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Combined far-field operators in electromagnetic inverse scattering theory

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SUMMARY

We consider the inverse scattering problem of determining the shape of a perfect conductor D from a knowledge of the scattered electromagnetic wave generated by a time-harmonic plane wave incident upon D. By using polarization effects we establish the validity of the linear sampling method for solving this problem that is valid for all positive values of the wave number. We also show that it suffices to consider incident directions and observation angles that are restricted to a limited aperture. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: electromagnetic inverse scattering; linear sampling method

1. INTRODUCTION

The linear sampling method is a technique for recovering the shape of a scattering object from a knowledge of the far-field pattern of the acoustic or electromagnetic scattered wave [1–5]. Its advantages over other methods for solving the same problem are that it is a linear algorithm that does not rely on any weak scattering assumptions and it is not necessary to know the boundary conditions on the scatterer *a priori*. The disadvantages compared to other methods are that it requires multi-static data distributed over the unit sphere and that the method fails at values of the wave number corresponding to eigenvalues of an associated interior boundary value problem. The purpose of this paper is to address both of these disadvantages for the special case of the scattering of an electromagnetic plane wave by a perfect conductor.

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The difficulty with eigenvalues in the linear sampling method is due to the fact that the far-field operator has eigenvalues (cf. Reference [6], Corollary 8.18). Hence, in order to overcome this problem, we must modify the far-field operator. The same problem occurs in the dual space method [6] for solving the inverse scattering problem and in the case of the scattering of electromagnetic waves by a perfect conductor a resolution to this difficulty was given in References [7,8] by considering a linear combination of far-field operators corresponding to different polarizations of the incident field. Since the linear sampling method had its origin in the dual space method by moving the origin to an arbitrary point $z \neq 0$, it is not unreasonable to expect that a similar modification should be possible for the linear sampling method. We show in this paper that this is indeed the case. However, the analysis is not as straightforward as might at first be expected. In particular, it is now necessary to rely on new approximation properties of Herglotz wave functions and electromagnetic Herglotz pairs as developed in References [9,10] as well as requiring a factorization of the modified far-field operator, neither of which was needed in References [7,8]. We note that the approach used here to eliminate the problem of eigenvalues in the linear sampling method relies heavily on the polarization of the incident and scattered fields and hence is only applicable to the case of electromagnetic scattering. For the case of acoustic waves a different method for eliminating eigenvalues in the linear sampling method is currently being investigated by Muniz [11].

As previously mentioned, a second problem with the linear sampling method is that it requires a large amount of multi-static data for its successful implementation. Indeed, perhaps due to the fact that in most discussions of the linear sampling method it is assumed that full aperture scattering data is available, some assume by 'large' it is meant 'full aperture' (cf. the Introduction to Reference [12]). As pointed out by the referee to Reference [12], this is definitely not the case. We will address this 'limited aperture' problem in Sections 2.3 and 3.2.

In what follows, for the sake of motivation and clarity, we will first consider the two-dimensional case of scattering by a perfectly conducting infinite cylinder. We will then proceed to a discussion of the full three-dimensional vector case corresponding to the scattering due to a perfectly conducting bounded obstacle in $\mathbb{R}^3$.

2. THE SCALAR CASE

2.1. Formulation of the direct and inverse scattering problem

In this section we consider the scattering of a plane time-harmonic electromagnetic wave by a perfectly conducting infinite cylinder. We assume that the axis of the cylinder coincides with the unit vector n_z on the z-axis and that the incident wave propagates in a direction perpendicular to the cylinder. Let D denote the bounded and simply connected cross-section of the cylinder with Lipschitz boundary ∂D and ν the unit outward normal to ∂D defined almost everywhere on ∂D. If we further assume that the incident electric wave is polarized parallel to the z-axis, then it is known that the electric field has only a component in the n_z direction. If $\tilde{E}e^{-i\omega t}$ is this component and we assume that the incident electric field is given by

$$E^{\mathrm{i}}(x,t) = \exp\, \mathrm{i}(kd\cdot x - \omega t)n_z$$

where d is the incident direction such that $d \cdot n_z = 0$, $k > 0$ is the wave number, ω is the frequency and $x \in \mathbb{R}^2$, then $\tilde{E}$ satisfies

$$\begin{aligned} &\text{(i)} \quad \Delta \tilde{E} + k^2 \tilde{E} = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D} \\ &\text{(ii)} \quad \tilde{E}(x) = \exp(\mathrm{i}kd \cdot x) + E(x) \\ &\text{(iii)} \quad \tilde{E}(x) = 0 \quad \text{for } x \in \partial D \\ &\text{(iv)} \quad \lim_{r \to \infty} \sqrt{r} \left(\frac{\partial E}{\partial r} - \mathrm{i}kE \right) = 0 \end{aligned} \tag{1}$$

where $r = |x|$ and the radiation condition (1) (iv) is satisfied uniformly in $\hat{x} = x/r$.

Now assume that the incident electric wave is polarized perpendicular to the z-axis. Then the magnetic field has only a component in the n_z direction, and if $\tilde{H}\mathrm{e}^{-\mathrm{i}\omega t}$ is this component and we assume that the incident electric field is given by

$$H^{\mathrm{i}}(x, t) = \exp \mathrm{i}(kd \cdot x - \omega t) n_z$$

then H satisfies

$$\begin{aligned} &\text{(i)} \quad \Delta \tilde{H} + k^2 \tilde{H} = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D} \\ &\text{(ii)} \quad \tilde{H}(x) = \exp(\mathrm{i}kd \cdot x) + H(x) \\ &\text{(iii)} \quad \frac{\partial \tilde{H}}{\partial \nu}(x) = 0 \quad \text{for } x \in \partial D \\ &\text{(iv)} \quad \lim_{r \to \infty} \sqrt{r} \left(\frac{\partial H}{\partial r} - \mathrm{i}kH \right) = 0 \end{aligned} \tag{2}$$

The existence and uniqueness of solutions to (1) and (2) are well known [13,6]. In particular, these solutions $\tilde{E}$ and $\tilde{H}$ belong to $H^1_{\mathrm{loc}}(\mathbb{R}^2 \backslash \bar{D})$ and the scattered fields E and H have the asymptotic behaviour [6]

$$E(x) = \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} E_\infty(\hat{x}; d) + O\left(\frac{1}{r^{3/2}} \right) \tag{3}$$

$$H(x) = \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} H_\infty(\hat{x}; d) + O\left(\frac{1}{r^{3/2}} \right) \tag{4}$$

where $E_\infty(\hat{x}; d)$ and $H_\infty(\hat{x}; d)$ are the *far-field patterns* corresponding to the Dirichlet problem (1) and the Neumann problem (2), respectively.

The *inverse obstacle scattering problem* we are concerned with is to determine D from a knowledge of both $E_\infty(\hat{x}; d)$ and $H_\infty(\hat{x}; d)$ for $\hat{x}$ and d on the unit circle Ω and fixed wave number k.

2.2. The linear sampling method for combined far-field data

The linear sampling method as introduced in Reference [3] and developed further in References [1,4,5] for solving the inverse obstacle scattering problem makes use of only the electric far-field E_∞. However, the price paid for using only the electric far-field pattern is that it is necessary to assume that k^2 is not an eigenvalue of the interior Dirichlet problem for Laplace's equation. In order to avoid this problem in this section of our paper we will assume that both E_∞ and H_∞ are known. In particular, we consider the combined *far-field operator* $F: L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Fg)(\hat{x}) := \lambda \int_\Omega E_\infty(\hat{x}; d) g(d)\, \mathrm{d}s(d) + \mu \int_\Omega H_\infty(\hat{x}; d) g(d)\, \mathrm{d}s(d), \quad \hat{x} \in \Omega \tag{5}$$

where $g \in L^2(\Omega)$ and $\lambda > 0$ and $\mu < 0$ are real numbers. Note that by superposition the first integral and the second integral in (5) are the far-field patterns of the exterior Dirichlet problem (1) and the Neumann problem (2), respectively, corresponding to the Herglotz wave function

$$V_g(x) = \int_\Omega \mathrm{e}^{\mathrm{i}kx \cdot d} g(d)\, \mathrm{d}s(d), \quad x \in \mathbb{R}^2 \tag{6}$$

with kernel $g \in L^2(\Omega)$ as incident field.

For $z \in \mathbb{R}^2$ we now consider the *far-field equation*

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z) \tag{7}$$

where $\Phi_\infty(\hat{x}, z) = (\mathrm{e}^{\mathrm{i}\pi/4}/\sqrt{8\pi k})\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}$ is the far-field pattern of the fundamental solution

$$\Phi(x, z) = H_0^{(1)}(k|x - z|) \tag{8}$$

with $H_0^{(1)}$ denoting a Hankel function of the first kind of order zero. The transmission problem associated with the far-field equation (7), which we shall refer to as problem (TP), is given a function $h \in H^{-1/2}(\partial D)$ find $E \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \backslash \bar{D})$ and $W \in H^1(D)$ such that

$$\begin{aligned}
&\text{(i)} && \Delta E + k^2 E = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D} \\
&\text{(ii)} && \Delta W + k^2 W = 0 \quad \text{in } D \\
&\text{(iii)} && E + W = 0, \quad \text{on } \partial D \\
&\text{(iv)} && \lambda \frac{\partial E}{\partial \nu} - \mu \frac{\partial W}{\partial \nu} = h \quad \text{on } \partial D \\
&\text{(v)} && \lim_{r\to\infty} \sqrt{r} \left(\frac{\partial E}{\partial r} - \mathrm{i}kE \right) = 0
\end{aligned} \tag{9}$$

In particular, for $z \in D$, (7) implies that if V_g is the Herglotz wave function with kernel g then by Rellich's lemma

$$\lambda E(x) + \mu H(x) = \Phi(x, z), \quad x \in \mathbb{R}^2 \backslash \bar{D}, \ z \in D$$

where E is the radiating solution to

$$\Delta E + k^2 E = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D}, \qquad E + V_g = 0 \quad \text{on } \partial D \tag{10}$$

and H is the radiating solution to

$$\Delta H + k^2 H = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D}, \qquad \frac{\partial H}{\partial \nu} + \frac{\partial V_g}{\partial \nu} = 0 \quad \text{on } \partial D \tag{11}$$

Since the Herglotz wave function is an entire solution to the Helmholtz equation one obtains that this E and $W = V_g$ satisfy (9) for $h = \partial\Phi/\partial\nu$. Conversely, if E and $W = V_g$ are a solution to (9) with $h = \partial\Phi/\partial\nu$ and H is the unique solution of (11) then $\lambda E(x) + \mu H(x)$ and $\Phi(x,z)$ are radiating solutions to the same Neumann boundary value problem for the Helmholtz equation, whence they are equal which means that the kernel g of V_g is the solution of the far-field equation (7).

The transmission problem (9) can be reformulated as a particular case of the transmission problem considered by Hähner in [14]. In particular, the approach of Reference [14] allows the boundary to be Lipschitz as far as H^1 regularity of the solutions is concerned. Hence from Reference [14] we have the following result.

Theorem 2.1
Let $h \in H^{-1/2}(\partial D)$, and let $\lambda > 0$ and $\mu < 0$ be real numbers. Then problem (TP) has a unique solution $E \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \backslash \bar{D})$, $W \in H^1(D)$ which satisfy

$$\|W\|_{H^1(D)} + \|E\|_{H^1(B_R \cap (\mathbb{R}^2 \backslash \bar{D}))} \leqslant C\|h\|_{H^{-1/2}(\partial D)} \tag{12}$$

where B_R is a disk of radius R containing D and $C > 0$ is a constant depending on R but not on h.

Definition 2.2
The operator $\mathscr{B} : H^{-1/2}(\partial D) \to L^2(\Omega)$ maps $h \in H^{-1/2}(\partial D)$ onto the far-field pattern $\lambda E_\infty + \mu H_\infty$ of the radiating solution $\lambda E + \mu H$ with H being the unique radiating solution of

$$\Delta H + k^2 H = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D}, \qquad \frac{\partial H}{\partial \nu} + \frac{\partial W}{\partial \nu} = 0 \quad \text{on } \partial D \tag{13}$$

and E and W the unique solution of (TP) with boundary condition h.

We note that E is the solution of (10) with $V_g = W$. Hence from (12), the well posedness of the exterior Neumann problem, and the fact that the operator which takes the restriction to $H^{1/2}(\partial B_R)$ of a radiating solution of the Helmholtz equation to the corresponding far-field pattern is injective and compact we obtain that $\mathscr{B} : H^{-1/2}(\partial D) \to L^2(\Omega)$ is a continuous linear operator that is injective and compact. (Note that if $\lambda E + \mu H = 0$ then $h = 0$ by (9) (iv) and (13).)

Theorem 2.3
The set $\mathscr{B}(H^{-1/2}(\partial D))$ is dense in $L^2(\Omega)$.

Proof
Let $\mathscr{B}^\top : L^2(\Omega) \to H^{1/2}(\partial D)$ be the dual operator of $\mathscr{B}$ such that

$$\langle \mathscr{B}h, g \rangle_{L^2(\Omega), L^2(\Omega)} = \langle h, \mathscr{B}^\top g \rangle_{H^{1/2}(\partial D), H^{-1/2}(\partial D)}$$

with $h \in H^{-1/2}(\partial D)$ and $g \in L^2(\Omega)$ and $\langle \cdot, \cdot \rangle$ the duality pairing. Define $U := \lambda E + \mu H$ where E, H are as above. Note that from (9)(iv) and (13) we have that U satisfies $(\partial U/\partial \nu)|_{\partial D} = h$. By superposition we can write

$$\langle \mathscr{B}h, g \rangle_{L^2(\Omega), L^2(\Omega)} = \gamma \int_{\partial D} \left(U \frac{\partial V_g}{\partial \nu} - V_g \frac{\partial U}{\partial \nu} \right) \mathrm{d}s = \gamma \int_{\partial D} \left(U \frac{\partial V_g}{\partial \nu} - h\, V_g \right) \mathrm{d}s$$

where $\gamma := e^{i\pi/4}/\sqrt{8\pi k}$ and V_g is the wave Herglotz function $V_g(x) = \int_\Omega e^{-ikx\cdot d} g(d)\, ds(d)$. Let $\tilde{U}$ be the unique radiating solution of the exterior problem

$$\Delta \tilde{U} + k^2 \tilde{U} = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D}, \qquad \frac{\partial \tilde{U}}{\partial \nu} = \frac{\partial V_g}{\partial \nu} \quad \text{on } \partial D$$

Applying Green's formula to the radiating solutions $U, \tilde{U}$ of the Helmoltz equation in $\mathbb{R}^2 \backslash \bar{D}$ we obtain

$$\langle \mathscr{B}h, g \rangle_{L^2(\Omega), L^2(\Omega)} = \gamma \int_{\partial D} \left(\tilde{U} \frac{\partial U}{\partial \nu} - hV_g \right) ds = \gamma \int_{\partial D} h \left(\tilde{U} - V_g \right) ds$$

whence

$$\mathscr{B}^\top g = \gamma(\tilde{U} - V_g)_{\partial D} \in H^{1/2}(\partial D)$$

Now let $\mathscr{B}^\top g = 0$. Then $\tilde{U}|_{\partial D} = V_g|_{\partial D}$ and from the boundary condition $\partial \tilde{U}/\partial \nu|_{\partial D} = \partial V_g/\partial \nu|_{\partial D}$. Therefore using Green's formula we can extend $\tilde{U}$ to a solution of the Helmholtz equation in $\mathbb{R}^2$ satisfying the Sommerfeld radiation condition at infinity. But this implies $\tilde{U} \equiv 0$ and consequently $V_g \equiv 0$, whence $g \equiv 0$. The injectivity of $\mathscr{B}^\top$ and the relation $\ker \mathscr{B}^\top = (\operatorname{range} \mathscr{B})^a$, where $(\cdot)^a$ denotes the polar (or annihilator) set (cf. Reference [13]), imply

$$\{g \in L^2(\Omega)\colon\ \langle g, h \rangle = 0 \quad \text{for all } h \in \operatorname{range} \mathscr{B}\} = \{0\} \tag{14}$$

whence the range of $\mathscr{B}$ is dense in $L^2(\Omega)$. This ends the proof. □

We now turn to our main goal, that is the study of the far-field equation (7) for various 'sampling' points $z \in \mathbb{R}^2$. To this end, let $\mathscr{S}$ be the operator which maps a function $h \in H^{-1/2}(\partial D)$ onto the component W of the solution to (TP) with boundary data h. From the *a priori* estimate (12) and the inequalities (which follow from the continuity of the Dirichlet to Neumann map, the trace theorem and (9)(iii))

$$\begin{aligned} \|h\|_{H^{-1/2}(\partial D)} &\leqslant \lambda \left\| \frac{\partial E}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} + |\mu| \left\| \frac{\partial W}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} \leqslant C(\|E\|_{H^{1/2}(\partial D)} + \|W\|_{H^1(D)}) \\ &= C(\|W\|_{H^{1/2}(\partial D)} + \|W\|_{H^1(D)}) \leqslant \tilde{C} \|W\|_{H^1(D)} \end{aligned} \tag{15}$$

where C and $\tilde{C}$ are positive constants, we conclude that the operator $\mathscr{S}$ is an isomorphism between $H^{-1/2}(\partial D)$ and the Hilbert space

$$\mathscr{L}(D) := \{W \in H^1(D);\ \Delta W + k^2 W = 0 \text{ in the distribution sense}\}$$

equipped with the $H^1(D)$ norm. Therefore by the bounded inverse theorem $\mathscr{S}^{-1}$ is continuous. In terms of the operator $\mathscr{B}$, using (10) and (11), we can now rewrite (7) as

$$(\mathscr{B}\mathscr{S}^{-1} V_g)(\hat{x}) = \Phi_\infty(\hat{x}, z) \tag{16}$$

where V_g is the Herglotz wave function given by (6).

We first assume that $z \in D$. In this case one sees that $\Phi_\infty(\hat{x}, z)$ is in the range $\mathscr{B}$ and in particular $\mathscr{B}((\partial\Phi(x,z)/\partial\nu)|_{\partial D}) = \Phi_\infty(\hat{x}, z)$. Let $W_z \in \mathscr{L}(D)$ and E_z be the solution of (9) with boundary data $h := (\partial\Phi(\cdot, z)/\partial\nu)|_{\partial D}$. Then from References [9,10] (see also Reference [15]) for every $\varepsilon > 0$ we can find a $g_\varepsilon(\cdot, z)$ such that the corresponding Herglotz wave function $V_{g_\varepsilon(\cdot,z)}$ satisfies

$$\|W_z - V_{g_\varepsilon(\cdot,z)}\|_{H^1(D)} \leqslant \varepsilon \tag{17}$$

Hence the continuity of $\mathscr{B}$ and $\mathscr{S}^{-1}$ implies that for a positive constant C

$$\|\Phi_\infty(\hat{x}, z) - (\mathscr{B}\mathscr{S}^{-1}V_{g_\varepsilon(\cdot,z)})(\hat{x})\|_{H^1(D)} = \|(\mathscr{B}\mathscr{S}^{-1}W_z)(\hat{x}) - (\mathscr{B}\mathscr{S}^{-1}V_{g_\varepsilon(\cdot,z)})(\hat{x})\|_{H^1(D)} \leqslant C\varepsilon$$

Furthermore, if the point z approaches the boundary ∂D than $\|(\partial\Phi(x,z)/\partial\nu)\|_{H^{-1/2}(\partial D)} \to \infty$ because for $z \in \partial D$ we have that Φ is not in $H^1(B_R \cap (\mathbb{R}^2 \backslash \bar{D}))$ for $B_R \supset D$ a disk of radius R. Hence (15) with $h = (\partial\Phi(\cdot, z)/\partial\nu)|_{\partial D}$ implies that the corresponding W satisfies

$$\lim_{z \to \partial D} \|W_z\|_{H^1(D)} = \infty$$

and from (17) we finally obtain that

$$\lim_{z \to \partial D} \|V_{g_\varepsilon(\cdot,z)}\|_{H^1(D)} = \infty \quad \text{and} \quad \lim_{z \to \partial D} \|g_\varepsilon(\cdot, z)\|_{L^2(\Omega)} = \infty$$

Now let us assume that $z \in \mathbb{R}^2 \backslash \bar{D}$. In this case $\Phi_\infty(\hat{x}, z)$ does not belong to the range of $\mathscr{B}$ because $\Phi(x,z)$ has a singularity at $z \in \mathbb{R}^2 \backslash \bar{D}$. But, from Theorem 2.3 and the injectivity and compactness of $\mathscr{B}$, by using Tikhonov regularization we can construct a regularized solution of the equation

$$(\mathscr{B}h)(\hat{x}) = \Phi_\infty(\hat{x}, z) \tag{18}$$

In particular, if $h_z^\alpha \in H^{-1/2}(\partial D)$ is the regularized solution of (18) corresponding to a 'noise level' δ and regularization parameter α (chosen by a regular regularization strategy, e.g. the Morozov discrepancy principle), we have

$$\|(\mathscr{B}h_z^\alpha)(\hat{x}) - \Phi_\infty(\hat{x}, z)\|_{L^2(\Omega)} < \delta \tag{19}$$

for an arbitrary small $\delta > 0$, and

$$\lim_{\alpha \to 0} \|h_z^\alpha\|_{H^{-1/2}(\partial D)} = \infty \tag{20}$$

Note that in this case we have that $\alpha \to 0$ as $\delta \to 0$. Let W_z^α and E_z^α be the solution of (TP) with boundary data $h := h_z^\alpha$. Thus we can rewrite (19) as

$$\|(\mathscr{B}\mathscr{S}^{-1}W_z^\alpha)(\hat{x}) - \Phi_\infty(\hat{x}, z)\|_{L^2(\Omega)} < \delta \tag{21}$$

and, by the same argument as in the case of $z \in D$, the Herglotz wave function $V_{g_\varepsilon^\alpha(\cdot,z)}$ which approximates $W_z^\alpha \in \mathscr{L}(D)$ with an arbitrary small $\varepsilon > 0$ satisfies

$$\|(\mathscr{B}\mathscr{S}^{-1}W_z^\alpha)(\hat{x}) - (\mathscr{B}\mathscr{S}^{-1}V_{g_\varepsilon^\alpha(\cdot,z)})(\hat{x})\|_{L^2(\Omega)} < \varepsilon \tag{22}$$

whence (22) combined with (21) gives

$$\|(\mathscr{B}\mathscr{S}^{-1}V_{g_\varepsilon^\alpha(\cdot,z)})(\hat{x}) - \Phi_\infty(\hat{x}, z)\|_{L^2(\Omega)} < \delta + \varepsilon \tag{23}$$

Finally from (20) and (15) with $h = h_z^\alpha$ we have that

$$\lim_{\alpha\to 0} \|W_z^\alpha\|_{H^1(D)} = \infty$$

whence

$$\lim_{\alpha\to 0} \|V_{g_\varepsilon^\alpha(\cdot,z)}\|_{H^1(D)} = \infty \quad \text{and} \quad \lim_{\alpha\to 0} \|g_\varepsilon^\alpha(\cdot,z)\|_{L^2(\Omega)} = \infty$$

We summarize these results in the following theorem.

Theorem 2.4

Let D be the bounded and simply connected cross-section of a perfectly conducting infinite cylinder with Lipschitz boundary ∂D. Then if F is the far-field operator defined by (5), where $\lambda > 0$ and $\mu < 0$ are real numbers, we have that

(1) if $z \in D$ then for every $\varepsilon > 0$ there exists a solution $g_\varepsilon(\cdot,z) \in L^2(\Omega)$ of the inequality

$$\|Fg_\varepsilon(\cdot,z) - \Phi_\infty(\cdot,z)\|_{L^2(\Omega)} < \varepsilon$$

Moreover, this solution satisfies

$$\lim_{z\to\partial D} \|g_\varepsilon(\cdot,z)\|_{L^2(\Omega)} = \infty \quad \text{and} \quad \lim_{z\to\partial D} \|V_{g_\varepsilon}(\cdot,z)\|_{H^1(D)} = \infty$$

where V_{g_ε} is the Herglotz wave function with kernel g_ε and

(2) if $z \in \mathbb{R}^2 \backslash \bar{D}$ then for every $\varepsilon > 0$ and $\delta > 0$ there exists a solution $g_\varepsilon^\delta(\cdot,z) \in L^2(\Omega)$ of the inequality

$$\|Fg_\varepsilon^\delta(\cdot,z) - \Phi_\infty(\cdot,z)\|_{L^2(\Omega)} < \varepsilon + \delta$$

such that

$$\lim_{\delta\to 0} \|g_\varepsilon^\delta(\cdot,z)\|_{L^2(\Omega)} = \infty \quad \text{and} \quad \lim_{\delta\to 0} \|V_{g_\varepsilon^\delta}(\cdot,z)\|_{H^1(D)} = \infty$$

where $V_{g_\varepsilon^\delta}$ is the Herglotz wave function with kernel g_ε^δ.

The importance of theorems such as this for solving the inverse scattering problem has been demonstrated in previous papers on the linear sampling method [1,2]. In particular, by using regularization methods to solve the far-field equation $Fg = \Phi_\infty(\cdot,z)$ for z on an appropriate grid containing D, an approximation to $g(\cdot,z)$ can be obtained and hence ∂D can be determined by those points where $\|g(\cdot,z)\|_{L^2(\Omega)}$ is not finite (for numerical examples in the scalar case see References [3,4]). Note however, in contrast to previous work on the linear sampling method for non-absorbing media, the above theorem makes no restriction on k not being an eigenvalue of the corresponding interior problem.

2.3. Limited aperture

In many cases of practical interest, the far-field data $E_\infty(\hat{x},d)$ and $H_\infty(\hat{x},d)$ is restricted to the case when $\hat{x}$ and d are on a subset Ω_0 of the unit sphere Ω, i.e. we are concerned with limited aperture scattering data. In order to handle this case, we note that from the proof of Theorem 2.4 the function $g_\varepsilon \in L^2(\Omega)$ is the kernel of a Herglotz wave function which approximates a solution to the Helmholtz equation in D with respect to the $H^1(D)$

norm. Therefore to treat the limited aperture case it is enough to show that a Herglotz wave function and its first derivative can be approximated uniformly on compact subsets of a disk B_R of radius R by a Herglotz wave function with kernel supported in a subset of Ω. This new Herglotz wave function and the kernel can now be used in place of V_{g_ε} and g_ε in Theorem 2.4. The above approximation property was previously established by Ochs [16]. Here we present a different proof of this result which can also be extended to the case of electromagnetic waves (Section 3.2). Assuming that k^2 is not a Dirichlet eigenvalue for the disk B_R (this is not a restriction since we can always find a disk containing D and having this property), it suffices to show that the set of functions

$$V_g(x) := \int_\Omega g(d)\mathrm{e}^{\mathrm{i}kx\cdot d}\,\mathrm{d}s(d), \quad g\in L^2(\Omega) \quad \text{with support in } \Omega_0\subseteq\Omega$$

for some subset $\Omega_0\subseteq\Omega$ is complete in $L^2(\partial B_R)$, where ∂B_R is the circle of radius R. Then we obtain our desired approximation property from Theorem 5.4 in Reference [6].

Let $\varphi\in L^2(\partial B_R)$ and suppose that for a fixed $\Omega_0\subset\Omega$ we have that

$$\int_{\partial B_R}\varphi(x)\left[\int_{\Omega_0}\bar{g}(d)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(d)\right]\mathrm{d}s(x)=0 \tag{24}$$

for every $g\in L^2(\Omega_0)$. Our aim is to show that $\varphi=0$. To this end, we interchange the order of integration to arrive at

$$\int_{\Omega_0}\bar{g}(d)\left[\int_{\partial B_R}\varphi(x)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(x)\right]\mathrm{d}s(d)=0 \tag{25}$$

for every $g\in L^2(\Omega_0)$, which implies that the far-field pattern $(S\varphi)_\infty$ of the single-layer potential

$$(S\varphi)(y) := \int_{\partial B_R}\varphi(x)\Phi(x,y)\,\mathrm{d}s(x), \qquad y\in\mathbb{R}^2\backslash\bar{B}_R, \quad \varphi\in L^2(\partial B_R)$$

satisfies

$$(S\varphi)_\infty(d) = \int_{\partial B_R}\varphi(x)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(x)\equiv 0, \quad d\in\Omega_0$$

By analyticity we can conclude that $(S\varphi)_\infty\equiv 0$ on Ω. Since the single-layer potential is a solution to the Helmholtz equation, from Rellich's lemma we obtain that $S\varphi\equiv 0$ in $\mathbb{R}^2\backslash\bar{B}_R$, whence by the continuity of $S\varphi$ across ∂B_R (cf. the proof of Theorem 5.5 in Reference [6]) we have that $S\varphi\equiv 0$ in B_R as well (because k^2 is not a Dirichlet eigenvalue for the Helmholtz equation in B_R). Finally by applying again the jump relation for the normal derivative of $S\varphi$ [6,13] we obtain that $\varphi\equiv 0$.

For an example of limited aperture reconstructions in the case when $\mu=0$, $\lambda=1$ and k^2 is not a Dirichlet eigenvalue see Figure 5 of Reference [17] (as shown above, the problem of Dirichlet eigenvalues can be avoided if we choose $\mu<0$!).

3. THE VECTOR CASE

3.1. The linear sampling method for combined far-field data

In this section we turn our attention to the scattering of a time-harmonic electromagnetic wave by a bounded perfectly conducting obstacle in $\mathbb{R}^3$. In particular, let $D \subset \mathbb{R}^3$ be a bounded domain such that $\mathbb{R}^3 \backslash \bar{D}$ is connected. The boundary ∂D of D is assumed to be a Lipschitz curvilinear polyhedron, and v denotes the unit outward normal defined almost everywhere on ∂D. After factoring out a term of the form $e^{-i\omega t}$ where ω is the frequency, we are then led to the following boundary value problem for the electric field E and magnetic field H References [6,18]:

$$\nabla \times E - ikH = 0 \quad \text{in } \mathbb{R}^3 \backslash \bar{D} \tag{26}$$

$$\nabla \times H + ikE = 0 \quad \text{in } \mathbb{R}^3 \backslash \bar{D} \tag{27}$$

$$v \times E = 0 \quad \text{on } \partial D \tag{28}$$

$$E = E^{i} + E^{s} \tag{29}$$

$$H = H^{i} + H^{s} \tag{30}$$

where the incident field E^i, H^i is given by

$$\begin{aligned} E^{i}(x; d, p) &= \frac{i}{k} \nabla \times \nabla \times p e^{ikd \cdot x} = ik(d \times p) \times d e^{ikd \cdot x} \\ H^{i}(x; d, p) &= \nabla \times p e^{ikd \cdot x} = ikd \times p e^{ikd \cdot x} \end{aligned} \tag{31}$$

and the scattered field E^s, H^s satisfies the Silver Müller radiation condition

$$\lim_{r \to \infty} (H^{s} \times x - rE^{s}) = 0 \tag{32}$$

uniformly in $\hat{x} = x/|x|$, where $r = |x|$, $k > 0$ is the wave number, $d \in \Omega$ is a unit vector giving the direction of the incident plane wave and $p \in \mathbb{R}^3$ is the polarization. Here Ω denotes the unit sphere in $\mathbb{R}^3$.

We define the spaces

$$H(\text{curl}, D) := \{U \in (L^2(D))^3 : \ \nabla \times U \in (L^2(D))^3\}$$

$$L_t^2(\partial D) := \{U \in (L^2(\partial D))^3 : v \cdot U = 0 \quad \text{on } \partial D\}$$

$$H_{\text{div}}^{-1/2}(\partial D) := \{U \in H^{-1/2}(\partial D), \ \text{div}_{\partial D}\, U \in H^{-1/2}(\partial D)\}$$

$$H_{\text{curl}}^{-1/2}(\partial D) := \{U \in H^{-1/2}(\partial D), \ \text{curl}_{\partial D}\, U \in H^{-1/2}(\partial D)\}$$

It is known that, for smooth boundary and $U \in H(\text{curl}, D)$, $v \times U \in H_{\text{div}}^{-1/2}(\partial D)$ and $v \times (U \times v) \in H_{\text{curl}}^{-1/2}(\partial D)$, and a duality pairing is defined between $H_{\text{div}}^{-1/2}(\partial D)$ and $H_{\text{curl}}^{-1/2}(\partial D)$ [19]. We remark that for Lipschitz boundaries the definition of the trace spaces of $U \to v \times U$ and $U \to v \times (U \times v)$ on ∂D needs a more careful investigation. In particular, these spaces are fully characterized (note different notations $H^{-1/2}_{\|\mathrm{div}}(\partial D)$ and $H^{-1/2}_{\perp\mathrm{curl}}(\partial D)$ are used!), the continuity and surjectivity of the trace operators is proved and the duality pairing is interpreted in Reference [20] (see also Reference [2]). However, for simplicity of our presentation, we will keep the same notations for the trace spaces of $U \to \nu \times U$ and $U \to \nu \times (U \times \nu)$, namely $H^{-1/2}_{\mathrm{div}}(\partial D)$ and $H^{-1/2}_{\mathrm{curl}}(\partial D)$.

In References [19,18] it is shown that the direct scattering problem has a unique solution $E, H \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$, and moreover the scattered field $E^{\mathrm{s}}, H^{\mathrm{s}}$ has the asymptotic behaviour [6]

$$E^{\mathrm{s}}(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left\{E_\infty(\hat{x}; d, p) + O\left(\frac{1}{|x|}\right)\right\}, \quad H^{\mathrm{s}}(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left\{H_\infty(\hat{x}; d, p) + O\left(\frac{1}{|x|}\right)\right\}$$

as $|x| \to \infty$, where $E_\infty(\cdot; d, p)$ and $H_\infty(\cdot; d, p)$ defined on the unit sphere Ω are the electric far-field pattern and the magnetic far-field pattern corresponding to the incident direction d and polarization p. Moreover, they satisfy [6]

$$H_\infty(\hat{x}; d, p) = \hat{x} \times E_\infty(\hat{x}; d, p)$$
$$\hat{x} \cdot H_\infty(\hat{x}; d, p) = \hat{x} \cdot E_\infty(\hat{x}; d, p) = 0$$

for all $\hat{x}, d \in \Omega$ and $p \in \mathbb{R}^3$. As in the scalar case, we will investigate the possibility of determining ∂D from a knowledge of both $E_\infty(\hat{x}; d, p)$ and $H_\infty(\hat{x}; d, p \times d)$, for $\hat{x}, d \in \Omega$ and $p \in \mathbb{R}^3$ by using the linear sampling method. Note that the electric far-field pattern $E_\infty(\hat{x}; d, p)$ and magnetic far-field pattern $H_\infty(\hat{x}; d, p \times d)$ correspond to an incident field propagating in the same direction d but polarized perpendicular to each other. We also note that H_∞ can be computed from a knowledge of E_∞.

We again consider the combined *far-field operator* $F : L^2_t(\Omega) \to L^2_t(\Omega)$ defined by

$$(Fg)(\hat{x}) := \lambda \int_\Omega E_\infty(\hat{x}; d, g(d))\,\mathrm{d}s(d) + \mu \int_\Omega H_\infty(\hat{x}; d, g(d) \times d)\,\mathrm{d}s(d) \tag{33}$$

where $g \in L^2_t(\Omega)$, and $\lambda > 0$ and $\mu < 0$ are real numbers. An *electromagnetic Herglotz pair* is defined to be a pair of vector fields of the form

$$E_g(x) = \int_\Omega \mathrm{e}^{\mathrm{i}kx \cdot d} g(d)\,\mathrm{d}s(d), \qquad H_g(x) = \frac{1}{\mathrm{i}k} \nabla \times E_g(x) \tag{34}$$

with kernel $g \in L^2_t(\Omega)$. One can easily see by superposition that Fg is a linear combination of the electric far-field pattern corresponding to the electromagnetic Herglotz pair with kernel $\mathrm{i}kg(d)$ as incident field, i.e. the electric far-field pattern E^{ext}_∞ of $E^{\mathrm{ext}}, H^{\mathrm{ext}}$ satisfying the exterior boundary value problem

$$\begin{aligned} \nabla \times E^{\mathrm{ext}} - \mathrm{i}kH^{\mathrm{ext}} &= 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \\ \nabla \times H^{\mathrm{ext}} + \mathrm{i}kE^{\mathrm{ext}} &= 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \\ \nu \times E^{\mathrm{ext}} + \mathrm{i}k\nu \times E_g &= 0 \quad \text{on } \partial D \\ \lim_{r \to \infty} (H^{\mathrm{ext}} \times x - rE^{\mathrm{ext}}) &= 0 \end{aligned} \tag{35}$$

and the magnetic far-field pattern corresponding to the electromagnetic Herglotz pair with kernel $\mathrm{i}kg(d)\times d$ as incident field, i.e. the magnetic far-field pattern $\tilde{H}_\infty$ of $\tilde{E}$, $\tilde{H}$ satisfying the exterior boundary value problem

$$\begin{aligned} \nabla\times\tilde{E}-\mathrm{i}k\tilde{H} &= 0 \quad \text{in } \mathbb{R}^3\backslash\bar{D} \\ \nabla\times\tilde{H}+\mathrm{i}k\tilde{E} &= 0 \quad \text{in } \mathbb{R}^3\backslash\bar{D} \\ v\times\nabla\times\tilde{H}+\mathrm{i}kv\times\nabla\times E_g &= 0 \quad \text{on } \partial D \\ \lim_{r\to\infty}(\tilde{H}\times x-r\tilde{E}) &= 0 \end{aligned} \tag{36}$$

We note that the boundary condition in (36) follows from the perfectly conducting boundary condition satisfied by $\tilde{E}$, the identity $\nabla\times\tilde{H}+\mathrm{i}k\tilde{E}=0$ and expressing the electric field of the Herglotz pair with kernel $g(d)\times d$ in terms of the electric field with kernel $g(d)$.

The *linear sampling method* for solving the inverse problem consists of solving the *far-field equation*

$$(Fg)(\hat{x})=E_{\mathrm{e},\infty}(\hat{x},z,q) \tag{37}$$

for a set of sampling points $z\in\mathbb{R}^3$ and three linearly independent polarizations $q\in\mathbb{R}^3$, where $E_{\mathrm{e},\infty}(\hat{x},z,q)$ is given by

$$E_{\mathrm{e},\infty}(\hat{x},z,q)=\frac{\mathrm{i}k}{4\pi}(\hat{x}\times q)\times\hat{x}\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z} \tag{38}$$

Note that $E_{\mathrm{e},\infty}$ is the electric far-field pattern of the electric dipole $E_\mathrm{e}(x,z,q):=(\mathrm{i}/k)\nabla_x\times\nabla_x\times q\Phi(x,z)$, with $\Phi(x,z):=(1/4\pi)(\mathrm{e}^{\mathrm{i}k|x-z|}/|x-z|)$ and $q\in\mathbb{R}^3$.

The transmission problem associated with the far-field equation (37), which we will refer to as the problem (TPM), is given a function $h\in H^{-1/2}_{\mathrm{div}}(\partial D)$ find $E^{\mathrm{int}},H^{\mathrm{int}}\in H(\mathrm{curl},D)$ and $E^{\mathrm{ext}},H^{\mathrm{ext}}\in H_{\mathrm{loc}}(\mathrm{curl},\mathbb{R}^3\backslash\bar{D})$ such that

$$\begin{gathered} \nabla\times E^{\mathrm{ext}}-\mathrm{i}kH^{\mathrm{ext}}=0, \qquad \nabla\times H^{\mathrm{ext}}+\mathrm{i}kE^{\mathrm{ext}}=0 \quad \text{in } \mathbb{R}^3\backslash\bar{D} \\ \nabla\times E^{\mathrm{int}}-\mathrm{i}kH^{\mathrm{int}}=0, \qquad \nabla\times H^{\mathrm{int}}+\mathrm{i}kE^{\mathrm{int}}=0 \quad \text{in } D \\ v\times E^{\mathrm{ext}}-v\times E^{\mathrm{int}}=0 \quad \text{on } \partial D \\ \lambda v\times\nabla\times E^{\mathrm{ext}}+\mu v\times\nabla\times E^{\mathrm{int}}=w \quad \text{on } \partial D \\ \lim_{r\to\infty}(H^{\mathrm{ext}}\times x-rE^{\mathrm{ext}})=0 \end{gathered} \tag{39}$$

In particular, for $z\in D$, (37) implies that

$$\lambda E^{\mathrm{ext}}+\mu\tilde{H}\equiv E_\mathrm{e} \quad \text{in } \mathbb{R}^3\backslash\bar{D}$$

where E^{ext} and $\tilde{H}$ are solutions of (35) and (36), respectively. Using the same argument as in the scalar case, it is now easy to see that $g\in L^2_t(\Omega)$ is a solution of the far-field equation if and only if there exists a solution to (TPM) with $h:=(1/\mathrm{i}k)v\times\nabla\times E_\mathrm{e}$ such that E^{int} coincides with the electric Herglotz wave function $-\mathrm{i}kE_g$ in D.

By lifting the boundary data $h \in H_{\mathrm{div}}^{-1/2}(\partial D)$ to a function in $H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ one can rewrite (TPM) as the problem considered by Kirsch and Monk [21], from which we have the following result.

Theorem 3.1
Let $h \in H_{\mathrm{div}}^{-1/2}(\partial D)$, and let $\lambda > 0$ and $\mu < 0$ be two different fixed real numbers. Then problem (TPM) has a unique solution $E^{\mathrm{int}}, H^{\mathrm{int}} \in H(\mathrm{curl}, D)$, $E^{\mathrm{ext}}, H^{\mathrm{ext}} \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$. Moreover the electric field satisfies

$$\|E^{\mathrm{int}}\|_{H(\mathrm{curl}, D)} + \|E^{\mathrm{ext}}\|_{H(\mathrm{curl}, B_R \cap \mathbb{R}^3 \setminus \bar{D})} \leqslant C \|h\|_{H_{\mathrm{div}}^{-1/2}(\partial D)} \tag{40}$$

where B_R is a ball of radius R containing D and $C > 0$ is a constant depending on R but not on h.

Definition 3.2
The operator $\mathscr{D}: H_{\mathrm{div}}^{-1/2}(\partial D) \to L_t^2(\Omega)$ maps $h \in H_{\mathrm{div}}^{-1/2}(\partial D)$ onto the far-field pattern $\lambda E_\infty^{\mathrm{ext}} + \mu \tilde{H}_\infty \in L_t^2(\Omega)$ of $\lambda E^{\mathrm{ext}} + \mu \tilde{H}$ where E^{ext} is the radiating electric field of the unique solution of (TPM) with boundary data h and $\tilde{H}$ is the unique solution of (36) with $\mathrm{i}kE_g$ replaced by $-E^{\mathrm{int}}$.

By the same argument as in the scalar case the operator $\mathscr{D}$ is compact and injective.

Theorem 3.3
The range of the operator $\mathscr{D}$ is dense in $L_t^2(\Omega)$.

Proof
As in the proof of Theorem 2.3 it is enough to show that the dual operator $\mathscr{D}^\top : L_t^2(\Omega) \to H_{\mathrm{curl}}^{-1/2}(\partial D)$ defined by

$$\langle \mathscr{D}h, g \rangle_{L_t^2(\Omega), L_t^2(\Omega)} = \langle h, \mathscr{D}^\top g \rangle_{H_{\mathrm{div}}^{-1/2}(\partial D), H_{\mathrm{curl}}^{-1/2}(\partial D)}$$

with $h \in H_{\mathrm{div}}^{-1/2}(\partial D)$ and $g \in L_t^2(\Omega)$, is injective. To this end we define $U := \lambda E^{\mathrm{ext}} + \mu \tilde{H}$ and observe that $v \times \mathrm{curl}\, U|_{\partial D} = h$. Then it is known [6] that $U_\infty = (\mathscr{D}h)(\hat{x})$ is given by

$$(\mathscr{D}h)(\hat{x}) = \frac{\mathrm{i}k}{4\pi} \hat{x} \times \int_{\partial D} \left\{ v(y) \times U(y) + \frac{1}{\mathrm{i}k} (v(y) \times \nabla \times U(y)) \times \hat{x} \right\} \mathrm{e}^{-\mathrm{i}k\hat{x} \cdot y} \, \mathrm{d}s(y)$$

The surface integrals are now understood as the duality pairing between $H_{\mathrm{div}}^{-1/2}(\partial D)$ and $H_{\mathrm{curl}}^{-1/2}(\partial D)$. By changing the order of integration, using the boundary condition for U and the relations

$$\nabla_y \times E_g(y) = \mathrm{i}k \int_\Omega [g(\hat{x}) \times \hat{x}] \mathrm{e}^{-\mathrm{i}k\hat{x} \cdot y} \, \mathrm{d}s(\hat{x})$$

$$\nabla_y \times \nabla_y \times E_g(y) = k^2 \int_\Omega [\hat{x} \times (g(\hat{x}) \times \hat{x})] \mathrm{e}^{-\mathrm{i}k\hat{x} \cdot y} \, \mathrm{d}s(\hat{x})$$

where $E_g = \int_\Omega g(\hat{x})\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\,\mathrm{d}s(\hat{x})$, we obtain

$$\langle \mathscr{D}h, g\rangle = \frac{1}{4\pi}\int_{\partial D} [(v\times U)\cdot(\nabla\times E_g) + h\cdot E_g]\,\mathrm{d}s \tag{41}$$

Next, let $\tilde{U}$ be the unique radiating solution of $\nabla\times\nabla\times\tilde{U} = k^2\tilde{U}$ in $\mathbb{R}^3\backslash\bar{D}$ such that $v\times\nabla\times\tilde{U} = v\times\nabla\times E_g$ on ∂D. Then applying Green's second vector theorem to U and $\tilde{U}$ and using the boundary condition for $\tilde{U}$ yield

$$\begin{aligned}\langle \mathscr{D}h, g\rangle &= \frac{1}{4\pi}\int_{\partial D} [-(v\times\nabla\times\tilde{U})\cdot U + h\cdot E_g]\,\mathrm{d}s\\ &= \frac{1}{4\pi}\int_{\partial D} [-(v\times\nabla\times U)\cdot\tilde{U} + h\cdot E_g]\,\mathrm{d}s = \frac{1}{4\pi}\int_{\partial D}(E_g - \tilde{U})\cdot h\,\mathrm{d}s\end{aligned}$$

whence $\mathscr{D}^\top g = v\times(E_g - \tilde{U})\times v|_{\partial D} \in H_{\mathrm{curl}}^{-1/2}(\partial D)$. Now it is easy to see that $\mathscr{D}^\top$ is injective. In particular $\mathscr{D}^\top g = 0$ implies that, on the boundary ∂D, $v\times\tilde{U} = v\times E_g$ and $v\times\nabla\times\tilde{U} = v\times\nabla\times E_g$ which gives $\tilde{U} = E_g = 0$ in $\mathbb{R}^3$ because $\tilde{U}$ is a radiating solution, whence $g = 0$. This ends the proof.

We are now at the position to study the far-field equation (37). Let $\mathscr{P}$ be the operator which maps $h\in H_{\mathrm{div}}^{-1/2}(\partial D)$ onto the interior electric component $E^{\mathrm{int}}\in H(\mathrm{curl}, D)$ of the solution to (TPM) with boundary data h. An argument similar to that used for $\mathscr{S}$ in the scalar case shows that $\mathscr{P}$ is in fact an isomorphism between $H_{\mathrm{div}}^{-1/2}(\partial D)$ and the Hilbert space

$$\mathscr{M}(D) := \{W\in H(\mathrm{curl}, D);\ \nabla\times\nabla\times W = k^2 W \text{ in the distribution sense}\}$$

equipped with the $H(\mathrm{curl}, D)$ norm. In particular, $\mathscr{P}^{-1}$ is well defined and continuous. The far-field equation (37) can now be rewritten as

$$(\mathscr{D}\mathscr{P}^{-1}E_g)(\hat{x}) = \frac{1}{\mathrm{i}k}E_{\mathrm{e},\infty}(\hat{x}, z, q) \tag{42}$$

We have now all the necessary ingredients to proceed exactly in the same way as in the scalar case to show that we can always find a solution to a perturbation of (37) and that this solution exhibits a specific behaviour. In particular if $z\in D$ we have showed that $\mathscr{D}((1/\mathrm{i}k)v\times\nabla\times E_{\mathrm{e}}|_{\partial D}) = E_{\mathrm{e},\infty}$. Let E^{int} be the interior electric field that solves (TPM) with boundary data $h := (1/\mathrm{i}k)v\times\nabla\times E_{\mathrm{e}}$. Then from the results of Reference [9] (for comments on nonsmooth boundaries see Reference [2]) for each $\varepsilon > 0$ we can find an electromagnetic Herglotz pair with kernel $g_\varepsilon(\cdot, z)\in L_t^2(\Omega)$ such that E_{g_ε} approximates $E^{\mathrm{int}}\in\mathscr{M}(D)$ with respect to the $H(\mathrm{curl}, D)$ norm, and moreover E_{g_ε} is an approximate solution to (42), i.e.

$$\left\|(\mathscr{D}\mathscr{P}^{-1}E_{g_\varepsilon})(\hat{x}) - \frac{1}{\mathrm{i}k}E_{\mathrm{e},\infty}(\hat{x}, z, q)\right\|_{L^2(\Omega)} \leqslant \varepsilon$$

Furthermore, as z approaches the boundary ∂D, the fact that $\|v\times\nabla\times E_{\mathrm{e}}\|_{H_{\mathrm{div}}^{-1/2}(\partial D)}\to\infty$ implies that

$$\lim_{z\to\partial D}\|E_{g_\varepsilon(\cdot,z)}\|_{H(\mathrm{curl}, D)} = \infty \quad \text{and} \quad \lim_{z\to\partial D}\|g_\varepsilon(\cdot, z)\|_{L_t^2(\Omega)} = \infty$$

For $z \in \mathbb{R}^3 \backslash \bar{D}$ we do not have that $E_{e,\infty}$ is in the range of $\mathscr{D}$ due to the singularity of the electric dipole E_e. However, from Theorem 3.3 by using a regularization argument and the approximation property of electromagnetic Herglotz pairs we can again find $E_{g_\varepsilon^\alpha}$ satisfying

$$\left\| (\mathscr{D}\mathscr{P}^{-1} E_{g_\varepsilon^\alpha(\cdot,z)})(\hat{x}) - \frac{1}{\mathrm{i}k} E_{e,\infty}(\hat{x},z,q) \right\|_{L^2(\Omega)} < \delta + \varepsilon \tag{43}$$

and

$$\lim_{\alpha\to 0} \|E_{g_\varepsilon^\alpha(\cdot,z)}\|_{H(\mathrm{curl},D)} = \infty \quad \text{and} \quad \lim_{\alpha\to 0} \|g_\varepsilon^\alpha(\cdot,z)\|_{L^2(\Omega)} = \infty$$

where arbitrary small $\varepsilon > 0$ measures the approximation by the Herglotz function, the arbitrary small $\delta > 0$ measures the perturbation of (42) to ensure a right-hand side in the range of $\mathscr{D}$ and $\alpha > 0$ is the regularization parameter corresponding to δ which additionally satisfies $\alpha \to 0$ as $\delta \to 0$.

We can now formulate a theorem similar to Theorem 2.4 which provides the mathematical bases of the linear sampling method for solving the inverse obstacle problem in electromagnetic scattering by a perfect conductor by using combined far-field data. Note that in contrast to Reference [22], no assumption is made on k not being a Maxwell eigenvalue.

Theorem 3.4
Let $D \subset \mathbb{R}^3$ be a bounded domain such that $\mathbb{R}^3 \backslash \bar{D}$ is connected and D has Lipschitz boundary ∂D. Then if F is the far-field operator defined by (33), where $\lambda > 0$ and $\mu < 0$ are real numbers, we have that

(1) If $z \in D$ then for every $\varepsilon > 0$ there exists a solution $g_\varepsilon(\cdot,z) = g_\varepsilon(\cdot,z,q) \in L_t^2(\Omega)$ of the inequality

$$\|Fg_\varepsilon(\cdot,z) - E_{e,\infty}(\cdot,z,q)\|_{L^2(\Omega)} < \varepsilon$$

Moreover, this solution satisfies

$$\lim_{z\to\partial D} \|g_\varepsilon(\cdot,z)\|_{L_t^2(\Omega)} = \infty \quad \text{and} \quad \lim_{z\to\partial D} \|E_{g_\varepsilon}(\cdot,z)\|_{H(\mathrm{curl},D)} = \infty$$

where E_{g_ε} is the electric component of the elecromagnetic Herglotz pair with kernel g_ε and

(2) If $z \in \mathbb{R}^3 \backslash \bar{D}$ then for every $\varepsilon > 0$ and $\delta > 0$ there exists a solution $g_\varepsilon^\delta(\cdot,z) = g_\varepsilon^\delta(\cdot,z,q) \in L_t^2(\Omega)$ of the inequality

$$\|Fg_\varepsilon^\delta(\cdot,z) - E_{e,\infty}(\cdot,z,q)\|_{L^2(\Omega)} < \varepsilon + \delta$$

such that

$$\lim_{\delta\to 0} \|g_\varepsilon^\delta(\cdot,z)\|_{L_t^2(\Omega)} = \infty \quad \text{and} \quad \lim_{\delta\to 0} \|E_{g_\varepsilon^\delta}(\cdot,z)\|_{H(\mathrm{curl},D)} = \infty$$

where $E_{g_\varepsilon^\delta}$ the electric component of the elecromagnetic Herglotz pair with kernel g_ε^δ.

3.2. Limited aperture

As we remarked in Section 2.3, to treat the case of limited aperture far-field data we only need to show that the set of electromagnetic Herglotz pairs can be approximated uniformly on compact subsets of a ball B_R of radius R by a Herglotz pair with kernel supported on a subset of Ω. According to Reference [6, Theorem 7.10] it suffices to show that the set of functions

$$E_g(x) := \int_\Omega g(d)\mathrm{e}^{\mathrm{i}kx\cdot d}\,\mathrm{d}s(d), \quad g \in L_t^2(\Omega) \quad \text{with support in } \Omega_0 \subseteq \Omega$$

for some subset $\Omega_0 \subseteq \Omega$ is complete in $L_t^2(\partial B_R)$, provided k is not a Maxwell eigenvalue for $B_R \supset D$ (which again is not a restriction since we can always find such a ball!).

To this end, let $a \in L_t^2(\partial B_R)$ and assume that for a fixed $\Omega_0 \subseteq \Omega$ we have that

$$\int_{\partial B_R} a(x)\left[\int_{\Omega_0} \bar{g}(d)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(d)\right]\mathrm{d}s(x) = 0 \tag{44}$$

for every $g \in L_t^2(\Omega_0)$. We want to show that $a = 0$. By interchanging the order of integration we arrive at

$$\int_{\Omega_0} \bar{g}(d)\left[\int_{\partial B_R} a(x)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(x)\right]\mathrm{d}s(d) = 0$$

for every $g \in L_t^2(\Omega_0)$, which implies that

$$d \times \int_{\partial B_R} a(x)\mathrm{e}^{-\mathrm{i}kx\cdot d}\,\mathrm{d}s(x) \times d \equiv 0, \quad d \in \Omega_0 \tag{45}$$

The left-hand side of (45) coincides with the far-field pattern $(\mathrm{Va})_\infty$ of the surface potential defined by

$$(\mathrm{Va})(y) := \frac{1}{k^2}\nabla_y \times \nabla_y \times \int_{\partial B_R} a(x)\Phi(x,y)\,\mathrm{d}s(x), \qquad y \in \mathbb{R}^3 \backslash \bar{B}_R, \quad a \in L_t^2(\partial B_R)$$

By analyticity we can conclude that $(\mathrm{Va})_\infty \equiv 0$ on Ω, which implies $(\mathrm{Va}) \equiv 0$ in $\mathbb{R}^3 \backslash \bar{B}_R$. The continuity of $v \times (\mathrm{Va})$ across ∂B_R, where v is the normal vector on ∂B_R, implies that $(\mathrm{Va}) \equiv 0$ in B_R as well since k is not a Maxwell eigenvalue for B_R (cf. References [6,23, p. 172]). Finally, by applying the jump relation for $v \times \nabla \times (\mathrm{Va})$ across ∂B_R [24,6] we obtain that $a \equiv 0$. This ends the proof. □

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