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Qualitative Methods in Inverse Electromagnetic Scattering Theory

Inverse scattering for anisotropic media.

In recent years, a new approach has been developed in the study of the inverse scattering problem for electromagnetic waves. In this approach, a weak scattering assumption has been avoided, and no use has been made of nonlinear optimization methods. Instead, a study is made of the analytic properties of the far-field operator, and the results of this study are used to determine the support of the scattering object together with an estimate of the material properties of the scatterer. This article introduces this new approach in inverse electromagnetic scattering theory, which is called the *qualitative* approach to inverse scattering theory.

INVERSE SCATTERING THEORY

Inverse scattering theory is central to such diverse application areas as medical imaging, geophysical exploration, and nondestructive testing. The growth of this field has been driven by the realization that the inverse scattering problem is both nonlinear and ill posed, thus presenting difficulties in the development of efficient inverse algorithms. Although linearized models continue to play an important role in many applications, the

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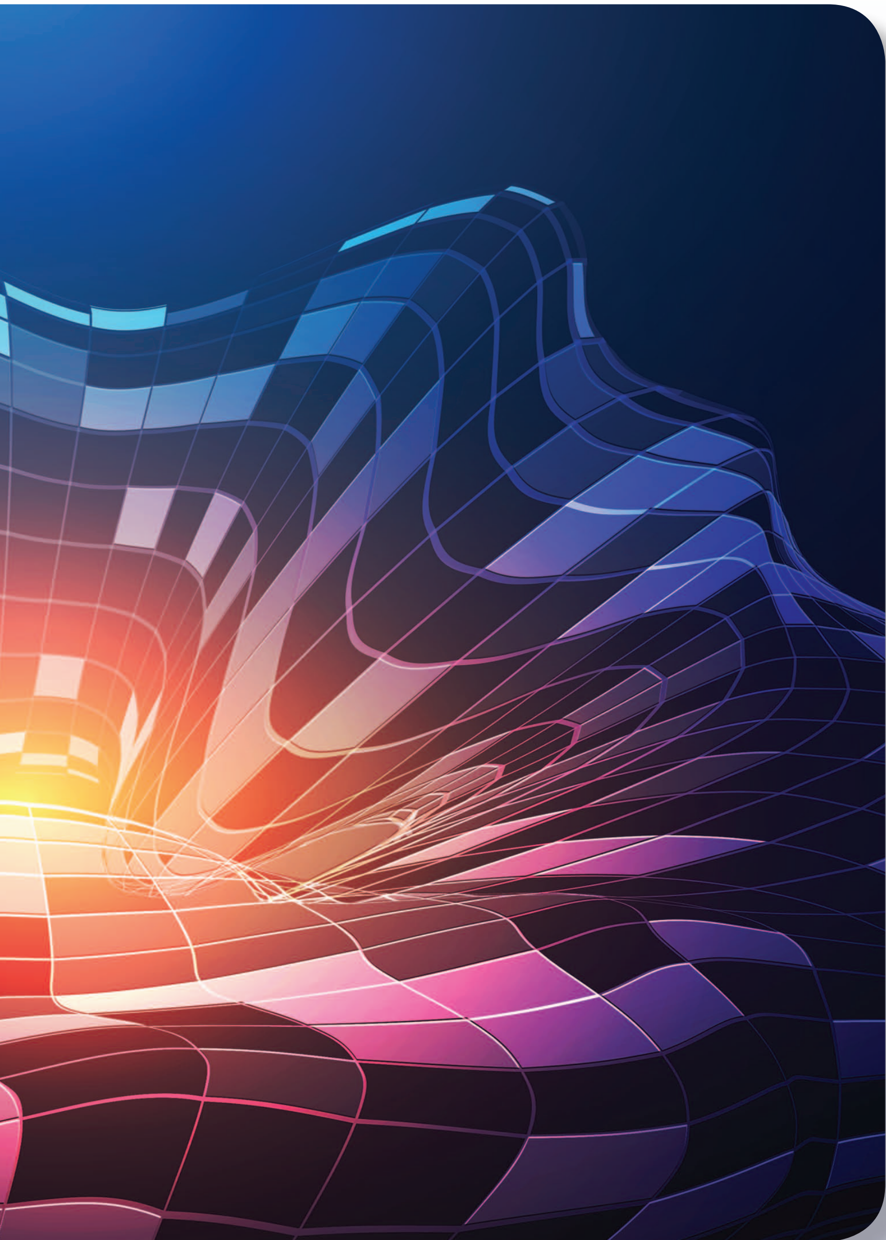
increased need to focus on problems in which multiple scattering effects can no longer be ignored has led to the nonlinearity of the inverse scattering problem playing a central role. In addition, the possibility of collecting large amounts of data over a limited region of space has led to the situation where the ill-posed nature of the inverse scattering problem becomes of central importance.

Initial efforts to deal with the nonlinear and ill-posed nature of the inverse scattering problem have focused on the use of nonlinear optimization methods, in particular Newton's method and various versions of what are now called *decomposition methods*. For a discussion of this approach to the inverse scattering problem together with numerous references, see [1]. Although efficient in many situations, the use of nonlinear optimization methods suffers from the need for strong a priori information to implement such an approach. Hence, to circumvent this difficulty, a recent trend in inverse scattering theory has focused on the development of the qualitative approach, in which the amount of a priori information needed is drastically reduced but at the expense of obtaining only limited information about the scatterer, such as connectivity, support, and an estimate on the values of the constitutive parameters. Examples of such an approach are the linear sampling method, the factorization method, and the theory of transmission eigenvalues.

The qualitative approach to inverse scattering theory was initiated by Colton and Kirsch in 1996 [2]. In this paper, they introduced a linear integral equation of the first kind, called the *far-field equation*, whose solution could be used as an indicator function to determine the support of the scattering obstacle. Obtaining the solution of the far-field equation is an ill-posed problem, and the determination of a regularized solution to this problem is complicated by the fact that, in general, the noise-free equation has no solution. In particular, the general theory of ill-posed problems is not immediately applicable. This problem was solved by Kirsch in 1999 [3]. Having determined the support of the scatterer by the aforementioned methods (called, respectively, the *linear sampling method* and *factorization method*), the next step in the qualitative approach to inverse scattering theory is to obtain estimates on the material properties of the scatterer. This was accomplished by Cakoni, Gintides, and Haddar in 2010 [4] using transmission eigenvalues first introduced by Kirsch [5] and Colton and Monk [6]. The development of the aforementioned themes is the subject of this article. For a more detailed discussion, together with numerical examples, refer to the recently published monograph [7] for scalar problems and to the monograph [8] for Maxwell's equations.



SCATTERING BY INHOMOGENEOUS MEDIA

We consider the scattering of an incident time harmonic electromagnetic field by a possibly anisotropic medium of compact support $\bar{D}$ (Figure 1). The known incident field E^i, H^i impinges on the inhomogeneous scatterer occupying the domain D. This creates a scattered field E^s, H^s outside D and a total field E, H in D. The unit normal vector ν points outward from D. Assuming $\exp(-i\omega t)$ dependence on time, this is modeled by the following set of Maxwell's equations, where E^i, H^i is the given

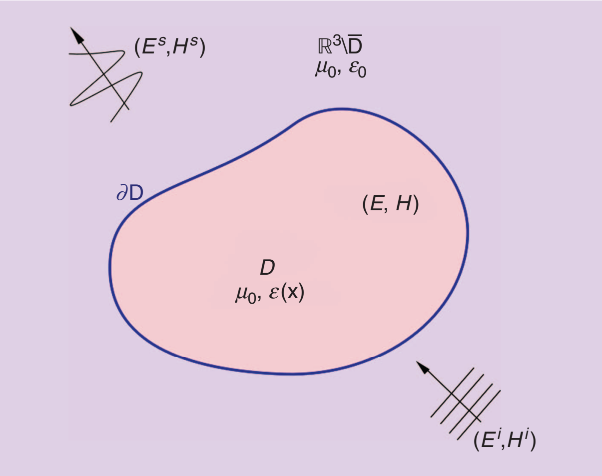


FIGURE 1. An illustration of the direct electromagnetic scattering problem for inhomogeneous media.

electromagnetic incident field, E^s, H^s is the scattered electromagnetic field, and E, H is the total field inside D:

$$\begin{aligned}
\operatorname{curl} E^s - i\omega\mu_0 H^s &= 0 \quad \text{in } \mathbb{R}^3\backslash\bar{D},\\
\operatorname{curl} H^s + i\omega\epsilon_0 E^s &= 0 \quad \text{in } \mathbb{R}^3\backslash\bar{D},\\
\operatorname{curl} E - i\omega\mu_0 H &= 0 \quad \text{in } D,\\
\operatorname{curl} H + (i\omega\epsilon(x) - \sigma(x))E &= 0 \quad \text{in } D,\\
\nu\times E &= \nu\times(E^s+E^i) \quad \text{on } \partial D,\\
\nu\times H &= \nu\times(H^s+H^i) \quad \text{on } \partial D,
\end{aligned}$$

$$\lim_{|x|\to\infty}(\sqrt{\mu_0}H^s\times x - \sqrt{\epsilon_0}|x|E^s) = 0, \text{ and}$$

$$\lim_{|x|\to\infty}(\sqrt{\epsilon_0}E^s\times x - \sqrt{\mu_0}\,|x|\,H^s) = 0.$$

Here, ω is the interrogation frequency, v is the unit outward normal to D, and the constants $\epsilon_0 > 0$ and $\mu_0 > 0$ are the electric permittivity and magnetic permeability of the vacuum (for simplicity, we consider the inhomogeneity embedded in vacuum, but the discussion here can be extended to a more complicated background; see [8] and [9]). In addition, the 3×3 real matrix valued functions $\epsilon(x)$ and $\sigma(x)$ are the electric permeability and conductivity inside the inhomogeneity (again, we take for simplicity the magnetic permeability of the medium to be the same as in the vacuum, but the discussion here is valid with slight modification if this is not the case; see [8] and [10]). Expressing the magnetic field in terms of the electric field, we can rewrite the aforementioned set of equations as follows:

$$\begin{aligned}
\operatorname{curl}\operatorname{curl} E^s - k^2E^s &= 0 \quad \text{in } \mathbb{R}^3\backslash\bar{D},\\
\operatorname{curl}\operatorname{curl} E - k^2\epsilon_r E &= 0 \quad \text{in } D,\\
\nu\times E &= \nu\times(E^s+E^i) \quad \text{on } \partial D,\\
\nu\times\operatorname{curl} E &= \nu\times(\operatorname{curl} E^s + \operatorname{curl} E^i) \quad \text{on } \partial D,
\end{aligned}$$

and

$$\lim_{|x|\to\infty}(\operatorname{curl} E^s\times x - ik|x|E^s) = 0,$$

where the above radiation condition, known as the *Silver–Muller radiation condition*, is assumed to hold uniformly with respect to $\hat{x} = x/|x|$, $k = \omega\sqrt{\epsilon_0\mu_0}$ is the wavenumber, and the 3×3 matrix valued function ϵ_r with $L^\infty(D)$-entries is given by

$$\epsilon_r(x) = \frac{\epsilon(x)}{\epsilon_0} + i\frac{\sigma(x)}{\omega\epsilon_0}.$$

The mathematical theory for Maxwell's equations makes use of spaces of functions with a well-defined curl in the last-squares sense. More precisely, for a generic open set O, $H(\operatorname{curl}, O)$ contains square-integrable functions with a square-integrable curl in O. For exterior problems on unbounded domains, we denote by $H_{\text{loc}}(\operatorname{curl}, O)$ the space of functions in $H(\operatorname{curl}; B)$ for any compact subset $B\subset O$ (see [11]). Assuming that E^i satisfies $\operatorname{curl}\operatorname{curl} E^i - k^2E^i = 0$ in $\mathbb{R}^3$ (possibly excluding one point for the case of point sources), and under the physical assumptions

$$\bar{\xi}\cdot\Re(\epsilon_r)\xi \geq \alpha\|\xi\|^2 \quad\text{and}\quad \bar{\xi}\cdot\Im(\epsilon_r)\xi\geq 0,$$

it can be shown that the direct scattering problem has a unique solution $E\in H(\operatorname{curl}, D)$ and $E^s\in H_{\text{loc}}(\operatorname{curl}, \mathbb{R}^3\backslash\bar{D})$.

In this article, our main interest will be in the inverse scattering problem where, from a knowledge of E^s on a surface Γ corresponding to several interrogating fields E^i and possibly a range of frequencies, we want to obtain the support D and estimates on ϵ_r. Methods for doing this are called *qualitative* in inverse scattering theory (compare with [7], [8], and [20]). To fix our ideas, we assume that the incident field is a plane wave given by

$$E^i(x; d, p, k) := ik(d\times p)\times de^{ikx\cdot d},$$

where $d\in\mathbb{S}^2$ is the direction of propagation, $\mathbb{S}^2 := \{x\in\mathbb{R}^3 : |x| = 1\}$, and $p\in\mathbb{R}^3$, $p\neq 0$ denotes the polarization. Then, the scattered electric field E^s has the asymptotic behavior

$$E^s(x; d, p, k) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}; d, p, k) + O\left(\frac{1}{|x|}\right)\right\},$$

as $|x|\to\infty$ uniformly with respect $\hat{x} = x/|x|$ (see [9]). The tangential function $E_\infty(\hat{x}; d, p, k)$ defined on the unit sphere $\mathbb{S}^2$ is the far-field pattern of the scattered field, and we assume that $E_\infty(\hat{x}; d, p, k)$ is known for all $\hat{x}, d\in\mathbb{S}^2$ and (possibly) $k\in[k_1, k_2]$. It is also possible to consider only $E_\infty(\hat{x}; d, p, k)$ to be known for $\hat{x}, d$ on an open subset of $\mathbb{S}^2$ (see [8] and [12]), but for the sake of simplicity, we shall not consider this situation in this article. In our investigation of the inverse scattering problem, our primary interest will be the far-field operator $F: L_t^2(\mathbb{S}^2)\to L_t^2(\mathbb{S}^2)$ ($L_t^2(\mathbb{S}^2)$ contains square-integrable tangential fields on $\mathbb{S}^2$), defined by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} E_\infty(\hat{x}; d, g(d), k)\,ds(d). \tag{1}$$

We note that F is a linear compact operator. Clearly, Fg is the far-field pattern of the scattered field corresponding to the incident field, being an electric Herglotz wave function with kernel g defined by

$$E_g(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d)\,ds(d) \quad g \in L_t^2(\mathbb{S}^2). \tag{2}$$

Note that F is related to scattering operator $\mathcal{S}$ (see, e.g., [9]) by

$$\mathcal{S} = I + \frac{1}{2\pi}F.$$

The basic mathematical property of the far-field operator is formulated in the following statement [4], [5]: the far-field operator $F: L_t^2(\mathbb{S}^2) \to L_t^2(\mathbb{S}^2)$ is injective and has a dense range if and only if there does not exist a nontrivial solution to the homogeneous interior transmission problem

$$\operatorname{curl}\operatorname{curl} E_0 - k^2 E_0 = 0 \quad \text{in } D, \tag{3}$$

$$\operatorname{curl}\operatorname{curl} E - k^2 \epsilon_r E = 0 \quad \text{in } D, \tag{4}$$

$$\nu \times E = \nu \times E_0 \quad \text{on } \partial D, \text{ and} \tag{5}$$

$$\nu \times \operatorname{curl} E = \nu \times curl\, E_0 \quad \text{on } \partial D, \tag{6}$$

such that $E_0 := E_g$ is an electric Herglotz wave function.

Values of $k \in \mathbb{C}$ for which the transmission eigenvalue problem (3)–(6) has a nontrivial solution are called *transmission eigenvalues*. Transmission eigenvalues are related to nonscattering frequencies. In particular, if k is a transmission eigenvalue and the eigenfunction E_0 that solves $\operatorname{curl}\operatorname{curl} E_0 - k^2 E_0 = 0$ in D can be extended outside D as a solution $\tilde{E}_0$ of the same equation, then the scattered field due to $\tilde{E}_0$ as an incident wave is identically zero.

By separating variables, it is easy to see that if D is a sphere centered at the origin with constant scalar relative permittivity ϵ_r, then at a transmission eigenvalue E_0 is an electric Herglotz wave function, and therefore it does not scatter by this inhomogeneity. Unfortunately, in general, such an extension of E_0 does not exist (see [13] for domains with a corner in the scalar case). However, since electric Herglotz wave functions are dense in the set (see [9])

$$\{U \in L^2(D): \operatorname{curl}\operatorname{curl} U - k^2 U = 0 \quad \text{in } D\},$$

at a transmission eigenvalue, it is possible to superimpose plane waves such that this superposition produces an arbitrarily small scattered field.

THE DETERMINATION OF THE SUPPORT

We now address the problem of determining the support D of the scattering obstacle from a knowledge of the far-field pattern $E_\infty(\hat{x}; d, p, k)$ for $\hat{x}, d \in \mathbb{S}^2$ and k fixed. Note that no knowledge of ϵ_r is assumed. The first issue to address is uniqueness. In [14] the following result is proven: If $D_1, \epsilon_{r,1}$ and $D_2, \epsilon_{r,2}$ give rise to the same far-field data, i.e., $E_\infty^{(1)}(\hat{x}; d, p, k) = E_\infty^{(2)}(\hat{x}; d, p, k)$ for all $d, \hat{x} \in \mathbb{S}^2$, for three linearly independent polarizations and one fixed k, then $D_1 = D_2$.

Our problem now is to determine the support D without any a priori knowledge of ϵ_r. The first step in this direction was the linear sampling method [2], [8], [15]. This method is based on solving the far-field equation

$$(Fg)(\hat{x}) = E_\infty(\hat{x}, z, q, k) \tag{7}$$

for $g \in L_t^2(\mathbb{S}^2), \hat{x} \in \mathbb{S}^2, z \in \mathbb{R}^3$, and for $q \in \mathbb{R}^3$ and $k > 0$ fixed, where

$$E_\infty(\hat{x}, z, q, k) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x} e^{-ik\hat{x}\cdot z}$$

is the far-field pattern of the electric dipole $E^e(x, z, q, k)$ given by

$$E^e(x, z, q, k) := \frac{1}{4\pi k^2} \operatorname{curl}\operatorname{curl} \Phi(x, z) q$$

with

$$\Phi(x, z) := \frac{e^{ik|x-z|}}{|x-z|}$$

denoting the radiating fundamental solution to the Helmholtz equation. It can be shown (see [8] and [16]) that, if k is not a transmission eigenvalue, then there exists a Herglotz kernel $g_{z,\epsilon} := g_{z,\epsilon,q,k}$ satisfying

$$\| F g_{z,\epsilon} - E_\infty(\hat{x}, z, q, k) \|_{L_t^2(\mathbb{S}^2)} < \epsilon, \tag{8}$$

such that

1) for $z \in D$, $\lim_{\epsilon \to 0} \| E_{g_{z,\epsilon}} \|_{L^2(D)}$ exists and
2) for $z \notin D$, $\lim_{\epsilon \to 0} \| E_{g_{z,\epsilon}} \|_{L^2(D)} = \infty$,

where $E_{g_{z,\epsilon}}$ is the electric Herglotz wave function given by (2) with $g := g_{z,\epsilon}$. Since the far-field equation is ill posed (the singular values of the compact far-field operator F decay exponentially), it is necessary to use regularization techniques. What is done in practice is to apply Tikhonov regularization [1], [8], [17] to the far-field equation and solve

$$(\alpha I + F^* F) g = F^* E_\infty(\hat{x}, z, q, k), \tag{9}$$

where $\alpha > 0$ is the regularization parameter and F^* is the adjoint operator to F. We denote by $g_{\alpha,z,q,k} \in L_t^2(\mathbb{S}^2)$ the resulting solution. In reality, F is replaced by the noisy measured far-field operator F^δ, whose kernel is now the measured far-field data $E_\infty^\delta(\hat{x}; d, p, k)$ with noise level $\delta > 0$. Using the Herglotz kernels, we can define the indicator function

$$\mathbb{I}(z) = \frac{1}{\| g_{z,\alpha,q_1,k} \|_{L^2}} + \frac{1}{\| g_{z,\alpha,q_2,k} \|_{L^2}} + \frac{1}{\| g_{z,\alpha,q_3,k} \|_{L^2}},$$

where q_1, q_2, and q_3 are three independent artificial polarization vectors and $z \in \mathbb{R}^3$ is an artificial source point. The artificial source point z is usually taken to lie at the vertices of a uniform grid in the region of $\mathbb{R}^3$, where the unknown scatterer is assumed to be. Using these computed values, the linear sampling method then visualizes the support D of the inhomogeneity as the surface

$$\mathbb{I}(z) = C$$

for some ad hoc constant C that signals the transition between small and large values of $\mathbb{I}$. The choice of the search grid for z is where the main a priori information for the linear sampling method is needed: we need to know the approximate size of the unknown scatterer and the approximate position to choose a search grid with sufficient resolution in the region to be scanned.

The assumptions on the material properties needed for the linear sampling method to work in addition to the ones stated in the introduction are that

$$\begin{cases} \text{either } \bar{\xi}\cdot\Re(\epsilon_r - I)\xi \geq \gamma\|\xi\|^2 \\ \quad\text{or} \quad \bar{\xi}\cdot\Re(I - \epsilon_r)\xi \geq \gamma\|\xi\|^2 \end{cases} \tag{10}$$

for some $\gamma > 0$ in an arbitrary neighborhood of the boundary ∂D, which guarantees that the interior transmission problem (3)–(6) is a compact perturbation of an invertible operator [19].

The linear sampling approach appears to work satisfactorily in that the numerical results confirm the viability of the method [8]. For example, in Figure 2 we show computational results from [18], where an inhomogeneous and disconnected scatter, coated with a thin, highly conducting layer, is reconstructed from far-field measurements. We use $k = 4, \epsilon_r = 2I$, and 92 incident waves and measurement points. The two balls are covered by a thin conducting layer with the impedance parameter set to 1 (see [18] for details).

That transmission eigenvalues have a profound effect on the reconstruction is shown in Figures 3 and 4. In Figure 3, it is clear that a scatterer is to be found centered at the origin and that the choice of threshold C will have relatively little effect on the predicted size of the scatterer due to the rapid rise in $\mathbb{I}(z)$ near the edges of the scatterer. In contrast,

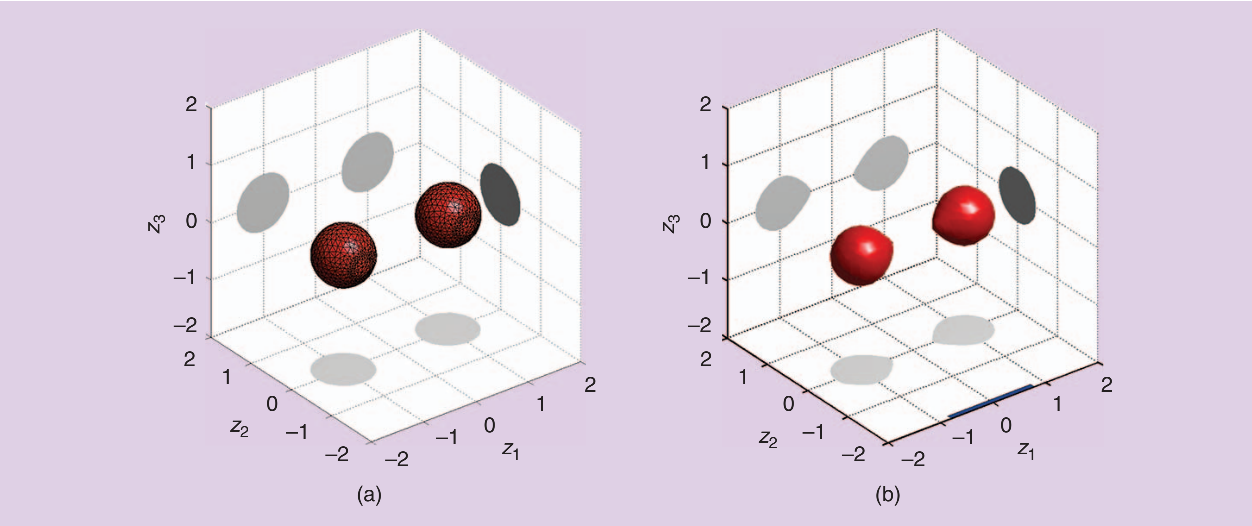


FIGURE 2. A part of Figure 6 from [18]. (a) The exact scatterer consists of two penetrable balls with a radius of 0.5, one unit apart. (b) The reconstruction uses a contour level for $\mathbb{I}(z)$, chosen using the computed Herglotz kernels and described in [18]. The wavelength of the incident field is shown as a thick line along the z_1 axis in the lower panel. (Figure courtesy of [18], reproduced with permission.)

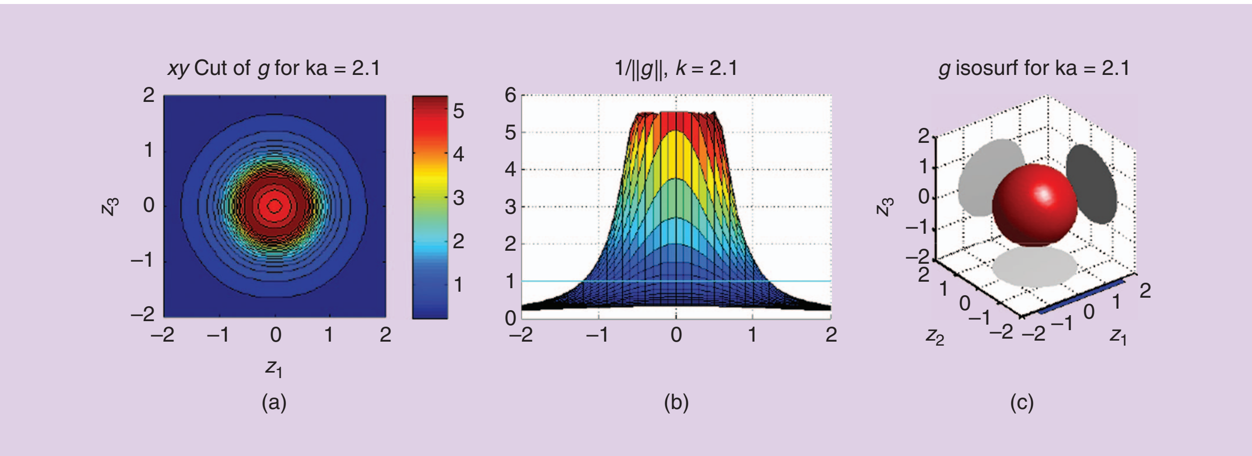


FIGURE 3. The reconstruction of a dielectric sphere of radius 1 with $\epsilon_r = 16I$ using the wavenumber $k = 2.1$ (not a transmission eigenvalue for the sphere). (a) A contour map of $\mathbb{I}(z)$ in the plane $z = 0$. (b) A mesh plot of $\mathbb{I}(z)$ along the line $z_1 = z_3 = 0$. (c) A reconstruction of the sphere using C chosen by the method described in [18] and marked as a blue line in (b).

when the same problem is solved at a transmission eigenvalue, as shown in Figure 4, we see that, although a plot of $\mathbb{I}(z)$ peaks inside the scatterer, it is difficult to predict the size of the scatterer.

A problem with the linear sampling method is that it is not clear whether the regularized solution $g_{\alpha,z,q,k}$ computed by Tikhonov regularization inherits the same properties as the approximate solution $g_{z,\epsilon}$ of the far-field equation satisfying (8). The unsatisfactory state of the mathematical justification of the linear sampling method stems from the fact that, even if k is not a transmission eigenvalue, in general there does not exist a solution of the far-field equation. Hence, trying to solve the regularized far-field equation with noisy data $E_\infty^\delta(\hat{x}; d, p, k)$ lies outside the general theory of ill-posed problems. Attempts to fill this mathematical gap started with Kirsch's factorization method [20]. For the case of Maxwell's equation, assuming that (10) holds in all D, in [21] and [22] it is shown that

$$z \in D \Leftrightarrow E_\infty(\hat{x}, z, q, k) \in \text{range}\,(F_\#)^{1/2},$$

where $F_\#$ is the operator given from the real and imaginary parts of F by $F_\# = |\Re(F)| + |\Im(F)|$, which is obviously known from the measurements.

In the case of a dielectric when $\Im(\epsilon_r) = 0$, the far-field operator is normal (see [1]), and in this case a simpler method can be derived. It can be shown that $z \in D$ if and only if $E_\infty(\hat{x}, z, q, k)$ is in the range of $(F^*F)^{1/4}$. This fact can be used as in [24] to show the following result. Assume that k is not a transmission eigenvalue and $\Im(\epsilon_r) = 0$. Let $g_{z,\alpha} := g_{z,\alpha,q,k}$ be the solution of the Tikhonov regularized far-field equation (9), and let $E_{g_{z,\alpha,q,k}}$ be the corresponding Herglotz wave function. Suppose that $\varphi_z \in L^2(\mathbb{S}^2)$ is the unique solution of $(F^*F)^{1/4}\varphi = E_\infty(\hat{x}, z, q, k)$ for $z \in D$. Then two assertions can be made:

1) for $z \in D$, $c\|\varphi_z\|_{L^2} \le \lim_{\alpha\to 0}|E_{g_{z,\alpha}}(z)| \le C\|\varphi_z\|_{L^2}$,
2) for $z \notin D$, $\lim_{\alpha\to 0}|E_{g_{z,\alpha}}(z)| = \infty$.

Since $\|\varphi_z\|_{L^2}$ for $z \in D$ is given in terms of the Picard's series, the previous result provides a convergence result for the linear sampling method and a rigorous characterization of D in terms of $|E_{g_{z,\alpha}}(z)|$ [see Figure 5, where we compare the use of $|E_{g_{z,\alpha}}(z)|$ instead of the most commonly used indicator function $\mathbb{I}(z)$].

Under the less restrictive assumption on the relative permittivity (10) and for limited aperture data, Audibert and Haddar [26] have recently developed the generalized linear sampling method, which provides a rigorously justified characterization of the support D of ϵ_r by modifying the far-field equation. More specifically, the far-field equation is replaced by the problem of finding a minimizing sequence of the functional

$$J_\alpha(g, E_\infty^z) := \alpha|(Bg, g)_{L^2}| + \|Fg - E_\infty^z\|,$$

where the operator $B: L_t^2(\mathbb{S}^2) \to L_t^2(\mathbb{S}^2)$ is given in terms of the far-field operator F and is such that $|(Bg, g)_{L^2}|$ is equivalent to $\|E_g\|_{L^2(D)}$. For the application of the generalized linear sampling method to electromagnetic scattering problems, we refer the reader to [16]. The linear sampling method and its more refined versions have been widely used for various electromagnetic inverse scattering problems with more complicated material structure than our reference problem in this article. For example, in Figure 6 we show an example from [9] of reconstructing a perfectly conducting torus in a conducting half-space from near-field measurements in the dielectric half-space above. This is a model problem for detecting buried objects. We emphasize that the support of the scattering medium can be determined by this method without making any use of the material properties of the scatterer (however, the background medium must be known).

ESTIMATES FOR THE MATERIAL PROPERTIES

Having determined the support D without knowing anything about the material properties, we would like to get

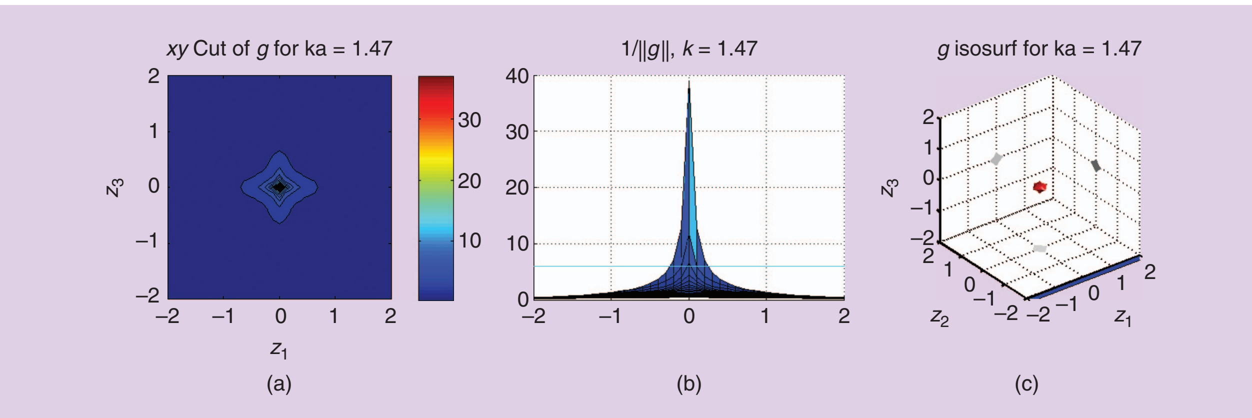


FIGURE 4. The reconstruction of a dielectric sphere of radius 1 with $\epsilon_r = 16I$ using the wavenumber $k = 1.47$ (a transmission eigenvalue for the sphere). (a) A contour map of $\mathbb{I}(z)$ in the plane $z_2 = 0$. (b) A mesh plot of $\mathbb{I}(z)$ along the line $x_1 = x_3 = 0$. (c) A reconstruction of the sphere using the same method for choosing the isosurface value C as in Figure 3 and marked in (b) as a light blue line.

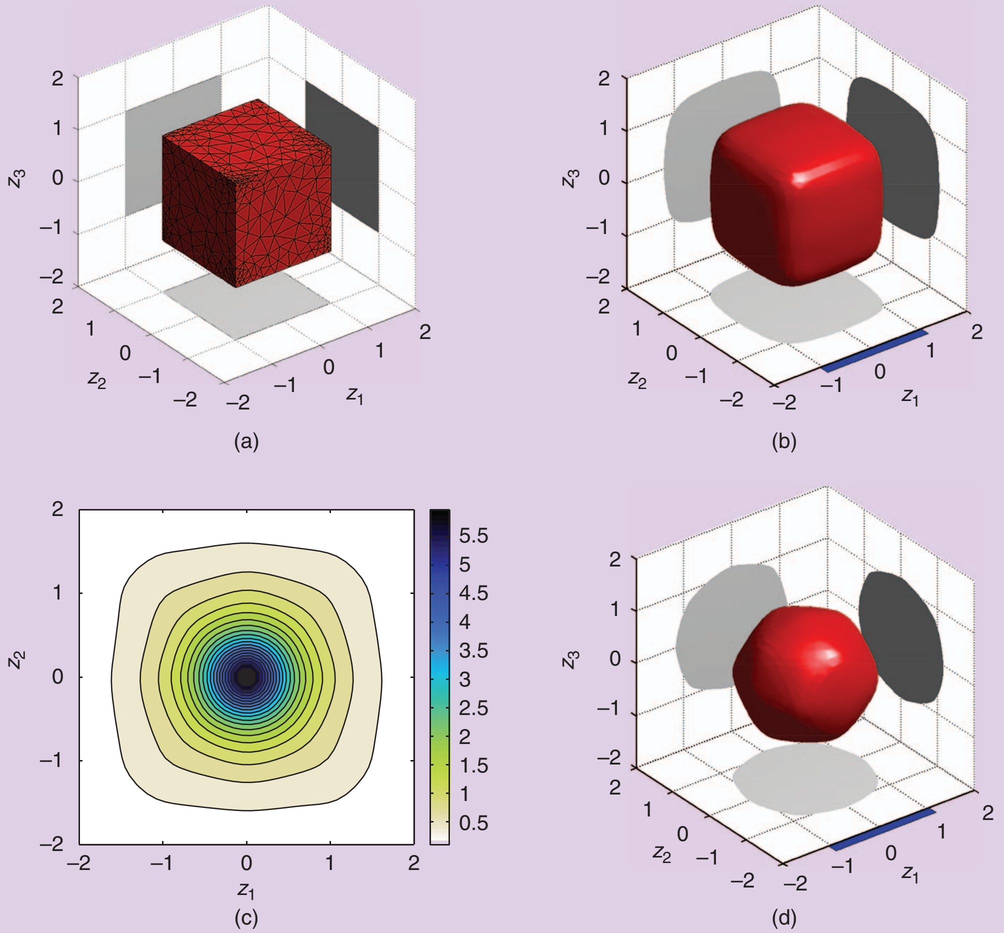


FIGURE 5. The reconstruction of a dielectric cube ($\epsilon_r = 2I$) with a conductive coating and an impedance parameter of 0.1 at $k = 3$ using the linear sampling method with 96 incidence and measurement directions (see [25] for details). (a) The exact scatterer. (b) The reconstruction using the linear sampling method. (c) The contours of $\mathbb{I}(z)$ in the plane $z_3 = 0$. (d) The reconstruction of the scatterer using the indicator $|E_{g_{z,\alpha,q,k}}(z)|$. In this example, the indicator $|E_{g_{z,\alpha,q,k}}(z)|$ gives the best reconstruction even though this example is not covered by the theory mentioned in this article because of the surface conducting layer. (Figure courtesy of [25], reproduced with permission.)

some information about the (possibly matrix-valued) relative permittivity ϵ_r. To this end, we appeal to the transmission eigenvalue problem

$$\begin{aligned} \text{curl curl } E_0 - k^2 E_0 &= 0 && \text{in } D, \\ \text{curl curl } E - k^2 \epsilon_r E &= 0 && \text{in } D, \\ \nu \times E &= \nu \times E_0 && \text{on } \partial D, \text{ and} \\ \nu \times \text{curl } E &= \nu \times \text{curl } E_0 && \text{on } \partial D. \end{aligned}$$

Even when ϵ_r is real, this is a nonself-adjoint eigenvalue problem [27]. Nevertheless, when ϵ_r is real, the existence of infinitely real transmission eigenvalues was proved in [4] (in particular cases, it is shown that complex transmission eigenvalues also exist [28]). To use transmission eigenvalues to obtain information about the material properties of the media, we must know whether the real transmission eigenvalues can indeed be determined from the scattering data (we only consider real transmission eigenvalues, since physically the wavenumber $k = \omega\sqrt{\epsilon_0 \mu_0}$ must be real). A positive answer to this question is given in [7], where it is shown that real transmission eigenvalues can be seen from the behavior of the regularized solution of the far-field equation

$$(Fg)(\hat{x}) = E_\infty(\hat{x}, z, q, k), \qquad z \in D,$$

the same equation used to determine the support D. The following result holds true. Let $g_{z,\alpha} := g_{z,\alpha,q,k}$ be the Tikhonov regularized solution of the far-field equation, i.e., the solution of

$$(\alpha I + F^* F) g = F^* E_\infty(\hat{x}, z, q, k), \qquad z \in D.$$

Let $E_{g_{z,\alpha}}$ be the electric Herglotz wave function with kernel g_α. Then for any ball $B \subset D, \|E_{g_{z,\alpha}}\|_{L^2(D)}$ is bounded as $\alpha \to 0$ for every $z \in B$ if and only if k is not a transmission eigenvalue.

The aforementioned results suggest that, if we keep $z \in D$ and solve the far-field equation varying the wavenumber k, the transmission eigenvalues will coincide with those values of k where $\|E_{g_{z,\alpha}}\|_{L^2(D)}$ or (mostly used in practice) $\|g_{z,\alpha}\|_{L^2(\mathbb{S}^2)}$ becomes large (for a simple example where transmission eigenvalues can be computed analytically, see Figure 7).

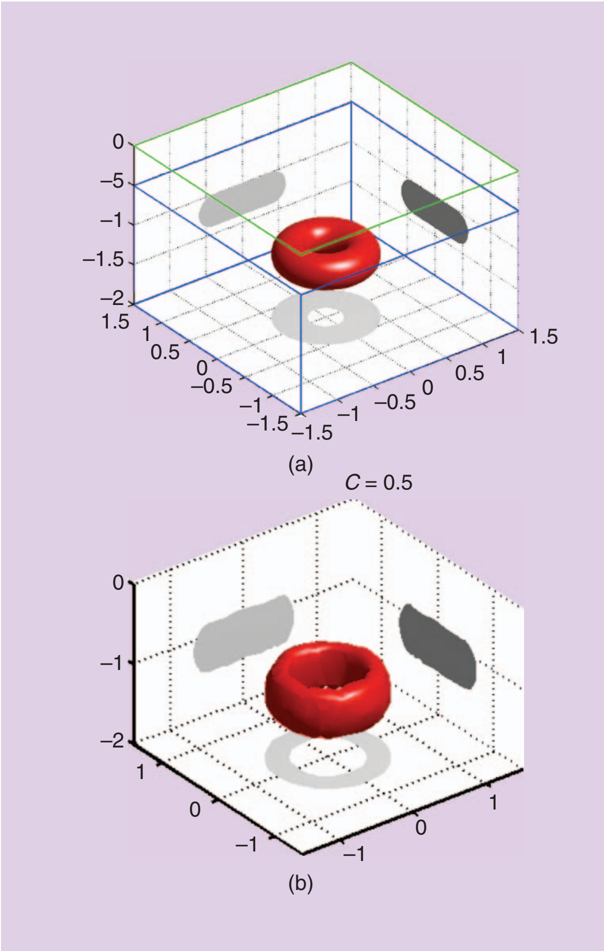


FIGURE 6. An example of the reconstruction of a buried object. In the air ($z_3 > 0$), the permittivity is $\epsilon_r = I$, whereas for $z_3 < 0$ it is set $\epsilon_r = (2 + 0.5i)I$. The torus is a perfect conductor. The point sources are located on a 25×25 array of size $3 \times 3\lambda$ at $z_3 = \lambda/2$, where λ is the wavelength of the incident field in the air. (a) An exact scatterer. (b) A reconstruction using the surface $\mathbb{I}(z) = 0.5 \max_z \mathbb{I}(z)$. (Figure courtesy of [9], reproduced with permission.)

As an alternative characterization of transmission eigenvalues, we remark that under the assumption that the far-field operator F is normal (e.g., if $\Im(\epsilon_r) = 0$ in D), the transmission eigenvalues can be observed in the phase behavior of the eigenvalues of the far-field operator. This was first proven for the scalar scattering problem in [22] and then for the electromagnetic scattering problem in [23]. We briefly describe this result. To this end, let us assume that $\bar{\xi} \cdot (\epsilon_r - I)\xi \geq \gamma \| \xi \|^2$ in D for some constant $\gamma > 0$. Since in this case the far-field operator F_k (we explicitly indicate here the dependence on k) is normal, it has an infinite number of eigenvalues $\lambda_j(k)$ accumulating to 0, such that $\Im(\lambda_j(k)) > 0$, and they lie on a circle in the complex plane with the center on the imaginary axis and touching tangentially the real axis at the origin. Let us write these eigenvalues as $\lambda_j(k) = r_j(k) e^{i\vartheta_j(k)}$. In [22], it is shown that, if k is not a transmission eigenvalue, then $\Re(\lambda_j(k)) > 0$ for $j \in \mathbb{N}$ is large enough, and thus $\vartheta_j(k) \to 0$ as $j \to \infty$. However, if we define $\vartheta_*(k) := \max_j \{\vartheta_j(k) \in [0, \pi)\}$ and

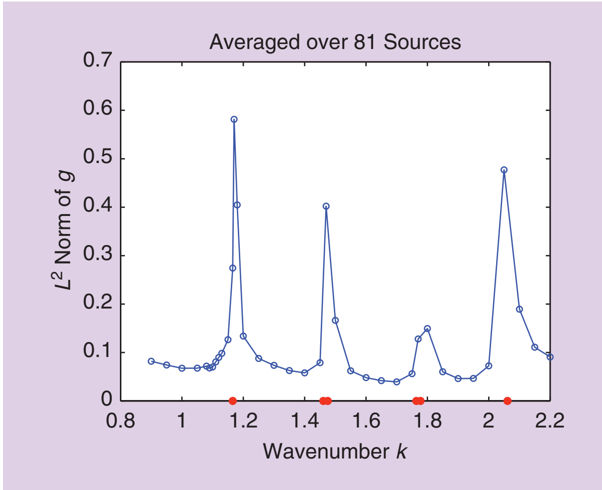


FIGURE 7. The average L^2 norm of $g_{z,\alpha}$ against the wavenumber k. We averaged over 81 sources randomly placed in a unit sphere with $\epsilon_r = 16I$. Red points indicate the values of k corresponding to analytically calculated transmission eigenvalues. There is an excellent correspondence between peaks in the norm of $g_{z,\alpha}$ and these eigenvalues.

$$\lim_{k \to k_0} \vartheta_*(k) = \pi,$$

then $k_0 > 0$ is a transmission eigenvalue. This relationship is referred to as an *inside–outside duality* (see [22] and the references therein).

Before examining what information transmission eigenvalues carry about the material properties of the scattering medium, we first briefly review the history of the transmission eigenvalue problem. The transmission eigenvalue problem in scattering theory was introduced by Kirsch [5] in 1986 and Colton and Monk [6] in 1988. The fact that transmission eigenvalues form a discrete set (for the scalar case of isotropic media) was shown by Colton, Kirsch, and Päivärinta [29] and Rynne and Sleeman [30] in 1989 and 1991, respectively, while for Maxwell's equations this was shown by Haddar and Monk [31] and Haddar [32] in 2002 and 2004, respectively. Finally (after a gap of 20 years), the existence of at least one transmission eigenvalue for the scalar case of isotropic media and large contrast was shown by Päivärinta and Sylvester in [33]. The existence of an infinite set of real transmission eigenvalues for general anisotropic media for both scalar and Maxwell's equations was given by Cakoni, Gintides, and Haddar in 2010 [4]. This paper also contained a basic monotonicity property of transmission eigenvalues, which opened the possibility of using transmission eigenvalues in nondestructive testing, as investigated in [34] and [35]. Since the appearance of these papers, there has been an explosion of interest in the transmission eigenvalue problem; see the recent monograph by Cakoni, Colton, and Haddar for further details and references [7].

In what follows, we will assume that $\Im(\epsilon_r) = 0$ in D and

$$\begin{cases} \text{either} & \bar{\xi} \cdot \Re(\epsilon_r - I)\xi \geq \gamma \| \xi \|^2 \\ \text{or} & \bar{\xi} \cdot \Re(I - \epsilon_r)\xi \geq \gamma \| \xi \|^2 \end{cases}$$

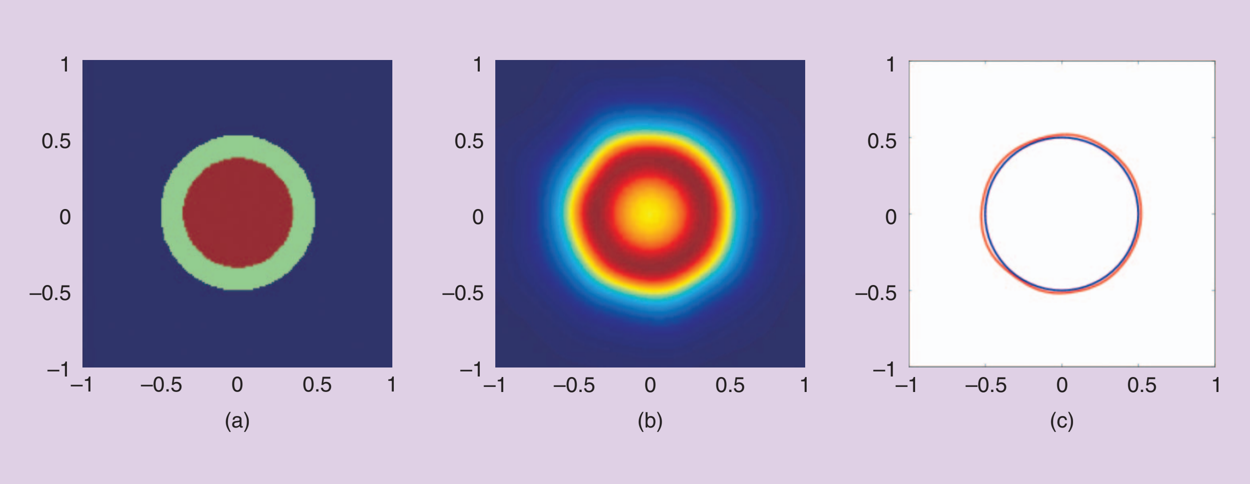


FIGURE 8. (a) The isotropic inhomogeneous scatterer is made of two equal-area regions. (b) The support of the scatter computed using the linear sampling method when $k = 4.2$, $n_e = 22$, and $n_i = 19$. (c) The exact boundary ∂D compared to the optimal reconstruction. (Figure courtesy of [35], reproduced with permission.)

TABLE 1. THE RESULTS FOR RECONSTRUCTING N_0 USING EITHER THE EXACT OR APPROXIMATE BOUNDARY FOR THE DOMAIN SHOWN IN FIGURE 8.

$\epsilon_{r,e}$	$\epsilon_{r,i}$	k_1	n_0—Exact Shape	n_0—Reconstructed Shape
8	8	2.98	8.07	7.61
11	5	3.27	7.05	6.69
22	19	1.76	20.28	18.86
67	61	0.97	64.11	59.42

Table courtesy of [35], reproduced with permission.

in $D \backslash \bar{D}_0$, where $D_0 \subset D$ is an open subset of D (possibly empty) where $\epsilon_r = I$ (i.e., we are allowing for the presence of voids in the media). We note that, if the inhomogeneous media is conducting [i.e., $\Im(\epsilon_r) > 0$ in an open subset of D] embedded in a nonconducting background (such as a vacuum), real transmission eigenvalues do not exist. In particular, setting

$$\epsilon_{r,\min} := \inf_D \inf_{|\xi|=1} \xi \cdot \epsilon_r \xi$$

and

$$\epsilon_{r,\max} := \sup_D \sup_{|\xi|=1} \xi \cdot \epsilon_r \xi,$$

we have the following. Assume that $\epsilon_{r,\min} > 1$. Then, there exists an infinite discrete set of real transmission eigenvalues k_j accumulating at $+\infty$ and satisfying

$$k_j(\epsilon_{r,\max}, B_1) \le k_j(\epsilon_{r,\max}, D) \le k_j(\epsilon_r(x), D) \text{ and } \le k_j(\epsilon_{r,\min}, D) \le k_j(\epsilon_{r,\min}, B_2),$$

where $B_2 \subset D \subset B_1$.

Given D (known a priori or determined by the linear sampling method) and after computing $k_1(\epsilon_r(x), D)$, which is the first real transmission eigenvalue corresponding to D with relative permittivity $\epsilon_r(x)$, from the far-field equation, one can now determine a constant n_0 such that the first transmission eigenvalue $k_1(n_0, D)$ of the isotropic media $n_0 I$ with support D satisfies $k_1(n_0, D) = k_1(\epsilon_r(x), D)$; hence, by the above result, we have $\epsilon_{r,\min} \le n_0 \le \epsilon_{r,\max}$. This computed n_0 will detect changes in the anisotropy of the scattering medium. If the original inhomogeneous media is isotropic [i.e., $\epsilon_r(x) = n(x) I$, with $n(x)$ a bounded scalar function], numerical results show that

$$n_0 \approx \frac{1}{|D|} \int_D n(x) dx.$$

Numerical evidence for this in the case of a transverse electric polarized wave in two dimensions can be found in [35] (see Figure 8 and Table 1), which we now describe. In these numerical experiments, a piecewise constant medium is considered inside a circle of radius 0.5. The two regions are of equal area, having relative permittivity $\epsilon_r = \epsilon_{r,i}$ in the inner region and $\epsilon_r = \epsilon_{r,e}$ in the outer annulus [see Figure 8(a)]. Two experiments are considered in [35]: 1) the domain is assumed known (a circle of radius 0.5), or 2) the domain is first constructed using the linear sampling method. Then, using transmission eigenvalues, an estimate is computed for n_0 as described previously. The results are shown in Table 1. For the exact geometry, the results show that n_0 is a rough estimate of the average permittivity.

For isotropic homogeneous media with constant relative permittivity $\epsilon_r > 1$, the first transmission eigenvalue $k_1(\epsilon_r)$ is strictly monotonically decreasing and is continuous with respect to ϵ_r. This implies that the first transmission eigenvalue uniquely determines the constant relative permittivity. Similar results can be obtained in the case when the maximum relative permittivity $\epsilon_{r,\max} < 1$ as well as for media with voids $D_0 \subset D$, where $\epsilon_r(x) = I$ [27] and [36]. This is also shown in the top row of Table 1.

CONCLUSIONS

We have surveyed the fundamental ideas and the state of the art of qualitative methods and transmission eigenvalues for solving the electromagnetic inverse scattering problem for anisotropic inhomogeneous media. A drawback of the qualitative approach is the amount of spatial multistatic data needed. A possible remedy could be to use time-domain data and develop linear sampling and factorization methods in the time domain. Preliminary results for the scalar problem in [37] are promising. A drawback of the use of transmission eigenvalues is that it needs data for a range of frequencies and that it does not work for media with absorption. It is possible to introduce a new eigenvalue problem by modifying the far-field equations with an artificial eigenvalue parameter. Such an idea was applied in [38].

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