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Asymptotic analysis of the transmission eigenvalue problem for a Dirichlet obstacle coated by a thin layer of non-absorbing media

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We consider the transmission eigenvalue problem for an impenetrable obstacle with Dirichlet boundary condition surrounded by a thin layer of non-absorbing inhomogeneous material. We derive a rigorous asymptotic expansion for the first transmission eigenvalue with respect to the thickness of the thin layer. Our convergence analysis is based on a Max–Min principle and an iterative approach which involves estimates on the corresponding eigenfunctions. We provide explicit expressions for the terms in the asymptotic expansion up to order 3.

Keywords: transmission eigenvalues; thin layers; asymptotic methods; inverse scattering.

1. Introduction

Transmission eigenvalues appear in the study of scattering by inhomogeneous media and are closely related to non-scattering frequencies (Cakoni & Haddar, 2012; Blasten *et al.*, 2014). Such eigenvalues provide information about material properties of the scattering media (Cakoni *et al.*, 2010b) and can be determined from scattering data (Cakoni *et al.*, 2010a; Kirsch & Lechleiter, 2013). Hence, they can play an important role in a variety of inverse problems in target identification and non-destructive testing (Giorgi & Haddar, 2012). The transmission eigenvalue problem is a non-self-adjoint and nonlinear problem that is not covered by the standard theory of eigenvalue problems for elliptic operators. In the past few years, transmission eigenvalues have become an important area of research in inverse scattering theory. Since the first proof of existence of transmission eigenvalues in Cakoni *et al.* (2010b) and Päivärinta & Sylvester (2008), the interest in the transmission eigenvalue problem has increased, resulting in a number of important advancements. For an update survey on the topic, we refer the reader to Cakoni & Haddar (2012).

In this paper, we consider the transmission eigenvalue problem corresponding to the scattering by an impenetrable obstacle with Dirichlet boundary condition coated by a thin layer of non-absorbing inhomogeneous material. The existence and discreteness of transmission eigenvalue problem is investigated in Cakoni *et al.* (2010a) (see also Lakshtanov & Vainberg, 2013). In the 2D case, this problem models the scattering of TE-polarized electromagnetic waves (written in terms of the electric field) by an infinitely long cylindrical prefect conductor coated by a thin layer of non-magnetic dielectric material. In the 3D case, it models the scattering of acoustic waves by a sound-soft object surrounded by acoustically non-absorbing material. It is well known (see, e.g. Bendali & Lemrabet, 1996) that the first-order approximation to the scattering problem for a coated perfect conductor is an exterior boundary value problem with impedance-type boundary condition where the impedance function depends inverse proportionally to the thickness of the layer, here denoted by δ. The corresponding 'non-scattering' frequencies for this approximate model become the eigenvalues of a non-coercive Robin eigenvalue problem, which is studied by the authors of this paper in Cakoni *et al.* (2013).

The main concern of this study is to develop a rigorous asymptotic expansion for transmission eigenvalues as $\delta \to 0$. Our asymptotic analysis is based on an iterative and constructive approach. We restrict ourselves here to the first transmission eigenvalue. As expected this transmission eigenvalue is close to the first Dirichlet eigenvalue up to order δ, result that is proved directly in this paper by using the Max–Min principle. Then, the main idea of our approach is, roughly speaking, having proved convergence of order k for the asymptotic expansion of the transmission eigenvalue, we next prove estimates of order k for the corresponding eigenfunctions by using standard approximation results for the eigenfunctions of the negative Laplacian with Dirichlet boundary conditions. Then, we deduce convergence at order $k+1$ for the eigenvalues by using the Max–Min principle. Similar techniques have already been successfully used in the asymptotic analysis of eigenvalue problems in various settings, see Dauge *et al.* (1999), Bonnaillie-Noël & Dauge (2006), Bonnaillie-Noël *et al.* (2007), Schmidt & Tordeux (2010) and Bendali *et al.* (2012), for example. Although our analysis can in principle be carried through for any order, for sake of simplicity we provide here explicit expressions only for the terms up to order 3 in the asymptotic expansion of the first transmission eigenvalue. The explicit construction of the asymptotic expansion is simplified by the fact that the first eigenvalue of the Dirichlet problem is simple. The extension of our analysis to higher order transmission eigenvalues is challenging, first because explicit construction of the asymptotic is complicated and second because one looses the characterization of the transmission eigenvalues in terms of a Max–Min principle.

From practical point of view, the second-order expansion provides in fact a formula for the thickness of the layer in terms of the first (measurable) transmission eigenvalue. Unfortunately, the refractive index of the layer does not appear in the first three terms of the asymptotic expansion. Of course, the refractive index will show in higher order terms but then the obtained reconstruction formula would be highly unstable with respect to noise in the transmission eigenvalue. A better model to capture both the thickness and the refractive index in the first-order term in the context of electromagnetic scattering is to write the problem in terms of the magnetic field, which would lead to Neumann boundary condition on the boundary of the inclusion. Unfortunately, the transmission eigenvalue problem for inhomogeneous media containing an inclusion with Neumann boundary condition is still open. Moreover, no Max–Min principle is available in this case which is the corner stone of our approach.

The structure of the paper is as follows. In the next section, we formulate the problem and recall some relevant results on the transmission eigenvalue problem for an inhomogeneous media containing an inclusion with Dirichlet boundary condition. In Section 3, we derive the formal asymptotic expansion for transmission eigenvalues and provide explicit formulas for the terms up to order 3. In Section 4, we provide technical elliptic *a priori* estimates and trace lemma with explicit dependance on δ that are needed to justify the formal asymptotic development. Section 5 is dedicated to the rigorous convergence proof of the asymptotic expansion derived in Section 3 for the first transmission eigenvalue. Finally, some classical convergence results for the spectrum of compact operators are given in Appendix A.

2. Formulation of the problem

We consider an impenetrable object coated with a thin layer of non-absorbing penetrable material with refractive index n which occupies the region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ where Ω is bounded and simply connected with smooth enough (to become precise later) boundary Γ. We denote by

$$\Omega_\delta = \{x \in \Omega \text{ such that } d(x, \Gamma) > \delta\},$$

and by

$$\Gamma_\delta = \{x \in \Omega \text{ such that } d(x, \Gamma) = \delta\}$$

its boundary. The simply connected domain Ω_δ (see Fig. 1) represents here the impenetrable object and $\Omega \setminus \overline{\Omega_\delta}$ represents the thin layer. The scattering of an incident wave u^i, which here for simplicity is assumed to be an entire solution of the Helmholtz equation (one could also consider the incident field to be a point source located outside Ω), by such a structure gives rise to a scattered field $u^s = u - u^i$, with u being the total field, that satisfies

$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } \Omega \setminus \overline{\Omega_\delta}, \\ \Delta u + k^2 u = 0 \quad \text{in } \Omega_{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}, \\ \left[\dfrac{\partial u}{\partial \boldsymbol{\nu}}\right] = 0, \quad [u] = 0 \text{ on } \Gamma, \\ u = 0 \quad \text{on } \Gamma_\delta, \\ \lim\limits_{R \to \infty} \displaystyle\int_{|x|=R} |\partial_r u^s - \mathrm{i} k u^s|^2 \, \mathrm{d}s = 0, \end{cases} \tag{2.1}$$

where k is the wave number, $n \in L^\infty(\Omega \setminus \overline{\Omega_\delta})$ is the index of refraction of the layer such that $n \geqslant n_0 > 0$, $\boldsymbol{\nu}$ is unitary normal to Γ directed inward to Ω and $[v] = v^+ - v^-$ denotes the jump of v across Γ where v^+ is the exterior trace of v and v^- is the interior trace of v on Γ. The corresponding transmission eigenvalue problem is to find the values of k_δ^2 such that there exists a non-trivial solution $(w_\delta, v_\delta) \in L^2(\Omega \setminus \overline{\Omega_\delta}) \times L^2(\Omega)$ to the following homogeneous coupled problem:

$$\begin{cases} \Delta w_\delta + k_\delta^2 n w_\delta = 0 \quad \text{in } \Omega \setminus \overline{\Omega_\delta}, \\ \Delta v_\delta + k_\delta^2 v_\delta = 0 \quad \text{in } \Omega, \\ \dfrac{\partial v_\delta}{\partial \boldsymbol{\nu}} = \dfrac{\partial w_\delta}{\partial \boldsymbol{\nu}}, \quad v_\delta = w_\delta \text{ on } \Gamma, \\ w_\delta = 0 \quad \text{on } \Gamma_\delta. \end{cases} \tag{2.2}$$

DEFINITION 2.1 The values $k_\delta^2 > 0$ for which (2.2) has a non-trivial solution $(w_\delta, v_\delta) \in L^2(\Omega \setminus \overline{\Omega_\delta}) \times L^2(\Omega)$ are called transmission eigenvalues, and the non-zero solutions w_δ and v_δ the associated eigenfunctions.

It is shown in Cakoni *et al.* (2010a) that the real transmission eigenvalues (the wavenumber k is related to the interrogating frequency) can be determine from measured far field (or near field) scattering data. Note that the transmission eigenvalue problem is non-self-adjoint and complex eigenvalues may occur but, from practical point of view as discussed in Section 1 and the fact that only real transmission eigenvalues are proved to exist, here we are interested only on real transmission eigenvalues (see, e.g.

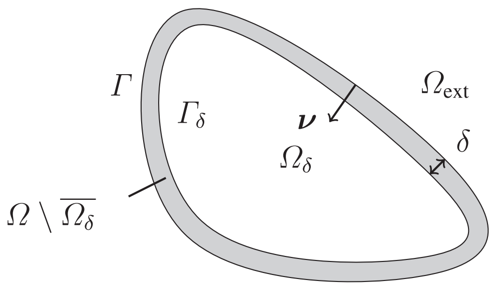

FIG. 1. The scattering layered object.

Cakoni & Haddar, 2012). Our main goal in this paper is to derive rigorous asymptotic expansions for transmission eigenvalues in terms of the thickness of the layer δ as $\delta \to 0$.

The transmission eigenvalue problem for an inhomogeneity containing an impenetrable inclusion with Dirichlet boundary condition is investigated in Cakoni *et al.* (2012) and Lakshtanov & Vainberg (2013) (our problem (2.2) is exactly of that form) where the discreteness and existence of real transmission eigenvalues is shown under appropriate assumptions on the refractive index n. For the sake of reader's convenience and later use, we summarize the main results from Cakoni *et al.* (2012).

The first step in the analysis of (2.2) consists in reformulating it as an eigenvalue problem for a forth-order equation. To this end, introducing

$$u_\delta = \begin{cases} w_\delta - v_\delta & \text{in } \Omega \setminus \overline{\Omega_\delta}, \\ -v_\delta & \text{in } \Omega_\delta, \end{cases} \tag{2.3}$$

we obtain that this u_δ satisfies

$$(\Delta + k_\delta^2) \frac{1}{1-n} (\Delta + k_\delta^2 n) u_\delta = 0 \quad \text{in } \Omega \setminus \overline{\Omega_\delta}. \tag{2.4}$$

Equation (2.4) together with the fact that u_δ must be in $H_0^1(\Omega)$ and satisfy the Helmholtz equation in Ω_δ suggest that to arrive at a variational formulation equivalent to the eigenvalue problem (2.2) we need to introduce the space

$$W_\delta := \left\{ u \in H_0^1(\Omega) \cap H_\Delta^1(\Omega \setminus \overline{\Omega_\delta}) \text{ such that } \frac{\partial u}{\partial \boldsymbol{\nu}} = 0 \text{ on } \Gamma \right\},$$

equipped with the norm

$$\|u\|_{W_\delta}^2 := \|u\|_{H^1(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega \setminus \overline{\Omega_\delta})}^2.$$

Then it is shown in Cakoni *et al.* (2012) that $k_\delta^2 > 0$ is a transmission eigenvalue (according to Definition 2.1) with associated eigenfunctions (w_δ, v_δ) if and only if u_δ defined by (2.3) solves

$$A_{k_\delta} u_\delta - k_\delta^4 B u_\delta = 0, \tag{2.5}$$

where the bounded linear self-adjoint operators $A_k : W_\delta \to W_\delta$ and $B : W_\delta \to W_\delta$ are given by

$$(A_k u, v)_{W_\delta} := \int_{\Omega\setminus\overline{\Omega_\delta}} \frac{1}{1-n}(\Delta u + k^2 u)(\Delta\bar{v} + k^2\bar{v})\,\mathrm{d}x + k^4\int_\Omega u\bar{v}\,\mathrm{d}x + k^2\int_\Omega \nabla u\cdot\nabla\bar{v}\,\mathrm{d}x,$$

$$(Bu, v)_{W_\delta} := 2\int_\Omega u\bar{v}\,\mathrm{d}x.$$

In the following, we denote $n_* = \inf_{\Omega\setminus\overline{\Omega_\delta}} n(x)$ and $n^* = \sup_{\Omega\setminus\overline{\Omega_\delta}} n(x)$. The operators A_k and B satisfy the following properties.

PROPOSITION 2.2 Assume that $0 < n_* < n(x) < n^* < 1$. Then B is a compact operator and there exists a constant $C > 0$ such that for all $\delta > 0$

$$|(A_k u, u)_{W_\delta}| \geqslant C\|u\|^2_{W_\delta}.$$

Proof. The proof can be found in Cakoni *et al.* (2012, Theorem 2.1). The fact that the coercivity constant C is independent of δ is clear in this proof. □

We remark that if k_δ and $u_\delta \neq 0$ satisfy (2.5), then (w_δ, v_δ) are obtained from u_δ by

$$w_\delta = \frac{1}{k_\delta^2(1-n)}(\Delta u_\delta + k_\delta^2 u_\delta) \quad \text{in } \Omega\setminus\overline{\Omega_\delta}, \tag{2.6}$$

$$v_\delta = \begin{cases} -u_\delta & \text{in } \Omega_\delta, \\ \dfrac{1}{k_\delta^2(1-n)}(\Delta u_\delta + k_\delta^2 n u_\delta) & \text{in } \Omega\setminus\overline{\Omega_\delta}. \end{cases} \tag{2.7}$$

The following result proved in Cakoni *et al.* (2012) is the starting point of our discussion.

THEOREM 2.3 Assume that $0 < n_* < n(x) < n^* < 1$. There exists an infinite discrete set of transmission eigenvalues and $+\infty$ is the only accumulation point.

At this point, we choose to normalize the w_δ and v_δ so that

$$\|u_\delta\|_{L^2(\Omega)} = 1.$$

The following regularity result for the eigenfunctions (w_δ, v_δ) holds true.

LEMMA 2.4 Assume that Γ is a C^{k+2}-boundary and $n \in C^{k+2}(\overline{\Omega}\setminus\Omega_\delta)$ with $k \geqslant 2$. Then $w_\delta \in H^k(\Omega\setminus\overline{\Omega_\delta})$ and $v_\delta \in H^k(\Omega)$.

Proof. First since $\Delta v_\delta = -k_\delta^2 v_\delta$, using interior elliptic regularity for the Laplacian, we have that $v_\delta \in C^\infty(\omega)$ for all open set $\omega \subset\subset \Omega$. Hence, its trace and its normal derivative trace on Γ_δ are in $H^{k+2-1/2}(\Gamma_\delta)$ and $H^{k+2-3/2}(\Gamma_\delta)$, respectively. Using the same argument but this time for the Laplace operator with homogeneous Dirichlet boundary condition on Γ_δ, we can conclude that the trace of the normal derivative of w_δ on Γ_δ is also in $H^{k+2-3/2}(\Gamma_\delta)$. Hence, we can easily obtain that on Γ_δ we have

$$\begin{cases} \Delta u_\delta \in H^{k+2-1/2}(\Gamma_\delta), \\ \dfrac{\partial}{\partial \boldsymbol{\nu}_\delta}\Delta u_\delta \in H^{k+2-3/2}(\Gamma_\delta), \end{cases}$$

where $\boldsymbol{\nu}_\delta$ is the unit normal to Γ_δ directed inward Ω_δ. Since u_δ satisfies (2.4) in $\Omega\setminus\overline{\Omega_\delta}$ with homogeneous boundary conditions on Γ, regularity results for the bilaplacian implies that u_δ is in $H^{k+2}(\Omega\setminus\overline{\Omega_\delta})$ (see, e.g. Agmond, 1965). We finally obtain the result by using (2.6) and (2.7). $\square$

From now on, we assume that the refractive index satisfies

$$0 < n_* < n(x) < n^* < 1.$$

This assumption ensure existence of the interior transmission eigenvalues (Theorem 2.3) but is more restrictive than the one proposed in Lakshtanov & Vainberg (2013) that allows n to be greater than 1 provided the thickness of the layer is sufficiently small. Nevertheless, when $n > 1$ the operator A_{k_δ} is sign indefinite and we loose the Max–Min principle which is the main ingredient of our approach.

3. Formal asymptotic expansion

3.1 *Preliminary material*

For the sake of simplicity, here we perform the asymptotic expansion in 2D case. The extension to 3D case is purely a technical issue and it is possible to obtain similar asymptotic expansions by using the same approach. Having limited ourselves to the 2D case and assuming that the boundary is C^{k+2}-smooth for $k \geqslant 2$, we can parametrize Γ as

$$\Gamma = \{x_\Gamma(s),\ s \in [0; s_0]\},$$

where the periodic function $x_\Gamma : [0; s_0] \to \mathbb{R}^2$ is in $C^{k+2}([0; s_0])$ for some $s_0 > 0$. Moreover, we can choose this parametrization such that the tangent vector $\boldsymbol{\tau}(s) := (\mathrm{d}x_\Gamma/\mathrm{d}s)(s)$ to the surface Γ at the arbitrary point $x_\Gamma(s)$ is a unit vector. Then denoting by $\boldsymbol{\nu}(s)$ the inward unit normal vector to Γ at the point $x_\Gamma(s)$ and we can define the curvature $\kappa(s)$ by

$$\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s}(s) = -\kappa(s)\boldsymbol{\nu}(s).$$

Based on this parametrization of the curve Γ, we obtain the following parametrization of the surface Γ_δ:

$$\Gamma_\delta = \{x_\Gamma(s) + \delta(s)\boldsymbol{\nu}(s),\ s \in [0; s_0]\}, \tag{3.1}$$

where $\delta \in C^\infty([0; s_0])$ is a periodic function of sufficiently small values. Let us define by

$$\eta_0 := \inf_{s\in[0,s_0]} \frac{1}{|\kappa(s)|},$$

and $\Omega_0 := \{x \in \mathbb{R}^2,\ \operatorname{dist}(x, \Gamma) \leqslant \eta_0\}$. Then the map

$$\begin{aligned}\varphi\,:\,[0, s_0] \times [-\eta_0, \eta_0] &\longrightarrow \Omega_0,\\ (s, \eta) &\longmapsto x_\Gamma(s) + \eta \boldsymbol{\nu}(s)\end{aligned}$$

is a C^{k+2}-diffeomorphism, in other words, for every point $x \in \Omega_0$ there exists a unique $(s, \eta) \in [0, s_0] \times [-\eta_0, \eta_0]$ such that

$$x = x_\Gamma(s) + \eta \boldsymbol{\nu}(s).$$

Next, for any function u defined on Ω_0 we can define $\tilde{u}$ in $[0, s_0] \times [-\eta_0, \eta_0]$ by

$$\tilde{u}(s, \eta) := u \circ \varphi(s, \eta), \tag{3.2}$$

and the gradient of u in the local coordinates (s, η) writes as

$$\nabla u = \frac{1}{(1 + \eta\kappa)} \frac{\partial}{\partial s} \tilde{u} \boldsymbol{\tau} + \frac{\partial}{\partial \eta} \tilde{u} \boldsymbol{\nu}.$$

Furthermore, using integration by parts we have that the divergence of a vector field $\vec{u} = u_\tau \boldsymbol{\tau} + u_n \boldsymbol{\nu}$ writes as

$$\operatorname{div} \vec{u} = \frac{1}{(1 + \eta\kappa)} \frac{\partial}{\partial s} \tilde{u}_\tau + \frac{1}{(1 + \eta\kappa)} \frac{\partial}{\partial \eta} (1 + \eta\kappa) \tilde{u}_n.$$

We finally denote by $J_{s,\eta} := |\det(\nabla\varphi(s, \eta))| = 1 + \eta\kappa(s)$ the Jacobian of the change of variables.

3.2 *Formal derivation of the asymptotic expansion*

Let us now turn our attention to the transmission eigenvalue problem (2.2). To be able to carry on our computations, we assume that the function δ used in (3.1) to define the interior boundary is of the form $\delta(s) = \delta_0 g(s)$ for some constant $\delta_0 > 0$ and some strictly positive C^∞-function g independent of δ_0 such that $|\delta_0 g(s)| < \eta_0$. To simplify the notation and since there is no ambiguity, we make no distinction between g as a function of local and global variables. Then, we postulate the following ansatz for the interior transmission eigenvalues and the associated eigenfunctions:

$$\begin{aligned} k_\delta^2 &= \sum_{j=0}^{\infty} \delta_0^j \lambda_j, \\ w_\delta(x) = \hat{w}_\delta(s, \xi) &= \sum_{j=0}^{\infty} \delta_0^j \hat{w}_j(s, \xi), \\ v_\delta(x) &= \sum_{j=0}^{\infty} \delta_0^j v_j(x), \end{aligned} \tag{3.3}$$

for $\xi = \eta/\delta_0$. We remark that the functions $\hat{w}_j$ are defined on $\mathcal{G} := \{(s, \xi) \in [0, s_0] \times [0, \max(g)],\ \xi \leqslant g(s)\}$ which is independent of δ_0 and we define $w_k(x) = \hat{w}_k(s, \eta/\delta_0)$. Using (2.2) and the expressions for the gradient and divergence operators in the local coordinates, we obtain that $(\hat{w}_\delta, v_\delta)$ satisfies

$$\frac{1}{(1+\xi\delta_0\kappa)}\frac{\partial}{\partial s}\frac{1}{(1+\xi\delta_0\kappa)}\frac{\partial}{\partial s}\hat{w}_\delta + \frac{1}{\delta_0^2(1+\xi\delta_0\kappa)}\frac{\partial}{\partial \xi}\frac{1}{(1+\xi\delta_0\kappa)}\frac{\partial}{\partial \xi}\hat{w}_\delta + k_\delta^2 n\hat{w}_\delta = 0 \quad \text{in } \mathcal{G}, \tag{3.4}$$

together with the boundary conditions

$$\begin{cases} \hat{w}_\delta(s, g(s)) = 0, & s \in [0, s_0], \\ \hat{w}_\delta(s, 0) = \tilde{v}_\delta(s, 0), & s \in [0, s_0], \\ \left.\dfrac{1}{\delta_0}\dfrac{\partial \hat{w}_\delta}{\partial \xi}\right|_{\xi=0} = \left.\dfrac{\partial \tilde{v}_\delta}{\partial \eta}\right|_{\eta=0}, & s \in [0, s_0], \end{cases}$$

where $\tilde{v}_\delta$ is defined by (3.2). Let us multiply (3.4) by $\delta_0^2(1+\xi\delta_0\kappa)^3$ to obtain

$$\sum_{k=0}^{5} \delta_0^k A_k \hat{w}_\delta = 0,$$

where the $(A_k)_{k=0,\dots,5}$ are differential operators of order 2 at maximum with the following expression for the few first terms:

$$\begin{aligned} A_0 &= \frac{\partial^2}{\partial \xi^2}, \\ A_1 &= 3\xi\kappa\frac{\partial^2}{\partial \xi^2} + \kappa\frac{\partial}{\partial \xi}, \\ A_2 &= 3\xi^2\kappa^2\frac{\partial^2}{\partial \xi^2} + 2\xi\kappa^2\frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial s^2} + \lambda_0 n, \\ A_3 &= \cdots . \end{aligned}$$

Hence, by equating the terms of same order in δ, the function $\hat{w}_k$ for $k \in \mathbb{N}$, solves

$$A_0\hat{w}_k = -\sum_{l=1}^{5} A_l \hat{w}_{k-l} \quad \text{in } \mathcal{G}, \tag{3.5}$$

$$\hat{w}_k(s, g(s)) = 0, \quad s \in [0, s_0], \tag{3.6}$$

$$\hat{w}_k(s, 0) = \tilde{v}_k(s, 0), \quad s \in [0, s_0], \tag{3.7}$$

$$\frac{\partial \hat{w}_k}{\partial \xi}(s, 0) = \frac{\partial \tilde{v}_{k-1}}{\partial \eta}(s, 0), \quad s \in [0, s_0], \tag{3.8}$$

with the convention that $\hat{w}_k = v_k = 0$ for negative k. The functions v_k also satisfy

$$\Delta v_k + \lambda_0 v_k = -\sum_{l=1}^{k} \lambda_l v_{k-l}. \tag{3.9}$$

Now we can easily obtain the formal expansion at any order by solving (3.5–3.9) recursively.

Order 0.
From (3.5), we have

$$\frac{\partial^2 \hat{w}_0}{\partial^2 \xi} = 0 \quad \text{in } \mathcal{G},$$

and using the boundary conditions (3.6) and (3.8) we obtain $\hat{w}_0 = 0$ on $\mathcal{G}$. Equation (3.7) together with (3.9) give that (λ_0, v_0) solves

$$\begin{cases} \Delta v_0 + \lambda_0 v_0 = 0 & \text{in } \Omega, \\ v_0 = 0 & \text{on } \Gamma, \end{cases} \tag{3.10}$$

and hence we define (λ_0, v_0) as being an eigenpair of the $-\Delta$ in Ω with Dirichlet boundary condition and $\|v_0\|_{L^2(\Omega)} = 1$. We remark that v_0 is not uniquely determined (since it can be any Dirichlet eigenfunction), but this will be made precise later in the convergence analysis. Nevertheless, we assume that λ_0 is simple, which is the case, for example, for the first Dirichlet eigenvalue of $-\Delta$ in a Lipschitz and connected domain. The latter assumption is necessary to simplify the formal analysis to come.

Order 1.
Having determined $\hat{w}_0$ and v_0, we iterate the process and obtain that $\hat{w}_1$ is the solution to

$$\frac{\partial^2 \hat{w}_1}{\partial \xi^2} = 0 \quad \text{in } \mathcal{G},$$

with boundary conditions (3.6) and (3.8). That gives

$$\hat{w}_1(s, \xi) = \frac{\partial \tilde{v}_0}{\partial \eta}(s, 0)\xi - g(s)\frac{\partial \tilde{v}_0}{\partial \eta}(s, 0) \quad \text{in } \mathcal{G}.$$

The function v_1 solves

$$\begin{cases} \Delta v_1 + \lambda_0 v_1 = -\lambda_1 v_0 & \text{in } \Omega, \\ v_1 = -g\dfrac{\partial v_0}{\partial \boldsymbol{\nu}} & \text{on } \Gamma. \end{cases} \tag{3.11}$$

Since λ_0 is a simple eigenvalue for the operator $-\Delta$ with a Dirichlet boundary condition, to ensure uniqueness of v_1 we have to constraint v_1 to be orthogonal to v_0 in $L^2(\Omega)$. This compatibility condition gives a unique definition for λ_1. By multiplying the first equation of (3.11) by v_0 and by integrating by part, we obtain

$$\lambda_1 = \int_\Gamma g \left| \frac{\partial v_0}{\partial \boldsymbol{\nu}} \right|^2 \mathrm{d}s. \tag{3.12}$$

Here we see the simplification due to the assumption that λ_0 is simple. If this does not hold, then the definition of λ_1 does not seem to be obvious.

Order 2. To obtain the next term in the asymptotic expansion, we iterate the process once more, which yields to the following equation for $\hat{w}_2$:

$$\frac{\partial^2 \hat{w}_2}{\partial \xi^2} + \kappa \frac{\partial \hat{w}_1}{\partial \xi} = 0 \quad \text{in } \mathcal{G}.$$

The boundary conditions (3.6) and (3.8) on Γ and Γ_δ, respectively, imply that $\hat{w}_2$ is given by

$$\hat{w}_2(s,\xi) = -\frac{\kappa(s)}{2}\frac{\partial \tilde{v}_0}{\partial \eta}(s,0)\xi^2 + \frac{\partial \tilde{v}_1}{\partial \eta}(s,0)\xi + \frac{\kappa(s)}{2}\frac{\partial \tilde{v}_0}{\partial \eta}(s,0)g(s)^2 - \frac{\partial \tilde{v}_1}{\partial \eta}(s,0)g(s) \quad \text{in } \mathcal{G},$$

where κ is the curvature defined in Section 3.1. From this, we deduce that v_2 solves

$$\begin{cases} \Delta v_2 + \lambda_0 v_2 = -\lambda_1 v_1 - \lambda_2 v_0 & \text{in } \Omega, \\ v_2 = \dfrac{\kappa}{2}\dfrac{\partial v_0}{\partial \boldsymbol{\nu}} g^2 - \dfrac{\partial v_1}{\partial \boldsymbol{\nu}} g & \text{on } \Gamma. \end{cases} \tag{3.13}$$

Once more, we have to constraint v_2 to be orthogonal to v_0 and this uniquely defines λ_2 as being

$$\lambda_2 = -\int_\Gamma \left(\frac{\kappa}{2}\frac{\partial v_0}{\partial \boldsymbol{\nu}} g^2 - \frac{\partial v_1}{\partial \boldsymbol{\nu}} g\right)\frac{\partial v_0}{\partial \boldsymbol{\nu}}\,\mathrm{d}s. \tag{3.14}$$

Order k. Now it becomes clear how to recursively obtain each of the terms in the asymptotic expansion. In particular, for $k > 1$ we assume that the functions $\hat{w}_l$ and v_l as well as the real numbers λ_l are well defined for $l < k$. Assume moreover that for all $0 < l < k$,

$$\int_\Omega v_l v_0\,\mathrm{d}x = 0,$$

and that $\|v_0\|_{L^2(\Omega)} = 1$. The first step consists in computing $\hat{w}_k$ by solving (3.5) together with the boundary conditions (3.6) and (3.8). This uniquely determines $\hat{w}_k$ which leads to an explicit formula for $\hat{w}_k$. Then, by (3.7) and (3.9), v_k is uniquely defined as being the solution to

$$\begin{cases} \Delta v_k + \lambda_0 v_k = -\displaystyle\sum_{l=1}^{k} \lambda_l v_{k-l} & \text{in } \Omega, \\ v_k = w_k & \text{on } \Gamma, \\ \displaystyle\int_\Omega v_k v_0\,\mathrm{d}s = 0, & \end{cases}$$

where the last equation uniquely defines λ_k as being

$$\lambda_k = -\int_\Gamma w_k \frac{\partial v_0}{\partial \boldsymbol{\nu}}\,\mathrm{d}s.$$

Of course, the asymptotic expansion obtained above is only formal at this point. The next section is dedicated to its convergence analysis.

4. Elliptic estimates for interior transmission problems in the presence of thin layers

The main ingredient to justify the formal expansion above is to establish explicit *a priori* estimates with respect to δ for the solutions of the interior transmission problem

$$\begin{cases} \Delta w_\delta = f_1 & \text{in } \Omega \setminus \overline{\Omega_\delta}, \\ \Delta v_\delta = f_2 & \text{in } \Omega, \\ \dfrac{\partial v_\delta}{\partial \boldsymbol{\nu}} - \dfrac{\partial w_\delta}{\partial \boldsymbol{\nu}} = f_3, \quad v_\delta - w_\delta = f_4 & \text{on } \Gamma, \\ w_\delta = 0 & \text{on } \Gamma_\delta. \end{cases}$$

These stability estimates are stated in the next section in Propositions 5.1 and 5.2. The proof of these propositions requires a few technical lemmas which we state and prove in this section.

We start by establishing some crucial elliptic regularity estimates with explicit dependence of the constants on δ. In the following, we denote by C a generic constant independent of δ. In the next lemma, we adopt the notation and the definitions of McLean (2000, Section 4), which we recall here in a simplified setting. Let $\mathcal{O}$ be a connected Lipschitz domain of $\mathbb{R}^2$ and denote by (x_1, x_2) the coordinates of a point x in some given basis. We define the operator $\mathcal{P}$ in $\mathcal{O}$ by

$$\mathcal{P} := -\sum_{i,j} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j},$$

where for $(i,j) \in \{1,2\}^2$ $a_{ij} \in C^1(\mathcal{O})$. We say that $\mathcal{P}$ is coercive if there exists a constant $C > 0$ such that for all $\xi \in \mathbb{R}^2$ and $x \in \mathcal{O}$

$$\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geqslant C|\xi|^2.$$

We call the conormal derivative the operator $\mathcal{B}_\nu$ given by

$$\mathcal{B}_\nu := \sum_i \nu_i \gamma_{\partial\mathcal{O}} \left(\sum_j a_{ij} \frac{\partial}{\partial x_j} \right),$$

where $\gamma_{\partial\mathcal{O}}$ is the trace operator on $\partial\mathcal{O}$ and ν_i for $i = 1, 2$ is the ith component of the inward normal vector to $\partial\mathcal{O}$.

Lemma 4.1 Consider $\delta > 0$, $g \in H^{1/2}(\mathbb{R})$ and $f \in L^2(\mathbb{R} \times (0,\delta))$. Let $\mathcal{P}$ be a coercive operator with coercivity constant independent of δ and $\mathcal{B}_\nu$ the associated conormal derivative. If $w \in H_0 := \{w \in H^1(\mathbb{R} \times (0,\delta)),\ w(x_1, \delta) = 0 \text{ for all } x_1 \in \mathbb{R}\}$ solves

$$\begin{cases} \mathcal{P} w = f & \text{in } \mathbb{R} \times (0,\delta), \\ \mathcal{B}_\nu w(x_1, 0) = g(x_1) & \text{for all } x_1 \in \mathbb{R}, \\ w(x_1, \delta) = 0 & \text{for all } x_1 \in \mathbb{R}, \end{cases}$$

then there exists a constant $C > 0$ independent of δ such that

$$\|w\|_{H^2(\mathbb{R}\times(0,\delta))} \leqslant C(\|f\|_{L^2(\mathbb{R}\times(0,\delta))} + \|g\|_{H^{1/2}(\mathbb{R})}).$$

REMARK 4.2 The novel important aspect in the above *a priori* estimate is to show that the constant is independent of δ, and to our knowledge such a result was not available in the literature.

Proof. Using Green's formula, we have that

$$\int_{\mathbb{R}\times(0,\delta)} \mathcal{P}w\psi \,\mathrm{d}x_1\,\mathrm{d}x_2 = \Phi(w,\psi) + \int_{\mathbb{R}} \mathcal{B}_\nu w(x_1,0)\psi(x_1,0)\,\mathrm{d}x_1$$

or

$$\int_{\mathbb{R}\times(0,\delta)} f\psi \,\mathrm{d}x_1\,\mathrm{d}x_2 = \Phi(w,\psi) + \int_{\mathbb{R}} g(x_1)\psi(x_1,0)\,\mathrm{d}x_1, \tag{4.1}$$

for all $\psi \in H_0$, where the bilinear form $\Phi(\cdot,\cdot)$ is defined by

$$\Phi(u,v) = \sum_{i,j}\int_{\mathbb{R}\times(0,\delta)} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for all } (u,v)\in H_0.$$

In order to obtain the desired regularity result, we apply the approach of the difference quotient in the direction x_1. To this end, for $h \in \mathbb{R}$ and all $(x_1,x_2)\in\mathbb{R}\times[0,\delta]$ we define the difference quotient by

$$\Delta_h u := \frac{u(x_1+h,x_2)-u(x_1,x_2)}{h}.$$

Straightforward algebraic calculations show that the following formulas hold true:

$$\int_{\mathbb{R}\times(0,\delta)} (\Delta_h u)v\,\mathrm{d}x_1\,\mathrm{d}x_2 = -\int_{\mathbb{R}\times(0,\delta)} u(\Delta_{-h}v)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{4.2}$$

$$|\Phi(\Delta_h u,v)+\Phi(u,\Delta_{-h}v)| \leqslant C\|v\|_{H^1(\mathbb{R}\times(0,\delta))}\|u\|_{H^1(\mathbb{R}\times(0,\delta))}, \tag{4.3}$$

for all u and v in H_0, and moreover there exists a constant $C>0$ independent of δ such that for all $u\in H_0$ and h sufficiently small

$$C\left\|\frac{\partial u}{\partial x_1}\right\|_{L^2(\mathbb{R}\times(0,\delta))} \leqslant \|\Delta_h u\|_{L^2(\mathbb{R}\times(0,\delta))} \leqslant \left\|\frac{\partial u}{\partial x_1}\right\|_{L^2(\mathbb{R}\times(0,\delta))} \tag{4.4}$$

(see McLean (2000, Lemma 4.13) for the proof of this last result). Substituting (4.2) and (4.3) in (4.1), we obtain that for $\psi = \Delta_{-h}\Delta_h w \in H_0$ and all h sufficiently small

$$\begin{aligned}|\Phi(\Delta_h w,\Delta_h w)| \leqslant{}& C\|w\|_{H^1(\mathbb{R}\times(0,\delta))}\|\Delta_h w\|_{H^1(\mathbb{R}\times(0,\delta))}\\ &+\left|\int_{\mathbb{R}} g(x_1)(\Delta_{-h}\Delta_h w)(x_1,0)\,\mathrm{d}x_1 + \int_{\mathbb{R}\times(0,\delta)} f(\Delta_{-h}\Delta_h w)\,\mathrm{d}x_1\,\mathrm{d}x_2\right|.\end{aligned} \tag{4.5}$$

From McLean (2000, Exercise 4.4), we have that or all $s\in\mathbb{R}$

$$\|\Delta_h u\|_{H^s(\mathbb{R})} \leqslant \left\|\frac{\partial u}{\partial x_1}\right\|_{H^s(\mathbb{R})}$$

provided it is known that $\partial u/\partial x_1 \in H^s(\mathbb{R})$. On the other hand for the boundary term in (4.5), we have

$$\int_{\mathbb{R}} |g \Delta_{-h}\Delta_h w| \, \mathrm{d}x_1 \leqslant \|g\|_{H^{1/2}(\mathbb{R})} \|\Delta_{-h}\Delta_h w\|_{H^{-1/2}(\mathbb{R})}$$
$$\leqslant \|g\|_{H^{1/2}(\mathbb{R})} \|\Delta_h w\|_{H^{1/2}(\mathbb{R})} \leqslant C\|g\|_{H^{1/2}(\mathbb{R})} \|\Delta_h w\|_{H^1(\mathbb{R}\times(0,\delta))}, \tag{4.6}$$

with a constant $C>0$ independent of δ and h (see the trace Lemma 4.5 for the last inequality). Finally, the coercivity of Φ together with (4.4–4.6) give

$$\|\Delta_h w\|_{H^1(\mathbb{R}\times(0,\delta))} \leqslant C(\|g\|_{H^{1/2}(\mathbb{R})} + \|w\|_{H^1(\mathbb{R}\times(0,\delta))} + \|f\|_{L^2(\mathbb{R}\times(0,\delta))}),$$

which in view of McLean (2000, Lemma 4.13), gives

$$\left\| \frac{\partial w}{\partial x_1} \right\|_{H^1(\mathbb{R}\times(0,\delta))} \leqslant C(\|g\|_{H^{1/2}(\mathbb{R})} + \|w\|_{H^1(\mathbb{R}\times(0,\delta))} + \|f\|_{L^2(\mathbb{R}\times(0,\delta))}).$$

To estimate the second-order derivative with respect to x_2, we recall that $\mathcal{P}w=f$ and since $\mathcal{P}$ is coercive there exists a constant $C>0$ that depends on the coefficients a_{ij} but not on δ such that

$$\left\| \frac{\partial^2 w}{\partial x_2^2} \right\|_{L^2(\mathbb{R}\times(0,\delta))} \leqslant C\left(\|f\|_{L^2(\mathbb{R}\times(0,\delta))} + \left\| \frac{\partial w}{\partial x_1} \right\|_{H^1(\mathbb{R}\times(0,\delta))} \right)$$
$$\leqslant C(\|g\|_{H^{1/2}(\mathbb{R})} + \|w\|_{H^1(\mathbb{R}\times(0,\delta))} + \|f\|_{L^2(\mathbb{R}\times(0,\delta))}).$$

Then the result is a consequence of the standard *a priori* estimate for $\|w\|_{H^1(\mathbb{R}\times(0,\delta))}$ making use of the coercivity of $\mathcal{P}$. □

From the above regularity result in a strip, it is now possible to obtain the same type of regularity result in $\Omega\setminus\overline{\Omega_\delta}$ first with homogeneous mixed boundary conditions which is stated in Lemma 4.3 and then with inhomogeneous mixed boundary conditions which is stated in Lemma 4.4.

Lemma 4.3 There exists a constant $C>0$ independent of δ such that for all $f\in L^2(\Omega\setminus\overline{\Omega_\delta})$ the unique solution $w\in H^1(\Omega\setminus\overline{\Omega_\delta})$ of

$$\begin{cases} -\Delta w = f & \text{in } \Omega\setminus\overline{\Omega_\delta}, \\ \dfrac{\partial w}{\partial \boldsymbol{\nu}} = 0 & \text{on } \Gamma, \\ w = 0 & \text{on } \Gamma_\delta, \end{cases}$$

satisfies the *a priori* estimate

$$\|w\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C\|f\|_{L^2(\Omega\setminus\overline{\Omega_\delta})}.$$

Proof. First from the standard *a priori* estimate for the Laplacian, we have that

$$\|w\|_{H^1(\Omega\setminus\overline{\Omega_\delta})} \leqslant C\|f\|_{L^2(\Omega\setminus\overline{\Omega_\delta})}, \tag{4.7}$$

with a constant $C>0$ independent of δ. To obtain H^2 estimates our approach is based on first locally straighten the boundary and then apply Lemma 4.1. To this end, since $\overline{\Omega\setminus\overline{\Omega_\delta}}$ is a compact set, there exists an integer n and a sequence $(\Omega_i)_{i=1,\dots,n}$ of bounded and connected domains of $\mathbb{R}^2$ such that

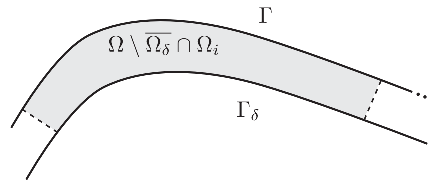

FIG. 2. Local covering of the layer.

$\overline{\Omega\setminus\overline{\Omega_\delta}} \subset \bigcup_{i=1}^{n} \Omega_i$ for all δ sufficiently small. Moreover, we take Ω_i such that there exists $s_i > 0$ and a $C^\infty([-s_i, s_i])$, $k \geqslant 0$, function x_Γ such that for all δ sufficiently small we have

$$(\Omega\setminus\overline{\Omega_\delta}) \cap \Omega_i = \{x_\Gamma(s) + \eta\boldsymbol{\nu}(s)\,, \forall (s,\eta) \in (-s_i, s_i) \times (0,\delta)\},$$

where $x_\Gamma(s) \in \Gamma$ for $s \in (-s_i, s_i)$ and $i = 1, \ldots, n$ (see Fig. 2). Thus, for all δ sufficiently small,

$$\begin{aligned} \varphi_i : \quad \overline{(\Omega\setminus\overline{\Omega_\delta}) \cap \Omega_i} &\longrightarrow \overline{\hat{\Omega}_i}, \\ x &\longmapsto (s,\eta) \end{aligned}$$

is a C^2-diffeormorphism, where $\hat{\Omega}_i := (-s_i, s_i) \times (0,\delta)$. Let $(\phi_i)_{i=1,\ldots,n}$ be a partition of unity such that $\phi_i \in C^\infty(\mathbb{R}^2)$, $\operatorname{supp}(\phi_i) \subset \Omega_i$ and $\sum_i \phi_i = 1$ in $\overline{\Omega\setminus\overline{\Omega_\delta}}$. Hence, if we define $w_i := \phi_i w \in H^1(\Omega\setminus\overline{\Omega_\delta})$, then

$$\|w\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \sum_{i=1}^{N} \|w_i\|_{H^2(\Omega\setminus\overline{\Omega_\delta})}, \tag{4.8}$$

and w_i is compactly supported in Ω_i. Furthermore, w_i solves

$$\begin{cases} -\Delta w_i = f_i & \text{in } \Omega\setminus\overline{\Omega_\delta}, \\ \dfrac{\partial w_i}{\partial \boldsymbol{\nu}} = g_i & \text{on } \Gamma, \\ w_i = 0 & \text{on } \Gamma_\delta, \end{cases}$$

with

$$f_i := f\phi_i + 2\nabla w \nabla\phi_i + w\Delta\phi_i \quad \text{and} \quad g_i := w\frac{\partial\phi_i}{\partial\boldsymbol{\nu}}.$$

In the following, for $U \in L^2((\Omega\setminus\overline{\Omega_\delta}) \cap \Omega_i)$ and $G \in H^s(\Gamma \cap \partial\Omega_i)$ we denote by $\tilde{U} \in L^2(\hat{\Omega}_i)$ and $\tilde{G} \in H^s((-s_i, s_i))$ the functions defined by $\tilde{U} := U \circ \varphi_i$ and $\tilde{G} := G \circ \varphi_i$, respectively. Since the $W^{2,\infty}$ norm of φ_i and φ_i^{-1} does not depend on δ, it is easy to see that there exists a constant $C > 0$ independent of δ such that for $U \in H^p((\Omega\setminus\overline{\Omega_\delta}) \cap \Omega_i)$

$$\frac{1}{C}\|\tilde{U}\|_{H^p(\hat{\Omega}_i)} \leqslant \|U\|_{H^p((\Omega\setminus\overline{\Omega_\delta})\cap\Omega_i)} \leqslant C\|\tilde{U}\|_{H^p(\hat{\Omega}_i)} \quad \text{for all integer } 0 < p \leqslant 2, \tag{4.9}$$

and for $G \in H^s(\Gamma \cap \partial\Omega_i)$

$$\frac{1}{C}\|\tilde{G}\|_{H^s((-s_i,s_i))} \leqslant \|G\|_{H^s(\Gamma\cap\partial\Omega_i)} \leqslant C\|\tilde{G}\|_{H^s((-s_i,s_i))} \quad \text{for all } 0 < s \leqslant 2. \tag{4.10}$$

With these notation, we can now prove using the calculations developed in Section 3.2 that $\tilde{w}_i \in H^1(\hat{\Omega}_i)$ solves

$$\begin{cases} \mathcal{P}\tilde{w}_i = \tilde{f}_i \quad \text{in } \hat{\Omega}_i, \\ (\mathcal{B}_\nu \tilde{w}_i)(s,0) = \tilde{g}_i(s) \quad \text{for all } s \in (-s_i, s_i), \\ \tilde{w}_i(s,\delta) = 0 \quad \text{for all } s \in (-s_i, s_i), \end{cases} \tag{4.11}$$

where $\tilde{f}_i(s,\eta) := (1+\eta\kappa(s))(f_i \circ \varphi_i)(s,\eta)$, $\tilde{g}_i(s,\eta) := -(1+\eta\kappa(s))(g_i \circ \varphi_i)(s,\eta)$ and $\mathcal{P}$ and $\mathcal{B}_\nu$ are as in Lemma 4.1 (note that the coercivity constant for $\mathcal{P}$ does not depend on δ). Furthermore, since $\tilde{w}_i, \tilde{f}_i$ and $\tilde{g}_i$ are equal to 0 in a vicinity of $-s_i$ and s_i we can extend them by 0 into $\mathbb{R} \times (0,\delta)$. For sake of simplicity, we do not change the notation for their extension and note that these extension also satisfy the system (4.11) for $s \in \mathbb{R}$. Hence, we can apply Lemma 4.1 to $\tilde{w}_i$ to obtain

$$\|\tilde{w}_i\|_{H^2(\mathbb{R}\times(0,\delta))} \leqslant C(\|\tilde{g}_i\|_{H^{1/2}(\mathbb{R})} + \|\tilde{f}_i\|_{L^2(\mathbb{R}\times(-s_i,s_i))}),$$

where C is independent of δ. Using (4.9) and (4.10) in (4.8) together with our first *a priori* estimate (4.7) finally proves the lemma. □

Next we obtain the same type of estimates as in Lemma 4.3 for inhomogeneous boundary condition on Γ. The challenge is to show that the lifting function is bounded independently of δ in appropriate norm.

Lemma 4.4 Let $g \in H^{1/2}(\Gamma)$ and $w \in H^1(\Omega \setminus \overline{\Omega_\delta})$ be the unique solution of

$$\begin{cases} \Delta w = 0 \quad \text{in } \Omega \setminus \overline{\Omega_\delta}, \\ \dfrac{\partial w}{\partial \boldsymbol{\nu}} = g \quad \text{on } \Gamma, \\ w = 0 \quad \text{on } \Gamma_\delta. \end{cases}$$

Then there exists a constant $C > 0$ such that for all $\delta > 0$

$$\|w\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C\|g\|_{H^{1/2}(\Gamma)}.$$

In addition, if $\|g\|_{H^{1/2}(\Gamma)} \leqslant C_g$ for some $C_g > 0$ and all $\delta > 0$, then

$$\|w\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \xrightarrow[\delta\to 0]{} 0.$$

Proof. We start by building an appropriate lifting of g which equals to 0 on Γ_δ. To this end, let us define $g_i = \phi_i g$ and its local counterpart $\tilde{g}_i$ using the same partition of unity and local parametrization as in Lemma 4.3. Then we can define an extension of $\tilde{g}_i$ to $\mathbb{R}$ denoted by $\overline{g}_i$ by

$$\overline{g}_i(s) := \begin{cases} \tilde{g}_i(s) & \text{if } s \in [-s_i, s_i], \\ 0 & \text{elsewhere.} \end{cases}$$

For any function $G \in L^1(\mathbb{R})$, let

$$\mathcal{F}G(\xi) := \int_{\mathbb{R}} G(s)\, \mathrm{e}^{-\mathrm{i}2\pi s\xi}\, \mathrm{d}s \quad \text{and} \quad \mathcal{F}^{-1}G(s) := \int_{\mathbb{R}} G(\xi)\, \mathrm{e}^{\mathrm{i}2\pi s\xi}\, \mathrm{d}\xi$$

be its Fourier transform and its inverse Fourier transform, respectively. Since Γ is of class C^2 and $g \in H^{1/2}(\Gamma)$, Plancherel's Theorem ensure the existence of a constant C independent of δ such that

$$\|(1+\xi^2)^{1/4}\mathcal{F}\overline{g}_i\|_{L^2(\mathbb{R})} \leqslant C\|g\|_{H^{1/2}(\Gamma)}. \tag{4.12}$$

For all $\xi \in \mathbb{R}$ and $\eta \in [0, \delta]$, let us define

$$w_i(\xi, \eta) := \frac{\mathcal{F}\overline{g}_i(\xi)}{|\xi|}\left(\frac{\sinh(|\xi|(\eta-\delta))}{\cosh(|\xi|\delta)}\right)$$

and

$$\overline{w}_i(s, \eta) := (\mathcal{F}^{-1}w_i)(s, \eta).$$

Then $\overline{w}_i$ satisfies $\overline{w}_i(s, \delta) = 0$ and $(\partial\overline{w}_i/\partial\eta)(s, 0) = \overline{g}_i(s)$ in $\mathbb{R}$. Moreover, there exists a constant $C > 0$ independent of δ such that for all $(\xi, \eta) \in \mathbb{R} \times (0, \delta)$ we have

$$|w_i(\xi,\eta)|^2 \leqslant C(\mathcal{F}\overline{g}_i(\xi))^2, \quad |\xi w_i(\xi,\eta)|^2 \leqslant C(\mathcal{F}\overline{g}_i(\xi))^2, \quad \left|\frac{\partial w_i}{\partial\eta}(\xi,\eta)\right|^2 \leqslant C(\mathcal{F}\overline{g}_i(\xi))^2.$$

Integrating the above inequalities over $\mathbb{R} \times (0, \delta)$ and using (4.12) and the Plancherel's Theorem, we have that there exists a constant $C > 0$ independent of δ such that

$$\|\overline{w}_i\|^2_{H^1(\mathbb{R}\times(0,\delta))} \leqslant C\delta\|g\|^2_{H^{1/2}(\Gamma)}.$$

Moreover, for all $(\xi, \eta) \in \mathbb{R} \times (0, \delta)$ we also have

$$\begin{aligned}
\xi^2 w_i(\xi,\eta) &= |\xi|\mathcal{F}\overline{g}_i(\xi)\frac{\sinh(|\xi|(\eta-\delta))}{\cosh(|\xi|\delta)}, \\
\xi\frac{\partial w_i}{\partial\eta}(\xi,\eta) &= |\xi|\mathcal{F}\overline{g}_i(\xi)\frac{\cosh(|\xi|(\eta-\delta))}{\cosh(|\xi|\delta)}, \\
\frac{\partial^2 w_i}{\partial\eta^2}(\xi,\eta) &= |\xi|\mathcal{F}\overline{g}_i(\xi)\frac{\sinh(|\xi|(\eta-\delta))}{\cosh(|\xi|\delta)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}}\int_0^\delta |\xi|^4|w_i(\xi,\eta)|^2\,\mathrm{d}\eta\,\mathrm{d}\xi &= \int_{\mathbb{R}}|\xi|^2|\mathcal{F}\overline{g}_i(\xi)|^2\int_0^\delta \frac{\sinh^2(|\xi|(\eta-\delta))}{\cosh^2(|\xi|\delta)}\,\mathrm{d}\eta\,\mathrm{d}\xi\\
&= \int_{\mathbb{R}}|\xi|^2|\mathcal{F}\overline{g}_i(\xi)|^2\int_0^\delta \frac{\cosh(2|\xi|(\eta-\delta))-1}{2\cosh(|\xi|\delta)^2}\,\mathrm{d}\eta\,\mathrm{d}\xi\\
&= \frac{1}{2}\int_{\mathbb{R}}|\xi|^2|\mathcal{F}\overline{g}_i(\xi)|^2\left(\frac{\sinh(|\xi|\delta)}{|\xi|\cosh(|\xi|\delta)}-\frac{\delta}{\cosh(|\xi|\delta)^2}\right)\mathrm{d}\xi\\
&\leqslant \frac{1}{2}\int_{\mathbb{R}}|\xi||\mathcal{F}\overline{g}_i(\xi)|^2\frac{\sinh(|\xi|\delta)}{\cosh(|\xi|\delta)}\,\mathrm{d}\xi\\
&\leqslant \frac{1}{2}\int_{\mathbb{R}}|\xi||\mathcal{F}\overline{g}_i(\xi)|^2\,\mathrm{d}\xi,
\end{aligned} \tag{4.13}$$

and hence an application of (4.12) yields

$$\int_{\mathbb{R}}\int_0^\delta |\xi|^4|w_i(\xi,\eta)|^2\,\mathrm{d}\eta\,\mathrm{d}\xi \leqslant C\|g\|^2_{H^{1/2}(\Gamma)}.$$

In a similar way, we obtain that

$$\int_{\mathbb{R}}\int_0^\delta \left|\xi\frac{\partial w_i}{\partial\eta}(\xi,\eta)\right|\mathrm{d}\eta\,\mathrm{d}\xi \leqslant C\|g\|^2_{H^{1/2}(\Gamma)} \quad\text{and}\quad \int_{\mathbb{R}}\int_0^\delta \left|\frac{\partial^2 w_i}{\partial\eta^2}(\xi,\eta)\right|\mathrm{d}\eta\mathrm{d}\xi \leqslant C\|g\|^2_{H^{1/2}(\Gamma)},$$

with a different constant $C>0$ independent of δ. Hence, once more application of Plancherel's Theorem implies

$$\|\overline{w}_i\|^2_{H^2(\mathbb{R}\times(0,\delta))} \leqslant C\|g\|^2_{H^{1/2}(\Gamma)}. \tag{4.14}$$

Next assume in addition that there exists $C_g>0$ such that $\|g\|_{H^{1/2}(\Gamma)} \leqslant C_g$ for all $\delta>0$, then for almost every $\xi\in\mathbb{R}$,

$$|\xi||\mathcal{F}\overline{g}_i(\xi)|^2\frac{\sinh(|\xi|\delta)}{\cosh(|\xi|\delta)} \xrightarrow[\delta\to 0]{} 0,$$

and since

$$|\xi|\mathcal{F}\overline{g}_i(\xi)|^2\frac{\sinh(|\xi|\delta)}{\cosh(|\xi|\delta)} \leqslant |\xi|(\mathcal{F}\overline{g}_i(\xi))^2,$$

the Lebesgue dominated convergence theorem with (4.13) implies that

$$\int_{\mathbb{R}}\int_0^\delta |\xi|^4|w_i(\xi,\eta)|^2\,\mathrm{d}\eta\,\mathrm{d}\xi \xrightarrow[\delta\to 0]{} 0.$$

In a similar way, we obtain that

$$\int_{\mathbb{R}}\int_0^\delta \left|\xi\frac{\partial w_i}{\partial\eta}(\xi,\eta)\right|\mathrm{d}\eta\,\mathrm{d}\xi \xrightarrow[\delta\to 0]{} 0 \quad\text{and}\quad \int_{\mathbb{R}}\int_0^\delta \left|\frac{\partial^2 w_i}{\partial\eta^2}(\xi,\eta)\right|\mathrm{d}\eta\,\mathrm{d}\xi \xrightarrow[\delta\to 0]{} 0,$$

whence, again from the Plancherel's Theorem,

$$\|\overline{w}_i\|^2_{H^2(\mathbb{R}\times(0,\delta))} \xrightarrow[\delta\to 0]{} 0. \tag{4.15}$$

Now we go back to the physical domain and define for $x \in \Omega\setminus\overline{\Omega_\delta}$

$$w_g(x) := \sum_{j \text{ s.t. } x\in\Omega_j} (\overline{w}_j \circ \varphi_j^{-1})(x).$$

Then w_g satisfies $(\partial w_g/\partial \boldsymbol{\nu})|_\Gamma = g$ together with $w_g|_{\Gamma_\delta} = 0$ and using the fact that Γ is of class C^2 and that $(\overline{w}_i)_i$ satisfy (4.14), we can claim that there exists a constant $C > 0$ independent of δ such that

$$\|w_g\|^2_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C\|g\|^2_{H^{1/2}(\Gamma)}. \tag{4.16}$$

In addition, if there exists $C_g > 0$ such that $\|g\|_{H^{1/2}(\Gamma)} \leqslant C_g$ for all $\delta > 0$, then from (4.15) we also have that

$$\|w_g\|^2_{H^2(\Omega\setminus\overline{\Omega_\delta})} \xrightarrow[\delta\to 0]{} 0. \tag{4.17}$$

Finally, we consider $W := w - w_g$ which solves

$$\begin{cases} \Delta W = -\Delta w_g \quad \text{in } \Omega\setminus\overline{\Omega_\delta}, \\ \dfrac{\partial W}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \Gamma, \\ W = 0 \quad \text{on } \Gamma_\delta. \end{cases}$$

Lemma 4.3 states that there exists a constant C independent of δ such that

$$\|W\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C\|w_g\|^2_{H^2(\Omega\setminus\overline{\Omega_\delta})}.$$

This last estimate together with (4.16) imply the existence of a constant C independent of δ such that

$$\|w\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C\|g\|_{H^{1/2}(\Gamma)},$$

and, if in addition there exists $C_g > 0$ such that $\|g\|_{H^{1/2}(\Gamma)} \leqslant C_g$ for all $\delta > 0$, (4.17) implies

$$\|w\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \xrightarrow[\delta\to 0]{} 0,$$

which ends the proof. □

Next we prove a trace theorem which displays explicit dependence on δ of the constant.

LEMMA 4.5 For $k = 1, 2$, there exists a constant $C > 0$ independent of δ such that for all $v \in H^k(\Omega\setminus\overline{\Omega_\delta})$ with $v|_{\Gamma_\delta} = 0$

$$\|v\|_{H^{k-1/2}(\Gamma)} \leqslant C\|v\|_{H^k(\Omega\setminus\overline{\Omega_\delta})}.$$

Proof. We prove the result for the domain $\mathbb{R}\times(0,\delta)$ and then use the partition of unity and change of variable introduced in the proof of Lemma 4.3 to obtain the desired result. To this end, we consider an arbitrary $v \in C^\infty(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$ and denote as before by $\mathcal{F}v$ its Fourier transform with respect to the first variable where ξ denotes the dual variable. Integrating by parts, we obtain that

$$(1+\xi^2)^{2(k-1/2)}|\mathcal{F}v(\xi,0)|^2 = (1+\xi^2)^{2(k-1/2)}|\mathcal{F}v(\xi,\delta)|^2 \\ - 2\Re\left(\int_0^\delta (1+\xi^2)^{k-1}\frac{\partial \mathcal{F}v}{\partial \eta}(\xi,\eta)(1+\xi^2)^k\mathcal{F}v(\xi,\eta)\,\mathrm{d}\eta\right),$$

for all $\xi \in \mathbb{R}$. Integrating this equality along $\mathbb{R}$, and using the Cauchy–Schwartz inequality and Plancherel's Theorem imply the existence of $C > 0$ independent of δ such that

$$\|v(s,0)\|_{H^{k-1/2}(\mathbb{R})} \leqslant C(\|v\|_{H^k(\mathbb{R}\times(0,\delta))} + \|v(s,\delta)\|_{H^{k-1/2}(\mathbb{R})}).$$

A density argument ensures that the above estimate holds for all $v \in H^2(\mathbb{R} \times (0,\delta))$, and thus we obtain the result for v whose trace is 0 on $\mathbb{R} \times \{\delta\}$. □

We conclude this section with the following technical trace lemma, which was already used in the proof of Lemma 4.1.

LEMMA 4.6 For any $w \in H^1(\Omega \setminus \overline{\Omega_\delta})$, we have that

$$\|w\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C(\delta^{1/2}\|w\|_{L^2(\Gamma)} + \delta\|w\|_{H^1(\Omega\setminus\overline{\Omega_\delta})}), \tag{4.18}$$

and for any function $w \in H^2(\Omega \setminus \overline{\Omega_\delta})$ such that $w|_{\Gamma_\delta} = 0$ we have that

$$\|w\|_{H^{1/2}(\Gamma)} \leqslant C\delta^{1/2}\|w\|_{H^2(\Omega\setminus\overline{\Omega_\delta})}, \tag{4.19}$$

where the constant $C > 0$ is independent of δ.

Proof. To prove (4.6), we take $\phi \in C_0^\infty(\mathbb{R}^2)$ and using local variables in the layer to obtain

$$\tilde{\phi}(s,\eta) = \tilde{\phi}(s,0) + \int_0^\eta \frac{\partial\tilde{\phi}}{\partial t}(s,t)\,\mathrm{d}t\,\mathrm{d}s \quad \text{for every } \eta \leqslant \delta,$$

where $\tilde{\phi}$ denotes the function in the new variables (s,η) (see Lemma 4.3 for the definition). The Cauchy–Schwartz inequality implies

$$|\tilde{\phi}(s,\eta)|^2 \leqslant C(|\tilde{\phi}(s,0)|^2 + \delta\|\phi\|^2_{H^1(\Omega\setminus\overline{\Omega_\delta})}),$$

whence we obtain the desired estimate for $\phi \in C_0^\infty(\mathbb{R}^2)$ by integrating this inequality over $\Gamma \times [0,\delta]$. A density argument gives the result for all $w \in H^1(\Omega \setminus \overline{\Omega_\delta})$.
To prove (4.19), we again take $\phi \in C_0^\infty(\mathbb{R}^2)$ and in a similar way as above use local coordinates in the layer. Thus, we obtain that for $s \in [0,s_0]$

$$|\tilde{\phi}(s,0)| \leqslant |\tilde{\phi}(s,\delta)| + \left|\int_0^\delta \frac{\partial\tilde{\phi}}{\partial\eta}(s,\eta)\,\mathrm{d}\eta\right| \leqslant |\tilde{\phi}(s,\delta)| + \delta^{1/2}\|\phi\|_{H^1(\Omega\setminus\overline{\Omega_\delta})},$$

which leads to

$$\|\phi\|_{L^2(\Gamma)} \leqslant C(\|\phi\|_{L^2(\Gamma_\delta)} + \delta^{1/2}\|\phi\|_{H^1(\Omega\setminus\overline{\Omega_\delta})}),$$

for some positive $C > 0$ independent of δ. Similarly, we also have

$$\|\nabla_\Gamma \phi\|_{L^2(\Gamma)} \leqslant C(\|\nabla_\Gamma \phi\|_{L^2(\Gamma_\delta)} + \delta^{1/2}\|\phi\|_{H^2(\Omega\setminus\overline{\Omega_\delta})}).$$

By density, the above two inequalities remain true for all $w \in H^2(\Omega\setminus\overline{\Omega_\delta})$ and by interpolation between $L^2(\Gamma)$ and $H^1(\Gamma)$ we obtain the desired result nothing that $w|_{\Gamma_\delta} = 0$. □

5. Convergence analysis

Our main goal in this section is to rigorously justify the asymptotic expansion formally obtained in the Section 3.2. To this end, for sake of simplicity of presentation and to avoid secondary technical difficulties we assume that the thickness of the thin layer is constant (i.e. $g \equiv 1$), that n is in $C^\infty(\overline{\Omega} \setminus \Omega_\delta)$ and that Γ is of class C^∞ as well. Moreover, we only perform the convergence analysis for the first transmission eigenvalue that we denote $\lambda_\delta^1 := (k_\delta^1)^2$. More specifically, in the following we justify the expansion

$$\lambda_\delta^1 = \lambda_0 + \delta\lambda_1 + \delta^2\lambda_2 + \mathcal{O}(\delta^3), \tag{5.1}$$

where λ_0 is the first Dirichlet eigenvalue for the $-\Delta$ operator in Ω, λ_1 and λ_2 are given in the previous section and $\mathcal{O}(x)$ stands for a generic function in $C^\infty(\mathbb{R}^+)$ such that

$$|\mathcal{O}(x)| \leqslant C|x|,$$

for some constant $C > 0$ independent of $x \in \mathbb{R}^+$.

Proposition 5.1 Let $v_\delta \in H^2(\Omega)$ and $w_\delta \in H^2(\Omega\setminus\overline{\Omega_\delta})$ be such that for some $s \geqslant 0$

$$\|\Delta w_\delta\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^s), \tag{5.2}$$

$$w_\delta = 0 \quad \text{on } \Gamma_\delta,$$

$$\|\Delta v_\delta\|_{L^2(\Omega)} \leqslant \mathcal{O}(\delta^s), \tag{5.3}$$

$$\left\|\frac{\partial v_\delta}{\partial \boldsymbol{\nu}} - \frac{\partial w_\delta}{\partial \boldsymbol{\nu}}\right\|_{H^{1/2}(\Gamma)} \leqslant \mathcal{O}(\delta^s), \tag{5.4}$$

$$\|v_\delta - w_\delta\|_{H^{3/2}(\Gamma)} \leqslant \mathcal{O}(\delta^s). \tag{5.5}$$

Then, for sufficiently small $\delta > 0$,

$$\|w_\delta\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^s) \quad \text{and} \quad \|w_\delta\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{s+1}).$$

Proof. First we prove by contradiction that $\|w_\delta\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^s)$. Assume to the contrary that the latter is not true, then we can state that (up to an extracted subsequence)

$$\gamma_\delta := \delta^s \|w_\delta\|^{-1}_{H^2(\Omega\setminus\overline{\Omega_\delta})} \underset{\delta\to 0}{\longrightarrow} 0.$$

Since w_δ is in $H^2(\Omega\setminus\overline{\Omega_\delta})$ and since $\|v_\delta - w_\delta\|_{H^{3/2}(\Gamma)} \leqslant \mathcal{O}(\delta^s)$, from classic elliptic regularity for the Laplacian with Dirichlet boundary condition there exists a constant C independent of δ such that

$$\|v_\delta\|_{H^2(\Omega)} \leqslant C(\|\Delta v_\delta\|_{L^2(\Omega)} + \|w_\delta\|_{H^{3/2}(\Gamma)} + \mathcal{O}(\delta^s)),$$

and by using (5.3), Lemma 4.5 and the fact that γ_δ is bounded when $\delta \to 0$ we deduce that

$$\frac{\|v_\delta\|_{H^2(\Omega)}}{\|w_\delta\|_{H^2(\Omega\setminus\overline{\Omega_\delta})}} \leqslant \frac{1}{\|w_\delta\|_{H^2(\Omega\setminus\overline{\Omega_\delta})}}(C\|w_\delta\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} + \mathcal{O}(\delta^s)) \leqslant C,$$

for $C > 0$ independent of δ. Hence, by (5.4) and the trace theorem we have that

$$\frac{1}{\|w_\delta\|_{H^2(\Omega\setminus\overline{\Omega_\delta})}}\left\|\frac{\partial w_\delta}{\partial \boldsymbol{\nu}}\right\|_{H^{1/2}(\Gamma)} \leqslant C.$$

Now by applying Lemmas 4.3 and 4.4 to the function $w_\delta/\|w_\delta\|_{H^2(\Omega\setminus\overline{\Omega_\delta})}$, we have that there exists a constant $C > 0$ independent of δ such that

$$1 \leqslant C\left(\frac{\|\Delta w_\delta\|_{L^2(\Omega\setminus\overline{\Omega_\delta})}}{\|w_\delta\|_{H^2(\Omega\setminus\overline{\Omega_\delta})}} + \varepsilon(\delta)\right),$$

where $\varepsilon(\delta) \to 0$ when δ goes to 0. Hence, by using (5.2) since $\gamma_\delta = \delta^s\|w_\delta\|^{-1}_{H^2(\Omega\setminus\overline{\Omega_\delta})}$ goes to 0 when δ goes to 0 we obtain

$$1 \leqslant \varepsilon(\delta) \quad \underset{\delta\to 0}{\longrightarrow} 0\,.$$

Thus, the first estimate of the lemma holds, and then an application of (4.18) together with (4.19) imply the second estimate. □

PROPOSITION 5.2 Let $w_\delta \in H^2(\Omega\setminus\overline{\Omega_\delta})$ be such that for some $s \geqslant 0$ we have

$$\|\Delta w_\delta\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^s)$$

and

$$w_\delta = 0 \ \text{ on } \Gamma_\delta, \quad \left\|\frac{\partial w_\delta}{\partial \boldsymbol{\nu}}\right\|_{H^{-1/2}(\Gamma)} \leqslant \mathcal{O}(\delta^{s+1/2}).$$

Then, for sufficiently small δ,

$$\|w_\delta\|_{H^1(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{s+1/2}) \quad \text{and} \quad \|w_\delta\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{s+1}).$$

Proof. First by (4.18) and Lemma 4.5, there exists $C > 0$ such that for sufficiently small $\delta > 0$ we have

$$\|w_\delta\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C\delta^{1/2}\|w_\delta\|_{H^1(\Omega\setminus\overline{\Omega_\delta})}. \tag{5.6}$$

An application of Green's identity in $\Omega\setminus\overline{\Omega_\delta}$ yields

$$\int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_\delta|^2\,\mathrm{d}x = \int_{\Omega\setminus\overline{\Omega_\delta}} -\Delta w_\delta w_\delta\,\mathrm{d}x - \int_\Gamma \frac{\partial w_\delta}{\partial \boldsymbol{\nu}} w_\delta\,\mathrm{d}s,$$

and using the assumptions of the lemma and (5.6) we finally obtain

$$\int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_\delta|^2 \leqslant \mathcal{O}(\delta^{s+1/2})\|w_\delta\|_{H^1(\Omega\setminus\overline{\Omega_\delta})},$$

which proves the first estimate. We obtain the second estimate by simply using (5.6). □

Now with help of the above propositions, we are able to prove the convergence of our asymptotic expansion.

5.1 *The convergence of the zero-order approximation*

We start with the convergence of the zero-order term in the expansion, which can be easily obtained from the expression satisfied by the first transmission eigenvalue.

THEOREM 5.3 Let λ_δ^1 be the first real interior transmission eigenvalue, then for sufficiently small $\delta > 0$,

$$\lambda_\delta^1 = \lambda_0 + \mathcal{O}(\delta).$$

Proof. We first observe that for $\lambda < \lambda_0$, where λ_0 is the first Dirichlet eigenvalue for $-\Delta$ in Ω, the operator $A_\lambda - \lambda^2 B$ defined in Section 2 (where we set $\lambda := k^2$) is injective. Indeed for all $u \in W_\delta$,

$$((A_\lambda - \lambda^2 B)u, u)_{W_\delta} = \int_{\Omega \setminus \overline{\Omega_\delta}} \frac{1}{1-n} |\Delta u + \lambda u| \, \mathrm{d}x - \lambda^2 \int_\Omega |u|^2 \, \mathrm{d}x + \lambda \int_\Omega |\nabla u|^2 \, \mathrm{d}x$$

$$\geqslant \lambda \|\nabla u\|^2_{L^2(\Omega)} \left(1 - \frac{\lambda}{\lambda_0}\right),$$

where we have used the Poincaré inequality. Hence, we necessarily have

$$\lambda_0 \leqslant \lambda_\delta^1. \tag{5.7}$$

On the other hand, it is possible to characterize λ_δ^1 via the Max–Min principle (see Cakoni *et al.*, 2012 for details) as

$$2(\lambda_\delta^1)^2 = \inf_{\substack{u \in W_\delta \\ \|u\|_{L^2(\Omega)}=1}} (A_{\lambda_\delta^1} u, u)_{W_\delta}$$

$$= \inf_{\substack{u \in W_\delta \\ \|u\|_{L^2(\Omega)}=1}} \int_{\Omega \setminus \overline{\Omega_\delta}} \frac{1}{1-n} |\Delta u + \lambda_\delta^1 u|^2 \, \mathrm{d}x + (\lambda_\delta^1)^2 + \lambda_\delta^1 \int_\Omega |\nabla u|^2 \, \mathrm{d}x.$$

Now if we take $u = u^1_{\delta,D}$ where $u^1_{\delta,D}$ the first Dirichlet eigenfunction for $-\Delta$ in Ω_δ associated with the eigenvalue $\lambda^1_{\delta,D}$ extended by 0 outside Ω_δ such that $\|u^1_{\delta,D}\|_{L^2(\Omega_\delta)} = 1$ (note that due to the zero boundary condition on Γ_δ the extension by zero of $u^1_{\delta,D}$ is in W_δ) we obtain

$$(\lambda_\delta^1)^2 \leqslant \lambda_\delta^1 \lambda^1_{\delta,D},$$

or equivalently

$$\lambda_\delta^1 \leqslant \lambda^1_{\delta,D}, \tag{5.8}$$

since λ_δ^1 is bounded below by λ_0. To conclude, we remark that the first Dirichlet eigenvalue for $-\Delta$ is Fréchet differentiable with respect to the shape (see Henrot, 2006). A consequence of this result is that there exists a constant $C > 0$ such that for all δ sufficiently small

$$|\lambda^1_{\delta,D} - \lambda_0| \leqslant C\delta.$$

This fact together with the lower and upper bounds (5.7) and (5.8), respectively, ends the proof. □

REMARK 5.4 Theorem 5.3 is the cornerstone of our analysis since it allows to uniquely define the Dirichlet eigenvalue which is the first term in the asymptotic expansion of the transmission eigenvalue. The other terms in (5.1) are then uniquely defined.

5.2 *The convergence of the first-order approximation*

In order to proceed with next order approximation, we must first prove some estimates for the first-order approximation of the corresponding eigenfunction. To this end, let us define

$$e_w^1 := w_\delta^1 - \delta w_1 \quad \text{in } \Omega \setminus \overline{\Omega_\delta}, \tag{5.9}$$

extended by 0 in Ω_δ and

$$e_v^1 := v_\delta^1 - v_0 \quad \text{in } \Omega, \tag{5.10}$$

where v_0 is a solution to (3.10) such that $\|v_0\|_{L^2(\Omega)} = 1$, $w_1(x) := \hat{w}_1(s,\xi)$ with $x = \varphi(s, \delta\xi)$ and (w_δ^1, v_δ^1) are the eigenfunctions corresponding to the first transmission eigenvalue λ_δ^1. We also extend w_δ^1 by 0 inside Ω_δ. We remark that since w_δ^1 and w_1 vanish on Γ_δ, the functions e_w^1 and w_δ^1 are continuous across the interface Γ_δ. Let us begin with a lemma that provides δ-explicit *a priori* estimates for w_δ^1 which will enable us to derive estimates for the first-order approximation of w_δ^1 and v_δ^1.

LEMMA 5.5 There exists a constant $C > 0$ such that for sufficiently small $\delta > 0$, we have

$$\|w_\delta^1\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C \quad \text{and} \quad \|w_\delta^1\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta).$$

Proof. We show that Proposition 5.1 applies to (v_δ^1, w_δ^1) for $s = 0$. To this end, since

$$2(\lambda_\delta^1)^2 = (A_{\lambda_\delta^1} u_\delta^1, u_\delta^1)_{W_\delta},$$

and since $A_{\lambda_\delta^1}$ is coercive with a coercivity constant independent of δ (see Proposition 2.2), there exists a constant C independent of δ such that

$$\|u_\delta^1\|_{H^1(\Omega)} + \|\Delta u_\delta^1\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C, \tag{5.11}$$

where u_δ^1 is defined by (2.3). A straightforward consequence of (5.11) is that there exists another constant C still independent of δ such that

$$\|w_\delta^1\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C \quad \text{and} \quad \|v_\delta^1\|_{L^2(\Omega)} \leqslant C.$$

Since $-\Delta w_\delta^1 = \lambda_\delta^1 n w_\delta^1$ and $-\Delta v_\delta^1 = \lambda_\delta^1 v_\delta^1$, by using Proposition 5.1 with $s = 0$ we have that there exists $C > 0$ such that

$$\|w_\delta^1\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C \quad \text{and} \quad \|w_\delta^1\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta),$$

which ends the proof. □

LEMMA 5.6 The following error estimates hold true for sufficiently small $\delta > 0$:

$$\|e_v^1\|_{H^1(\Omega)} \leqslant \mathcal{O}(\delta) \tag{5.12}$$

and

$$\|e_w^1\|_{H^1(\Omega)} \leqslant \mathcal{O}(\delta), \quad \|e_w^1\|_{L^2(\Omega)} \leqslant \mathcal{O}(\delta^{3/2}), \quad \|w_\delta^1\|_{L^2(\Omega)} \leqslant \mathcal{O}(\delta^{3/2}). \tag{5.13}$$

Proof. The idea of the proof is to establish that v_δ^1 is a quasi-Dirichlet eigenfunction for $-\Delta$ in the domain Ω and then apply Lemma A.2. From the first estimate of Lemma 5.5 and the inequality (4.19), the trace of w_δ^1 on Γ satisfies

$$\|w_\delta^1\|_{H^{1/2}(\Gamma)} \leqslant \mathcal{O}(\delta^{1/2}).$$

Let us define $\theta_{w_\delta^1}$ a lifting of w_δ^1 in $H^1(\Omega)$ such that $\theta_{w_\delta^1}|_\Gamma = w_\delta^1|_\Gamma$ and that $\|\theta_{w_\delta^1}\|_{H^1(\Omega)} \leqslant \mathcal{O}(\delta^{1/2})$. Let us set $\overline{v}_\delta^1 := v_\delta^1 - \theta_{w_\delta^1}$ and show that $\overline{v}_\delta^1$ is close to v_0. Indeed, for all $\psi \in H_0^1(\Omega)$ we have that

$$\begin{aligned}\left|\int_\Omega \nabla\overline{v}_\delta^1 \cdot \nabla\psi - \lambda_0\overline{v}_\delta^1\psi \,\mathrm{d}x\right| &= \left|(\lambda_\delta^1 - \lambda_0)\int_\Omega v_\delta^1\psi \,\mathrm{d}x - \int_\Omega (\nabla\theta_{w_\delta^1}\cdot\nabla\psi - \lambda_0\theta_{w_\delta^1}\psi)\,\mathrm{d}x\right| \\ &\leqslant \mathcal{O}(\delta^{1/2})\|\psi\|_{H^1(\Omega)}.\end{aligned}$$

Moreover, since $\|u_\delta^1\|_{L^2(\Omega)} = 1$, Lemma 5.5 implies that

$$\big|\|\overline{v}_\delta^1\|_{L^2(\Omega)} - 1\big| \leqslant \mathcal{O}(\delta^{1/2}), \tag{5.14}$$

and hence by virtue of Lemma A.2 there exists $C > 0$ and v_0 solution to (3.10) with $\|v_0\|_{L^2(\Omega)} = 1$ such that for all δ sufficiently small we have

$$\|\overline{v}_\delta^1 - v_0\|_{H^1(\Omega)} \leqslant \mathcal{O}(\delta^{1/2}). \tag{5.15}$$

Note that this last inequality uniquely determines the function v_0. From the definition of $\overline{v}_\delta^1$ and the above bound on the lifting, (5.15) takes the form

$$\|e_v^1\|_{H^1(\Omega)} \leqslant \mathcal{O}(\delta^{1/2}). \tag{5.16}$$

But since w_1 solves (3.5), we have

$$\Delta w_1 = \frac{1}{(1+\delta\xi\kappa)^3}\left(\sum_{k=0}^{5}\delta^{k-2}A_k\hat{w}_1 - \lambda_\delta^1 n\hat{w}_1\right),$$

whence

$$\|\Delta w_1\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C(\delta^{-1/2}\|A_1\hat{w}_1\|_{L^2(\mathcal{G})} + \delta^{1/2}\|\hat{w}_1\|_{L^2(\mathcal{G})}) \leqslant C\delta^{-1/2}.$$

Therefore,

$$\|\Delta e_w^1\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{1/2}), \tag{5.17}$$

and in addition we also have by (5.16)

$$\|\Delta e_v^1\|_{L^2(\Omega)} \leqslant \mathcal{O}(\delta^{1/2}), \qquad \left\|\frac{\partial e_v^1}{\partial \boldsymbol{\nu}} - \frac{\partial e_w^1}{\partial \boldsymbol{\nu}}\right\|_{H^{1/2}(\Gamma)} = 0 \tag{5.18}$$

and

$$\|e_v^1 - e_w^1\|_{H^{3/2}(\Gamma)} = \|\delta w_1\|_{H^{3/2}(\Gamma)} \leqslant \mathcal{O}(\delta).$$

An application of Proposition 5.1 now implies

$$\|e_w^1\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{1/2}),$$

and then thanks to (4.19) applied e_w^1, we can improve the bound on w_δ^1 as follows:

$$\|w_\delta^1\|_{H^{1/2}(\Gamma)} \leqslant \mathcal{O}(\delta),$$

since $\|w_1\|_{H^{1/2}(\Gamma)} \leqslant C$ for all $\delta > 0$. Now repeating the previous steps of the proof allows us to improve the bound on e_v^1 as follows:

$$\|e_v^1\|_{H^1(\Omega)} \leqslant \mathcal{O}(\delta).$$

Finally, this last inequality together with (5.17) and the fact that

$$\frac{\partial e_v^1}{\partial \boldsymbol{\nu}} = \frac{\partial e_w^1}{\partial \boldsymbol{\nu}} \quad \text{on } \Gamma$$

yield the desired bounds for e_w^1 thanks to Proposition 5.2. □

As a consequence of the error estimates derived in Lemma 5.6, we can now obtain the desired first-order convergence result which is stated in the theorem below.

THEOREM 5.7 The following asymptotic expansion for the first transmission eigenvalue λ_δ^1 holds true for sufficiently small $\delta > 0$:

$$\lambda_\delta^1 = \lambda_0 + \delta\lambda_1 + \mathcal{O}(\delta^2),$$

where λ_0 is the first Dirichlet eigenvalue for $-\Delta$ in Ω and λ_1 is defined by (3.12).

Proof. Let $u_\delta^1 \in W_\delta$ be defined by (2.3) with v_δ^1 and w_δ^1 normalized such that $\|u_\delta^1\|_{L^2(\Omega)} = 1$. Then from Cakoni *et al.* (2012), we have that

$$(\lambda_\delta^1)^2 = \int_{\Omega\setminus\overline{\Omega_\delta}} \frac{1}{1-n} |\Delta u_\delta^1 + \lambda_\delta^1 u_\delta^1|^2 \,\mathrm{d}x + \lambda_\delta^1 \int_\Omega |\nabla u_\delta^1|^2 \,\mathrm{d}x,$$

and using the definition (2.3) of u_δ^1 and the equations for v_δ^1 this becomes

$$\lambda_\delta^1 = \int_{\Omega\setminus\overline{\Omega_\delta}} \lambda_1^\delta (1-n) |w_\delta^1|^2 \,\mathrm{d}x + \int_\Omega |\nabla u_\delta^1|^2 \,\mathrm{d}x. \tag{5.19}$$

From (5.13), the first term in (5.19) is of order δ^3, hence we need to develop only the second term. To this end, we can write

$$\begin{aligned}
\int_\Omega |\nabla u_\delta^1|^2 \,\mathrm{d}x &= \int_\Omega |\nabla (e_w^1 - e_v^1 - v_0 + \delta w_1)|^2 \,\mathrm{d}x \\
&= \int_\Omega |\nabla e_w^1|^2 \,\mathrm{d}x + \int_\Omega |\nabla e_v^1|^2 \,\mathrm{d}x + \int_\Omega |\nabla v_0|^2 \,\mathrm{d}x + \delta^2 \int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_1|^2 \,\mathrm{d}x \\
&\quad - 2\int_\Omega \nabla v_0 \nabla (w_\delta^1 - e_v^1) \,\mathrm{d}x + 2\delta \int_\Omega \nabla (e_w^1 - e_v^1) \nabla w_1 \,\mathrm{d}x - 2\int_\Omega \nabla e_v^1 \nabla e_w^1 \,\mathrm{d}x.
\end{aligned} \tag{5.20}$$

Recall that from Lemma 5.6, we have that

$$\int_\Omega |\nabla e_v^1|^2 \,\mathrm{d}x \leqslant \mathcal{O}(\delta^2), \quad \int_\Omega |\nabla e_w^1|^2 \,\mathrm{d}x \leqslant \mathcal{O}(\delta^2) \quad \text{and} \quad \int_\Omega \nabla e_v^1 \nabla e_w^1 \,\mathrm{d}x \leqslant \mathcal{O}(\delta^2).$$

Furthermore, from the definition of v_0 we have that

$$\int_{\Omega} |\nabla v_0|^2 \, \mathrm{d}x = \lambda_0,$$

and from the definition of w_1 we have that

$$\delta^2 \int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_1|^2 \, \mathrm{d}x = \delta^2 \int_0^{s_0} \left(\left| \frac{\partial v_0}{\partial \boldsymbol{\nu}} \right|^2 \int_0^{\delta} \frac{1}{\delta^2} J_{s,\eta} \, \mathrm{d}\eta \right) \mathrm{d}s + \mathcal{O}(\delta^2) = \delta\lambda_1 + \mathcal{O}(\delta^2).$$

In addition, we also know that $\|u_\delta^1\|_{L^2(\Omega)} = 1$ that is

$$1 = \int_{\Omega} |u_\delta^1|^2 \, \mathrm{d}x = \int_{\Omega} |u_\delta^1 + v_0|^2 - 2(u_\delta^1 + v_0)v_0 + |v_0|^2 \, \mathrm{d}x.$$

The estimates of Lemma 5.6 together with the definitions (5.9) and (5.10) give that

$$\|u_\delta^1 + v_0\|_{L^2(\Omega)} \leqslant \mathcal{O}(\delta),$$

and hence since $\|v_0\|_{L^2(\Omega)} = 1$ we have

$$\left| \int_{\Omega} v_0(u_\delta^1 + v_0) \, \mathrm{d}x \right| \leqslant \mathcal{O}(\delta^2). \tag{5.21}$$

Recalling that v_0 is a Dirichlet eigenfunction for $-\Delta$ with corresponding eigenvalue λ_0, from (5.21) we now obtain

$$\begin{aligned}\left| \int_{\Omega} \nabla v_0 \nabla (w_\delta^1 - e_v^1) \, \mathrm{d}x \right| &= \left| \int_{\Omega} \nabla v_0 \nabla (u_\delta + v_0) \, \mathrm{d}x \right| \\ &= \lambda_0 \left| \int_{\Omega} v_0 (u_\delta + v_0) \, \mathrm{d}x \right| \leqslant \mathcal{O}(\delta^2).\end{aligned}$$

Finally, noting that

$$\int_{\Omega\setminus\overline{\Omega_\delta}} \nabla (e_w^1 - e_v^1) \nabla w_1 \, \mathrm{d}x = \int_{\Omega\setminus\overline{\Omega_\delta}} -\Delta (e_w^1 - e_v^1) w_1 \, \mathrm{d}x - \int_{\Gamma} \frac{\partial}{\partial \boldsymbol{\nu}} (e_w^1 - e_v^1) w_1 \, \mathrm{d}s,$$

and using $\partial e_w^1/\partial \boldsymbol{\nu} = \partial e_v^1/\partial \boldsymbol{\nu}$ we obtain

$$\int_{\Omega\setminus\overline{\Omega_\delta}} \nabla (e_w^1 - e_v^1) \nabla w_1 \, \mathrm{d}x = \int_{\Omega\setminus\overline{\Omega_\delta}} -\Delta (e_w^1 - e_v^1) w_1 \, \mathrm{d}x. \tag{5.22}$$

But, (5.17) and (5.18) imply

$$\|\Delta (e_w^1 - e_v^1)\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C\delta^{1/2}.$$

This last estimate together with equality (5.22) and $\|w_1\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant C\delta^{1/2}$ gives

$$\int_{\Omega\setminus\overline{\Omega_\delta}} \nabla (e_w^1 - e_v^1) \nabla w_1 \, \mathrm{d}x \leqslant C\delta.$$

We have estimated all the terms in (5.20) and thus the expression (5.19) for λ_δ^1 finally yields the estimate

$$\lambda_\delta^1 = \lambda_0 + \delta\lambda_1 + \mathcal{O}(\delta^2). \qquad \square$$

5.3 *The convergence of the second-order approximation*

The goal of this section is twofold. We complete the rigorous justification of the asymptotic expansion (5.1), and present a constructive procedure how to iteratively obtain the converges of any order in the asymptotic expansion of transmission eigenvalues. To this end, before proceeding with the convergence of the eigenvalues we need to improve the rate of convergence of the corresponding eigenfunctions. This is possible by adding a correction term to the eigenfunctions v_δ^1 and w_δ^1. Let us consider the following error functions:

$$e_w^2 := w_\delta^1 - C_\delta^1(\delta w_1 + \delta^2 w_2) \quad \text{in } \Omega\setminus\overline{\Omega_\delta},$$

extended by 0 in Ω_δ and

$$e_v^2 := v_\delta^1 - C_\delta^1(v_0 + \delta v_1) \quad \text{in } \Omega,$$

where $w_2(x) := \hat{w}_2(s,\xi)$ and v_1 are defined in Section 3.2 and $C_\delta^1 := \|u_\delta^1 + \delta v_1\|_{L^2(\Omega)}$. As before, the error e_w^2 is continuous across the interface Γ_δ since it vanishes on Γ_δ. We remark that since $\|u_\delta^1\|_{L^2(\Omega)} = 1$ we have that

$$C_\delta^1 = 1 + \mathcal{O}(\delta). \tag{5.23}$$

We now proceed as in Lemma 5.6 to give a first estimate on e_w^2 which on its turn provides a H^1 bound for e_v^2 and then iterate the procedure to obtain the optimal bounds for e_w^2 and e_v^2.

LEMMA 5.8 The following *a priori* estimates hold for $\delta > 0$ sufficiently small:

$$\|e_w^2\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta) \quad \text{and} \quad \|e_w^2\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^2).$$

Proof. First of all, from the definition of w_1 and w_2 we have for $\delta > 0$ sufficiently small

$$\|w_1\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{1/2}), \quad \|w_2\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{1/2}). \tag{5.24}$$

Moreover, from Section 3.2

$$\Delta(w_1 + \delta w_2) = \frac{1}{(1+\delta\xi\kappa)^3}\left(\sum_{k=0}^{5}\delta^{k-2}A_k(\hat{w}_1 + \delta\hat{w}_2) - \lambda_\delta^1 n(\hat{w}_1 + \delta\hat{w}_2)\right),$$

and since $A_0\hat{w}_1 = 0$ and $A_0\hat{w}_2 + A_1\hat{w}_1 = 0$ we now have

$$\Delta(w_1 + \delta w_2) = \frac{1}{(1+\delta\xi\kappa)^3}\left(A_1\hat{w}_2 - \lambda_\delta^1 n(\hat{w}_1 + \delta\hat{w}_2) + \sum_{k=2}^{5}\delta^{k-2}A_k(\hat{w}_1 + \delta\hat{w}_2)\right),$$

which yields in view of (5.13) and (5.24)

$$\|\Delta e_w^2\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{3/2}). \tag{5.25}$$

Next,

$$\|e_w^2 - e_v^2\|_{H^{3/2}(\Gamma)} = \|C_\delta^1 \delta^2 w_2\|_{H^{3/2}(\Gamma)} \leqslant \mathcal{O}(\delta^2),$$
$$\frac{\partial e_w^2}{\partial \boldsymbol{\nu}} = \frac{\partial e_v^2}{\partial \boldsymbol{\nu}} \quad \text{on } \Gamma$$

and

$$\|\Delta e_v^2\|_{L^2(\Omega)} \leqslant \mathcal{O}(\delta),$$

whence by applying Lemma 5.1 with $s = 1$ we obtain the result. □

Similarly to the previous section, we are now able to obtain convergence rates for the eigenfunctions.

LEMMA 5.9 The following error estimates hold for $\delta > 0$ sufficiently small

$$\|e_v^2\|_{H^1(\Omega)} \leqslant C\delta^{3/2}$$

and

$$\|e_w^2\|_{H^1(\Omega)} \leqslant \mathcal{O}(\delta^{3/2}), \quad \|e_w^2\|_{L^2(\Omega)} \leqslant \mathcal{O}(\delta^{5/2}).$$

Proof. We proceed exactly as in the proof of Proposition 5.6. To this end, let us define $\mathrm{v}_2 := (C_\delta^1)^{-1}(v_\delta^1 - \delta v_1)$ and then since $C_\delta^1 = 1 + \mathcal{O}(\delta)$, we have that

$$\begin{aligned}
\mathrm{v}_2|_\Gamma &= (C_\delta^1)^{-1}(v_\delta^1|_\Gamma - \delta v_1|_\Gamma - \delta^2 v_2|_\Gamma) + (C_\delta^1)^{-1}\delta^2 v_2|_\Gamma \\
&= (C_\delta^1)^{-1}(v_\delta^1|_\Gamma - C_\delta^1(\delta v_1|_\Gamma - \delta^2 v_2|_\Gamma)) + \mathcal{O}(\delta^2) \\
&= (C_\delta^1)^{-1} e_w^2|_\Gamma + \mathcal{O}(\delta^2).
\end{aligned}$$

Using (4.19) together with Lemma 5.8, we see that

$$\|\mathrm{v}_2\|_{H^{1/2}(\Gamma)} \leqslant \mathcal{O}(\delta^{1/2})\|e_w^2\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} + \mathcal{O}(\delta^2) \leqslant \mathcal{O}(\delta^{3/2}).$$

Let us denote by θ_{v_2} a lifting of v_2 in Ω such that $\theta_{\mathrm{v}_2}|_\Gamma = \mathrm{v}_2|_\Gamma$ and $\|\theta_{\mathrm{v}_2}\|_{H^1(\Omega)} \leqslant \mathcal{O}(\delta^{3/2})$ then consider $\overline{\mathrm{v}}_2 := \mathrm{v}_2 - \theta_{\mathrm{v}_2}$. Obviously, $\overline{\mathrm{v}}_2 \in H_0^1(\Omega)$ and for all $\psi \in H_0^1(\Omega)$ we have

$$\int_\Omega \nabla\overline{\mathrm{v}}_2 \cdot \nabla\psi - \lambda_0 \overline{\mathrm{v}}_2 \psi \, \mathrm{d}x = \int_\Omega \nabla \mathrm{v}_2 \cdot \nabla\psi - \lambda_0 \mathrm{v}_2 \psi \, \mathrm{d}x - \int_\Omega \nabla\theta_{\mathrm{v}_2} \cdot \nabla\psi + \lambda_0 \theta_{\mathrm{v}_2} \psi \, \mathrm{d}x. \tag{5.26}$$

We can estimate the second term easily by using the bound on the lifting and the Cauchy–Schwarz inequality

$$\left| \int_\Omega \nabla\theta_{\mathrm{v}_2} \cdot \nabla\psi + \lambda_0 \theta_{\mathrm{v}_2} \psi \, \mathrm{d}x \right| \leqslant \mathcal{O}(\delta^{3/2})\|\psi\|_{H^1(\Omega)}.$$

Next we consider the first term in (5.26) containing v_2. For all $\psi \in H_0^1(\Omega)$ from Theorem 5.7 and Lemma 5.6, we can write

$$\begin{aligned}\int_\Omega \nabla(v_\delta^1 - \delta v_1)\cdot\nabla\psi\,\mathrm{d}x &= \lambda_\delta^1 \int_\Omega v_\delta^1\psi\,\mathrm{d}x - \delta\lambda_0\int_\Omega v_1\psi\,\mathrm{d}x - \delta\lambda_1\int_\Omega v_0\psi\,\mathrm{d}x\\ &= (\lambda_0+\delta\lambda_1)\int_\Omega v_\delta^1\psi\,\mathrm{d}x - \delta\lambda_0\int_\Omega v_1\psi\,\mathrm{d}x - \delta\lambda_1\int_\Omega v_0\psi\,\mathrm{d}x + \mathcal{O}(\delta^2)\|\psi\|_{H^1(\Omega)}\\ &= \lambda_0\int_\Omega (v_\delta^1-\delta v_1)\psi\,\mathrm{d}x + \mathcal{O}(\delta^2)\|\psi\|_{H^1(\Omega)}.\end{aligned}$$

Next by using (5.23), we obtain that there exists another constant C still independent of δ such that

$$\left|\int_\Omega (\nabla \mathrm{v}_2\cdot\nabla\psi - \lambda_0\mathrm{v}_2\psi)\,\mathrm{d}x\right| \leqslant C\delta^2\|\psi\|_{H^1(\Omega)},$$

for all $\psi \in H_0^1(\Omega)$. We can now apply Lemma A.2 to $\overline{v}_2$ since by (5.13) and the definition of C_δ^1, we have

$$\|\mathrm{v}_2\|_{L^2(\Omega)} = (C_\delta^1)^{-1}\|u_\delta^1+\delta v_1\|_{L^2(\Omega)} + \mathcal{O}(\delta^{3/2}) = 1+\mathcal{O}(\delta^{3/2}),$$

to obtain

$$\|\overline{v_2} - v_0\|_{H^1(\Omega)} \leqslant \mathcal{O}(\delta^{3/2}).$$

From the bound on the lifting θ_{v_2} and (5.23) the latter becomes

$$\|v_\delta^1 - C_\delta^1 v_0 - C_\delta^1\delta v_1\|_{H^1(\Omega)} \leqslant O(\delta^{3/2}).$$

Now it is clear that we can improve the bound on e_w^2 by applying Lemma 5.1 with $s=\frac{3}{2}$, since

$$\|\Delta e_v^2\|_{L^2(\Omega)} \leqslant \mathcal{O}(\delta^{3/2}). \tag{5.27}$$

Therefore, we arrive at the following improved estimate:

$$\|e_w^2\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{3/2}), \quad \|e_w^2\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{5/2}). \qquad \square$$

REMARK 5.10 As in Lemma 5.6, we can improve the bound on e_v^2 if we choose

$$C_\delta^1 = \|u_\delta^1 + \delta v_1 - \delta w_1\|_{L^2(\Omega)}.$$

Indeed, in this case

$$\|\overline{v}_2\|_{L^2(\Omega)} = 1+\mathcal{O}(\delta^2),$$

and since $\|e_w^2\|_{H^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{3/2})$ we deduce that $\|e_v^2\|_{H^{1/2}(\Gamma)} \leqslant \mathcal{O}(\delta^2)$ which implies that

$$\|e_v^2\|_{H^1(\Omega)} \leqslant \mathcal{O}(\delta^2).$$

The estimates obtained in Lemma 5.9 lead to the following result.

THEOREM 5.11 The following expansion for the first transmission eigenvalue holds true for $\delta > 0$ sufficiently small

$$\lambda_\delta^1 = \lambda_0 + \delta\lambda_1 + \delta^2\lambda_2 + \mathcal{O}(\delta^3),$$

where λ_0 is the first Dirichlet eigenvalue for $-\Delta$ in Ω, and λ_1 and λ_2 are defined by (3.12) and (3.14), respectively.

Proof. Similarly to the proof of Theorem 5.7, we expand the definition of λ_δ^1 by using the approximate eigenfunctions

$$w_{\text{app}}^2 := C_\delta^1(\delta w_1 + \delta^2 w_2) \quad \text{and} \quad v_{\text{app}}^2 := C_\delta^1(v_0 + \delta v_1),$$

and we extend w_{app}^2 by 0 inside Ω_δ. From the characterization (5.19) of λ_δ^1 and the bound (5.13), we have

$$\lambda_\delta^1 = \int_\Omega |\nabla u_\delta^1|^2 \,\mathrm{d}x + \mathcal{O}(\delta^3).$$

This writes

$$\begin{aligned}
\int_\Omega |\nabla u_\delta^1|^2 \,\mathrm{d}x &= \int_\Omega |\nabla(e_w^2 - e_v^2 - v_{\text{app}}^2 + w_{\text{app}}^2)|^2 \,\mathrm{d}x \\
&= \int_\Omega |\nabla e_w^2|^2 \,\mathrm{d}x + \int_\Omega |\nabla e_v^2|^2 \,\mathrm{d}x + \int_\Omega |\nabla v_{\text{app}}^2|^2 \,\mathrm{d}x + \int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_{\text{app}}^2|^2 \,\mathrm{d}x \\
&\quad - 2\int_\Omega \nabla v_{\text{app}}^2 \nabla(w_\delta^1 - e_v^2) \,\mathrm{d}x + 2\int_{\Omega\setminus\overline{\Omega_\delta}} \nabla(e_w^2 - e_v^2)\nabla w_{\text{app}}^2 \,\mathrm{d}x - 2\int_\Omega \nabla e_v^2 \nabla e_w^2 \,\mathrm{d}x. \quad (5.28)
\end{aligned}$$

There are several terms to evaluate which we consider one by one in the following.

Step 1: *Computation of the $\mathcal{O}(\delta^3)$ terms*. From Lemma 5.9, we can see easily

$$\int_\Omega |\nabla e_w^2|^2 \,\mathrm{d}x = \mathcal{O}(\delta^3), \qquad \int_\Omega |\nabla e_v^2|^2 \,\mathrm{d}x = \mathcal{O}(\delta^3) \quad \text{and} \quad \int_\Omega \nabla e_v^2 \nabla e_w^2 \,\mathrm{d}x = \mathcal{O}(\delta^3).$$

Furthermore,

$$\int_{\Omega\setminus\overline{\Omega_\delta}} \nabla(e_w^2 - e_v^2)\nabla w_{\text{app}}^2 \,\mathrm{d}x = -\int_{\Omega\setminus\overline{\Omega_\delta}} \Delta(e_w^2 - e_v^2) w_{\text{app}}^2 \,\mathrm{d}x - \int_\Gamma \frac{\partial}{\partial \boldsymbol{\nu}}(e_w^2 - e_v^2) w_{\text{app}}^2 \,\mathrm{d}x,$$

and recalling that $\partial e_v^2/\partial\boldsymbol{\nu} = \partial e_w^2/\partial\boldsymbol{\nu}$ we obtain

$$\int_{\Omega\setminus\overline{\Omega_\delta}} \nabla(e_w^2 - e_v^2)\nabla w_{\text{app}}^2 \,\mathrm{d}x = -\int_{\Omega\setminus\overline{\Omega_\delta}} \Delta(e_w^2 - e_v^2) w_{\text{app}}^2 \,\mathrm{d}x.$$

But (5.25) and (5.27) give

$$\|\Delta(e_w^2 - e_v^2)\|_{L^2(\Omega\setminus\overline{\Omega_\delta})} \leqslant \mathcal{O}(\delta^{3/2}),$$

which complemented with (5.24) gives

$$\left|\int_{\Omega\setminus\overline{\Omega_\delta}} \nabla(e_w^2 - e_v^2)\nabla w_{\text{app}}^2 \,\mathrm{d}x\right| \leqslant \mathcal{O}(\delta^3).$$

Step 2: *Computation of* $\int_\Omega |\nabla v_{\mathrm{app}}^2|^2\,\mathrm{d}x$. From its definition, we have after integration by part:

$$
\begin{aligned}
\int_\Omega |\nabla v_{\mathrm{app}}^2|^2\,\mathrm{d}x &= (C_\delta^1)^2 \left(\int_\Omega |\nabla v_0|^2\,\mathrm{d}x + 2\delta \int_\Omega \nabla v_0 \cdot \nabla v_1\,\mathrm{d}x + \delta^2 \int_\Omega |\nabla v_1|^2\,\mathrm{d}x \right) \\
&= (C_\delta^1)^2 \left(\lambda_0 + 2\delta\lambda_0 \int_\Omega v_0 v_1\,\mathrm{d}x + 2\delta\lambda_1 \right. \\
&\quad \left. + \delta^2 \int_\Omega (\lambda_0 |v_1|^2 + \lambda_1 v_0 v_1)\,\mathrm{d}x + \delta^2 \int_\Gamma \frac{\partial v_1}{\partial \boldsymbol{\nu}} \frac{\partial v_0}{\partial \boldsymbol{\nu}}\,\mathrm{d}s \right) \\
&= \lambda_0 \|v_{\mathrm{app}}^2\|_{L^2(\Omega)}^2 + 2(C_\delta^1)^2\delta\lambda_1 + \delta^2 \int_\Gamma \frac{\partial v_1}{\partial \boldsymbol{\nu}} \frac{\partial v_0}{\partial \boldsymbol{\nu}}\,\mathrm{d}s.
\end{aligned}
$$

Step 3: *Computation of* $\int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_{\mathrm{app}}^2|^2\,\mathrm{d}x$. To this end, we first write

$$
\begin{aligned}
\int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_{\mathrm{app}}^2|^2\,\mathrm{d}x &= (C_\delta^1)^2\delta^2 \int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_1|^2\,\mathrm{d}x \\
&\quad + 2(C_\delta^1)^2\delta^3 \int_{\Omega\setminus\overline{\Omega_\delta}} \nabla w_1 \cdot \nabla w_2\,\mathrm{d}x + (C_\delta^1)^2\delta^4 \int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_2|^2\,\mathrm{d}x.
\end{aligned} \tag{5.29}
$$

From the definition of w_1 and by using local coordinates, we have

$$
\begin{aligned}
\delta^2 \int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_1|^2\,\mathrm{d}x &= \delta^3 \int_0^{s_0} \int_0^1 \frac{1}{\delta^2} \left| \frac{\partial \hat{w}_1}{\partial \xi}(s,\xi) \right|^2 (1 + \delta\kappa\xi)\,\mathrm{d}\xi\, ds + \mathcal{O}(\delta^3) \\
&= \delta\lambda_1 + \delta^2 \frac{\kappa}{2}\lambda_1 + \mathcal{O}(\delta^3).
\end{aligned}
$$

Similarly, using the definition of w_2 we have

$$
\begin{aligned}
\delta^3 \int_{\Omega\setminus\overline{\Omega_\delta}} \nabla w_1 \cdot \nabla w_2\,\mathrm{d}x &= \delta^4 \int_0^{s_0} \int_0^1 \frac{1}{\delta^2} \frac{\partial \hat{w}_1}{\partial \xi}(s,\xi) \frac{\partial \hat{w}_2}{\partial \xi}(s,\xi)\,\mathrm{d}\xi\, ds + \mathcal{O}(\delta^3) \\
&= \delta^2 \int_0^{s_0} \int_0^1 \left(-\kappa \frac{\partial v_0}{\partial \boldsymbol{\nu}} \xi + \frac{\partial v_1}{\partial \boldsymbol{\nu}} \right) \frac{\partial v_0}{\partial \boldsymbol{\nu}}\,\mathrm{d}\xi\,\mathrm{d}s + \mathcal{O}(\delta^3) \\
&= -\delta^2 \frac{\kappa}{2}\lambda_1 + \delta^2 \int_\Gamma \frac{\partial v_1}{\partial \boldsymbol{\nu}} \frac{\partial v_0}{\partial \boldsymbol{\nu}}\,\mathrm{d}s + \mathcal{O}(\delta^3).
\end{aligned}
$$

For the last term of (5.29), we simply have

$$
\delta^4 \int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_2|^2\,\mathrm{d}x \leqslant \mathcal{O}(\delta^3).
$$

Next we need to estimate the constant $(C_\delta^1)^2$. Indeed

$$(C_\delta^1)^2 = \int_\Omega |u_\delta^1 + \delta v_1|^2 \, \mathrm{d}x = \int_\Omega |u_\delta^1|^2 + 2\delta \int_\Omega u_\delta^1 v_1 \, \mathrm{d}x + \delta^2 \int_\Omega |v_1|^2 \, \mathrm{d}x$$
$$= 1 + \delta^2 \int_\Omega |v_1|^2 \, \mathrm{d}x + 2\delta \int_\Omega (u_\delta^1 + v_0) v_1 \, \mathrm{d}x = 1 + \mathcal{O}(\delta^2), \tag{5.30}$$

since $\|u_\delta^1 + v_0\|_{L^2(\Omega)} \leqslant \mathcal{O}(\delta)$. Plugging everything into (5.29), we finally obtain

$$\int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla w_{\mathrm{app}}^2|^2 \, \mathrm{d}x = \delta\lambda_1 + \delta^2 \left(2\int_\Gamma \frac{\partial v_1}{\partial \boldsymbol{\nu}} \frac{\partial v_0}{\partial \boldsymbol{\nu}} \, \mathrm{d}s - \frac{\kappa}{2}\lambda_1 \right) + \mathcal{O}(\delta^3).$$

Step 4: *Computation of* $\int_\Omega \nabla v_{\mathrm{app}}^2 \cdot \nabla(w_\delta^1 - e_\nu^2) \, \mathrm{d}x$. To this end, we make use of the equation satisfied by v_{app}^2 together with the normalization of u_δ^1 to simplify it. First we can write

$$\int_\Omega \nabla v_{\mathrm{app}}^2 \cdot \nabla(w_\delta^1 - e_\nu^2) \, \mathrm{d}x = -\int_\Omega \Delta v_{\mathrm{app}}^2 (w_\delta^1 - e_\nu^2) \, \mathrm{d}x - \int_\Gamma \frac{\partial v_{\mathrm{app}}^2}{\partial \boldsymbol{\nu}} v_{\mathrm{app}}^2 \, \mathrm{d}s$$
$$= C_\delta^1 \int_\Omega (\lambda_0 v_0 + \delta\lambda_0 v_1 + \delta\lambda_1 v_0)(w_\delta^1 - e_\nu^2) \, \mathrm{d}x$$
$$- (C_\delta^1)^2 \int_\Gamma \left(\frac{\partial v_0}{\partial \boldsymbol{\nu}} + \delta \frac{\partial v_1}{\partial \boldsymbol{\nu}} \right) \delta v_1 \, \mathrm{d}s.$$

From (5.13) and Lemma 5.9, we have that

$$C_\delta^1 \delta^2 \int_\Omega \lambda_1 v_1 (w_\delta^1 - e_\nu^2) \, \mathrm{d}x = \mathcal{O}(\delta^3),$$

whence

$$\int_\Omega \nabla v_{\mathrm{app}}^2 \cdot \nabla(w_\delta^1 - e_\nu^2) \, \mathrm{d}x = (\lambda_0 + \delta\lambda_1) \int_\Omega v_{\mathrm{app}}^2 (w_\delta^1 - e_\nu^2) \, \mathrm{d}x + (C_\delta^1)^2 \delta\lambda_1 + \delta^2 \int_\Gamma \frac{\partial v_1}{\partial \boldsymbol{\nu}} \frac{\partial v_0}{\partial \boldsymbol{\nu}} \, \mathrm{d}s + \mathcal{O}(\delta^3).$$

We now use the normalization of u_δ^1 to obtain

$$\|u_\delta^1\|_{L^2(\Omega)}^2 = 1 = \int_\Omega |u_\delta^1 + v_{\mathrm{app}}^2 - v_{\mathrm{app}}^2|^2 \, \mathrm{d}x$$
$$= \int_\Omega |u_\delta^1 + v_{\mathrm{app}}^2|^2 \, \mathrm{d}x - 2\int_\Omega v_{\mathrm{app}}^2 (u_\delta^1 + v_{\mathrm{app}}^2) \, \mathrm{d}x + \int_\Omega |v_{\mathrm{app}}^2|^2 \, \mathrm{d}x$$
$$= \mathcal{O}(\delta^3) - 2\int_\Omega v_{\mathrm{app}}^2 (u_\delta^1 + v_{\mathrm{app}}^2) \, \mathrm{d}x + \int_\Omega |v_{\mathrm{app}}^2|^2 \, \mathrm{d}x. \tag{5.31}$$

Hence, since $u_\delta^1 + v_{\mathrm{app}}^2 = w_\delta^1 - e_\nu^2$, we have that

$$-2\int_\Omega \nabla v_{\mathrm{app}}^2 \cdot \nabla(w_\delta^1 - e_\nu^2) \, \mathrm{d}x = (\lambda_0 + \delta\lambda_1)(1 - \|v_{\mathrm{app}}^2\|_{L^2(\Omega)}^2) - 2(C_\delta^1)^2\delta\lambda_1 - 2\delta^2 \int_\Gamma \frac{\partial v_1}{\partial \boldsymbol{\nu}} \frac{\partial v_0}{\partial \boldsymbol{\nu}} \, \mathrm{d}s + \mathcal{O}(\delta^3).$$

The expansion (5.30) and the definition of v^2_{app} yield

$$|1 - \|v^2_{\mathrm{app}}\|^2_{L^2(\Omega)}| \leqslant \mathcal{O}(\delta^2),$$

and consequently

$$-2\int_\Omega \nabla v^2_{\mathrm{app}} \cdot \nabla(w^1_\delta - e^2_v)\,\mathrm{d}x = \lambda_0(1 - \|v^2_{\mathrm{app}}\|^2_{L^2(\Omega)}) - 2(C^1_\delta)^2\delta\lambda_1 - 2\delta^2\int_\Omega \frac{\partial v_1}{\partial \boldsymbol{\nu}}\frac{\partial v_0}{\partial \boldsymbol{\nu}}\,\mathrm{d}s + \mathcal{O}(\delta^3).$$

Finally, we have all the necessary estimates to reach the conclusion. Plugging the estimates obtained in Steps 1–4 into (5.28) leads to the desired final estimate:

$$\lambda^1_\delta = \lambda_0 + \delta\lambda_1 + \delta^2\lambda_2 + \mathcal{O}(\delta^3). \qquad \square$$

Remark 5.12 Although we stop at the order 2 the analysis of Section 5.3 is constructive and can in principle be carried through iteratively to any order of approximation. Note that in order to prove the k order of convergence for the transmission eigenvalue, we need to prove the $(k-1)$ order of convergence for the corresponding eigenfunctions. Also the convergence procedure is not limited to only the first eigenvalue. All these generalizations rely upon the ability to compute explicitly the terms in the asymptotic expansion of the transmission eigenvalues and on the estimate for the zero-order approximation of the eigenvalue.

Remark 5.13 In principle, the second-order asymptotic expansion of the first transmission eigenvalue can be used to estimate the thickness of the layer provided that Ω is known. In particular,

$$\delta \approx \frac{\lambda^1_\delta - \lambda_0}{\lambda_1},$$

where λ^1_δ can be computed from the scattering data (see Cakoni *et al.*, 2010a) and λ_0 and λ_1 can be computed.

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Appendix A. Perturbation of an eigenvalue problem

We recall here some known results about the convergence of eigenvalues for self-adjoint, positive and compact operators. The proof of the following fundamental result can be found in Oleinik *et al.* (1992, Section 3).

Theorem A.1 Assume that $A : H \to H$ is a linear self-adjoint positive and compact operator on an Hilbert space H. Let $u \in H$ be such that $\|u\|_H = 1$ and $\lambda, r > 0$ such that

$$\|Au - \lambda u\|_H \leqslant r.$$

Then there is an eigenvalue λ_i of the operator A satisfying

$$|\lambda - \lambda_i| \leqslant r.$$

Furthermore, for any $r^* > r$ there exists $u^* \in H$ with $\|u^*\|_H = 1$ belonging to the eigenspace associated with all the eigenvalues of the operator A lying in $[\lambda - r^*, \lambda + r^*]$ that satisfies

$$\|u - u^*\|_H \leqslant \frac{2r}{r^*}.$$

Based on this general result, we can obtain the following lemma for the Laplace operator with Dirichlet boundary conditions which is used in our asymptotical analysis in the main body of the paper.

LEMMA A.2 Let λ_i be a simple eigenvalue of the negative Laplacian with Dirichlet boundary conditions in Ω. Assume that it exists $u \in H_0^1(\Omega)$ and r such that

$$\left| \int_\Omega \nabla u \cdot \nabla v - \lambda_i u v \,\mathrm{d}x \right| \leqslant r \|v\|_{H_0^1(\Omega)} \quad \forall v \in H_0^1(\Omega) \tag{A.1}$$

and

$$\left| \|u\|_{L^2(\Omega)} - 1 \right| \leqslant r. \tag{A.2}$$

Then it exists an eigenfunction u_i associated with the eigenvalue λ_i and normalized as $\|u_i\|_{L^2(\Omega)} = 1$ such that

$$\|u - u_i\|_{H^1(\Omega)} \leqslant Cr,$$

for some constant $C > 0$ independent of r and u.

Proof. The proof is essentially based on Theorem A.1. We define the operator $A : H_0^1(\Omega) \to H_0^1(\Omega)$ by

$$(Au, v)_{H_0^1(\Omega)} := (u, v)_{L^2(\Omega)} \quad \forall (u, v) \in H_0^1(\Omega),$$

where for all $(u, v) \in H_0^1(\Omega)$,

$$(u, v)_{H_0^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x,$$

and $(u, v)_{L^2(\Omega)}$ is the usual L^2 scalar product. Obviously, A is a self-adjoint positive and compact operator on $H_0^1(\Omega)$, and hence it has a discrete spectrum $(1/\lambda_i)_{i=1,\ldots,\infty}$ such that

$$0 < \lambda_1 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots \underset{i \to \infty}{\longrightarrow} \infty,$$

and their associated eigenfunctions $u_i \in H_0^1(\Omega)$ satisfy

$$-\Delta u_i = \lambda_i u_i.$$

By hypothesis, the function $\tilde{u} := u / \|u\|_{H_0^1(\Omega)}$ satisfies

$$\left\| A\tilde{u} - \frac{1}{\lambda_i} \tilde{u} \right\|_{H_0^1(\Omega)} \leqslant \frac{r}{\lambda_i \|u\|_{H_0^1(\Omega)}}.$$

Therefore, since λ_i is supposed to be simple, the second part of Theorem A.1 ensures the existence of an eigenfunction $\tilde{u}_i \in H_0^1(\Omega)$ of A associated with $1/\lambda_i$ and normalized as $\|\tilde{u}_i\|_{H_0^1(\Omega)} = 1$, such that

$$\|\tilde{u} - \tilde{u}_i\|_{H_0^1(\Omega)} \leqslant C_1 \frac{r}{\|u\|_{H_0^1(\Omega)}}, \tag{A.3}$$

for some constant $C_1 > 0$ that only depends on λ_i and on the distance between $1/\lambda_i$ and the closest eigenvalue of A. To end the proof, we must renormalize this last inequality.

Let us introduce $u_i := \tilde{u}_i / \|\tilde{u}_i\|_{L^2(\Omega)}$, then (A.3) gives

$$\|u - u_i\|_{H_0^1(\Omega)} \leqslant C_1 r + \|u_i - \|u\|_{H_0^1(\Omega)} \tilde{u}_i\|_{H_0^1(\Omega)}. \tag{A.4}$$

By using the definition of u_i, the second term in this expression becomes

$$\|u_i - \|u\|_{H_0^1(\Omega)} \tilde{u}_i\|_{H_0^1(\Omega)} \leqslant |\sqrt{\lambda_i} - \|u\|_{H_0^1(\Omega)}| \leqslant |\lambda_i - \|u\|^2_{H_0^1(\Omega)}| \frac{1}{\sqrt{\lambda_i}}, \tag{A.5}$$

but from (A.1) and (A.2) we have

$$\begin{aligned} |\lambda_i - \|u\|^2_{H_0^1(\Omega)}| &\leqslant |\lambda_i - \lambda_i \|u\|^2_{L^2(\Omega)}| + r\|u\|_{H_0^1(\Omega)} \\ &\leqslant \lambda_i |1 - \|u\|_{L^2(\Omega)}||1 + \|u\|_{L^2(\Omega)}| + r\|u\|_{H_0^1(\Omega)} \\ &\leqslant \lambda_i r(2 + r) + r\|u\|_{H_0^1(\Omega)}. \end{aligned} \tag{A.6}$$

The H_0^1-norm of u can be controlled by using (A.1) and Poincarre's inequality:

$$\|u\|_{H_0^1(\Omega)} \leqslant r + \lambda_i \sqrt{\lambda_0}(1 + r).$$

This last inequality together with (A.4–A.6) give the result for a constant C that only depends on λ_0 and λ_i. □