_\tau \epsilon^{d/2},$$

that is,

$$\|u_\epsilon - u_0\|_{H_0^2(D)} \le C_\tau \epsilon^{d/2},$$

where C_τ is independent of ϵ and $C_\tau = C_0 + C_1\tau + C_2\tau^2$ with C_i independent of τ for $i = 0, 1, 2$.

Proof. Since $\mathbb{A}_{\tau,\epsilon} u_\epsilon = \mathbb{A}_{\tau,0} u_0 = f$, we have that for $\phi \in H_0^2(D)$,

$$\begin{aligned} 0 &= (\mathbb{A}_{\tau,\epsilon} u_\epsilon - \mathbb{A}_{\tau,0} u_0, \phi)_{H_0^2(D)} \\ &= (\mathbb{A}_{\tau,\epsilon}(u_\epsilon - u_0), \phi)_{H_0^2(D)} + (\mathbb{A}_{\tau,\epsilon} u_0 - \mathbb{A}_{\tau,0} u_0, \phi)_{H_0^2(D)}. \end{aligned} \tag{19}$$

From the definitions (11) of $\mathbb{A}_{\tau,\epsilon}$ and $\mathbb{A}_{\tau,0}$, we have

$$(\mathbb{A}_{\tau,\epsilon}u_0 - \mathbb{A}_{\tau,0}u_0, \phi)_{H_0^2(D)} = \int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right)\Delta u_0 \Delta\phi \,\mathrm{d}x + \tau\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right) u_0 \Delta\phi \,\mathrm{d}x + \tau\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right)\Delta u_0 \phi \,\mathrm{d}x + \tau^2\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right) u_0 \phi \,\mathrm{d}x. \tag{20}$$

Since we assume that $u_0 \in C^{2,\alpha}(D)$ by the fact that $n_\epsilon - n_0$ is zero outside of $W_\epsilon = \epsilon B$, we have a bound on the first term,

$$\begin{aligned} &\left|\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right)\Delta u_0 \Delta\phi \,\mathrm{d}x\right| \\ \leq\ & C_0 \left|\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right|_\infty \|\Delta u_0\|_\infty \int_{\epsilon B} |\Delta\phi| \,\mathrm{d}x \\ \leq\ & C_0 \|\chi_{\epsilon B}\|_{L^2(D)} \|\phi\|_{H_0^2(D)} \\ \leq\ & C_0 \epsilon^{d/2} \|\phi\|_{H_0^2(D)}, \end{aligned} \tag{21}$$

and the second

$$\begin{aligned} \left|\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right) u_0 \Delta\phi \,\mathrm{d}x\right| &\leq \left|\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right|_\infty \|u_0\|_\infty \int_{\epsilon B} |\Delta\phi| \,\mathrm{d}x \\ &\leq C_1 \epsilon^{d/2} \|\phi\|_{H_0^2(D)}. \end{aligned} \tag{22}$$

For the other terms, we can obtain the desired convergence rate from Sobolev embedding theorem

$$\begin{aligned} \left|\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right)\Delta u_0 \phi \,\mathrm{d}x\right| &\leq \left|\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right|_\infty \|\Delta u_0\|_\infty \int_{\epsilon B} |\phi| \,\mathrm{d}x \\ &\leq C_1 \epsilon^d \|\phi\|_{L^\infty(D)} \\ &\leq C_1 \epsilon^d \|\phi\|_{H_0^2(D)} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \left|\int_D \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) u_0 \phi \,\mathrm{d}x\right| &\leq C_2 \left|\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right|_\infty \|u_0\|_\infty \int_{\epsilon B} |\phi| \,\mathrm{d}x \\ &\leq C_2 \epsilon^d \|\phi\|_{H_0^2(D)}. \end{aligned} \tag{24}$$

Thus, we have shown that

$$(\mathbb{A}_{\tau,\epsilon}(u_\epsilon - u_0), \phi)_{H_0^2(D)} \leq C_\tau \epsilon^{d/2} \|\phi\|_{H_0^2(D)} \tag{25}$$

where $C_\tau = C_0 + C_1\tau + C_2\tau^2$. By choosing $\phi = u_\epsilon - u_0$, the desired result follows from the coercivity of the bilinear form $\mathcal{A}_{\tau,\epsilon}$

$$C\|u_\epsilon - u_0\|^2_{H_0^2(D)} \leq C\epsilon^{d/2} \|u_\epsilon - u_0\|_{H_0^2(D)}. \tag{26}$$

Note that the coercivity constant C for $\mathcal{A}_{\tau,\epsilon}$ on $H_0^2(D)$ is independent of ϵ and τ provided we assume $\tau > \tau_0 := \lambda_1(D)/(\alpha + 1)$ (see (13) in [6]). $\square$

We now construct an appropriate corrector function and use it to improve the estimates for the convergence of $\mathbb{A}^{-1}_{\tau,\epsilon}$. To do this, we will rescale the problem to one in which the inhomogeneity shape is fixed, similar to what was done in [8]. In what follows, we identify n_0 with its constant extension to $\mathbb{R}^d$, which is well defined and smooth by the assumption (1). Define

$$y = x/\epsilon, \quad \tilde{D} = \frac{D}{\epsilon} \tag{27}$$

and

$$\tilde{n}(y) = n_\epsilon(x) = \begin{cases} n_1 & y \in B \\ n_0 & y \in \mathbb{R}^d \setminus B. \end{cases} \tag{28}$$

Let the function $\tilde{v}_B \in H_0^2(\tilde{D})$ solve

$$\int_{\tilde{D}} \frac{1}{\tilde{n}-1} \Delta_y \tilde{v}_B \Delta_y \phi \, \mathrm{d}y = \int_B \Delta_y \phi(y) \, \mathrm{d}y \tag{29}$$

for any $\phi \in H_0^2(\tilde{D})$. The existence and uniqueness of $\tilde{v}_B$ follows from the Riesz representation theorem. Since the domain $\tilde{D}$ is increasing with ϵ, we also define a limiting function v_B on all of $\mathbb{R}^d$. To do this, we introduce the Sobolev space following [2]:

$$W_0^2(\mathbb{R}^d) = \left\{ \begin{array}{ll} u \in D'(\mathbb{R}^d): & 0 \le |m| \le k, \rho^{|m|-2} (\ln \omega)^{-1} D^{|m|} u \in L^2(\mathbb{R}^d) \\ & k+1 \le |m| \le 2, \rho^{|m|-2} D^{|m|} u \in L^2(\mathbb{R}^d) \end{array} \right.$$

where weights are given by $\rho := (1+|x|^2)^{1/2}$ and $\omega := (2+|x|^2)$; indices $k = 1$ if $d = 2$ or $k = -1$ if $d = 3$. This space is equipped with the usual H^2 norm with the indicated weights. Then the function $v_B \in W_0^2(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} \frac{1}{\tilde{n}-1} \Delta_y v_B \Delta_y \phi \, \mathrm{d}y = \int_B \Delta_y \phi(y) \, \mathrm{d}y \tag{30}$$

for all $\phi \in W_0^2(\mathbb{R}^d)$. As shown in [8], we may choose v_B such that it satisfies the following decay conditions at infinity:

$$v_B(y) = o(|y|^{2-d/2}), \quad \nabla \cdot v_B(y) = o(|y|^{1-d/2}), \quad D^2 v_B(y) = o(|y|^{-d/2}). \tag{31}$$

Remark 3.2. By its definition (30), v_B is a weak solution to the partial differential equation:

$$\Delta_y \frac{1}{\tilde{n}-1} \Delta_y v_B = \Delta_y \chi_B. \tag{32}$$

Due to the decay conditions (31), the Laplacian is invertible [2] and we have the identity

$$\Delta_y v_B = (\tilde{n}-1)\chi_B. \tag{33}$$

Recall that $u_0 := \mathbb{A}_{\tau,0}^{-1} f$ as in the previous proof (18). We define the correction $\tilde{u}^{(1)}$ to be

$$\tilde{u}^{(1)} := -\left(\frac{1}{n_1 - 1} - \frac{1}{n_0(0) - 1} \right) \big(\Delta u_0(0) + \tau u_0(0) \big) \tilde{v}_B(x/\epsilon). \tag{34}$$

Notice, this function is in $H_0^2(D)$ since $\tilde{v}_B(y) \in H_0^2(\tilde{D})$ thus it will act as a correction term in the $H_0^2(D)$ norm. However, since $\tilde{v}_B$ depends on ϵ, to derive an asymptotic formula we also define the correction term $u^{(1)} \in H^2(D)$

$$u^{(1)} := -\left(\frac{1}{n_1 - 1} - \frac{1}{n_0(0) - 1} \right) \big(\Delta u_0(0) + \tau u_0(0) \big) v_B(x/\epsilon). \tag{35}$$

where v_B is defined by (30). Note that these two corrections are close to each other due to the lemmas in the appendix, in particular Lemma 5.1. The following lemma and its corollary describe precisely the error when these corrections are introduced.

Lemma 3.3. *Let $d = 2, 3$. For $\mathbb{A}_{\tau,\epsilon}$ and $\tilde{u}^{(1)}$ defined by (11) and (34) respectively, if $f \in H_0^2(D)$ and $\mathbb{A}_{\tau,0}^{-1} f \in C^{2,\alpha}(D)$ for some $\alpha > 0$, then*

$$\|\mathbb{A}_{\tau,\epsilon}^{-1} f - \mathbb{A}_{\tau,0}^{-1} f - \epsilon^2 \tilde{u}^{(1)}\|_{H_0^2(D)} \le C_\tau o(\epsilon^{d/2}),$$

that is,

$$\|u_\epsilon - u_0 - \epsilon^2 \tilde{u}^{(1)}\|_{H_0^2(D)} \le C_\tau o(\epsilon^{d/2}),$$

where $C_\tau = \sum_{i=0}^{2} C_i \tau^i$ for C_i independent of τ.

Corollary 3.4. *Let $u^{(1)}$ defined by (35), if the conditions of Lemma 3.3 hold, then*

$$\|\Delta u_\epsilon - \Delta u_0 - \epsilon^2 \Delta u^{(1)}\|_{L^2(D)} \le C_\tau o(\epsilon^{d/2}),$$

where $C_\tau = \sum_{i=0}^{2} C_i \tau^i$ for C_i independent of τ.

Proof of Corollary 3.4. The proof follows from the above Lemma 3.3 and Lemma 5.1 (originally from [8]). □

Proof of Lemma 3.3. Starting from (19), we add and subtract $\mathbb{A}_{\tau,\epsilon} \epsilon^2 \tilde{u}^{(1)}$ to the right hand side to have

$$0 = (\mathbb{A}_{\tau,\epsilon}(u_\epsilon - u_0 - \epsilon^2 \tilde{u}^{(1)}), \phi) + (\mathbb{A}_{\tau,\epsilon} u_0 - \mathbb{A}_{\tau,0} u_0 + \mathbb{A}_{\tau,\epsilon} \epsilon^2 \tilde{u}^{(1)}, \phi)_{H_0^2(D)}. \tag{36}$$

As in the previous proof, we have the estimate

$$\begin{aligned}(\mathbb{A}_{\tau,\epsilon} u_0 - \mathbb{A}_{\tau,0} u_0 + \mathbb{A}_{\tau,\epsilon} \epsilon^2 \tilde{u}^{(1)}, \phi)_{H_0^2(D)} &= \int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right) \Delta u_0 \Delta \phi \, \mathrm{d}x \\ &\quad + \tau \int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right) u_0 \Delta \phi \, \mathrm{d}x \\ &\quad + (\mathbb{A}_{\tau,\epsilon} \epsilon^2 \tilde{u}^{(1)}, \phi)_{H_0^2(D)} + O\left(\epsilon^d \|\phi\|_{H_0^2(D)}\right).\end{aligned} \tag{37}$$

First we will show that

$$\begin{aligned}&(\mathbb{A}_{\tau,\epsilon} \epsilon^2 \tilde{u}^{(1)}, \phi)_{H_0^2(D)} \\ &= -\left(\frac{1}{n_1 - 1} - \frac{1}{n_0(0) - 1}\right) (\Delta u_0(0) + \tau u_0(0)) \int_{\epsilon B} \Delta \phi(x) \, \mathrm{d}x + o\left(\epsilon^2 \|\phi\|_{H_0^2(D)}\right).\end{aligned} \tag{38}$$

We begin by considering

$$\begin{aligned}\epsilon^2 (\mathbb{A}_{\tau,\epsilon} \tilde{v}_B(\cdot/\epsilon), \phi)_{H_0^2(D)} &= \epsilon^2 \left(\frac{1}{n_\epsilon - 1} \Delta \tilde{v}_B(\cdot/\epsilon), \Delta \phi\right)_{L^2(D)} \\ &\quad + \epsilon^2 \tau \left(\Delta \tilde{v}_B(\cdot/\epsilon), \frac{1}{n_\epsilon - 1} \phi\right)_{L^2(D)} + \epsilon^2 \tau \left(\frac{1}{n_\epsilon - 1} \tilde{v}_B(\cdot/\epsilon), \Delta \phi\right)_{L^2(D)} \\ &\quad + \epsilon^2 \tau^2 \left(\left(\frac{1}{n_\epsilon - 1} + 1\right) \tilde{v}_B(\cdot/\epsilon), \phi\right)_{L^2(D)}.\end{aligned} \tag{39}$$

For the second term in (39), observe that by using Cauchy-Schwarz and $\Delta_y = \epsilon^2\Delta_x$ for $y = x/\epsilon$,

$$\begin{aligned}
&\left|\epsilon^2\tau\left(\Delta\tilde{v}_B(\cdot/\epsilon) - \Delta v_B(\cdot/\epsilon), \frac{1}{n_\epsilon - 1}\phi\right)_{L^2(D)}\right| \\
&\leq \tau\left\|\frac{1}{n_\epsilon - 1}\right\|_{L^\infty(D)} \|\Delta_y\tilde{v}_B - \Delta_y v_B\|_{L^2(D)}\|\phi\|_{L^2(D)} \\
&\leq C\tau o(\epsilon^{d/2})\|\phi\|_{H^2_0(D)} \qquad (40)
\end{aligned}$$

where the last line follows from Lemma 5.1. We now use the smoothness of n_0 to replace it with its value at the center of the ball. Since $n_\epsilon - n_0$ only has support on ϵB, by Taylor's theorem we have for $\zeta_x = \epsilon\xi_x$,

$$\begin{aligned}
(41)\quad &\epsilon^2\tau\left(\Delta v_B(\cdot/\epsilon), \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right)\phi\right)_{L^2(D)} \\
&= \epsilon^2\tau\left(\Delta v_B(\cdot/\epsilon), \left(\frac{1}{n_1 - 1} - \frac{1}{n_0(0) - 1} + \epsilon\nabla\left(\frac{1}{n_0(\zeta_x) - 1}\right)(\xi_x)\right)\phi\right)_{L^2(\epsilon B)}.
\end{aligned}$$

We will first bound the term without the gradient of n_0. For convenience define

$$\delta n = \frac{1}{n_1 - 1} - \frac{1}{n_0(0) - 1}. \qquad (42)$$

We calculate

$$\begin{aligned}
&\epsilon^2\tau\left|\left(\Delta v_B(\cdot/\epsilon), \left(\frac{1}{n_1 - 1} - \frac{1}{n_0(0) - 1}\right)\phi\right)_{L^2(\epsilon B)}\right| \\
&= -\epsilon\delta n\tau\left(\nabla_y v_B(\cdot/\epsilon), \nabla\phi\right)_{L^2(\epsilon B)} + \epsilon^2\tau\delta n(\partial_\nu v_B(\cdot/\epsilon), \phi)_{L^2(\partial(\epsilon B))} \\
&\leq C\epsilon\|\nabla_y v_B(\cdot/\epsilon)\|_{L^2(\epsilon B)}\|\Delta\phi\|_{L^2(D)} + \epsilon^2\tau\delta n(\partial_\nu v_B(\cdot/\epsilon), \phi)_{L^2(\partial(\epsilon B))}. \qquad (43)
\end{aligned}$$

After the change of variables $y = x/\epsilon$,

$$\begin{aligned}
&\leq C\epsilon^{d/2+1}\|\nabla_y v_B\|_{L^2(B)}\|\Delta\phi\|_{L^2(D)} + \epsilon^2\tau\delta n(\partial_\nu v_B(\cdot/\epsilon), \phi)_{L^2(\partial(\epsilon B))} \\
&\leq C\epsilon^{d/2+1}\|\Delta\phi\|_{L^2(D)} + \epsilon\tau\delta n(\partial_{\nu_y} v_B(\cdot/\epsilon), \phi)_{L^2(\partial(\epsilon B))} \qquad (44)
\end{aligned}$$

because $v_B \in H^2(B)$. To bound the second term, we change variables and use continuity of v to show

$$\begin{aligned}
\epsilon\tau\delta n(\partial_{\nu_y} v_B(\cdot/\epsilon), \phi)_{L^2(\partial(\epsilon B))} &\leq \epsilon\tau\delta n\|\phi\|_{L^\infty(\partial(\epsilon B))}\|\partial_{\nu_y} v_B(\cdot/\epsilon)\|_{L^1(\partial(\epsilon B))} \\
&\leq \epsilon\tau\delta n\|\phi\|_{L^\infty(D)}\|\partial_{\nu_y} v_B(y)\|_{L^1(\partial B)}\epsilon^{d-1} \\
&\leq C\tau\epsilon^d\tau\delta n\|\phi\|_{H^2_0(D)}\|\partial_{\nu_y} v_B(y)\|_{L^1(\partial B)} \\
&= O\left(\epsilon^d\|\phi\|_{H^2_0(D)}\right) \qquad (45)
\end{aligned}$$

since v_B is bounded in $H^2(B)$. Estimating the term in (41) with the gradient of n_0, we obtain

$$\begin{aligned}
&\epsilon^2\tau\left(\Delta v_B(\cdot/\epsilon), \epsilon\nabla\left(\frac{1}{n_0(\zeta_x) - 1}\right)\cdot(\xi_x)\phi\right)_{L^2(\epsilon B)} \\
&\leq C\tau\epsilon\left\|\nabla\left(\frac{1}{n_0 - 1}\right)\right\|_{L^\infty(D)}\|\phi\|_{L^\infty(D)}\int_{\epsilon B}|\Delta_y v_B|\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq C\tau\epsilon\|\phi\|_{H_0^2(D)}\|\chi_{\epsilon B}\|_{L^2(D)}\|\Delta_y v_B\|_{L^2(D)} \\
&\leq C\tau\epsilon^{d/2+1}\|\phi\|_{H_0^2(D)}. \qquad (46)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\epsilon^2\tau\left(\Delta v_B(\cdot/\epsilon), \frac{1}{n_\epsilon - 1}\phi\right)_{L^2(D)} \\
=&\tau\left(\Delta v_B(\cdot/\epsilon), \frac{1}{n_0 - 1}\phi\right)_{L^2(D)} + O\left(\epsilon^{d/2+1}\|\phi\|_{H_0^2(D)}\right). \qquad (47)
\end{aligned}
$$

Because $n_0 \in C^\infty(D)$ and $\phi \in H_0^2(D)$, one clearly has

$$
\left\|\Delta\left(\frac{1}{n_0 - 1}\phi\right)\right\|_{L^2(D)} \leq C\|\Delta\phi\|_{L^2(D)}. \qquad (48)
$$

Therefore, we compute the bound

$$
\begin{aligned}
\left|\epsilon^2\tau\left(\Delta v_B(\cdot/\epsilon), \frac{1}{n_0 - 1}\phi\right)_{L^2(D)}\right| &= \left|\epsilon^2\tau\left(v_B(\cdot/\epsilon), \Delta\left(\frac{1}{n_0 - 1}\phi\right)\right)_{L^2(D)}\right| \\
&\leq C\epsilon^2\|v_B(\cdot/\epsilon)\|_{L^\infty(D)}\left\|\Delta\left(\frac{1}{n_0 - 1}\phi\right)\right\|_{L^2(D)} \\
&\leq C\epsilon^2\|v_B\|_{L^\infty(\mathbb{R}^d)}\|\phi\|_{H_0^2(D)} \\
&\leq C\epsilon^2\|\phi\|_{H_0^2(D)} \qquad (49)
\end{aligned}
$$

since $v_B \in L^\infty(\mathbb{R}^d)$ for $d = 2, 3$. Now, we will estimate the third term of (39) using Cauchy-Schwarz

$$
\begin{aligned}
\left|\epsilon^2\tau\left(\frac{1}{n_\epsilon - 1}\tilde{v}_B(\cdot/\epsilon), \Delta\phi\right)_{L^2(D)}\right| &\leq C\left\|\frac{1}{n_\epsilon - 1}\right\|_{L^\infty(D)}\|\epsilon^2\tilde{v}_B(\cdot/x)\|_{L^2(D)}\|\Delta\phi\|_{L^2(D)} \\
&= o\left(\epsilon^{d/2}\|\phi\|_{H_0^2(D)}\right) \qquad (50)
\end{aligned}
$$

by Proposition 5.2. For the last term we may do similarly to obtain

$$
\begin{aligned}
&\left|\epsilon^2\tau^2\left(\left(\frac{1}{n_\epsilon - 1} + 1\right)\tilde{v}_B(\cdot/\epsilon), \phi\right)_{L^2(D)}\right| \\
&\qquad \leq C\left\|\frac{1}{n_\epsilon - 1} + 1\right\|_{L^\infty(D)}\|\epsilon^2\tilde{v}_B(\cdot/x)\|_{L^2(D)}\|\phi\|_{L^2(D)} \\
&\qquad = o\left(\epsilon^{d/2}\|\phi\|_{H_0^2(D)}\right).
\end{aligned}
$$

Combining the above inequalities, we may write

(51)

$$
\epsilon^2(\mathbb{A}_{\tau,\epsilon}\tilde{v}_B(\cdot/\epsilon), \phi)_{H_0^2(D)} = \epsilon^2\left(\frac{1}{n_\epsilon - 1}\Delta\tilde{v}_B(\cdot/\epsilon), \Delta\phi\right)_{L^2(D)} + o\left(\epsilon^{d/2}\|\phi\|_{H_0^2(D)}\right).
$$

Observe by definition of $\tilde{v}_B(x/\epsilon)$ combined with a change of variables $y = x/\epsilon$,

$$\begin{aligned}\epsilon^2\left(\frac{1}{n_\epsilon-1}\Delta\tilde{v}_B(\cdot/\epsilon),\Delta\phi\right)_{L^2(D)} &= \epsilon^2\int_D \Delta\tilde{v}_B(x/\epsilon)\Delta\phi\,\mathrm{d}x\\ &= \frac{1}{\epsilon^2}\int_{\tilde{D}}\Delta_y\tilde{v}_B(y)\Delta_y\phi(\epsilon y)\,\epsilon^d\mathrm{d}y\\ &= \frac{1}{\epsilon^2}\int_B \Delta_y\phi(\epsilon y)\,\epsilon^d\mathrm{d}y\\ &= \int_{\epsilon B}\Delta_x\phi(x)\,\mathrm{d}y. \end{aligned}\tag{52}$$

Thus, we have the asymptotic formula

$$\begin{aligned}&(\mathbb{A}_{\tau,\epsilon}\epsilon^2\tilde{u}^{(1)},\phi)_{H_0^2(D)}\\ &\quad= -\left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right)\big(\Delta u_0(0)+\tau u_0(0)\big)\int_{\epsilon B}\Delta\phi(x)\,\mathrm{d}x\\ &\qquad\qquad + o\left(\epsilon^{d/2}\|\phi\|_{H_0^2(D)}\right). \end{aligned}\tag{53}$$

Using the support of $n_\epsilon - n_0$ and the definition of $\mathbb{A}_{\tau,\epsilon}\epsilon^2\tilde{u}^{(1)}$ in (37), we have that

$$\begin{aligned}&(\mathbb{A}_{\tau,\epsilon}u_0-\mathbb{A}_{\tau,0}u_0+\mathbb{A}_{\tau,\epsilon}\epsilon^2\tilde{u}^{(1)},\phi)_{H_0^2(D)}\\ &\quad=\int_{\epsilon B}\left(\frac{1}{n_1-1}-\frac{1}{n_0-1}\right)\Delta u_0\Delta\phi\,\mathrm{d}x+\tau\int_{\epsilon B}\left(\frac{1}{n_1-1}-\frac{1}{n_0-1}\right)u_0\Delta\phi\,\mathrm{d}x\\ &\qquad-\int_{\epsilon B}\left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right)\big(\Delta u_0(0)+\tau u_0(0)\big)\Delta\phi\,\mathrm{d}x\\ &\qquad\qquad+o(\epsilon^{d/2}\|\phi\|_{H_0^2(D)}). \end{aligned}\tag{54}$$

Recall that by assumption we have $u_0 = \mathbb{A}_{\tau,0}^{-1}f \in C^{2,\alpha}(D)$ for some $\alpha > 0$ and constant C, so we will be able to estimate the remaining terms of (54).

$$\begin{aligned}&\left|\int_{\epsilon B}\left(\left(\frac{1}{n_1-1}-\frac{1}{n_0-1}\right)\Delta u_0-\left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right)\Delta u_0(0)\right)\Delta\phi\,\mathrm{d}x\right|\\ &\qquad\le C\epsilon^\alpha\int_{\epsilon B}|\Delta\phi|\,\mathrm{d}x\\ &\qquad\le C\epsilon^\alpha\|\chi_{\epsilon B}\|_{L^2(D)}\|\phi\|_{H_0^2(D)}\\ &\qquad\le C\epsilon^{d/2+\alpha}\|\phi\|_{H_0^2(D)}. \end{aligned}\tag{55}$$

Similarly, we will use that u_0 is Lipschitz on D to show

$$\begin{aligned}&\left|\int_{\epsilon B}\left(\left(\frac{1}{n_1-1}-\frac{1}{n_0-1}\right)u_0-\left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right)u_0(0)\right)\Delta\phi\,\mathrm{d}x\right|\\ &\qquad\le C\epsilon\int_{\epsilon B}|\Delta\phi|\,\mathrm{d}x\\ &\qquad\le C\epsilon\|\chi_{\epsilon B}\|_{L^2(D)}\|\phi\|_{H_0^2(D)}\\ &\qquad\le C\epsilon^{d/2+1}\|\phi\|_{H_0^2(D)}. \end{aligned}$$

Choose $\phi = u_\epsilon - u_0 - \epsilon^2\tilde{u}^{(1)}$. By the argument of the past lemmas and since the coercivity of constant associated with $\mathbb{A}_{\tau,\epsilon}$ is independent of ϵ and $\tau \ge \tau_0$, we have that

$$\|u_\epsilon - u_0 - \epsilon^2 \tilde{u}^{(1)}\|^2_{H^2_0(D)} = o\left(\epsilon^{d/2}\|u_\epsilon - u_0 - \epsilon^2\tilde{u}^{(1)}\|_{H^2_0(D)}\right). \tag{56}$$

□

We can now combine the previous lemma and the technical lemma in the last section to prove the convergence of $\mathbb{B}\mathbb{A}^{-1}_{\tau,\epsilon}f$.

Lemma 3.5. *Let $d = 2, 3$. For $\mathbb{A}_{\tau,\epsilon}$ and $\mathbb{B}$ defined by (11). If $f \in H^2_0(D)$ and $\mathbb{A}^{-1}_{\tau,0}f \in C^{2,\alpha}(D)$ for some $\alpha > 0$, then*

$$\|\mathbb{B}(\mathbb{A}^{-1}_{\tau,\epsilon} - \mathbb{A}^{-1}_{\tau,0})f\|_{H^2_0(D)} \leq C_\tau o(\epsilon^{d/2})$$

where $C_\tau = \sum_{i=0}^2 C_i\tau^i$ for C_i independent of τ.

Proof. By the definition of the operator $\mathbb{B}$, we have that for $\phi \in H^2_0(D)$

$$\begin{aligned}(\mathbb{B}(\mathbb{A}^{-1}_{\tau,\epsilon} - \mathbb{A}^{-1}_{\tau,0})f, \phi)_{H^2_0(D)} &= (\nabla(\mathbb{A}^{-1}_{\tau,\epsilon} - \mathbb{A}^{-1}_{\tau,0})f, \nabla\phi)_{L^2(D)} \\ &= ((\mathbb{A}^{-1}_{\tau,\epsilon} - \mathbb{A}^{-1}_{\tau,0})f, \Delta\phi)_{L^2(D)} \\ &\leq \|(\mathbb{A}^{-1}_{\tau,\epsilon} - \mathbb{A}^{-1}_{\tau,0})f\|_{L^2(D)}\|\phi\|_{H^2_0(D)}. \end{aligned} \tag{57}$$

Choosing $\phi = \mathbb{B}(\mathbb{A}^{-1}_{\tau,\epsilon} - \mathbb{A}^{-1}_{\tau,0})f$, the inequality becomes

$$\|\mathbb{B}(\mathbb{A}^{-1}_{\tau,\epsilon} - \mathbb{A}^{-1}_{\tau,0})f\|_{H^2_0(D)} \leq \|(\mathbb{A}^{-1}_{\tau,\epsilon} - \mathbb{A}^{-1}_{\tau,0})f\|_{L^2(D)}. \tag{58}$$

From the previous lemma we have

$$\|\mathbb{A}^{-1}_{\tau,\epsilon}f - \mathbb{A}^{-1}_{\tau,0}f - \epsilon^2\tilde{u}^{(1)}\|_{L^2(D)} = o(\epsilon^{d/2}). \tag{59}$$

We note by Proposition 5.2 the correction term $\epsilon^2\tilde{u}^{(1)}$ defined by (34) is $o(\epsilon^{d/2})$. □

3.2. **Norm convergence of $\mathbb{A}^{-1}_{\tau,\epsilon}\mathbb{B}$.** We now show operator norm convergence of $\mathbb{A}^{-1}_{\tau',\epsilon}\mathbb{B}$ for τ' in an open set containing τ.

Lemma 3.6. *Let $d = 2, 3$, $\tau > \tau_0 > 0$, and let $\mathbb{A}_{\tau,\epsilon}$ and $\mathbb{B}$ be defined by (11). Then, there exists an α with $0 < \alpha < 1$ and an open bounded set U containing τ such that for all $\tau' \in \overline{U}$*

$$\|\mathbb{A}^{-1}_{\tau',\epsilon}\mathbb{B} - \mathbb{A}^{-1}_{\tau',0}\mathbb{B}\|_{\mathcal{L}(H^2_0(D))} \leq C\epsilon^\alpha$$

for a C independent of τ' but depending on choice of U.

Proof. Let $f, \phi \in H^2_0(D)$ and note that from (19) we have that

$$(\mathbb{A}_{\tau,\epsilon}(u_\epsilon - u_0), \phi)_{H^2_0(D)} = -(\mathbb{A}_{\tau,\epsilon}u_0 - \mathbb{A}_{\tau,0}u_0, \phi)_{H^2_0(D)}$$

hence we estimate the term in the right hand side. To this end, we define $u_\epsilon = \mathbb{A}^{-1}_{\tau,\epsilon}\mathbb{B}f$ for $\epsilon \geq 0$ as in Lemma 3.1. We begin by using (20) from the proof of Lemma 3.1,

$$\begin{aligned}(\mathbb{A}_{\tau,\epsilon}u_0 - \mathbb{A}_{\tau,0}u_0, \phi)_{H^2_0(D)} = &\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right)\Delta u_0\Delta\phi\,\mathrm{d}x \\ &+ \tau\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right)u_0\Delta\phi\,\mathrm{d}x \\ &+ \tau\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right)\Delta u_0\phi\,\mathrm{d}x \\ &+ \tau^2\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right)u_0\phi\,\mathrm{d}x\end{aligned}$$

$$=I + II + III + IV. \tag{60}$$

We are immediately able to bound II, III and IV

$$II, III \le C\tau \|\mathbb{A}_{\tau,0}^{-1}\mathbb{B}\| \epsilon^{d/2} \|f\|_{H_0^2(D)} \|\phi\|_{H_0^2(D)} \tag{61}$$

and

$$IV \le C\tau^2 \|\mathbb{A}_{\tau,0}^{-1}\mathbb{B}\| \epsilon^{d/2} \|f\|_{H_0^2(D)} \|\phi\|_{H_0^2(D)} \tag{62}$$

where the estimates follow from (22), (23), (24) and the Sobolev embedding of $H_0^2(D)$ into $C^{0,\alpha}(D)$. For the remaining term I, we need to use the fact that u_0 is more regular then $|f|$ due to the presence of compact $\mathbb{B}$. Because $\mathbb{B}f$ solves

$$\Delta\Delta\mathbb{B}f = -\Delta f, \tag{63}$$

standard elliptic regularity [1] yields that $\mathbb{B}f \in H^4(D)$ for $f \in H_0^2(D)$. Furthermore, by looking at the variational form for $\mathbb{A}_{\tau,0}$, one finds that given $u = \mathbb{B}f$, $\mathbb{A}_{\tau,0}^{-1}u$ also solves a fourth order elliptic equation:

$$\Delta\Delta\mathbb{A}_{\tau,0}^{-1}u = \Delta\left(\frac{1}{n_0 - 1}\Delta u\right) + \tau\Delta\left(\frac{1}{n_0 - 1}u\right) + \tau\frac{1}{n_0 - 1}\Delta u + \tau^2\left(\frac{1}{n_0 - 1} + 1\right)u, \tag{64}$$

Therefore, if $f \in H_0^2(D)$, elliptic regularity again implies that $\mathbb{B}f \in H^4(D)$ and hence that $\mathbb{A}_{\tau,0}^{-1}\mathbb{B}f \in H^4(D)$. Therefore, we have the bound

$$\|\mathbb{A}_{\tau,0}^{-1}\mathbb{B}f\|_{H^4(D)} \le \|\mathbb{A}_{\tau,0}^{-1}\|_{\mathcal{L}(H^4(D))} \|\mathbb{B}\|_{\mathcal{L}(H^4(D), H_0^2(D))} \|f\|_{H_0^2(D)}. \tag{65}$$

The Uniform Boundedness Principle applied to the set $\{\mathbb{A}_{\tau,0}^{-1} : \tau \in \overline{U},\ \overline{U} \text{ compact}\}$ yields a bound in $H^4(D)$ uniform in τ. That is, for some C independent of τ,

$$\|\mathbb{A}_{\tau,0}^{-1}\mathbb{B}f\|_{H^4(D)} \le C\|f\|_{H_0^2(D)}. \tag{66}$$

Therefore, we may compute a bound for the integral I,

$$\begin{aligned}
&\left|\int_D \left(\frac{1}{n_\epsilon - 1} - \frac{1}{n_0 - 1}\right)\Delta u_0 \Delta\phi \,\mathrm{d}x\right| \\
&\le \left|\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right| \|\Delta u_0\|_{L^2(\epsilon B)} \|\Delta\phi\|_{L^2(D)} \\
&\le \left|\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right| \int_D \chi_{\epsilon B}(\Delta u_0)^2 \,\mathrm{d}x.
\end{aligned} \tag{67}$$

Let $1 < p < 2$ and let p^* be its Sobolev conjugate so that they satisfy $1/p^* = 1/p - 1/d$. Recall that $L^2(D) \subset L^p(D)$ and therefore $H^1(D) \subset W^{1,p}(D) \subset L^{p^*}(D)$. Let q be the Hölder dual to $\hat{p} := p^*/2$ that is $1/q + 1/\hat{p} = 1$. It is important to notice that by our choice of p, $1/q > 0$. We may now calculate

$$\begin{aligned}
\left|\int_D \chi_{\epsilon B}(\Delta u_0)^2 \,\mathrm{d}x\right| &\le C\|(\Delta u_0)^2\|_{L^{\hat{p}}(D)} |\epsilon B|^{1/q} = C\|\Delta u_0\|_{L^{p^*}(D)}^2 |\epsilon B|^{1/q} \\
&\le C\|\Delta u_0\|_{H^1(D)}^2 \epsilon^{1/q} \le C\|u_0\|_{H^3(D)}^2 \epsilon^{1/q} \le C\|f\|_{H_0^2(D)}^2 \epsilon^{1/q}
\end{aligned} \tag{68}$$

through $u_0 = \mathbb{A}_{\tau,0}^{-1}\mathbb{B}f$ and (65). This along with (67) and (68) implies

$$I \le C\epsilon^{1/(2q)} \|f\|_{H_0^2(D)} \|\phi\|_{H_0^2(D)}. \tag{69}$$

We conclude the proof using (60) and the uniform coercivity of $\mathcal{A}_{\tau,\epsilon}$ for $\tau > \tau_0$ as in the end of the proof of Lemma 3.1. The bounds derived will hold for any $\tau' \in U$

by the same argument and since $\overline{U}$ is compact we can choose the constant C in the final estimate independent of τ'. $\square$

3.3. **An asymptotic formula.** Having established the order of convergence of the operators involved in the transmission eigenvalue problem, we next proceed with an asymptotic formula for the operator $\mathbb{A}_{\tau,\epsilon}$, where we explicitly display the first term in the asymptotic expansion. Such a formula is later used to obtain the correction term for the eigenvalues.

Lemma 3.7. *Let $d = 2, 3$, $f \in H_0^2(D)$ and $\phi \in H_0^2(D) \bigcap C^{2,\alpha}(D)$. If $\mathbb{A}_{\tau,0}^{-1} f \in C^{2,\alpha}(D)$ for some $\alpha > 0$, then*

$$\left((\mathbb{A}_{\tau,\epsilon} - \mathbb{A}_{\tau,0})(u_0 + \epsilon^2 \tilde{u}^{(1)}), \phi\right)_{H_0^2(D)} = \epsilon^d |B| \frac{n_0(0) - n_1}{(n_0(0) - 1)^2}$$
$$(\Delta u_0(0)\Delta\phi(0) + \tau(u_0(0)\Delta\phi(0) + \Delta u_0(0)\phi(0)) + \tau^2 u_0(0)\phi(0)) + o(\epsilon^d)$$

for $\tilde{u}^{(1)}$ defined by (34).

Proof. By definition and using the support of $n_\epsilon - n_0$,

$$\begin{aligned}
&((\mathbb{A}_{\tau,\epsilon} - \mathbb{A}_{\tau,0})(u_0 + \epsilon^2 \tilde{u}^{(1)}), \phi)_{H_0^2(D)} \\
&= \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) \Delta(u_0 + \epsilon^2 \tilde{u}^{(1)}) \Delta\phi \, \mathrm{d}x \\
&\quad + \tau \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) (u_0 + \epsilon^2 \tilde{u}^{(1)}) \Delta\phi \, \mathrm{d}x \\
&\quad + \tau \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) \Delta(u_0 + \epsilon^2 \tilde{u}^{(1)}) \phi \, \mathrm{d}x \\
&\quad + \tau^2 \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) (u_0 + \epsilon^2 \tilde{u}^{(1)}) \phi \, \mathrm{d}x \\
&= I + II + III + IV. \qquad (70)
\end{aligned}$$

We consider first II and IV which are the most straightforward. After a change of variables, the term containing u_0 in II is

$$\begin{aligned}
&\tau\epsilon^d \int_B \left(\frac{1}{n_1(\epsilon y) - 1} - \frac{1}{n_0(\epsilon y) - 1}\right) u_0(\epsilon y) \Delta\phi(\epsilon y) \, \mathrm{d}y \\
&\qquad = \tau\epsilon^d |B| \left(\frac{1}{n_1 - 1} - \frac{1}{n_0(0) - 1}\right) u_0(0)\Delta\phi(0) + o(\epsilon^d), \qquad (71)
\end{aligned}$$

because the integrand is continuous, whereas the term containing $\tilde{u}^{(1)}$ is

$$\begin{aligned}
&\tau\epsilon^d \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) \epsilon^2 \tilde{u}^{(1)} \Delta\phi \, \mathrm{d}y \\
&\qquad \leq \tau \left\| \frac{1}{n_1 - 1} - \frac{1}{n_0 - 1} \right\|_{L^\infty(D)}^2 \|\Delta\phi\|_{L^\infty(D)} \int_{\epsilon B} \epsilon^2 \tilde{v}_B \, \mathrm{d}x \\
&\qquad \leq C\tau \|\chi_{\epsilon B}\|_{L^2(D)} \|\epsilon^2 \tilde{v}_B\|_{L^2(D)} \leq C\tau\epsilon^{d/2} o(\epsilon^{d/2}) \qquad (72)
\end{aligned}$$

by Proposition 5.2. Therefore,

$$II = \tau\epsilon^d |B| \left(\frac{1}{n_1 - 1} - \frac{1}{n_0(0) - 1}\right) u_0(0)\Delta\phi(0) + o(\epsilon^d), \qquad (73)$$

and the same reasoning yields

$$IV = \tau^2\epsilon^d|B|\left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right)u_0(0)\phi(0)+o(\epsilon^d). \tag{74}$$

Now let us consider I:

$$\begin{aligned}
&\left|\int_{\epsilon B}\left(\frac{1}{n_1-1}-\frac{1}{n_0-1}\right)\Delta(\epsilon^2\tilde{u}^{(1)}-\epsilon^2u^{(1)})\Delta\phi\,\mathrm{d}x\right|\\
&\le C\left\|\frac{1}{n_1-1}-\frac{1}{n_0-1}\right\|^2_{L^\infty(D)}\|\Delta\phi\|_{L^\infty(D)}\|\chi_{\epsilon B}\|_{L^2(D)}\|\Delta_y(\tilde{v}_B-v_B)\|_{L^2(D)}\\
&\le C\|\Delta\phi\|_{L^\infty(D)}\epsilon^{d/2}o(\epsilon^{d/2})\\
&= o(\epsilon^d)
\end{aligned} \tag{75}$$

by Lemma 5.1 and the fact that $\phi\in C^{2,\alpha}(D)$. Next, we recall that

$$\delta n = \left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right). \tag{76}$$

We now change of variables, note that $\Delta_y=\epsilon^2\Delta$, and recall our $C^{2,\alpha}$ assumption on u_0 so that we have

$$\begin{aligned}
I =&\epsilon^d\int_B\left(\frac{1}{n_1-1}-\frac{1}{n_0(\epsilon y)-1}\right)\Delta(u_0(\epsilon y)+\epsilon^2u^{(1)}(y))\Delta\phi(\epsilon y)\,\mathrm{d}y\\
=&\epsilon^d\int_B\left(\frac{1}{n_1-1}-\frac{1}{n_0(\epsilon y)-1}\right)\Delta u_0(\epsilon y)\Delta\phi(\epsilon y)\,\mathrm{d}y\\
&-\epsilon^d\delta n(\Delta u_0(0)+\tau u_0(0))\int_B\left(\frac{1}{n_1-1}-\frac{1}{n_0(\epsilon y)-1}\right)\Delta_y v_B(y)\Delta\phi(\epsilon y)\mathrm{d}y+o(\epsilon^d)\\
=&\epsilon^d|B|\left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right)\Delta u_0(0)\Delta\phi(0)\\
&-\epsilon^d(\Delta u_0(0)+\tau u_0(0))\left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right)^2\Delta\phi(0)\int_B\Delta_y v_B(y)\mathrm{d}y+o(\epsilon^d)
\end{aligned} \tag{77}$$

by the Lebesgue Dominated Convergence Theorem. Similarly,

$$\begin{aligned}
III =&\tau\epsilon^d|B|\left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right)\Delta u_0(0)\phi(0)\\
&-\tau\epsilon^d\left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right)^2(\Delta u_0(0)+\tau u_0(0))\phi(0)\int_B\Delta_y v_B(y)\mathrm{d}y\\
&+o(\epsilon^d).
\end{aligned} \tag{78}$$

Due to Remark 3.2,

$$\int_B\Delta_y v_B(y)\mathrm{d}y=\int_B(\tilde{n}(y)-1)\,\mathrm{d}y=(n_1-1)|B| \tag{79}$$

since n_1 is constant. The previous equality implies

$$\delta n\left(|B|-\left(\frac{1}{n_1-1}-\frac{1}{n_0(0)-1}\right)\int_B\Delta_y v_B(y)\mathrm{d}y\right)=|B|\frac{n_0(0)-n_1}{(n_0(0)-1)^2} \tag{80}$$

for δn defined by (76). By collecting terms we have the formula in the statement of the theorem. □

4. **A correction formula for transmission eigenvalues.** Now we turn our attention to the transmission eigenvalue problem. In what follows let u be the solution to

$$\int_D \frac{1}{n_0 - 1}(\Delta u + \tau u)(\Delta \phi + \tau n_0 \phi)\,\mathrm{d}x = 0 \quad \text{for all } \phi \in H_0^2(D). \tag{81}$$

normalized with respect to the A-inner product, that is, the A-normalized eigenfunction of the transmission problem for the unperturbed media, and we define u_ϵ to be

$$u_\epsilon = \mathbb{A}_{\tau,\epsilon}^{-1}\mathbb{B}u = T_\epsilon(\tau)u \qquad \text{for } \epsilon \geq 0. \tag{82}$$

Recall that $u_0 = T_0(\tau)u = u/\tau$. The unperturbed eigenfunction u is known to be $H^4(D)\bigcap H_0^2(D)$ due to standard elliptic regularity results [1]. We will assume throughout this section that τ is simple, i.e., the eigen-space is one-dimensional. To derive the eigenvalue correction, we will apply a nonlinear eigenvalue correction theorem which was a corollary of [15]. For the convenience of the reader, we restate it (Corollary 4.1 of [14]) here:

Theorem 4.1 (Nonlinear Eigenvalue Correction [14])**.** *Let X be a Banach space and $\{T_\epsilon(\lambda) : X \to X\}$ a set of compact operator valued functions of λ which are analytic in a region U of the complex plane, such that $T_\epsilon(\lambda) \to T_0(\lambda)$ in norm as $\epsilon \to 0$ uniformly for $\lambda \in U$. Let $\lambda_0 \neq 0$, $\lambda_0 \in U$ be a simple nonlinear eigenvalue of T_0,*

$$\lambda_0 T_0(\lambda_0)\phi = \phi,$$

define $DT_0(\lambda_0)$ to be the derivative of T_0 with respect to λ evaluated at λ_0, and let ϕ be the normalized eigenfunction and ϕ^* its dual. Then for any ϵ small enough there exists λ_ϵ, a simple nonlinear eigenvalue of T_ϵ, such that if

$$\lambda_0^2 \langle DT_0(\lambda_0)\phi, \phi^*\rangle \neq -1$$

we have the following formula

$$\lambda_\epsilon - \lambda_0 = \lambda_0^2 \frac{\langle (T_0(\lambda_0) - T_\epsilon(\lambda_0))\phi, \phi^*\rangle}{1 + \lambda_0^2\langle DT_0(\lambda_0)\phi, \phi^*\rangle} + O\left(\sup_{\lambda\in U} \|(T_\epsilon(\lambda) - T_0(\lambda))\big|_{R(E)}\|\|(T_\epsilon^*(\lambda) - T_0^*(\lambda))\big|_{R(E)^*}\|\right)$$

where $R(E)$ is the space spanned by ϕ and $R(E)^*$ is its dual or the space spanned by ϕ^*.

We will apply this theorem using the A-inner product defined in the second section, and thus u will satisfy $\|u\|_A = 1$. We note that this inner product depends on τ, and therefore we keep track of how all constants depend on τ. The lemmas from the previous section are sufficient to prove the following:

Lemma 4.2. *Let $d = 2, 3$, $T_\epsilon(\tau)$ defined by (16) for $\epsilon \geq 0$, and u be the solution to the variational problem (8). Then,*

$$\|T_\epsilon(\tau)u - T_0(\tau)u\|_A \leq C_\tau \epsilon^{d/2}$$

and

$$\|T_\epsilon^*(\tau)u - T_0^*(\tau)u\|_A = C'_\tau o(\epsilon^{d/2})$$

where $C_\tau = \sum_{i=0}^{3} C_i\tau^i$ *and* $C'_\tau = \sum_{i=0}^{3} C'_i\tau^i$ *and* C_i, C'_i *are independent of* τ. *Furthermore, there exists open set* $U \subset \mathbb{R}_+$ *containing* τ *with* $\overline{U}$ *compact such that*

$$T_\epsilon(\tau') \to T_0(\tau')$$

uniformly in the operator norm for all $\tau' \in \overline{U}$.

Proof. The first is obvious from Lemma 3.1 because the bilinear form $\mathcal{A}_{\tau,0}$ is bounded

$$\mathcal{A}_{\tau,0}(\phi,\phi) \le \|\mathbb{A}_{\tau,0}\|_{\mathcal{L}(H_0^2(D))}\|\phi\|^2_{H_0^2(D)}. \tag{83}$$

To derive a finer estimate on the constant, we compute

$$\begin{aligned}\mathcal{A}_{\tau,0}(\phi,\phi) &= \int_D \frac{1}{n_0-1}|\Delta\phi+\tau^2\phi| + \tau^2|\phi|\,\mathrm{d}x\\ &\le \max\left\{1, \left\|\frac{1}{n_0-1}\right\|_{L^\infty(D)}\right\}(1+C\tau^2)\|\phi\|^2_{H_0^2(D)}\\ &\le (\tilde{C}_0+\tilde{C}_1\tau^2)\|\phi\|^2_{H_0^2(D)}\end{aligned} \tag{84}$$

for some $\tilde{C}_0, \tilde{C}_1$ independent of τ. Therefore, combining the previous inequality with the C_τ in Lemma 3.1 and computing:

$$(\tilde{C}_0+\tilde{C}_1\tau^2)^{1/2} \le C(C_0^{1/2}+C_1^{1/2}\tau) \tag{85}$$

we have the first bound in the statement of the theorem. To prove the faster convergence of the adjoint, notice that for a fixed ϵ, the adjoint of $T_\epsilon(\tau)$ (of course with respect to the A-inner product) is

$$T_\epsilon(\tau)^* = \mathbb{A}_{\tau,0}^{-1}\mathbb{B}\mathbb{A}_{\tau,\epsilon}^{-1}\mathbb{A}_{\tau,0} \tag{86}$$

since both $\mathbb{A}_{\tau,\epsilon}$ and $\mathbb{B}$ are self adjoint with respect to the standard inner product [6]. So we may compute using coercivity,

$$\begin{aligned}(T_\epsilon^*(\tau)u - T_0^*(\tau)u,\phi)_A &= \left(\mathbb{B}(\mathbb{A}_{\tau,\epsilon}^{-1}-\mathbb{A}_{\tau,0}^{-1})\mathbb{A}_{\tau,0}u,\phi\right)_{H_0^2(D)}\\ &\le \|\mathbb{B}(\mathbb{A}_{\tau,\epsilon}^{-1}-\mathbb{A}_{\tau,0}^{-1})\mathbb{A}_{\tau,0}u\|_{H_0^2(D)}\|\phi\|_{H_0^2(D)}\\ &\le \frac{1}{C}\|\mathbb{B}(\mathbb{A}_{\tau,\epsilon}^{-1}-\mathbb{A}_{\tau,0}^{-1})\mathbb{A}_{\tau,0}u\|_{H_0^2(D)}\|\phi\|_A.\end{aligned} \tag{87}$$

Let us choose $\phi = T_\epsilon^*(\tau)u - T_0^*(\tau)u$. Since $\mathbb{A}_{\tau,0}^{-1}\mathbb{B}u = \frac{1}{\tau}u \in H_0^2(D)\bigcap H^4(D)$, Lemma 3.5 yields the desired result. The last assertion in the statement of the lemma follows directly from Lemma 3.6 and (84). □

Next, we compute an asymptotic expansion for the inner product that appears in the eigenvalue correction theorem.

Lemma 4.3. *Let* $d = 2, 3$, $T_\epsilon(\tau)$ *defined by (16) for* $\epsilon \ge 0$, *and* u *be the solution to the variational problem (81). Then*

$$\begin{aligned}&((T_0(\tau)-T_\epsilon(\tau))u,u)_A\\ &\quad= \epsilon^d\frac{|B|}{\tau}\frac{n_0(0)-n_1}{(n_0(0)-1)^2}\left(\Delta u(0)^2+2\tau u(0)\Delta u(0)+\tau^2u^2(0)\right)+o(\epsilon^d)\\ &\quad= \epsilon^d\frac{|B|}{\tau}\frac{n_0(0)-n_1}{(n_0(0)-1)^2}(\Delta u(0)+\tau u(0))^2+o(\epsilon^d).\end{aligned}$$

Proof. Due to Lemma 3.7, it is sufficient to show that

$$((T_0(\tau) - T_\epsilon(\tau))u, u)_A = ((\mathbb{A}_{\tau,\epsilon} - \mathbb{A}_{\tau,0})(u_0 + \epsilon^2 \tilde{u}^{(1)}), u)_{H_0^2(D)} + o(\epsilon^d)$$

where $u_0 = \frac{1}{\tau}u$ and $\tilde{u}^{(1)}$ is defined by (34). Because $u_\epsilon = T_\epsilon(\tau)u$, we notice that $\mathbb{A}_{\tau,\epsilon}u_\epsilon = \mathbb{A}_{\tau,0}u_0 = \mathbb{B}u$ so that

$$\begin{aligned} (u_\epsilon - u_0, \phi)_A =& ((\mathbb{A}_{\tau,0} - \mathbb{A}_{\tau,\epsilon})u_\epsilon, \phi)_{H_0^2(D)} + (\mathbb{A}_{\tau,\epsilon}u_\epsilon - \mathbb{A}_{\tau,0}u_0, \phi)_{H_0^2(D)} \\ =& ((\mathbb{A}_{\tau,0} - \mathbb{A}_{\tau,\epsilon})u_\epsilon, \phi)_{H_0^2(D)}. \end{aligned} \tag{88}$$

We must now construct an approximation for this term. Let $z_\epsilon := u_\epsilon - u_0 - \epsilon^2 \tilde{u}^{(1)} \in H_0^2(D)$. We claim

$$((\mathbb{A}_{\tau,\epsilon} - \mathbb{A}_{\tau,0})z_\epsilon, \phi)_{H_0^2(D)} = o(\epsilon^d). \tag{89}$$

Indeed, using the support of $n_\epsilon - n_0$,

$$\begin{aligned} &((\mathbb{A}_{\tau,\epsilon} - \mathbb{A}_{\tau,0})z_\epsilon, \phi)_{H_0^2(D)} \\ &\quad = \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) \Delta z_\epsilon \Delta \phi \,\mathrm{d}x + \tau \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) z_\epsilon \Delta \phi \,\mathrm{d}x \\ &\quad + \tau \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) \Delta z_\epsilon \phi \,\mathrm{d}x + \tau^2 \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) z_\epsilon \phi \,\mathrm{d}x. \end{aligned}$$

The second term we may discard using Sobolev embedding and Lemma 3.3:

$$\begin{aligned} &\left| \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) z_\epsilon \Delta \phi \,\mathrm{d}x \right| \\ &\qquad \le \left\| \frac{1}{n_1 - 1} - \frac{1}{n_0 - 1} \right\|_{L^\infty(D)} \|z_\epsilon\|_{L^\infty(D)} \|\Delta \phi\|_{L^\infty} \int_{\epsilon B} \mathrm{d}x \\ &\qquad \le C \|z_\epsilon\|_{H_0^2(D)} \epsilon^d \\ &\qquad = o(\epsilon^{\frac{3d}{2}}). \end{aligned} \tag{90}$$

Similarly, we can calculate the last term by the same approach

$$\left| \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) z_\epsilon \phi \,\mathrm{d}x \right|$$

$$\le \left\| \frac{1}{n_1 - 1} - \frac{1}{n_0 - 1} \right\|_{L^\infty(D)} \|z_\epsilon\|_{L^\infty(D)} \|\phi\|_{L^\infty} \epsilon^d \tag{91}$$

$$= o(\epsilon^{\frac{3d}{2}}). \tag{92}$$

The remaining terms follow a similar technique

$$\begin{aligned} &\left| \int_{\epsilon B} \left(\frac{1}{n_1 - 1} - \frac{1}{n_0 - 1}\right) \Delta z_\epsilon \Delta \phi \,\mathrm{d}x \right| \\ &\qquad \le \left\| \frac{1}{n_1 - 1} - \frac{1}{n_0 - 1} \right\|_{L^\infty(D)} \|\Delta \phi\|_{L^\infty(D)} \int_D \chi_{\epsilon B} |\Delta z_\epsilon| \,\mathrm{d}x \\ &\qquad \le C \|\chi_{\epsilon B}\|_{L^2(D)} \|z_\epsilon\|_{H_0^2(D)} \\ &\qquad \le C \epsilon^{d/2} o(\epsilon^{d/2}) \end{aligned} \tag{93}$$

and

$$\left|\int_{\epsilon B}\left(\frac{1}{n_1-1}-\frac{1}{n_0-1}\right)\Delta z_\epsilon \phi \,\mathrm{d}x\right|$$
$$\leq \left\|\frac{1}{n_1-1}-\frac{1}{n_0-1}\right\|_{L^\infty(D)}\|\phi\|_{L^\infty(D)}\int_D \chi_{\epsilon B}|\Delta z_\epsilon|\,\mathrm{d}x$$
$$= o(\epsilon^d). \tag{94}$$

By applying Lemma 3.7,

$$((T_0(\tau)-T_\epsilon(\tau))u,\phi)_A = \epsilon^d|B|\frac{n_0(0)-n_1}{(n_0(0)-1)^2} \tag{95}$$
$$(\Delta u_0(0)\Delta\phi(0)+\tau(u_0(0)\Delta\phi(0)+\Delta u_0(0)\phi(0))+\tau^2 u_0(0)\phi(0))+o(\epsilon^d).$$

Since $u_0(0)=\frac{1}{\tau}u(0)$, we have completed the proof by taking $\phi = u$. $\square$

For the denominator in the correction theorem, we must compute the derivative of $T_0(\tau)$ with respect to τ, $DT_0(\tau)$. In fact this derivative is $DT_0(\tau)u = v$ where $v \in H_0^1(D)$ solves

(96)
$$\Delta\Delta\mathbb{A}_{\tau,0}v = \Delta\left(\frac{1}{n_0-1}\mathbb{A}_{\tau,0}^{-1}\mathbb{B}u\right)+\frac{1}{n_0-1}\Delta\mathbb{A}_{\tau,0}^{-1}\mathbb{B}u + 2\tau\left(\frac{1}{n_0-1}+1\right)\mathbb{A}_{\tau,0}^{-1}\mathbb{B}u.$$

This calculation is rigorously justified in the appendix.

Theorem 4.4. *Let $d = 2,3$, u be a solution to (81), and $T_\epsilon(\tau)$ defined by (16) for $\epsilon \geq 0$. For u chosen such that $\|u\|_A = 1$ we have that*

$$\tau_\epsilon - \tau = \epsilon^d\tau|B|\frac{n_0(0)-n_1}{(n_0(0)-1)^2}\frac{(\Delta u(0)+\tau u(0))^2}{1+\tau^2(DT_0(\tau)u,u)_A}+o(\epsilon^d)$$

if the denominator is nonzero, where

$$(DT_0(\tau)u,u)_A = -2\frac{1}{\tau}\left(\frac{1}{n_0-1}u,\Delta u+\tau u\right)_{L^2(D)} - 2\,(u,u)_{L^2(D)}\,.$$

Proof. Define $u_\epsilon = T_\epsilon(\tau)u$. We note that the condition $\|u\|_A = 1$ can be restated as

$$1 = \mathcal{A}_{\tau,0}(u,u) = \tau(\mathbb{A}_{\tau,0}\mathbb{A}_{\tau,0}^{-1}\mathbb{B}u,u)_{H_0^2(D)} = \tau\|\nabla u\|^2_{L^2(D)} \tag{97}$$

hence we can assume that $\|\nabla u\|^2_{L^2(D)} = \frac{1}{\tau}$. Therefore, Lemma 4.2 grants us the operator convergence needed for application of the nonlinear eigenvalue theorem (Theorem 4.1) and a compact $\overline{U}$ on which the convergence holds. Hence constants in Lemma 4.2 may be made uniform for $\tau \in U$, yielding

$$O\left(\sup_{\lambda\in U}\|(T_\epsilon(\lambda)-T_0(\lambda))\big|_{R(E)}\|\|(T^*_\epsilon(\lambda)-T^*_0(\lambda))\big|_{R(E)^*}\|\right) = o(\epsilon^d). \tag{98}$$

in the case of a simple eigenvalue. Applying the theorem gives the formula

$$\tau_\epsilon - \tau = \tau^2\frac{((T_0(\tau)-T_\epsilon(\tau))u,u)_A}{1+\tau^2(DT_0(\tau)u,u)_A}+o(\epsilon^d)$$
$$= \tau\epsilon^d|B|\frac{n_0(0)-n_1}{(n_0(0)-1)^2}\frac{(\Delta u(0)+\tau u(0))^2}{1+\tau^2(DT_0(\tau)u,u)_A}+o(\epsilon^d) \tag{99}$$

after substituting in the result of Lemma 4.3. The value for $(DT_0(\tau)u,u)_A$ in the statement of the theorem is given by (115) and Proposition 5.3. $\square$

As discussed in Section 3, the arguments used in the case of a single inhomogeneity carry over to multiple inhomogeneities. In particular, define

$$n(y) = \begin{cases} n_i & y \in B_i, i=1,\dots,m \\ n_0 & y \in \mathbb{R}^d \setminus B \end{cases}$$

and for $i=1,\dots,m$ define the functions $\tilde{v}_{B_i}$ and v_{Bi} as before by

$$\int_{\tilde{D}} \frac{1}{\tilde{n}-1} \Delta_y \tilde{v}_{B_i} \Delta_y \phi \,\mathrm{d}y = \int_{B_i} \Delta_y \phi(y)\,\mathrm{d}y \tag{100}$$

for any $\phi \in H_0^2(\tilde{D})$ and

$$\int_{\mathbb{R}^d} \frac{1}{\tilde{n}-1} \Delta_y v_{Bi} \Delta_y \phi \,\mathrm{d}y = \int_{B_i} \Delta_y \phi(y)\,\mathrm{d}y \tag{101}$$

for $\phi \in W_0^2(\mathbb{R}^d)$. We may then define the correction terms $\tilde{u}^{(1)}$ and $u^{(1)}$ by

$$\tilde{u}^{(1)} = -\sum_{i=1}^{m} \left(\frac{1}{n_i - 1} - \frac{1}{n_0(z_i)-1} \right) \big(\Delta u_0(z_i) + \tau u_0(z_i)\big) \tilde{v}_{B_i}(x/\epsilon) \tag{102}$$

and

$$u^{(1)} = -\sum_{i=1}^{m} \left(\frac{1}{n_i - 1} - \frac{1}{n_0(z_i)-1} \right) \big(\Delta u_0(z_i) + \tau u_0(z_i)\big) v_{Bi}(x/\epsilon) \tag{103}$$

where we note that z_i is the center of B_i. This yields our final theorem.

Theorem 4.5. *Let $d=2,3$, u be a solution to (81), and $T_\epsilon(\tau)$ defined by (16) for $\epsilon \geq 0$. Then for u chosen such that $\|u\|_A = 1$,*

$$\tau_\epsilon - \tau = \epsilon^d \tau \sum_{i=1}^{m} |B_i| \frac{n_0(z_i) - n_i}{(n_0(z_i)-1)^2} \frac{(\Delta u(z_i) + \tau u(z_i))^2}{1 + \tau^2 (DT_0(\tau)u,u)_A} + o(\epsilon^d).$$

where

$$(DT_0(\tau)u,u)_A = -2\frac{1}{\tau} \left(\frac{1}{n_0 - 1} u, \Delta u + \tau u \right)_{L^2(D)} - 2\,(u,u)_{L^2(D)}\,.$$

We conclude by remarking that the formula in Theorem 4.5 can be potentially used to obtain information about the small penetrable inclusion, more specifically the location and refractive index. We note that the transmission eigenvalues for the perturbed media τ_ϵ can be measured from scattering data, whereas the transmission eigenvalues τ and eigenvectors u for the unperturbed media can be computed since $n_0(x)$ is known.

5. **Appendix: Technical lemmas.**

In this section, we will collect the technical lemmas that are necessary for several of the results in this paper. The first lemma involves the asymptotic behavior of $\tilde{v}_B$ whereas the second computes the derivative of $T_0(\tau)$.

5.1. **Convergence of $\tilde{v}_B$.** Our goal is to prove $\epsilon^2\tilde{v}_B$ converges to 0 at the appropriate rate in the $L^2(D)$ norm. To do so, we will show that $\epsilon^2(\tilde{v}_B(x/\epsilon) - v_B(x/\epsilon)) \to 0$ in $L^2(D)$. Afterwards, we can use that $\epsilon^2 v_B$ itself converges to 0 yielding the desired result. Before we prove this, we restate a lemma from [8].

Lemma 5.1. [8] *Let $d = 2, 3$, $\tilde{v}_B$ be the solution to (29) and v_B be the solution to (30). Then we have that*

$$\|\Delta_y v_B - \Delta_y \tilde{v}_B\|_{L^2(D)} = o(\epsilon^{d/2}).$$

The proof of this lemma in [8] is for the constant n_0 case, but the identical proof holds for n_0 smooth with property (1). Since $v_B(x/\epsilon)$ is not in $H_0^2(D)$, we require a separate bound on its L^2 norm.

Proposition 5.2. *Let $d = 2, 3$. For $\tilde{v}_B$ defined by (29),*

$$\|\epsilon^2\tilde{v}_B(\cdot/\epsilon)\|_{L^2(D)} = o(\epsilon^{d/2}).$$

Proof. Recall v_B is defined by (30). We aim to bound the L^2 norm of $\epsilon^2(\tilde{v}_B(x/\epsilon) - v_B(x/\epsilon))$ by the L^2 norm of its Laplacian, which we know from Lemma 5.1 is $o(\epsilon^{d/2})$. To do so, we will construct a correction term which allows us to use Poincaré's inequality. We define v_1 to be the solution to

$$\begin{cases} \Delta_y v_1 = 0 & \text{in } D \\ v_1 = -v_B(\cdot/\epsilon) & \text{in } \partial D \end{cases} \tag{104}$$

for $d = 2, 3$. Since $v_B \in W_0^2(\mathbb{R}^d)$, $v_B \in L^\infty(\mathbb{R}^d)$ and in particular $v_B \in C^0(\partial D)$. Since the domain is C^2, v_1 is a classical solution and so $v_1 \in C^2(D)\bigcap C^0(\overline{D})$ [10]. From the maximum principle for harmonic functions,

$$\|v_1\|_{L^\infty(D)} \le \|v_1(x/\epsilon)\|_{L^\infty(\partial D)}. \tag{105}$$

The decay conditions of v_B (31) imply that $v_B(y) = o(|y|^{2-d/2}) = o(\epsilon^{d/2-2})$ for $y = x/\epsilon$ and $x \in \partial D$; furthermore the estimate is uniform in ϵ. Using the maximum principle,

$$\begin{aligned} \epsilon^2\|v_1\|_{L^2(D)} \le& \epsilon^2|D|^{1/2}C\|v_B(\cdot/\epsilon)\|_{L^\infty(\partial D)} \\ \le& \epsilon^2|D|^{1/2}Co(\epsilon^{d/2-2}) \\ =& Co(\epsilon^{d/2}). \end{aligned} \tag{106}$$

For convenience, define $V_\epsilon := \epsilon^2(\tilde{v}_B(x/\epsilon) - v_B(x/\epsilon) - v_1(x/\epsilon))$ and assume to the contrary that $\|V_\epsilon\|_{L^2(D)}/\epsilon^{d/2}$ does not converge to 0. Therefore there exists a subsequence $\epsilon_k \to 0$ and $C > 0$ such that

$$C \le \frac{\|V_{\epsilon_k}\|_{L^2(D)}}{\epsilon_k^{d/2}} \tag{107}$$

Noting that by construction $V_{\epsilon_k} \in H_0^1(D)$, Poincaré's Inequality yields

$$1 \le \frac{\|V_{\epsilon_k}\|_{L^2(D)}}{C\epsilon_k^{d/2}} \le \frac{\|V_{\epsilon_k}\|^2_{L^2(D)}}{C^2\epsilon_k^d} \le \tilde{C}_p\frac{\|\nabla V_{\epsilon_k}\|^2_{L^2(D)}}{C^2\epsilon_k^d}. \tag{108}$$

As $v_1 \in C^2(D)$, we have that $V_{\epsilon_k} \in H^2(D)\bigcap H_0^1(D)$; thus integration by parts is valid, yielding

$$\|\nabla V_{\epsilon_k}\|^2_{L^2(D)} = -(V_{\epsilon_k}, \Delta V_{\epsilon_k})_{L^2(D)} \le \|V_{\epsilon_k}\|_{L^2(D)}\|\Delta V_{\epsilon_k}\|_{L^2(D)} \tag{109}$$

Substituting this into (108), we conclude

$$\frac{\|V_{\epsilon_k}\|_{L^2(D)}}{C\epsilon^{d/2}} \leq \frac{\|V_{\epsilon_k}\|_{L^2(D)}}{\epsilon^{d/2}} \frac{\|\Delta V_{\epsilon_k}\|_{L^2(D)}}{C\epsilon^{d/2}}. \tag{110}$$

Because $\epsilon^2\Delta = \Delta_y$ and $\Delta v_1 = 0$,

$$1 \leq \frac{\|\Delta V_{\epsilon_k}\|_{L^2(D)}}{C\epsilon_k^{d/2}} = \frac{\|\Delta_y \tilde{v}_B - \Delta_y v_B\|_{L^2(D)}}{C\epsilon_k^{d/2}} = o(1) \tag{111}$$

from Lemma 5.1, which is a contradiction. Thus $\|V_k\|_{L^2(D)} = o(\epsilon^{d/2})$. To conclude, we must justify why each correction $\epsilon^2 v_1$ and $\epsilon^2 v_B$ are at least $o(\epsilon^{d/2})$ in the L^2 norm. First, (106) implies the claim for v_1, and finally, $v_B \in C^0(\mathbb{R}^d)$ implies

$$\epsilon^2\|v_B(\cdot/\epsilon)\|_{L^2(D)} \leq \epsilon^2|D|^{1/2}\|v_B\|_{L^\infty(\mathbb{R}^n)}. \tag{112}$$

□

5.2. **Derivative of $T_0(\tau)$ with respect to τ.** To apply the theorem, we must have the derivative of $T_0(\tau) = \mathbb{A}_{\tau,0}^{-1}\mathbb{B}$ with respect to τ evaluated at a function u. However, since $\mathbb{B}$ does not depend on τ, this problem is equivalent to the derivative of $\mathbb{A}_{\tau,0}^{-1}$ evaluated at $\mathbb{B}u$. Thus it is only necessary to compute $D\mathbb{A}_{\tau,0}^{-1}$. With that in mind, for $u \in H_0^2(D)$ we define the solution map L_τ to variational problem:

$$\Delta\Delta\mathbb{A}_{\tau,0}L_\tau u = \Delta\left(\frac{1}{n_0-1}\mathbb{A}_{\tau,0}^{-1}u\right) + \frac{1}{n_0-1}\Delta\mathbb{A}_{\tau,0}^{-1}u + 2\tau\left(\frac{1}{n_0-1}+1\right)\mathbb{A}_{\tau,0}^{-1}u \tag{113}$$

which exists and is bounded due to Riesz Representation. Further, define for $u \in H_0^2(D)$,

$$u_\tau = \mathbb{A}_{\tau,0}^{-1}u. \tag{114}$$

Notice that by construction,

$$\begin{aligned}(L_\tau u, \phi)_A =&(\mathbb{A}_{\tau,0}L_\tau u, \phi)_{H_0^2(D)}\\ =&\left(\frac{1}{n_0-1}u_\tau, \Delta\phi\right)_{L^2(D)} + \left(\frac{1}{n_0-1}\Delta u_\tau, \phi\right)_{L^2(D)}\\ &+ 2\tau\left(\left(\frac{1}{n_0-1}+1\right)u_\tau, \phi\right)_{L^2(D)}\end{aligned} \tag{115}$$

Proposition 5.3. *Let $d = 2, 3$ and $\tau > 0$. Then $-L_\tau$ as defined by (113) is the derivative of $\mathbb{A}_{\tau,0}^{-1}$ with respect to τ. That is, $D\mathbb{A}_{\tau,0}^{-1} = -L_\tau$.*

Proof. Observe since $\mathbb{A}_{\tau+h,0}u_{\tau+h} = \mathbb{A}_{\tau,0}u_\tau = u$,

$$\begin{aligned}(\mathbb{A}_{\tau+h,0}(u_{\tau+h} - u_\tau + hL_\tau u), \phi) =&(\mathbb{A}_{\tau+h,0}u_{\tau+h} - \mathbb{A}_{\tau+h,0}u_\tau + h\mathbb{A}_{\tau+h,0}L_\tau u, \phi)_{H_0^2(D)}\\ =&(\mathbb{A}_{\tau,0}u_\tau - \mathbb{A}_{\tau+h,0}u_\tau + h\mathbb{A}_{\tau+h,0}L_\tau u_\tau, \phi)_{H_0^2(D)}\\ =&-(2th + h^2)\int_D\left(\frac{1}{n_0-1}+1\right)u_\tau\phi\,\mathrm{d}x\\ &- h\int_D\frac{1}{n_0-1}(u_\tau\Delta\phi + \Delta u_\tau\phi)\,\mathrm{d}x\\ &+ h(\mathbb{A}_{\tau+h,0}L_\tau u, \phi)_{H_0^2(D)}.\end{aligned} \tag{116}$$

From the definition of the bilinear form, there exists a constant depending on τ and D such that

$$
\begin{aligned}
(\mathbb{A}_{\tau+h,0}u,\phi)_{H_0^2(D)} &= (\mathbb{A}_{\tau,0}u,\phi)_{H_0^2(D)} + h\left(\Delta u + \tau u, \frac{1}{n_0-1}\phi\right)_{L^2(D)} \\
&\quad + h\left(\frac{1}{n_0-1}u, \Delta\phi + \tau\phi\right)_{L^2(D)} + 2h(\tau+h)\left(\left(\frac{1}{n_0-1}+1\right)u,\phi\right)_{L^2(D)} \\
&= (\mathbb{A}_{\tau,0}u,\phi)_{H_0^2(D)} + O(h\|u\|_{H_0^2(D)}\|\phi\|_{H_0^2(D)}) \qquad (117)
\end{aligned}
$$

where the above estimate uses that H_0^2 is embedded in C^0. Using the above inequality and (115),

$$
\begin{aligned}
(\mathbb{A}_{\tau+h,0}L_\tau u,\phi)_{H_0^2(D)} &= \left(\frac{1}{n_0-1}u_\tau, \Delta\phi\right)_{L^2(D)} + \left(\frac{1}{n_0-1}\Delta u_\tau, \phi\right)_{L^2(D)} \\
&\quad + 2\tau\left(\left(\frac{1}{n_0-1}+1\right)u_\tau,\phi\right)_{L^2(D)} + O\left(h\|u_\tau\|_{H_0^2(D)}\|\phi\|_{H_0^2(D)}\right). \qquad (118)
\end{aligned}
$$

Substituting this into (116) yields

$$
\begin{aligned}
(\mathbb{A}_{\tau+h,0}(u_{\tau+h}-u_\tau+hL_\tau u),\phi) &= -h^2\int_D\left(\frac{1}{n_0-1}+1\right)u_\tau\phi \mathrm{d}x + O(h^2\|\phi\|_{H_0^2(D)}) \\
&\le Ch^2\left(\frac{1}{n_0-1}+1\right)\|u_\tau\|_{L^2(D)}\|\phi\|_{H_0^2(D)} \\
&\quad + O\left(h^2\|u_\tau\|_{H_0^2(D)}\|\phi\|_{H_0^2(D)}\right). \qquad (119)
\end{aligned}
$$

Of course, we have the bound

$$
\|u_\tau\|_{L^2(D)} \le C\|u_\tau\|_{H_0^2(D)} \le C\|A_{\tau,0}^{-1}\|_{\mathcal{L}(H_0^2(D))}\|u\|_{H_0^2(D)} \qquad (120)
$$

Choosing $\phi = u_{\tau+h} - u_\tau + hL_\tau u$, we have by coercivity that

$$
C\|u_{\tau+h} - u_\tau + hL_h u\|_{H_0^2(D)} = O\left(h^2\|u\|_{H_0^2(D)}\right) \qquad (121)
$$

where C can be chosen to be independent of τ. To finish, we divide by $h\|u\|_{H_0^2(D)}C$ and take the supremum over $u \in H_0^2(D)$,

$$
\frac{\|\mathbb{A}_{\tau+h,0}^{-1} - \mathbb{A}_{\tau,0}^{-1} + hL_\tau\|_{\mathcal{L}(H_0^2(D))}}{h} = O(h). \qquad (122)
$$

Therefore the Frechet derivative $D\mathbb{A}_{\tau,0}^{-1}(\tau) = -L_\tau$. □

REFERENCES

[1] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand-Reinhold, Princeton, NJ, 1965.

[2] C. Amrouche and M. Fontes, Biharmonic problem in exterior domains of $\mathbb{R}^n$: An approach with weighted Sobolev spaces, *J. Math. Anal. Appl.,* **304** (2005), 552–571.

[3] F. Cakoni, D. Colton and H. Haddar, On the determination of Dirichlet or transmission eigenvalues from far field data, *C. R. Acad. Sci. Paris,* **348** (2010), 379–383.

[4] F. Cakoni, D. Colton and P. Monk, On the use of transmission eigenvalues to estimate the index of refraction from far field data, *Inverse Problems,* **23** (2007), 507–522.

[5] F. Cakoni, D. Gintides and H. Haddar, The existence of an infinite discrete set of transmission eigenvalues, *SIAM J. Math. Anal.,* **42** (2010), 237–255.

[6] F. Cakoni and H. Haddar, On the existence of transmission eigenvalues in an inhomogeneous medium, *Appl. Anal.,* **88** (2009), 475–493.

[7] ______, Transmission eigenvalues in inverse scattering theory, in *Inverse Problems and Applications: Inside Out. II*, Math. Sci. Res. Inst. Publ., 60, Cambridge Univ. Press, Cambridge, 2013, 529–580.

[8] F. Cakoni and S. Moskow, Asymptotic expansions for transmission eigenvalues for media with small inhomogeneities, *Inverse Problems,* **29** (2013), 104014, 18pp.

[9] D. Colton, L. Päivärinta and J. Sylvester, The interior transmission problem, *Inverse Problems and Imaging,* **1** (2007), 13–28.

[10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, reprint of 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[11] G. Giorgi and H. Haddar, Computing estimates of material properties from transmission eigenvalues, *Inverse Problems,* **28** (2012), 055009, 23pp.

[12] I. Godberg and P. Lancaster, *Indefinite Linear Algebra and Applications*, L. Rodman, 2005.

[13] A. Kirsch and A. Lechleiter, The inside-outside duality for scattering problems by inhomogeneous media, *Inverse Problems,* **29** (2013), 104011, 21pp.

[14] S. Moskow, Nonlinear eigenvalue approximation for compact operators, preprint, 2014.

[15] J. E. Osborn, Spectral approximation for compact operators, *Math. Comput.,* **29** (1975), 712–725.

[16] L. Päivärinta and J. Sylvester, Transmission eigenvalues, *SIAM J. Math. Anal*, **40** (2008), 738–753.

[17] L. Robbiano, Spectral analysis of the interior transmission eigenvalue problem, *Inverse Problems,* **29** (2013), 104014, 28pp.

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