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The Linear Sampling Method in Inverse Electromagnetic Scattering

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Preface

The inverse scattering problem for electromagnetic waves is an area of major importance in applied mathematics. In particular, one can argue that the invention of radar is one of the most important inventions of the twentieth century. However, since radar is based on a weak scattering approximation and typically ignores polarization effects, it is of limited use for many target identification problems involving complex environments in which multiple scattering and/or polarization effects can no longer be ignored. For this reason considerable effort has been made in recent years to avoid the incorrect models inherent in the use of weak scattering approximations and instead to develop target identification algorithms without invoking such approximations. Initial efforts in this direction focused on nonlinear optimization techniques. However, although these techniques were successful in certain applications, it soon became apparent that they relied too heavily on strong a priori information about the scatterer and were numerically expensive as well. This then led to the search for target identification algorithms that, while avoiding incorrect model assumptions, were nevertheless easy to implement and required little a priori information. One result of this search has been the introduction of a class of methods collectively known as qualitative methods in inverse scattering theory (cf. [22]).

Qualitative methods in inverse scattering theory are characterized by the fact that, although they avoid the problems inherent in the use of weak scattering approximations or nonlinear optimization techniques, they typically recover less information than the latter two methods. In particular, with essentially no a priori assumptions about the material properties or geometry of the scatterer, the qualitative approach to the inverse scattering problem typically recovers the support of the scatterer as well as partial information on the scatterer's material properties. Furthermore, since the inversion algorithm is linear (even though the inverse scattering problem itself is nonlinear), the implementation of a given qualitative method is very rapid and easy to carry out (however, the implementation of a given qualitative method typically requires more data than the use of a nonlinear optimization scheme).

The oldest and most developed of the qualitative methods in inverse scattering theory is the linear sampling method (LSM), first introduced by Colton and Kirsch [49] in 1996 for the scalar case, and it is this approach (for the vector case) that will be the main focus of this book. For qualitative methods in electromagnetic inverse scattering theory other than the LSM, we refer the reader to Chapter 5 of the recent monograph by Kirsch and Grinberg [77] as well as to the article [69].

The basic material for this book was originally presented by one of us (Peter Monk) at the NSF-CBMS Regional Conference on Numerical Methods in Forward and Inverse Electromagnetic Scattering held at the Colorado School of Mines from June 3 to June 7 in 2002. Since that time the book [93] has appeared, which treats the forward problem in considerable detail. Hence in this book we have focused almost entirely on the inverse problem. In addition to the LSM, we have included in our presentation a discussion of uniqueness theorems and of the derivation of various inequalities on the material properties of the scattering object from a knowledge of the far field pattern of the scattered wave. Throughout our narrative the approximation properties of Herglotz wave functions and the behavior of solutions to a novel interior boundary value problem called the "interior transmission problem" play a central role.

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Chapter 1
Inverse Scattering in Two Dimensions

1.1 Introduction

Before launching into the rather complex problem of inverse electromagnetic scattering, we shall start by discussing a simpler reduced problem. This idealized model will serve to illustrate several issues to be faced when trying to solve inverse scattering problems.

Suppose we wish to solve the problem of finding the shape and location of a scatterer consisting of finitely many parallel infinite cylinders embedded in a background medium. One could consider, for example, long parallel metal rods in concrete (although we will provide no further discussion of this case). We assume that it is possible to probe the scatterers by a known incident field due to line sources also parallel to the axis of the scatterers and placed far from the objects to be imaged. The incident field will propagate through the background medium and interact in some way with the scatterers depending on their material makeup. This interaction will result in a scattered field that carries information about the unknown scatterers. We then assume that the scattered field is measured far from the object (usually at the same places as the sources of the incident field). From this scattered field data we wish to infer the position, shape, and, perhaps, properties of the scatterers.

Mathematically, let us suppose that the axes of the sources and scatterers are parallel to the x_3 axis and that the electric field $E(x,t)$, where $x = (x_1, x_2, x_3)$ denotes spatial position and t denotes time, is polarized so that

$$E(x,t) = (0, 0, \mathcal{E}(x_1, x_2, t))^T .$$

Then, under appropriate assumptions on the background medium, Maxwell's equations imply that $\mathcal{E}$ satisfies the scalar wave equation in the plane outside the scatterers:

$$\frac{1}{c^2}\frac{\partial^2 \mathcal{E}}{\partial t^2} = \Delta \mathcal{E}.$$

Here $c = c(x_1, x_2)$ is the local speed of light in the background medium, and, of course,

$$\Delta \mathcal{E} = \frac{\partial^2 \mathcal{E}}{\partial x_1^2} + \frac{\partial^2 \mathcal{E}}{\partial x_2^2}.$$

In this book we will always consider monochromatic waves so that we assume

$$\mathcal{E}(x_1,x_2,t) = \Re\left(u(x_1,x_2)e^{-i\omega t}\right),$$

where ω is the *temporal frequency* and u is independent of time (but now complex valued). The temporal period is then $2\pi/\omega$, and the frequency

$$f = \frac{\omega}{2\pi} > 0.$$

The field u satisfies the time-harmonic wave equation (we shall now write $u = u(x)$, where, in this section only, $x = (x_1,x_2)$)

$$\Delta u + \frac{\omega^2}{c^2}u = 0 \tag{1.1}$$

everywhere away from the scatterers.

To make progress in identifying the scatterers we have to assume that the background speed $c(x)$ is known. That the background speed has to be taken into account is in accord with everyday experience. For instance, if we look at a pebble at the bottom of a still pond of water, we will incorrectly judge its position because we assume a constant background of air. In fact the water usually appears shallower than it really is, which is one reason amongst many to check the depth of water using other means if you wish to, say, drive your car through it. To correctly estimate the depth of the pond, we need to take into account the change of refractive index (or speed of light) at the air–water interface. We shall simply assume that $c(x)$ is known (an alternative is to make the determination of $c(x)$ part of the inverse problem, but it is not obvious how to do this with the methods we shall discuss).

In this book we will, in fact, consider only one case where the background wave speed c is not constant (see Chapter 6), and in the present chapter we will assume $c(x) = c_0$, where c_0 is the speed of light in a vacuum. In this case it is convenient to define the wave number as

$$k = \frac{\omega}{c_0}$$

so that (1.2) becomes the familiar Helmholtz equation (satisfied everywhere outside the scatterers)

$$\Delta u + k^2 u = 0. \tag{1.2}$$

The probing or incident field, denoted by u^i, is the field that would propagate if no scatterer were present. Thus it is assumed to be a smooth solution of the background equation (1.2), at least far from the source antenna (and in particular near the scatterers). Under our assumptions that c is constant and that the scatterers are far from the antenna, we can assume that, near the scatterers, the incident field is a plane wave given by

$$u^i = \exp(ikx\cdot d),$$

where $d = (d_1,d_2)$ and $|d| = 1$. The vector d gives the direction of propagation of the wave, and its wavelength λ is given by

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = \frac{c}{f}.$$

This formula is key to understanding the “popular science" inverse problem of determining the speed of light using a microwave oven and a packet of mini-marshmallows. (We aren’t sure who first proposed this experiment, but for more information search “mini-marshmallow speed of light” using your favorite Internet search engine.)[1]

If scatterers are not present, the incident field would propagate throughout the plane $\mathbb{R}^2$, or, in other words, satisfy (1.2) for all x. However, if scatterers are present, the incident field will interact with the scatterers to produce a scattered field, denoted by u^s. This field also satisfies (1.2) away from the scatterer. Physically we measure the total field u given by

$$u = u^i + u^s. \tag{1.3}$$

The scattered field originates at the scatterer and propagates outwards. This physical consideration then motivates the requirement that the scattered field satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{1.4}$$

where $r = |x|$ and this limit is uniform in $\hat{x} = x/|x|$. If c is not constant, more complex radiation conditions may be required.

Equations (1.2), (1.3), and (1.4) do not uniquely determine E^s since we have not specified how the incident field interacts with the scatterer. But such knowledge should not be needed to solve the inverse scattering problem of determining the shape (we don’t need to know what we are looking at to see something!). Of course the details of the scattering mechanism will determine how well we can find the shape of the scatterer; if the scatterer is almost transparent, we will have more difficulty determining its shape compared to a strongly reflecting object.

In this introductory discussion we shall assume that the scattered field can be measured far from the scatterer (typically at the same location as the source antennas). It can be shown [50] that the scattered field has the asymptotic expansion far from the scatterer given by

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(\frac{1}{r}\right)$$

as $r \to \infty$. The function u_∞ is called the far field pattern of the scattered field and depends on the incident direction d and the measurement direction $\hat{x} = x/|x|$ (as well as on the wave number k and on the scatterers!).

While, in reality, we can have only finitely many receiving and transmitting antennas, we shall assume for now that $u_\infty(\hat{x}, d)$ is known (“measured”) for all $\hat{x} \in \Omega$ and $d \in \Omega$, where $\Omega = \left\{\hat{x} \;\middle|\; |\hat{x}| = 1\right\}$. From this data, for fixed k, we wish to reconstruct the location and shape of the scatterers. It can be shown that exact knowledge of $u_\infty(\hat{x}, d)$ does indeed uniquely determine the boundary of the scatterers in a wide variety of cases [50]. But, as we shall see, this inverse problem is both nonlinear and ill-posed. In particular, the uniqueness result does not imply the continuous dependence of the reconstruction on the far field data.

[1]The microwave oven needs to be primitive (i.e., have no mode stirrer); otherwise you need to remove the turntable. If you don’t have proper experimental training (and firefighting skills) please just eat the mini-marshmallows.

1.2 Classical Inversion Techniques

We shall now give a brief description of several classical methods for attacking the inverse scattering problem. In these methods we start by assuming a priori knowledge of the general nature of the scatterer. We might, for example, assume that the scatterer is penetrable and that it has an internal speed of light (and possibly absorption) that differs from the known background. Suppose $c(x_1, x_2) \neq c_0$ in the scatterer. Letting $n(x) = c_0^2/c^2$, we see that $n(x) = 1$ outside the scatterer and $n(x) \neq 1$ inside. In this case a complete set of equations for solving the forward problem (i.e., assuming $n(x)$ is known) is to find u and u^s that satisfy

$$\begin{aligned} \Delta u + k^2 n(x) u &= 0 \quad \text{in } \mathbb{R}^2, \\ u &= u^i + u^s \quad \text{in } \mathbb{R}^2, \\ r^{1/2}\left(\frac{\partial u^s}{\partial r} - iku^2\right) &\to 0 \quad \text{as } r \to \infty. \end{aligned} \tag{1.5}$$

It can be shown that for a wide class of functions $n(x)$, this system has an appropriately defined solution depending continuously on the data (i.e., the forward problem is well posed) [50]. Moreover, the far field pattern is given by

$$u_\infty(\hat{x}, d) = -e^{i\pi/4}\sqrt{\frac{k^3}{8\pi}} \int\limits_{\mathbb{R}^2} e^{-ik\hat{x}\cdot y} m(y) u(y)\, ds(y), \tag{1.6}$$

where $m = 1 - n$. Turning to the inverse problem, we assume that u_∞ is known but m is unknown, so equation (1.6) provides a nonlinear equation for m (of course u depends on m).

A very effective first approach to the inverse problem is to assume that the scattered field is small in comparison to the incident field so that $u \simeq u^i$. Then (1.6) becomes the weak scattering, or Born, approximation to the far field pattern

$$u_\infty(\hat{x}, d) \simeq -e^{i\pi/4}\sqrt{\frac{k^3}{6\pi}} \int\limits_{\mathbb{R}^2} e^{-ik(\hat{x}-d)\cdot y} m(y)\, ds(y), \tag{1.7}$$

where we have used the assumption that u^i is a plane wave in the direction d. If $\hat{x}$, d are varied over Ω, we see that $u_\infty(\hat{x}, d)$ gives an approximation to the Fourier transform of m for various values of the transform parameter $\xi = k(\hat{x} - d)$. Note in particular that $|\xi| \leq 2k$ so that the entire Fourier transform is not available. Equation (1.7) thus gives rise to the problem of computing the inverse Fourier transform of u_∞ (to determine m), but with incomplete or band-limited data. This band-limited inversion is ill posed, but the solution can be approximated by using a regularization method [97].

The Born approximation (1.7) is very popular [57], [10] because it is computationally efficient and often very successful. However, the method rests crucially on the weak scattering assumption (essentially linearization about the incoming field). If, for example, there is multiple scattering (e.g., waves bouncing around a partially enclosed cavity such as an aircraft engine inlet in three dimensions) or if m is too large, the approximation of

(1.6) by (1.7) will not be accurate, and a poor reconstruction may result (see, for example, [103]). There are, of course, several approaches for improving this situation, including the so-called distorted Born approximation (see, for example, [42]).

If strong scattering, or multiple scattering, is expected, a common alternative is to explicitly indicate that u depends on the unknown function m. For any suitable $\tilde{m}$ (not necessarily the true m), we denote by $u(x,\tilde{m})$ the total field at x that solves (1.5) with $n = 1-\tilde{m}$. This then has the following far field pattern $u_\infty(\hat{x},d,\tilde{m})$ given by (1.6):

$$u_\infty(\hat{x},d,\tilde{m}) = -e^{i\pi/4}\sqrt{\frac{k^3}{8\pi}}\int\limits_{\mathbb{R}^2} e^{-ik\hat{x}\cdot y}\tilde{m}(y)u(y,\tilde{m})\,ds(y).$$

Again, temporarily let $u_\infty^{\text{meas}}(\hat{x},d)$ denote the measured data far field pattern. We can now select a suitable admissible set A of possible functions $\tilde{m}$ (for example, constraining $\tilde{m} < 1$ and perhaps putting bounds on derivatives of $\tilde{m}$). An optimal best fit to the data is then found by solving

$$m^* = \operatorname*{argmin}_{\tilde{m}\in A}\int\limits_{\Omega}\int\limits_{\Omega}\left|u_\infty^{\text{meas}}(\hat{x},d) - u_\infty(\hat{x},d,\tilde{m})\right|^2 ds(d)\,ds(\hat{x}). \tag{1.8}$$

Such problems may be solved by constrained optimization techniques (see, for example, [11, page 173] and [60]). Usually these techniques are iterative, and it is necessary to solve the forward problem for different $\tilde{m}$ during the iterations. This must be done many times during the solution process, which can make this optimization approach rather slow. Examples of this approach include the work of Kleinman and coworkers using integral equation techniques [105], [79] and, more recently, the use of fast solvers and sophisticated preconditioned iterative techniques by Hohage [70]. For an alternative approach see [2]. On the one hand, these techniques can handle a wide variety of data (by generalizing (1.8)) and constraints. On the other hand, they remain expensive and prone to problems with stopping at local minima.

The next classical approach applies to a special class of scatterers at high frequency. Suppose that the scatterer is perfectly reflecting such that a Dirichlet boundary condition is satisfied on the boundary. If D denotes the bounded scatterer with boundary ∂D, then we now have that u and u^s satisfy

$$\begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ u &= u^i + u^s \quad \text{in } \mathbb{R}^2 \setminus D, \\ u &= 0 \quad \text{on } \partial D, \\ r^{1/2}\left(\frac{\partial u^s}{\partial r} - iku^s\right) &\to 0 \quad \text{as } r \to \infty. \end{aligned}$$

In this case, using the Dirichlet boundary condition, we have the expression [50]

$$u_\infty(\hat{x},d) = \frac{-e^{i\pi/4}}{\sqrt{8\pi k}}\int\limits_{\partial D}\frac{\partial u}{\partial \nu}(y)e^{-ik\hat{x}\cdot y}\,ds(y).$$

Again this is a nonlinear equation for ∂D in terms of u_∞. We can attempt a least squares optimization method similar to the previous case (see for example [11] and [100]). Newton methods are also used (see, for example, [82]). Both optimization and Newton methods are computationally expensive but are applicable with reduced measurement data (e.g., just one incident wave) compared to the qualitative methods that are the main focus of this book.

We now assume that the scatterer is smooth and convex, and also that ka is large, where a is the diameter of the inscribed circle to D. In this special case, we may approximate $\partial u/\partial \nu$ by the high-frequency Kirchoff approximation [50]. To write down this approximation, we define the illuminated zone by

$$\partial D_d^+ = \{x \in \partial D \mid \nu \cdot d < 0\},$$

where ν is the outward normal to D. The shadow region is then

$$\partial D_d^- = \{x \in \partial D \mid \nu \cdot d > 0\}.$$

The Kirchoff approximation, which is based on assuming that the shadow is total and that reflection in the illuminated zone can be approximated by locally linear scatterers, states that

$$\frac{\partial u}{\partial \nu} \simeq \begin{cases} 2\frac{\partial u^i}{\partial \nu}(x,d) & \text{if } x \in \partial D_d^+, \\ 0 & \text{if } x \in \partial D_d^-, \end{cases}$$

where, of course, $u^i(x,d) = \exp(ikx \cdot d)$. Thus

$$u_\infty(\hat{x},d) \simeq \frac{-2e^{i\pi/4}}{\sqrt{8\pi k}} \int\limits_{\partial D_d^+} \frac{\partial u^i}{\partial \nu}(y,d) e^{-ik\hat{x}\cdot y}\, ds(y).$$

Using this expression we obtain [91]

$$e^{i\pi/4} u_\infty(\hat{x},d) + e^{-i\pi/4} u_\infty(-\hat{x},-d) \approx \frac{-2}{\sqrt{8\pi k}} \int\limits_{\partial D} \frac{\partial u^i}{\partial \nu}(y,d) e^{-ik\hat{x}\cdot y}\, ds(y).$$

Hence, via Green's theorem, and using the fact that u^i is a plane wave,

$$e^{i\pi/4} u_\infty(\hat{x},d) + e^{-i\pi/4} u_\infty(-\hat{x},-d) \approx \frac{2k^{2/3}}{\sqrt{8\pi}} \int\limits_D (1 - d\cdot\hat{x}) e^{ik(d-\hat{x})\cdot y}\, ds(y).$$

This formula becomes even simpler if we consider "back scattered" data, in which case $d = -\hat{x}$ (the data is measured only at the transmitter):

$$e^{i\pi/4} u_\infty(\hat{x},-\hat{x}) + e^{-i\pi/4} \overline{u_\infty(-\hat{x},\hat{x})} \approx \frac{4k^{2/3}}{\sqrt{8\pi}} \int\limits_D e^{-2ik\hat{x}\cdot y}\, ds(y). \tag{1.9}$$

Thus, knowledge of back scattered data for all $\hat{x} \in \Omega$ gives an approximation to the Fourier transform of the characteristic function of D for the transform parameter $\xi = -2k\hat{x}$. Varying $\hat{x} \in \Omega$ and k over $k_{min} \le k \le k_{max}$ gives the Fourier transform for $2k_{\min} \le |\xi| \le 2k_{\max}$, and we can determine D by using an approximate band-limited inverse Fourier transform. For details see [91]. This method is interesting because it requires only back scattered data (which is more easily measured than multistatic data, where we assume knowledge of $u_\infty(\hat{x},d)$ for $\hat{x}\in\Omega$, $d\in\Omega$). In addition, it makes critical use of data for a range of wave numbers. However, it is limited by the need to have a convex scatterer and high-frequency data. For a more general physical optics method, and an example in which the method in this book succeeds where the physical optics method fails, see [13].

In our brief review of classical inverse scattering, we have seen several methods that either are based on asymptotic simplifications (e.g., the Born or Kirchoff approximation) that could limit their domain of applicability, or involve nonlinear optimization. In addition each method requires specific a priori data; for example we must typically know what type of scatterer we are trying to reconstruct. The next method we shall discuss, and the method that is the subject of this book, avoids some of these difficulties, but as we shall see has its own limitations and disadvantages.

1.3 The Linear Sampling Method

From the limited survey of classical scattering techniques in the previous section, we see that on the one hand, very efficient linearized methods based on band-limited Fourier transforms can be constructed. On the other hand, this efficiency is gained at the expense of needing very strong a priori data.

An alternative is offered by the linear sampling method (LSM). This method is relatively rapid (compared to optimization approaches), requires very limited a priori data, and involves only the solution of linear ill-posed problems. But it requires substantially more input data than either optimization approaches or asymptotic methods.

The LSM was discovered (invented?), via numerical experiments, by Andreas Kirsch while waiting for a flight to Germany at JFK International Airport in New York.[2] The method was first described in the paper of Colton and Kirsch [49].

The LSM is based on the "far field equation" of finding a function $g_z\in L^2(\Omega)$ such that

$$\int_\Omega u_\infty(\hat{x},d)g_z(d)\,ds(d)=\Phi_\infty(\hat{x},z), \tag{1.10}$$

where $\Phi_\infty(\hat{x},z)$ is the far field pattern of the field due to a point source located at the auxiliary point z. In two dimensions this field is given, in the near field, by

$$\Phi(x,z)=\frac{i}{4}H_0^{(1)}(k\,|x-y|)$$

where $H_0^{(1)}$ is the Hankel function of first kind of order zero. Its far field pattern is

$$\Phi_\infty(\hat{x},z)=\frac{e^{i\pi/4}}{\sqrt{8\pi k}}e^{-ik\hat{x}\cdot z}.$$

We shall argue that the function

$$\psi(z)=\|g_z\|^{-1}_{L^2(\Omega)} \tag{1.11}$$

[2]This historical fact is not intended to suggest that airport waiting rooms provide an ideal venue for mathematics, but it does attest to the computational power of laptop computers.

can serve as an indicator function for the scatterer D as z varies over $\mathbb{R}^2$. In fact we shall see that $\psi(z) \sim 0$ outside D.

Note that (1.10) is, in general, ill posed since u_∞ is a smooth function (in fact analytic) of $\hat{x}$ and d. Thus we shall have to use regularization techniques when we approximate the solution of (1.10) numerically [54]. In addition, this ill-posedness calls into question the existence of solutions to (1.10), and the resolution of this problem calls for significant analysis.

To see how this method might work, suppose for now that (1.10) has a solution for some $z \in \mathbb{R}^2$. Then by Rellich's uniqueness lemma (see Theorem 2.4), since both sides of (1.10) are far field patterns, we have

$$\int_\Omega u^s(x,d)g_z(d)\,ds(d) = \Phi(x,z), \quad x \in \mathbb{R}^2 \setminus D, \tag{1.12}$$

since the left-hand side of (1.10) is the far field pattern of the left-hand side of (1.12).

We immediately see that if $z \in \mathbb{R}^2 \setminus \overline{D}$, (1.12) cannot hold, since the left-hand side of (1.12) is bounded in L^2 on compact subsets of $\mathbb{R}^2 \setminus \overline{D}$, whereas the norm of Φ is unbounded if the subset contains z. Thus (1.10) does not have a solution if $z \in \mathbb{R}^2 \setminus \overline{D}$. Suppose now that (1.10) has a solution for each $z \in D$. Assuming Dirichlet boundary data and D, fix $x \in \partial D$, and let z approach x from inside D. The $L^2(D)$ norm of the right-hand side of (1.12) blows up and hence so must the norm of the left-hand side of (1.12). Since the scattered field is bounded, we must have $\|g_z\|_{L^2(\Omega)} \to \infty$ as $z \to x$. Thus we can hope that, after using a regularization scheme to stabilize the ill-posed problem (1.10), the function $\psi(z)$ in (1.11) will be approximately zero for $z \notin D$ and approach zero as z approaches ∂D from inside. The computation of g_z thus gives a qualitative way to visualize D.

The foregoing heuristic argument fails in general because very few domains are such that (1.10) admits an exact solution. The mathematical justification of the LSM then starts by showing that there is an approximate solution (to arbitrary accuracy) of (1.10) that has the desired blowup as z approaches ∂D. This analysis, which is different in detail for each type of scatterer, will be the main mathematical content of this book. The missing link in the full justification for the method is showing that this approximate g_z is in fact the one computed during the regularized solution of (1.10). In some cases (including the one considered in this chapter for example) this is known [7].[3] Indeed, the difficulty of fully justifying the LSM approach to inverse scattering may well be one reason why Andreas Kirsch has developed the more sophisticated "Factorization" method. This method has a stronger mathematical foundation [77] but is more difficult to extend to general scattering problems.

We can now summarize the main components of the LSM.

(1) Vary z over a grid in the region where D is sought.

(2) For each z, approximately solve (1.10) by some discrete regularization technique and compute an approximate g_z.

(3) Plot the indicator function $\psi(z)$ given in (1.11) and extract information about the scatterer. For example we might choose a contour value C and use the level curve $\psi(z) = C$

[3]The LSM has been the subject of some controversy; see [85] and [102].

to approximate the boundary ∂D. This requires a good choice of C, which is still broadly an open problem.

Step (1) requires a priori data on the size and approximate location of the scatterer. The grid for z must be fine enough that some grid points lie in or close to the scatterer. In addition D needs to be within the search domain if it is to be detected! A uniform grid for the sampling point z is not necessarily the most efficient strategy. A multilevel approach can be found in [45], [83].

Step (2) requires the numerical approximation of (1.10) by regularization techniques, which we discuss in the next section.

For step (3) several approaches have been tried to extract data from the indicator function $\psi(z)$. We have advocated "calibrating" the method by computing forward data for known objects (e.g., circles) of a size similar to the object to be reconstructed, and at the wave number to be used in practice. Using this data, the best choice of C (for the circle) can be computed. We then use this choice of C for more general objects [45]. Aramini et al. [6] suggest using a "deformable model" approach to adjust the contour based on $\psi(z)$ itself. Another approach, assuming a known scatterer is near the scatterers to be identified, is discussed in [84].

1.4 Regularization of the LSM

Equation (1.10) is ill posed due to the analyticity of u_∞. Thus we need to regularize it in order to obtain a reliable indicator (in the original paper [49] regularization was not used, and the use of regularization accounts for some of the improvement in computational results seen since then). We follow the "classical" approach described in [44], [54]. In this approach a different regularization parameter is used for each z. More recently, Brignone et al. [12] proposed using a single regularization parameter for all z. They term this approach "no-sampling linear sampling" since the method is applied at the continuous level for all z. This method is attractive because it avoids recomputing the regularization parameter for each z. Nevertheless we obtain reasonable performance with the classical approach and, since the numerical results we have reported (and will reproduce later in this book) are computed via the classical approach, we shall describe it next.

The method starts by using a discrete data matrix u_∞^δ that approximates u_∞ at certain data points. In the examples we shall present here we choose N equally spaced directions on the unit circle,

$$d_j = (\cos\theta_j, \sin\theta_j), \quad \theta_j = 2\pi j/N,\ j = 1,\dots,N.$$

Thus we have available an $N \times N$ matrix A^δ with

$$A^\delta_{\ell,m} = u^\delta_\infty(d_\ell, d_m), \quad 1 \le \ell,\ m \le N,$$

that approximates the exact data matrix A defined by

$$A_{\ell,m} = u_\infty(d_\ell, d_m).$$

We assume

$$\left\| A - A^\delta \right\| \le \delta,$$

where we use the spectral norm. This approximation may be due, in practice, to measurement error. In our numerical tests, we first choose a test scatterer D and then compute a synthetic approximation $u_\infty^{\text{comp}}(d_\ell, d_m)$ to $u_\infty(d_\ell, d_m)$ by a finite element method. We further perturb this data by random noise to avoid "inverse crimes" [50] using

$$u_{\ell,m}^\delta(d_\ell, d_m) = u_\infty^{\text{comp}}(d_\ell, d_m)(1 + \epsilon \xi_{\ell,m}), \tag{1.13}$$

where $\epsilon > 0$ is a parameter and $\xi_{\ell,m}$ is a uniformly distributed random number between -1 and 1. The actual value of δ used in our code is $\delta = \left\| A^{\text{comp}} - A^\delta \right\|$ and so may underestimate the total error in the data since it ignores discretization error from the finite element method.

A natural question is how large to choose N. It must be chosen depending on the wave number and size of the object. In [40] it is recommended to choose, at least,

$$N > 2ka, \tag{1.14}$$

where a is the radius of the circumscribing circle for D (which we have to assume a priori known approximately, as we have seen). This estimate is in accord with our experience.

We now approximate (1.10) using the trapezoidal rule to approximate the integral on the left-hand side, and we seek to compute $\vec{g}_z \in \mathbb{C}^N$ such that

$$A^\delta \vec{g}_z = b_z, \tag{1.15}$$

where $b_z \in \mathbb{C}^N$ and is given by $(b_z)_\ell = h^{-1}\Phi_\infty(d_\ell, z)$, $1 \le \ell \le N$, where $h = 2\pi/N$ arises from the weight in the trapezoidal rule. At the present time, there is no theory for predicting how well the solution of (1.15) approximates the solution of (1.10).

Because (1.10) is ill posed, we expect at best that (1.15) will be highly ill conditioned as N increases, and this is seen in practice. The approach we shall use to handle this is from [54]. Thus we actually use Tikhonov regularization to approximate the solution of (1.15) by solving

$$\left(\alpha_z I + (A^\delta)^* A^\delta\right) \vec{g}_z^\alpha = (A^\delta)^* \vec{b}_z, \tag{1.16}$$

where $\alpha_z > 0$ is the regularization parameter yet to be determined, and $(A^\delta)^*$ is the conjugate transpose of A^δ. We note that regularization with $\alpha_z I$ is an expedient and is not justified theoretically or by extensive numerical testing.

In our computations we choose α_z by the Morozov principle. So α_z is chosen such that

$$\left\| A^\delta \vec{g}_z^\alpha - b_z \right\|^2 = \delta^2 \left\| \vec{g}_z^\alpha \right\|^2, \tag{1.17}$$

where the norm is the ℓ^2 norm. This can be easily calculated using the singular value decomposition of A^δ. Let

$$A^\delta = USV^*,$$

where U and V are $N \times N$ unitary matrices and S is a diagonal matrix with

$$S_{p,p} = \sigma_p \ge 0, \quad p = 1, 2, \ldots, N.$$

Then, using the fact that U is unitary,

$$\left\| A^\delta \vec{g}_z^\alpha - \vec{b}_z \right\| = \left\| SV^* \vec{g}_z^\alpha - U^* \vec{b}_z \right\|,$$

and since V is unitary,

$$\left\|\vec{g}_z^\alpha\right\| = \left\|V^*\vec{g}_z^\alpha\right\|.$$

Furthermore (1.16) may be rewritten as

$$\left(\alpha_z I + VS^2V^*\right)\vec{g}_z^\alpha = VSU^*\vec{b}_z$$

or

$$\left(\alpha_z I + S^2\right)\left(V^*\vec{g}_z\right) = SU^*\vec{b}_z.$$

Letting $\vec{v}^\alpha = V^*\vec{g}_z^\alpha$ we have

$$v_p^\alpha = \left(\alpha_z + \sigma_p^2\right)^{-1}\sigma_p\left(U^*\vec{b}_z\right)_p, \quad p = 1,\dots,N.$$

So

$$\left\|A^\delta\vec{g}_z^\alpha - \vec{b}_z\right\| = \sum_{p=1}^{N}\left(\frac{\alpha_z}{\alpha_z+\sigma_p^2}\right)^2\left|\left(U^*b_z\right)_p\right|^2,$$

$$\|\vec{g}_z\|^2 = \sum_{p=1}^{N}\left(\frac{\sigma_p}{\alpha_z+\sigma_p^2}\right)^2\left|\left(U^*b_z\right)_p\right|^2.$$

The Morozov principle is then to seek α_z to be the zero of

$$f(\alpha) = \sum_{p=1}^{N}\frac{\alpha^2-\delta^2\sigma_p^2}{\left(\sigma_p^2+\alpha\right)^2}\left|\left(U^*b_z\right)\right|^2.$$

Note that $f(0) < 0$ and if the singular values are ordered by

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N \geq 0,$$

we have

$$f(\delta\sigma_1) \geq 0.$$

Since $f'(\alpha) > 0$, if $\alpha > 0$, there is a unique root that can be computed by a combination of bisection and secant iterations (in two dimensions we use the MATLAB `fzero` function). Note that this application of regularization theory is rather nonstandard since the kernel $u_\infty^\delta(\hat{x},d)$ is approximate and the right-hand side is exact (the opposite is usually assumed). See [72], [73] for theory for this approach. We will return to conditions required for Tikhonov regularization during our detailed mathematical discussion later.

1.5 Numerical Results in Two Dimensions

We now present some two-dimensional examples that will suggest to the reader that the LSM can provide useful information. We hope these results will sustain the reader through the rather more challenging material in the succeeding sections. Three-dimensional results will be discussed in Sections 3.5, 5.4, and 6.4.

As we have already mentioned, most of our results are for synthetic data. We choose a domain D, predict u_∞ using a finite element method (or any other convenient code), add random noise, and then solve the inverse problem. Expressions of delight and amazement follow when we recognize our initial figure!

We start with an example that is very special but allows us to avoid numerical issues to a large extent: we consider scattering by a circular cylinder. This study is motivated by a similar study in three dimensions in [43]. We follow this with two more challenging problems involving disconnected scatterers that show how the LSM can easily handle disjoint and possibly different scatterers. Finally we show an example using measured data kindly supplied by Professor F. Simonetti, Imperial College, London. This shows how the method can be applied to near field scattering with a limited aperture of sources and receivers.

1.5.1 Scattering by a Circular Cylinder

Suppose $D = \left\{x \in \mathbb{R}^2 \mid |x| < a\right\}$ is a disc of radius a. In this case the solution of (1.2)–(1.4), together with the Dirichlet boundary condition that $u = 0$ on ∂D, can be expressed using a series of special functions. In particular the following representation of the far field pattern can be derived (cf. [50]):

$$u_\infty(\theta,\phi) = -\sigma \sum_{n=-\infty}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)} \exp(in(\theta-\phi)),$$

where $\sigma = e^{i\pi/4}/\sqrt{8\pi k}$, and J_n and $H_n^{(1)}$ are, respectively, the Bessel and Hankel functions of the first kind of order n. The incident wave has the direction vector $d = (\cos\phi, \sin\phi)$ and the measurement direction $\hat{x} = (\cos\theta, \sin\theta)$. Expressing the function g_z as a trigonometric series with unknown coefficients $\left\{g_{z,m}\right\}_{m=-\infty}^{\infty}$, we have

$$g_z(\phi) = \sum_{m=-\infty}^{\infty} g_{z,m} \exp(im\phi),$$

and so the far field operator can be written as

$$\int_0^{2\pi} E_\infty(\theta,\phi) g_z(\phi)\, d\phi = -2\pi\sigma \sum_{n=-\infty}^{\infty} g_{z,n} \frac{J_n(ka)}{H_n^{(1)}(ka)} \exp(in\theta).$$

From [50], the right-hand side of (1.9) can also be expanded as

$$e^{-ikz\cdot\hat{x}} = \sigma \sum_{n=-\infty}^{\infty} J_n(kr_z) i^n \exp(in(\theta-\psi)),$$

where $z = (r_z\cos\psi, r_z\sin\psi)$. Formally we can then solve the far field equation to obtain

$$g_{z,m} = -i^n \frac{J_m(kr_z)}{J_m(ka)} H_m^{(1)}(ka), \quad -\infty < m < \infty.$$

If $r_z = 0$, we have $g_{z,n} = 0$ for $n \neq 0$, so the expansion for g_z converges in that case. Unfortunately this is the only case when the series converges, but it can be argued that the series diverges faster for $r_z > a$ than for $r_z < a$ (see [43] in the case of $\mathbb{R}^3$). To obtain a stable scheme when $r_z \neq 0$, we seek a regularized solution of the far field equation. One possibility is to argue as in [43] and choose a spectral cutoff ε. Then we use only the $|n| \leq N_\varepsilon$ modes for which the Fourier coefficients of the far field operator are "significant"; N_ε is chosen so that

$$\left|\frac{J_n(ka)}{H_n^{(1)}(ka)}\right| \geq \varepsilon \quad \text{for } |n| \leq N_\varepsilon,$$
$$\left|\frac{J_{N_\varepsilon+1}(ka)}{H_{N_\varepsilon+1}^{(1)}(ka)}\right| < \varepsilon.$$

The idea is that low-amplitude, high-frequency components cannot be measured accurately (due to their low amplitude) but have a disproportionate effect on instability and so should not be allowed to influence the solution. In this case Collino, Fares, and Haddar [43] suggest using the indicator $\psi(z) = \log_{10} \|g_z^{N_\varepsilon}\|_{L^2(\Omega)}$, where $g_z^{N_\varepsilon} = \sum_{n=-N_\varepsilon}^{N_\varepsilon} g_{z,n} \exp(in\theta)$.

Traditionally we have not used spectral cutoff regularization. Instead we use the Tikhonov/Morozov technique described earlier. We inject some noise into the data by defining

$$u_{\infty,n}^\varepsilon = \frac{J_n(ka)}{H_n^{(1)}(ka)}\left(1 + \frac{\varepsilon \xi_n}{\sqrt{N_\varepsilon}}\right),$$

where ξ_n is a random number drawn uniformly from the interval $[-1,1]$ and $\varepsilon > 0$ is a parameter. We choose $\varepsilon = 0.01$, giving a relative L_2 error for the coefficients of E_∞ of 0.9%. Then, using the known error in our approximate far field pattern to compute δ in the Morozov scheme, we can solve the Tikhonov problem to get

$$g_{z,n}^\varepsilon = -i^n \frac{\overline{u_{\infty,n}^\varepsilon} J_n(kr_z)}{\left|u_{\infty,n}^\varepsilon\right|^2 + \gamma^2} e^{-in\phi}$$

and $g_z^\varepsilon = \sum_{n=-N_k}^{N_k} g_{z,n}^\varepsilon \exp(in\theta)$. In the results in Figure 1.1 we choose $N_k = 14, 36, 138$ when $k = 3, 12, 60$. The Tikhonov parameter γ is computed by the Morozov principle.

In Figure 1.1, we consider spectral regularization and show $1/\|g_z\|_{L^2(\Omega)}$ and $\log_{10} \|g_z\|_{L^2(\Omega)}$ as functions of $r_z = |z|$ along the x-axis. Either choice of indicator provides a characterization of the domain. When $k = 1$ the indicator is a simple function of r_z but varies relatively gradually for r_z near $r_z = 1$. This implies that it is difficult to make a precise prediction about the radius of the target, but the presence of the target is clear. As k increases, the gradient of the indicator increases near $r_z = 1$, implying a more accurate reconstruction, but by $k = 60$ the indicator is highly oscillating, and in particular there are some oscillations for $r_z > 1$ making a reliable prediction more complicated.

In Figure 1.2 we show results using the Tikhonov/Morozov scheme with the same parameters as for Figure 1.1. The main conclusions are the same as for the spectral cutoff. However, when $k = 60$ the oscillations in the indicator for $r_z > 1$ are greatly reduced, and hence a more reliable indication of the exterior of D is possible. In all cases there are more artifacts inside D than in the exterior of D.

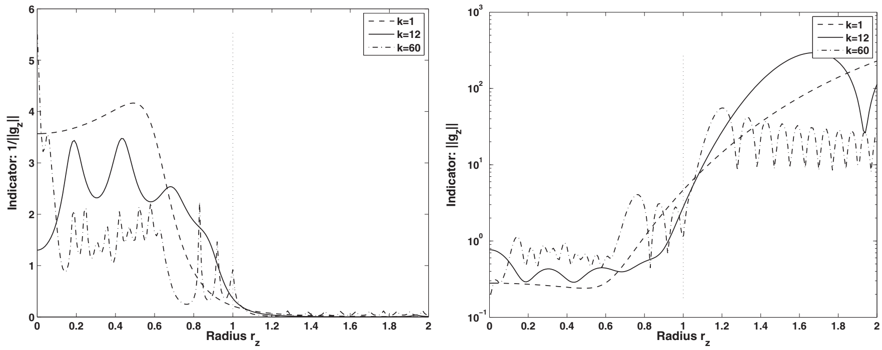


Figure 1.1. *Reconstructions using spectral regularization. We plot the indicator function along the real axis using (on the left) the reciprocal indicator and (on the right) the logarithmic indicator. Curves for* $k = 1, 12$, *and* 60 *are shown. The reciprocal indicator gives a better indication of the presence of the scatterer.*

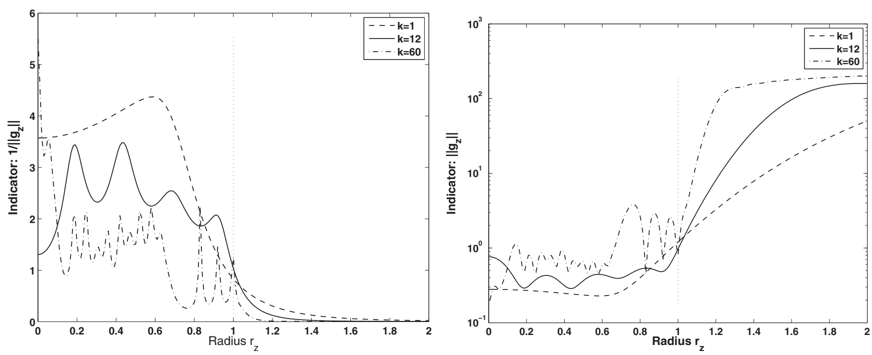
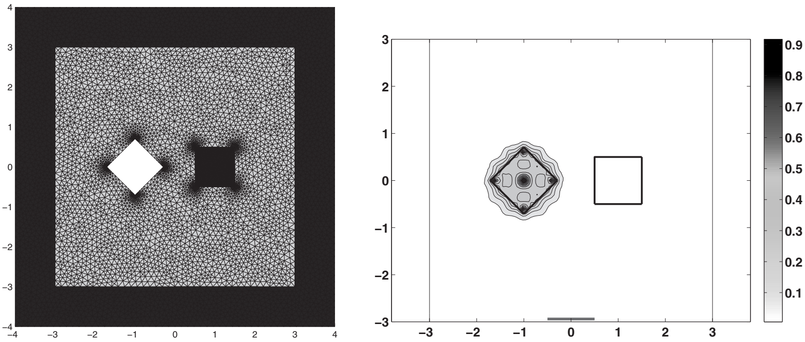


Figure 1.2. *Reconstructions using Morozov/Tikhonov regularization. We plot the indicator function along the real axis using (on the left) the reciprocal indicator and (on the right) the logarithmic indicator. Curves for* $k = 1, 12$, *and* 60 *are shown. As* k *increases, the indicators become more oscillatory, but the gradient at* $r = 1$ *increases, thus giving a more precise estimate of the radius.*

1.5.2 Two Scatterers Using Synthetic Data

For the synthetic experiments in this subsection, the far field pattern is computed via a cubic finite element code for a given domain D. We use the mesh shown in Figure 1.3(a) that is refined near the corners of the two scatterers. As can be seen from the mesh, the left-hand scatterer is impenetrable (Dirichlet boundary condition) while the right-hand scatterer has an index of refraction possibly differing from the background. We can then use the data, perturbed by noise (as described previously in Section 1.4 with parameter $\epsilon = 0.01$), in our inverse solver.



(a) The mesh (b) Reconstruction

Figure 1.3. (a) *The finite element mesh used to generate synthetic data for the two scatterer problem. This makes clear that the left-hand scatterer is impenetrable but the right-hand scatterer is penetrable.* (b) *The reconstruction obtained if we apply the LSM when $n(x) = 1$ in the penetrable scatterer. Here we plot contours of the function $\psi(x)$ using the reciprocal indicator* (1.11). *The single impenetrable square is clearly imaged. The thick bar at the lower edge of* (b) *shows the wavelength $2\pi/k$.*

We assume a priori that the scatterer has an internal radius of approximately unity and is located within the search domain $[-3,3]^2$ (using a uniform 101×101 grid of z values). The wave number k varies from experiment to experiment, and we have used 90 incoming wave directions and 90 far field values from directions uniformly distributed on the unit circle.

In Figure 1.3(b) we show results when $n(x) = 1$ in the right-hand square. This means that no scatterer is present there, and the LSM correctly reconstructs only the left-hand square (here we plot the indicator function $\psi(z)$ given by (1.11)). The wave number is $k = 20$, and so the wavelength is $\lambda = \pi/10$ and is shown as a thick bar along the bottom of the plot. The square is about 2 wavelengths in width, and so the boundary is sharply defined, but there is also considerable interior structure to the indicator function, as we would expect from our previous results for the unit circle.

Our next results are shown in Figure 1.4, where we have now set $n(x) = 4$ in the right-hand square. Of course the very same LSM as used in the previous example is used here as well, but now the LSM reconstructs two scatterers (or at least an extended dumbbell). We show results for different wave numbers. Here the scatterers are always at most one wavelength apart and hence difficult to distinguish. Surprisingly the higher wave number results do not distinguish the objects as well as the lower wave numbers.

In Figure 1.5 we show results using the two squares ($n(x) = 4$ in the right-hand square) but now we have moved the squares further apart. As expected, the squares can now be distinguished as two objects more easily. At lower wave numbers we obtain a reconstruction free of internal oscillation, but with less fidelity than at higher wave numbers.

This example shows that the LSM can easily handle disconnected scatterers (i.e., the same method is used regardless of the type and number of scatterers) but the quality of the reconstruction will depend on the interaction of the scatterers.

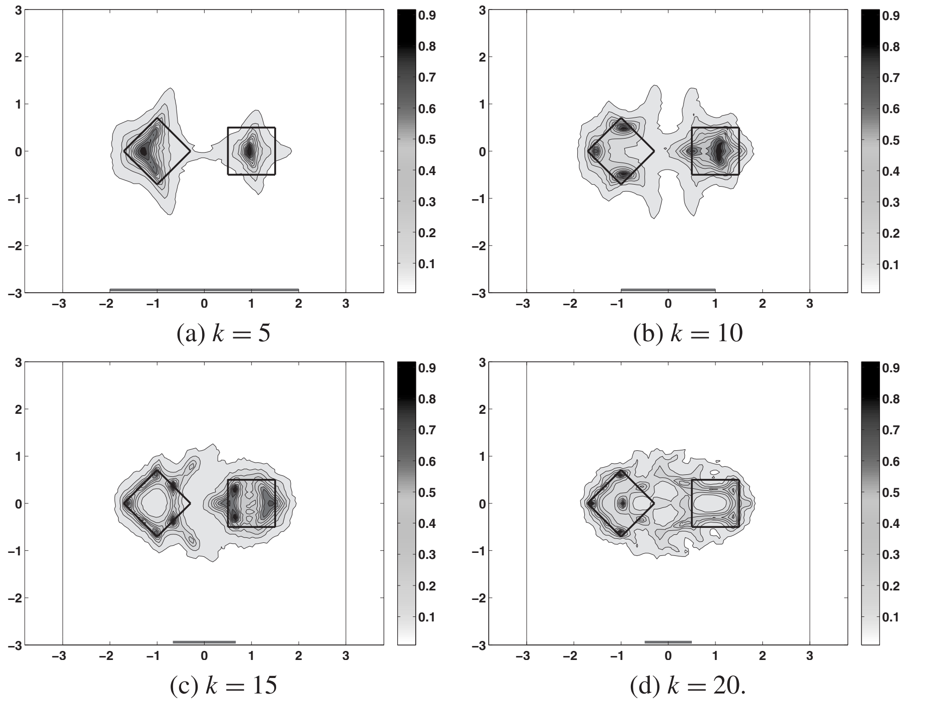


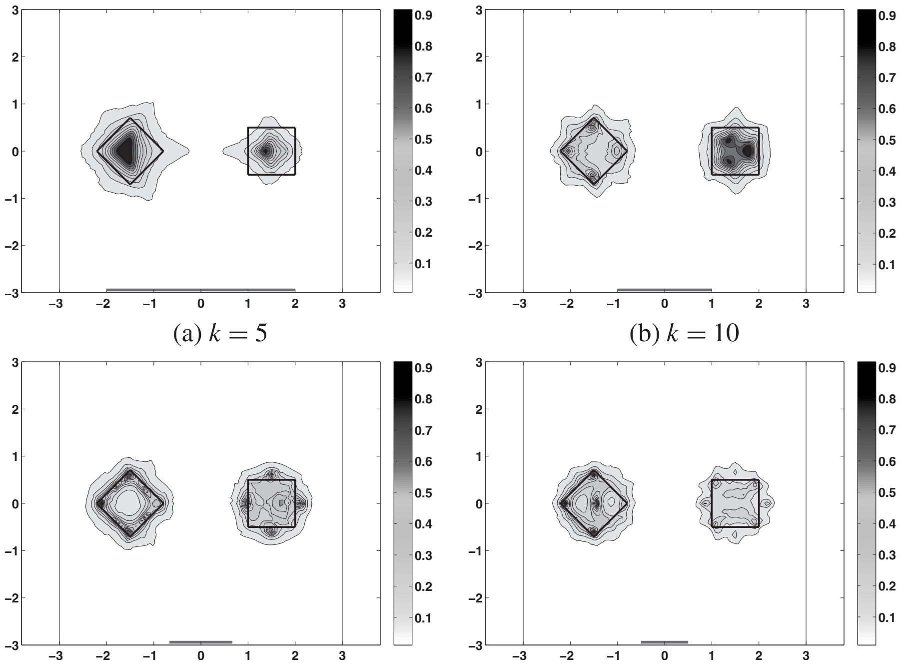
Figure 1.4. *Reconstruction of the two squares shown in Figure* 1.3(a)*. We show* $\psi(z)$ *given by* (1.11) *for various* k*. The same LSM is used for these reconstructions as was used for Figure* 1.3*, but the presence of the second scatterer is now visible (surprisingly, the higher wave number results are less able to distinguish the two squares).*

1.5.3 Real Data

In this section we describe some results of applying the LSM to real measured data. This data was kindly supplied by Professor F. Simonetti, Imperial College, London.

The problem exemplifies the power of the LSM because it involves a fluid structure interaction, a problem not discussed so far. However, exterior to the solid scatterers, the pressure field in the fluid satisfies the Helmholtz equation, and hence an extension of the simple LSM outlined in this chapter will apply (see [94] for details).

Long cylindrical polyoxymethylene (POM) copolymer rods are suspended in water (with sound speed $c = 1480$ m/s). Sound pulses are launched by transducers (32 elements spaced 1.5 mm apart and each 1 mm wide) located along the y-axis from -23.25 mm to $+23.25$ mm. The pressure wave travels through the fluid and interacts with the POM scatterers (the wave actually penetrates the scatterers via fluid–solid interaction). The scattered field is measured at the same transducers. The mathematical theory of the LSM in this case is covered in [94], but since measurements are carried out in the fluid where the Helmholtz equation again governs the time-harmonic pressure fields, we can use a standard LSM modified for near field measurements. We write down the analogue of the far field op-



(a) $k = 5$ (b) $k = 10$

(c) $k = 15$ (d) $k = 20$.

Figure 1.5. *In this example we use the same squares as in Figure* 1.4 *but move them further apart. As expected, the LSM more clearly detects two squares, and the fidelity of the reconstruction generally improves as k increases.*

erator. Let $p^s(x, y)$ denote the pressure field at position x due to a source at position y. Assuming that the sources and receiver are located on a line segment L, the appropriate near field equation is to seek $g_z \in L^2(L)$ such that

$$\int_L p^s(x, y) g_z(y)\, dy = \Phi(x, z)$$

for all $x \in L$. Here Φ is given by (1.3). After discretization by the trapezoidal rule, this equation needs to be solved by regularization. We do not know statistics for the error in the measurements, but we experimented with the Morozov parameter δ to give clean reconstructions. The data is, in fact, measured in the time domain and hence, using the fast Fourier transform, we have access to measurements at many frequencies. We chose two frequencies for the results in Figure 1.6, again on the basis of obtaining artifact free results. Note that we do not need to specify the scattering mechanism in our inversion scheme, so no properties of POM are needed. We point out that we did not take into account interaction between other transducers and the pressure field (except as point sources) or account for the finite size of the transducers. This is likely reasonable since the scatterer is placed far from the transducers (43 mm when the wavelength is roughly 1.3 mm).

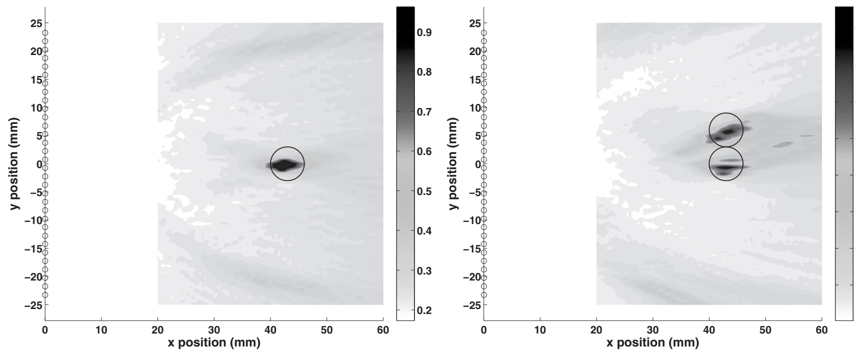


Figure 1.6. *Results of reconstructing one POM rod (left panel) at* 2.1*MHz and two POM rods (right panel) (at* 1.7*MHz). We plot* $1/\|\vec{g}_z\|$ *as a function of* z *(normalized to a maximum value of unity). The position of the transducers is shown by small circles on the* y*-axis, and the true boundary of the solid rods is shown as a larger circle or circles in the search domain. Reproduced from* [94] *with permission.*

Figure 1.6 shows our results and confirms that, despite the use of limited aperture data, the LSM produces a prediction of the number and location of the scatterers even when they touch.

Chapter 2
Maxwell's Equations

In this chapter we start our investigation of the inverse scattering problem for Maxwell's equations by presenting some basic results for solutions of the time-harmonic Maxwell's equations that will be needed in subsequent chapters. Since the focus of this book is the inverse scattering problem rather than Maxwell's equations in general, we shall make no effort at completeness. In particular, we will provide little or no proof of the results given but instead will refer the reader to either [50] or [93]. For more information on modeling of electromagnetic phenomena see [106].

2.1 The Scattering of Electromagnetic Waves

We begin by considering electromagnetic wave propagation in a source free isotropic medium in $\mathbb{R}^3$ with constant electric permittivity ε_0, magnetic permeability μ_0, and electric conductivity σ_0. The electromagnetic wave is then described by the electric field $\mathcal{E}$ and the magnetic field $\mathcal{H}$ satisfying the time-domain *Maxwell's equations*

$$\operatorname{curl}\mathcal{E} + \mu_0 \frac{\partial \mathcal{H}}{\partial t} = 0,$$

$$\operatorname{curl}\mathcal{H} - \varepsilon_0 \frac{\partial \mathcal{E}}{\partial t} = \sigma_0 \mathcal{E}.$$

In particular, for time-harmonic electromagnetic waves of the form

$$\mathcal{E}(x,t) = \Re\left\{\left(\varepsilon_0 + \frac{i\sigma_0}{\omega}\right)^{-1/2} E(x) e^{-i\omega t}\right\},$$

$$\mathcal{H}(x,t) = \Re\left\{\mu_0^{-1/2} H(x) e^{-i\omega t}\right\}$$

for $x \in \mathbb{R}^3$ and frequency $\omega > 0$, we see that the complex valued, space dependent fields E and H satisfy the time-harmonic Maxwell's equations

$$\begin{aligned} \operatorname{curl} E - ikH &= 0, \\ \operatorname{curl} H + ikE &= 0, \end{aligned} \tag{2.1}$$

where the wave number k is given by

$$k^2 = \left(\varepsilon_0 + \frac{i\sigma_0}{\omega}\right)\mu_0\omega^2$$

with the sign of k chosen such that $\Im k \geq 0$.

Now consider the scattering of a time-harmonic wave by an obstacle surrounded by a homogeneous medium with vanishing conductivity $\sigma_0 = 0$, i.e., $k > 0$. We first consider the scattering of an incoming wave E^i, H^i by a perfect conductor D, where D is a bounded domain such that $\mathbb{R}^3 \setminus \overline{D}$ is connected and ∂D is piecewise smooth. We assume that E^i, H^i is a solution of Maxwell's equations (2.1) in all of $\mathbb{R}^3$ and that the total field E, H is defined by

$$\begin{aligned} E &= E^i + E^s, \\ H &= H^i + H^s, \end{aligned} \tag{2.2}$$

where E^s, H^s is the scattered field satisfying the *Silver–Müller radiation condition*

$$\lim_{r\to\infty} \left(H^s \times x - rE^s\right) = 0 \tag{2.3}$$

uniformly for all directions $\hat{x} = x/|x|$, where $r = |x|$. We require that E, H satisfy Maxwell's equations (2.1) in $\mathbb{R}^3 \setminus \overline{D}$ and the perfectly conducting boundary condition

$$\nu \times E = 0 \quad \text{on } \partial D, \tag{2.4}$$

where ν is the unit outward normal to the boundary ∂D. If we define

$$H(\text{curl}; D) := \left\{u \in (L^2(D))^3 : \text{curl}\, u \in (L^2(D))^3\right\} \tag{2.5}$$

and $H_{\text{loc}}(\text{curl}; \mathbb{R}^3 \setminus \overline{D})$ to be the space of functions $u \in H(\text{curl}; B_R \setminus \overline{D})$ for every ball B_R : $\{x : |x| < R\}$ containing D in its interior, then it can be shown [93] that there is a unique solution $E, H = \frac{1}{ik}\text{curl}\, E$ of the scattering problem (2.1)–(2.4), where $E \in H_{\text{loc}}(\text{curl}; \mathbb{R}^3 \setminus \overline{D})$ and the boundary condition (2.4) is interpreted by the trace theorem for $H(\text{curl})$ functions [93].

For scattering by an obstacle that is not perfectly conducting but does not allow the electromagnetic wave to penetrate deeply into the obstacle, the boundary condition (2.4) is replaced by the *impedance boundary condition*

$$\nu \times (\text{curl}\, E) - i\lambda(\nu \times E) \times \nu = 0 \quad \text{on } \partial D, \tag{2.6}$$

where $\lambda = \lambda(x)$ is a positive continuous function defined on ∂D. For technical reasons connected to the trace properties of functions in $H(\text{curl}; D)$, we must now look for a solution $E, H = \frac{1}{ik}\text{curl}\, E$ of (2.1)–(2.3), (2.6) where E is in the space

$$X_{\text{loc}}(\mathbb{R}^3 \setminus \overline{D}) := \left\{u \in H_{\text{loc}}(\text{curl}; \mathbb{R}^3 \setminus \overline{D}) : \nu \times u\Big|_{\partial D} \in L^2_t(\partial D)\right\}, \tag{2.7}$$

where $L^2_t(\partial D)$ is the space of square integrable tangential vector fields defined on ∂D [93]. In general a subscript t on a function space denotes that the fields are tangential to the relevant surface almost everywhere.

Scattering by a penetrable obstacle D with constant magnetic permeability μ_0 but a variable electric permittivity $\varepsilon = \varepsilon(x) > 0$ and electric conductivity $\sigma(x) > 0$ for $x \in D$ leads to a scattering problem for the time-harmonic Maxwell's equations

$$\left.\begin{aligned} \operatorname{curl} E - ikH = 0 \\ \operatorname{curl} H + ikn(x)E = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3, \tag{2.8}$$

where $k^2 = \varepsilon_0\mu_0\omega^2$ (i.e., we have assumed that $\sigma_0 = 0$ in the background medium) and $n = n(x)$ is given by

$$n(x) := \frac{1}{\varepsilon_0}\left(\varepsilon(x) + i\frac{\sigma(x)}{\omega}\right),$$

and ε_0, μ_0 are the (constant) permittivity and permeability, respectively, of the homogeneous medium $\mathbb{R}^3 \setminus \overline{D}$. The function $\sqrt{n}$ is referred to as the refractive index. We assume that n is piecewise continuous in $\mathbb{R}^3$ and note that our assumptions imply that $n(x) = 1$ for $x \in \mathbb{R}^3 \setminus \overline{D}$. It can be shown that there exists a unique solution to the scattering problem (2.2), (2.3), (2.8) such that E and H are in $H_{\text{loc}}(\text{curl}; \mathbb{R}^3 \setminus \overline{D})$ [78]. Similar results hold for the case of an anisotropic medium, i.e., when $n = n(x)$ is a 3×3 matrix [78].

2.2 The Stratton–Chu Formulae and Their Application

We again (and throughout this book) assume that D is a bounded domain such that $\mathbb{R}^3 \setminus \overline{D}$ is connected and ∂D is smooth. The Stratton–Chu formulae are representation formulas for solutions of Maxwell's equations. To state these formulae (for their derivation see [93] or [50]), define the radiating fundamental solution to the Helmholtz equation by

$$\Phi(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y. \tag{2.9}$$

This satisfies

$$\Delta_x \Phi(x,y) + k^2\Phi(x,y) = -\delta(x-y)$$

together with the Sommerfeld radiation condition. Then if $E, H \in C^1(D) \cap C(\overline{D})$ is a solution of Maxwell's equations (2.1) in D, the *first Stratton–Chu formula* is

$$\begin{aligned} E(x) = \operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\Phi(x,y)\,ds(y) \\ + \frac{1}{ik}\operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\Phi(x,y)\,ds(y), \quad x \in D, \end{aligned} \tag{2.10}$$

$$\begin{aligned} H(x) = -\operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\Phi(x,y)\,ds(y) \\ - \frac{1}{ik}\operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\Phi(x,y)\,ds(y), \quad x \in D, \end{aligned} \tag{2.11}$$

where ν is the unit outward normal to D. An immediate consequence of (2.10) and (2.11) is the following theorem.

Theorem 2.1. *Any continuously differentiable solution of Maxwell's equations is an analytic function of its Cartesian components.*

Now assume that $E, H \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ is a radiating solution to Maxwell's equations (2.1) in $\mathbb{R}^3 \setminus \overline{D}$; i.e., E, H satisfy the Silver–Müller radiation condition (2.3). Then we have the *second Stratton–Chu formula*

$$\begin{aligned} E(x) = \operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\Phi(x,y)\,ds(y) \\ - \frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \overline{D}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} H(x) = \operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\Phi(x,y)\,ds(y) \\ + \frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \overline{D}, \end{aligned} \tag{2.13}$$

where again ν is the outward normal to D (i.e., inward to $\mathbb{R}^3 \setminus \overline{D}$).

For p a constant vector, the fields

$$\begin{aligned} E_m(x) &:= \operatorname{curl}_x p\, \Phi(x,y), \\ H_m(x) &:= \frac{1}{ik} \operatorname{curl} E_m(x) \end{aligned} \tag{2.14}$$

represent the electromagnetic field generated by a *magnetic dipole* located at the point y and satisfy Maxwell's equations for $x \neq y$. Similarly

$$\begin{aligned} H_e(x) &:= \operatorname{curl}_x p\, \Phi(x,y), \\ E_e(x) &:= \frac{i}{k} \operatorname{curl} H_e(x) \end{aligned} \tag{2.15}$$

represent the electromagnetic field generated by an *electric dipole*. The Stratton–Chu formulae obviously give representations of solutions to Maxwell's equations in terms of electric and magnetic dipoles distributed over the boundary. In this sense, the fields (2.14) and (2.15) my be considered as radiating fundamental solutions to Maxwell's equations. Straightforward calculations show that the Cartesian components of the fundamental solutions (2.14) and (2.15) satisfy the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r \left(\frac{\partial u}{\partial r} - iku \right) = 0$$

uniformly for $y \in \partial D$. Hence from the second Stratton–Chu formula we see that the Cartesian components of solutions to Maxwell's equations satisfying the Silver–Müller radiation condition also satisfy the Sommerfeld radiation condition. It is also easily verified, in the same way, that the Cartesian components of a continuously differentiable solution of Maxwell's equations satisfy the *Helmholtz equation*

$$\Delta u + k^2 u = 0.$$

Solutions of Maxwell's equations which are defined in all of $\mathbb{R}^3$ are called *entire solutions*. In particular, each Cartesian component of an entire solution of Maxwell's equations is a solution of the Helmholtz equation in all of $\mathbb{R}^3$. Hence, if u is such a component,

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm}\, j_n(kr) Y_n^m(\hat{x}),$$

where j_n is a spherical Bessel function, Y_n^m is an orthonormalized spherical harmonic and

$$a_{nm}\, j_n(kr) = \int\limits_{|x|=1} u(r\hat{x}) Y_n(\hat{x})\, ds(\hat{x})$$

for $\hat{x} = x/|x|$. But if u satisfies the Sommerfeld radiation condition, then the above formula implies that $a_{nm} = 0$ for $n \geq 0, -n \leq m \leq n$ (since j_n does not satisfy the radiation condition) and hence $u(x) = 0$ for $x \in \mathbb{R}^3$. We thus have the following theorem.

Theorem 2.2. *An entire solution of Maxwell's equations satisfying the Silver–Müller radiation condition must vanish identically.*

We will now exploit the relationship between radiating solutions of the Helmholtz equation and radiating solutions of Maxwell's equations to derive two basic results for solutions of Maxwell's equations in exterior domains. To this end, we recall *Rellich's lemma* for the Helmholtz equation (see [50, Lemma 2.11]).

Theorem 2.3 (Rellich's lemma). *Let $u \in C^2(\mathbb{R}^3 \setminus \overline{D})$ be a solution to the Helmholtz equation satisfying*

$$\lim_{r\to\infty} \int\limits_{|x|=r} |u(x)|^2\, ds = 0.$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

Since radiating solutions of the Helmholtz equation have the asymptotic behavior (see [50, Theorem 2.5])

$$u(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}) + O\left(\frac{1}{r^2}\right),$$

Rellich's lemma implies the following theorem.

Theorem 2.4. *Let $u \in C^2(\mathbb{R}^3 \setminus \overline{D})$ be a radiating solution to the Helmholtz equation for which the* far field pattern u_∞ *vanishes identically. Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.*

We now return to radiating solutions of Maxwell's equations. If E, H are radiating solutions to Maxwell's equations, then from the second Stratton–Chu formula it follows that

$$\begin{aligned} E(x) &= \frac{e^{ikr}}{r} E_\infty(\hat{x}) + O\left(\frac{1}{r^2}\right), \quad r \to \infty, \\ H(x) &= \frac{e^{ikr}}{r} H_\infty(\hat{x}) + O\left(\frac{1}{r^2}\right), \quad r \to \infty, \end{aligned} \tag{2.16}$$

uniformly in $\hat{x}$, where

$$
\begin{aligned}
E_\infty(\hat{x}) &= \frac{ik}{4\pi}\hat{x} \times \int_{\partial D} \{\nu(y) \times E(y) + (\nu(y) \times H(y)) \times \hat{x}\} e^{-ik\hat{x}\cdot y}\, ds(y), \\
H_\infty(\hat{x}) &= \frac{ik}{4\pi}\hat{x} \times \int_{\partial D} \{\nu(y) \times H(y) - (\nu(y) \times E(y)) \times \hat{x}\} e^{-ik\hat{x}\cdot y}\, ds(y).
\end{aligned}
\tag{2.17}
$$

E_∞ and H_∞ are known as the *electric far field pattern* and *magnetic far field pattern*, respectively, and they clearly satisfy

$$H_\infty = \hat{x} \times E_\infty \tag{2.18}$$

and

$$\hat{x} \cdot E_\infty = \hat{x} \cdot H_\infty = 0. \tag{2.19}$$

From Theorem 2.4 and our previous discussion we can now conclude that the following theorem holds.

Theorem 2.5. *Let $E, H \in C^1(\mathbb{R}^3 \setminus \overline{D})$ be a radiating solution to Maxwell's equations for which either the electric or the magnetic far field pattern vanishes identically. Then $E = H = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.*

Finally, using the identity (see [50, p. 161])

$$\lim_{r\to\infty} \int_{|x|=r} \left(|H \times \nu|^2 + |E|^2\right) ds = 2\Re \int_{\partial D} \nu \times E \cdot \overline{H}\, ds$$

for radiating solutions of Maxwell's equations, we have from Rellich's lemma that the following theorem is valid.

Theorem 2.6. *Let $E, H \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a radiating solution to Maxwell's equations satisfying*

$$\Re \int_{\partial D} \nu \times E \cdot \overline{H}\, ds \le 0.$$

Then $E = H = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

We conclude this section with a basic property (cf. [50, Theorem 6.28]) of the electric far field pattern corresponding to the scattering of an incident electromagnetic plane wave

$$
\begin{aligned}
E^i(x, d, p) &= \frac{i}{k} \operatorname{curl}\operatorname{curl} p e^{ikx\cdot d}, \\
H^i(x, d, p) &= \operatorname{curl} p e^{ikx\cdot d}
\end{aligned}
\tag{2.20}
$$

by an obstacle, where the constant unit vector d gives the direction of propagation and the constant vector p gives the polarization. A similar result also applies for the magnetic far field pattern. In the next theorem, $\Omega := \{x : |x| < 1\}$ is the unit sphere.

Theorem 2.7. *The electric far field pattern $E_\infty(\hat{x}) = E_\infty(\hat{x}, d, p)$ for the scattering of plane electromagnetic waves by an obstacle or penetrable medium satisfies the* reciprocity relation

$$q \cdot E_\infty(\hat{x}, d, p) = p \cdot E_\infty(-d, -\hat{x}, q)$$

for all $\hat{x}, d \in \Omega$ and all $p, q \in \mathbb{R}^3$.

From (2.17) we see that E_∞ is an analytic function of $\hat{x}$ on Ω. Hence from Theorem 2.7 we have the following theorem.

Theorem 2.8. *The electric far field pattern $E_\infty(\hat{x}) = E_\infty(\hat{x}, d, p)$ is an analytic function of $\hat{x}$ and d on the unit sphere Ω.*

2.3 Vector Wave Functions and Electromagnetic Herglotz Pairs

Let $Y_n^m, -n \le m \le n$, be an orthonormal system of spherical harmonics of order $n > 0$. Then the tangential fields on the unit sphere

$$U_n^m(\hat{x}) := \frac{1}{\sqrt{n(n+1)}} \operatorname{Grad} Y_n^m(\hat{x}),$$
$$V_n^m(\hat{x}) := \hat{x} \times U_n^m(\hat{x}),$$

where Grad denotes the surface gradient, are called *vector spherical harmonics* of order n. It can be shown (see [50, Theorem 6.23]) that the vector spherical harmonics are a complete orthonormal system in the space

$$L_t^2(\Omega) := \left\{ a : \Omega \to \mathbb{C}^3 \middle| a \in L^2(\Omega), a \cdot \nu = 0 \right\},$$

where Ω is again the unit sphere in $\mathbb{R}^3$.

Now let $j_n(t)$ be a spherical Bessel function and let $h_n^{(1)}(t)$ be a spherical Hankel function of the first kind. The function $j_n(t)$ is an entire function of t with the asymptotic behavior (cf. [50, Section 2.4])

$$j_n(t) = \frac{1}{t} \cos\left(t - \frac{n\pi}{2} - \frac{\pi}{2}\right) \left\{1 + O\left(\frac{1}{t}\right)\right\}, \quad t \to \infty, \tag{2.21}$$

whereas $h_n^{(1)}(t)$ is analytic for $t \neq 0$ and has the asymptotic behavior

$$h_n^{(1)}(t) = \frac{1}{t} e^{i\left(t - \frac{n\pi}{2} - \frac{\pi}{2}\right)} \left\{1 + O\left(\frac{1}{t}\right)\right\}, \quad t \to \infty. \tag{2.22}$$

From this it is easy to see that

$$M_n^m(x) := \operatorname{curl} \left\{ x j_n(k|x|) Y_n^m(\hat{x}) \right\},$$

and $\frac{1}{ik} \operatorname{curl} M_n^m(x)$ defines an entire solution to Maxwell's equations and

$$N_n^m(x) := \operatorname{curl} \left\{ x h_n^{(1)}(k|x|) Y_n(\hat{x}) \right\},$$

and $\frac{1}{ik}\operatorname{curl} N_n^m(x)$ defines a radiating solution to Maxwell's equations in $\mathbb{R}^3 \setminus \{0\}$. A short computation shows that

$$\begin{aligned} M_n^m(x) &= j_n(k|x|)\,\operatorname{Grad} Y_n^m(\hat{x}) \times \hat{x}, \\ N_n^m(x) &= h_n^{(1)}(k|x|)\,\operatorname{Grad} Y_n^m(\hat{x}) \times \hat{x}. \end{aligned} \tag{2.23}$$

The functions M_n^m and N_n^m are known as *spherical vector wave functions*. For future use we state the following *vector addition theorem* (see [50, Theorem 6.27]).

Theorem 2.9. *We have*

$$\begin{aligned} \Phi(x,y)p = ik \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=-n}^{n} N_n^m(x) \overline{M_n^m(y)} \cdot p \\ + \frac{i}{k} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=-n}^{n} \operatorname{curl} N_n^m(x) \operatorname{curl} \overline{M_n^m(y)} \cdot p \\ + \frac{i}{k} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \operatorname{grad} h_n^{(1)}(k|x|) Y_n^m(\hat{x}) \operatorname{grad} j_n(k|x|) \overline{Y_n^m(\hat{x})} \cdot p, \end{aligned}$$

where the series and its term-by-term derivatives are uniformly convergent for fixed y with respect to x and, conversely, for fixed x with respect to y on compact subsets of $|x| > |y|$.

We now consider a special case of entire solutions of Maxwell's equations which will play a special role throughout this book. In particular let $g \in L_t^2(\Omega)$, where $L_t^2(\Omega)$ is the space of square integrable tangential vector fields on the unit sphere Ω. Then we define an *electromagnetic Herglotz pair* to be a pair of vector fields of the form

$$\begin{aligned} E_g(x) &:= \int_{\Omega} e^{ikx\cdot d} g(d)\, ds(d), \\ H_g(x) &:= \frac{1}{ik} \operatorname{curl} E(x) \end{aligned} \tag{2.24}$$

for $x \in \mathbb{R}^3$. The vector field g is called the *Herglotz kernel* of the pair E_g, H_g. It is easily seen that the property of the kernel g to be tangential is equivalent to $\operatorname{div} E_g = 0$ in $\mathbb{R}^3$ and that an electromagnetic Herglotz pair is an entire solution of Maxwell's equations. By using the asymptotic formula (2.21) it is possible to show (see [50, Theorem 6.30]) that an entire solution E, H of Maxwell's equations possesses the growth property

$$\sup_{R>0} \frac{1}{R} \int_{|x|<R} \left(|E(x)|^2 + |H(x)|^2 \right) dx < \infty$$

if and only if it is an electromagnetic Herglotz pair. Using this result it is not difficult to see that M_n^m and $\frac{1}{ik}\operatorname{curl} M_n^m$ provide examples of electromagnetic Herglotz pairs. Finally, we note that $E_g = H_g = 0$ if and only if $g = 0$ [50].

In closing, we recall that for a given scattering problem with incident field given by (2.20) and corresponding far field pattern E_∞, H_∞, the far field pattern corresponding to

$$\tilde{E}^i(x) = \int_\Omega E^i(x; d, g(d))\, ds(d),$$

$$\tilde{H}^i(x) = \int_\Omega H^i(x; d, g(d))\, ds(d)$$

for $g \in L^2_t(\Omega)$ is given by

$$\tilde{E}_\infty(\hat{x}) = \int_\Omega E_\infty(\hat{x}; d, g(d))\, ds(d),$$

$$\tilde{H}_\infty(\hat{x}) = \int_\Omega H_\infty(\hat{x}; d, g(d))\, ds(d).$$

In particular, for $g \in L^2_t(\Omega)$ we can write

$$\tilde{E}^i(\hat{x}) = ik \int_\Omega g(d) e^{ikx\cdot d}\, ds(d),$$

$$\tilde{H}^i(\hat{x}) = \operatorname{curl} \int_\Omega g(d) e^{ikx\cdot d}\, ds(d);$$

i.e., $\tilde{E}^i$, $\tilde{H}^i$ is an electromagnetic Herglotz pair with kernel ikg.

Chapter 3
The Inverse Scattering Problem for Obstacles

In this chapter we consider the inverse scattering problem of determining the shape of a perfect conductor, or the shape and surface impedance λ of an imperfect conductor, from knowledge of the far field pattern of the scattered wave arising from the incident plane wave given by (2.20). In order to consider both problems at the same time, we will in fact consider the direct scattering problem to be the mixed boundary value problem of finding E and H satisfying

$$\begin{aligned} \operatorname{curl} E - ikH &= 0, \\ \operatorname{curl} H + ikE &= 0 \end{aligned} \tag{3.1}$$

in $\mathbb{R}^3 \setminus \overline{D}$ (where D satisfies the same hypothesis as in Chapter 2). Moreover, we assume that the boundary $\partial D = \partial D_P \cup \Pi \cup \partial D_I$ is split into two disjoint parts ∂D_P and ∂D_I having Π as their possible common boundary in ∂D and that each part ∂D_P and ∂D_I can be written as the union of a finite number of open smooth faces. Let ν denote the unit outward normal defined almost everywhere on Γ. Then on the boundary $\partial D = \overline{\partial D_P} \cup \overline{\partial D_I}$ we have that

$$\begin{aligned} \nu \times E &= 0 \quad \text{on } \partial D_P, \\ \nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu &= 0 \quad \text{on } \partial D_I, \end{aligned} \tag{3.2}$$

where $\lambda = \lambda(x) \geq \lambda_0 > 0$ is the surface impedance and is assumed to be a continuous function defined on ∂D_I. The total field is again given by

$$\begin{aligned} E &= E^i + E^s, \\ H &= H^i + H^s, \end{aligned} \tag{3.3}$$

where E^s, H^s is the scattered field satisfying the Silver–Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{3.4}$$

uniformly in $\hat{x} = x/|x|$, where $r = |x|$ and the incident field is the plane wave (2.20), i.e.,

$$\begin{aligned} E^i(x) &:= \frac{i}{k} \operatorname{curl}\operatorname{curl} p e^{ikx\cdot d} = ik(d \times p) \times d e^{ikx\cdot d}, \\ H^i(x) &:= \operatorname{curl} p e^{ikx\cdot d} = ikd \times p e^{ikx\cdot d}. \end{aligned} \tag{3.5}$$

In particular, the case of a perfect conductor corresponds to the case when $\partial D_I = \emptyset$, and the case of an imperfect conductor corresponds to the case when $\partial D_P = \emptyset$. In [27] it is shown that there exists a unique solution $E, H = \frac{1}{ik}\operatorname{curl} E$ to (3.1)–(3.5) with E in the space

$$X(\mathbb{R}^3 \setminus \overline{D}, \partial D_I) := \left\{ u \in H_{\text{loc}}(\text{curl}; \mathbb{R}^3 \setminus \overline{D}) : \nu \times u\Big|_{\partial D_I} \in L_t^2(\partial D_I) \right\}$$

and (3.2) is interpreted in terms of an appropriate trace theorem.

As mentioned above, the inverse scattering problem that we will consider in this chapter is to determine D and λ (if $\partial D_I \neq \emptyset$) from knowledge of the electric far field pattern E_∞ corresponding to (3.1)–(3.5) for a fixed wave number $k > 0$. We will begin in the next section with a uniqueness theorem due to Kress [81] showing that D and λ are uniquely determined from E_∞. We then continue our presentation by establishing certain denseness properties of electromagnetic Herglotz pairs, and we use these results to derive the *linear sampling method* (LSM) for determining D and λ from knowledge of the electric far field pattern E_∞. We conclude this chapter by considering the problem of limited aperture far field data.

3.1 A Uniqueness Theorem

Before presenting our promised uniqueness theorem, we first make a few preliminary observations. Since by Rellich's lemma the electric far field pattern uniquely determines the scattered field in the exterior of the scatterer, the question of uniqueness for the inverse problem is equivalent to the question of whether or not the total field can satisfy a boundary condition of the form (3.2) for two different domains D_1 and D_2. We can immediately exclude the case when $\overline{D}_1 \cap \overline{D}_2 = \emptyset$ since in this case E^s, H^s is an entire solution to Maxwell's equations satisfying the Silver–Müller radiation condition, and therefore by Theorem 2.2 must be identically zero. But then $E = E^i$ and $H = H^i$, and for E^i, H^i given by (3.5) this is impossible since E^i, H^i cannot satisfy a boundary condition of the form (3.2). Thus we can assume that $\overline{D}_1 \cap \overline{D}_2 \neq \emptyset$.

In the proof of our uniqueness theorem, we will make use of a *mixed reciprocity relation* due to Potthast [101]. In particular, let $E_{e,\infty}(\cdot, y, p)$ be the far field pattern due to the incident field being an electric dipole (2.15), and let E^s be the scattered field due to the incident field being the plane wave (3.5). Then we have the identity

$$q \cdot E^s(z, d, p) = 4\pi p \cdot E_{e,\infty}(-d, z, q) \tag{3.6}$$

for all $z \in \mathbb{R}^3 \setminus \overline{D}$, all incident directions $d \in \Omega$, and all polarizations $p, q \in \mathbb{R}^3$.

Finally, we note that if the electric far field patterns corresponding to the incident field (3.5) coincide for two linearly independent polarizations p_i such that p_i is tangential to $\Omega, i = 1, 2$, then by linearity they coincide for all $p \in \mathbb{R}^3$. Furthermore, if (3.2) is satisfied for two surface impedances λ_1 and λ_2 defined on ∂D_{I_1} and ∂D_{I_2}, respectively, then $\lambda_1 = \lambda_2$ implies that $\partial D_{I_1} = \partial D_{I_2}$.

Theorem 3.1. *Assume that D_1 and D_2 are two scatterers corresponding to the boundary condition* (3.2) *(where ∂D_I may be the empty set) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively, such that the electric far field patterns coincide for a fixed wave number k, all incident directions $d \in \Omega$, all $p \in \mathbb{R}^3$, and all observation directions $\hat{x}$. Then $D_1 = D_2$ and $\lambda_1 = \lambda_2$.*

Proof. Since the far field patterns coincide, by Theorem 2.5 we have that $E_1^s(\cdot,d,p) = E_2^s(\cdot,d,p)$ in the unbounded component G of the complement of $\overline{D}_1 \cap \overline{D}_2$ for all $d \in \Omega$ and $p \in \mathbb{R}^3$, where E_i^s, $i = 1,2$, is the scattered field corresponding to D_i. Then, for the incident field being an electric dipole, from the mixed reciprocity relation (3.6) we can conclude that $E_{1,e,\infty}(\cdot,z,q) = E_{2,e,\infty}(\cdot,z,q)$ for all $z \in G$ and all polarizations $q \in \mathbb{R}^3$. By again using Theorem 2.5 we can conclude that for the corresponding scattered waves we have that

$$E_{1,e}^s(x,z,q) = E_{2,e}^s(x,z,q) \tag{3.7}$$

for all $x,z \in G$ and all polarizations q.

Now assume that $D_1 \neq D_2$. Then, without loss of generality, there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \overline{D}_2$. In particular, we have that $z_n := x^* + \frac{1}{n}\nu(x^*) \in G$ for sufficiently large n. Then from the well-posedness of the direct scattering problem corresponding to the obstacle D_2 we have that as $n \to \infty$ the boundary condition (3.2) is assumed for $E_{2,e}^s(x,x^*,q)$ at $x = x^*$. On the other hand, for $q \perp \nu(x^*)$, this boundary condition is not satisfied for $E_{1,e}^s(x^*,z_n,q)$ as $z_n \to x^*$ since the electric dipole becomes unbounded as the source location z_n tends to x^*. This contradicts (3.7), and hence $D_1 = D_2$.

Now, setting $D = D_1 = D_2$ and $E = E_1 = E_2$ we assume that E satisfies (3.2) for different surface impedances $\lambda_1 \neq \lambda_2$ with corresponding domains of definition $\partial D_I^{(1)}$ and $\partial D_I^{(2)}$. Assume that $\partial D_I^{(1)} \neq \emptyset$. Then $\partial D_I^{(1)} = \partial D_I^{(2)}$, since if this were not true, then $\nu \times E = \nu \times \operatorname{curl} E = 0$ on a surface $\Gamma \subset \partial D$ and by the second Stratton–Chu formula,

$$\tilde{E}(x) := \begin{cases} 0, & x \in D, \\ E(x), & x \in \mathbb{R}^3 \setminus \overline{D}, \end{cases}$$

$$\tilde{H}(x) := \frac{1}{ik} \operatorname{curl} \tilde{E}(x)$$

defines a solution of Maxwell's equations in $\mathbb{R}^3 \setminus (\partial D \setminus \Gamma)$. By Theorem 2.1 we can now conclude that $E(x) = 0$ for $x \in \mathbb{R}^3 \setminus (\partial D \setminus \Gamma)$; i.e., the scattered field E^s, H^s is an entire solution of Maxwell's equations. By Theorem 2.2 this is a contradiction. Hence $\partial D_P^{(1)} = \partial D_P^{(2)}$ and thus $\partial D_I = \partial D_I^{(1)} = \partial D_I^{(2)}$. This implies that

$$(\lambda_1 - \lambda_2)\nu \times E = 0 \quad \text{on } \partial D_I,$$

and hence on the open set $U := \{x \in \partial D_I : \lambda_1(x) \neq \lambda_2(x)\}$ we have that $\nu \times E = 0$ on ∂D_I. The boundary condition (3.2) now implies that $\nu \times \operatorname{curl} E = 0$ on ∂D_I and, as before, this is a contradiction. The case when $\partial D_c^{(1)} = \emptyset$ can be treated analogously. □

A challenging open problem is whether or not the electric far field pattern corresponding to one incident plane wave determines the scatterer. Recent progress in this direction has been obtained by Kress [81], Liu [86], and Liu, Zhang, and Zou [88] (see also [87]).

We note that in Theorem 3.1 the electric far field pattern can be replaced by the magnetic far field pattern. This follows immediately from (2.18) and (2.19).

3.2 Approximation Properties of Electromagnetic Herglotz Pairs

Recall from (2.24) that E_g, H_g represents an electromagnetic Herglotz pair. In order to establish the LSM for solving the inverse scattering problem, it is necessary to first derive certain approximation properties of Herglotz wave functions. We will do this in this section. In particular we will need to consider the following two function spaces: $H(\mathrm{curl}, D)$ introduced in (2.5) with the norm

$$\|u\|^2_{H(\mathrm{curl},D)} := \|u\|^2_{L^2(D)} + \|\mathrm{curl}\, u\|^2_{L^2(D)}, \tag{3.8}$$

and

$$X(D, \partial D_I) := \left\{ u \in H(\mathrm{curl}, D) : \nu \times u\Big|_{\partial D_I} \in L^2_t(\partial D_I) \right\}$$

equipped with the norm

$$\|u\|^2_{X(D,\partial D_I)} := \|u\|^2_{H(\mathrm{curl},D)} + \|\nu \times u\|^2_{L^2(\partial D_I)}. \tag{3.9}$$

Recall that the trace $\nu \times u|_{\partial D}$ of a function $u \in H(\mathrm{curl}, D)$ is in

$$H^{-1/2}(\mathrm{Div}, \partial D) := \left\{ u \in H_t^{-1/2}(\partial D) : \mathrm{Div}\, u \in H^{-1/2}(\partial D) \right\}, \tag{3.10}$$

where Div denotes the surface divergence, whereas the corresponding trace space of $X(D, \partial D_I)$ on ∂D_P is given by [93]

$$\begin{aligned} Y(\partial D_P) := \Big\{ f \in \left(H^{-1/2}(\partial D_P) \right)^3 &: \exists u \in H_0(\mathrm{curl}, B_R), \\ \nu \times u\Big|_{\partial D_I} &\in L^2_t(\partial D_I) \text{ and } f = \nu \times u\Big|_{\partial D_P} \Big\}, \end{aligned} \tag{3.11}$$

where the ball $B_R := \{x : |x| < R\}$ contains D and $H_0(\mathrm{curl}, B_R)$ is the space of functions u in $H(\mathrm{curl}, B_R)$ such that $\nu \times u|_{\partial B_R} = 0$. Then we set

$$X_0(B_R \setminus \overline{D}; \Gamma_I) = X(B_R \setminus \overline{D}; \Gamma_I) \cap H_0(\mathrm{curl}; B_R).$$

Note that $H^{-1/2}(\mathrm{Div}, \partial D)$ is a Banach space with norm

$$\|u\|^2_{H^{-1/2}(\mathrm{Div},\partial D)} := \|u\|^2_{H^{-1/2}(\partial D)} + \|\,\mathrm{Div}\, u\|^2_{H^{-1/2}(\partial D)}, \tag{3.12}$$

whereas $Y(\partial D_P)$ is a Banach space with respect to the norm

$$\|f\|^2_{Y(\partial D_P)} := \inf \left\{ \|u\|^2_{H(\mathrm{curl},B_R)} + \|\nu \times u\|^2_{L^2(\partial D_I)} \right\}, \tag{3.13}$$

where the infimum is taken over all functions $u \in H_0(\mathrm{curl}, B_R)$ such that $\nu \times u|_{\partial D_I} \in L^2_t(\partial D_I)$ and $f = \nu \times u|_{\partial D_P}$. It can be shown (cf. [27]) that $\|\cdot\|_{Y(\partial D_P)}$ is equivalent to both of the norms

$$|||f|||_1 := \sup_{\phi \in X(D,\Gamma_I)} \frac{\left|\langle f, \phi \rangle_1\right|}{\|\phi\|_{X(D,\partial D_I)}} \quad \text{and} \quad |||f|||_2 := \sup_{\phi \in X_0(B_R \setminus \overline{D}, \Gamma_I)} \frac{\left|\langle f, \phi \rangle_2\right|}{\|\phi\|_{X(B_R \setminus \overline{D}, \partial D_I)}},$$

where for $u \in H_0(\mathrm{curl}, B_R)$ such that $\nu \times u|_{\partial D_I} \in L^2_t(\partial D_I)$ and $f = \nu \times u|_{\partial D_P}$,

$$\langle f, \phi \rangle_1 := \int_D (\mathrm{curl}\, u \cdot \phi - u \cdot \mathrm{curl}\, \phi)\, dv - \int_{\partial D_I} \nu \times u \cdot \phi\, ds, \quad \phi \in X(D, \partial D_I), \tag{3.14}$$

$$\langle f, \phi \rangle_2 := \int_{B_R \setminus D} (\mathrm{curl}\, u \cdot \phi - u \cdot \mathrm{curl}\, \phi)\, dv + \int_{\partial D_I} \nu \times u \cdot \phi\, ds, \quad \phi \in X_0(B_R \setminus \overline{D}, \partial D_I).$$

In particular $Y(\partial D_P)$ is a Hilbert space. The dual space $Y(\partial D_P)'$ of $Y(\partial D_P)$ with respect to the duality pairing defined by (3.14) contains all $\varphi \in Y(\partial D_P)'$ that can be extended to a function $\tilde{\varphi} \in H^{-1/2}(\mathrm{Curl}, \partial D)$ defined on the whole boundary and satisfying $\tilde{\varphi}|_{\partial D_I} \in L^2_t(\partial D_I)$.

For future reference we note that the dual space of $H^{-1/2}(\mathrm{Div}, \partial D)$ is

$$H^{-1/2}(\mathrm{Curl}, \partial D) := \left\{ u \in H_t^{-1/2}(\partial D) : \mathrm{Curl}\, u \in H^{-1/2}(\partial D) \right\}, \tag{3.15}$$

where Curl denotes the surface curl and the dual space $Y(\partial D_P)'$ of $Y(\partial D_P)$ is the set of functions that can be extended to a function $\varphi \in H^{-1/2}(\mathrm{Curl}, \partial D)$ such that $\varphi|_{\partial D_I} \in L^2_t(\partial D_I)$.

We now show that the set of Herglotz wave functions E_g for $g \in L^2_t(\Omega)$ is dense in $\overline{M(D)}$, where

$$M(D) := \left\{ E \in C^2(D) \cap C^1(\overline{D}) : \mathrm{curl}\,\mathrm{curl}\, E = k^2 E \quad \text{in } D \right\}$$

and the closure of $M(D)$ is taken in $H(\mathrm{curl}, D)$, i.e., with respect to the norm (3.8) [51]. To this end, we define the Herglotz operator $\mathbb{H} : L^2_t(\Omega) \to \overline{M(D)}$ by

$$(\mathbb{H}g)(x) := \int_\Omega e^{ikx\cdot d} g(d)\, ds(d), \quad x \in D.$$

Lemma 3.2. *For all $g \in L^2_t(\Omega)$ and $E \in \overline{M(D)}$ we have that*

$$(\mathbb{H}g, E)_{H(\mathrm{curl}, D)} = (g, \mathbb{H}^* E), \tag{3.16}$$

where $\mathbb{H}^* : \overline{M(D)} \to L^2_t(\Omega)$ is given by

$$(\mathbb{H}^* E)(d) := d \times \left\{ (1 + k^2) \int_D e^{-ikx\cdot d} E(x)\, dx - \int_{\partial D} e^{-ikx\cdot d} \nu(x) \times \mathrm{curl}\, E(x)\, ds(x) \right\} \times d, \quad d \in \Omega,$$

with $\nu \times \mathrm{curl}\, E$ to be interpreted as the tangential trace of $\mathrm{curl}\, E \in H(\mathrm{curl}, D)$*, and where the second integral is understood as the duality pairing between* $H^{-1/2}(\mathrm{Div}, \partial D)$ *and* $H^{-1/2}(\mathrm{Curl}, \partial D)$.

Proof. For $g \in L^2_t(\Omega)$ and $E \in M(D)$ from the divergence theorem we have that

$$(\mathbb{H}g, E)_{H(\mathrm{curl},D)} = (\mathbb{H}g, E)_{L^2(D)} + (\mathbb{H}g, \mathrm{curl}\,\mathrm{curl}\,E)_{L^2(D)} - (\mathbb{H}g, \nu \times \mathrm{curl}\,E)_{L^2(\partial D)},$$

and hence (3.16) follows from $\mathrm{curl}\,\mathrm{curl}\,E = k^2 E$ and interchanging the order of integration. The lemma now follows by a denseness argument. ∎

We note that $\mathbb{H}^* E$ coincides with the far field pattern of the combined volume and surface potential defined by

$$V(z) := \frac{1}{k^2}(1+k^2)\,\mathrm{curl}\,\mathrm{curl} \int_D \Phi(z,x)E(x)\,dx - \frac{1}{k^2}\mathrm{curl}\,\mathrm{curl} \int_{\partial D} \Phi(z,x)\nu(x) \times \mathrm{curl}\,E(x)\,ds(x) \tag{3.17}$$

for $z \in \mathbb{R}^3 \setminus \overline{D}$. Using $\mathrm{curl}\,\mathrm{curl} = -\Delta + \mathrm{grad}\,\mathrm{div}$, the divergence theorem, and $\mathrm{Div}[\nu \times \mathrm{Curl}\,E] = -k^2 \nu \cdot E$ for $E \in M(D)$, we have that

$$\mathrm{curl}\,\mathrm{curl} \int_D \Phi(z,x)E(x)\,dx = k^2 \int_D \Phi(z,x)E(x)\,dx - \mathrm{grad} \int_{\partial D} \Phi(z,x)\nu(x)\cdot E(x)\,ds(x)$$

and

$$\mathrm{curl}\,\mathrm{curl} \int_{\partial D} \Phi(z,x)\nu(x) \times \mathrm{curl}\,E(x)\,ds(x) = k^2 \int_{\partial D} \Phi(z,x)\nu(x) \times \mathrm{curl}\,E(x)\,ds(x) - k^2\,\mathrm{grad} \int_{\partial D} \Phi(z,x)\nu(x)\cdot E(x)\,ds(x)$$

for $z \in \mathbb{R}^3 \setminus \overline{D}$. Substituting into (3.17) now gives

$$V(z) = (1+k^2) \int_D \Phi(z,x)E(x)\,dx - \int_{\partial D} \Phi(z,x)\nu(x) \times \mathrm{curl}\,E(x)\,ds(x) - \frac{1}{k^2}\,\mathrm{grad} \int_{\partial D} \Phi(z,x)\nu(x)\cdot E(x)\,ds(x) \tag{3.18}$$

for $z \in \mathbb{R}^3 \setminus \overline{D}$ and $E \in M(D)$. It can be shown [51] that the mapping $E \to V$ is in fact bounded from $H(\mathrm{curl}, D)$ into $H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{D})$.

Lemma 3.3. *For $E \in \overline{M(D)}$ we have that*

$$\|E\|^2_{H(\mathrm{curl},D)} = \int_{\partial D} \left\{ \overline{E}_t \cdot \nu \times \mathrm{curl}\,V - V_t \cdot \nu \times \mathrm{curl}\,\overline{E} \right\} ds,$$

where E_t and V_t denote the tangential components of E and V, respectively, on ∂D and the integral is understood in the same sense of duality pairing.

Proof. By a denseness argument it suffices to establish the above identity for $E \in M(D)$. In this case we can extend the identity (3.18) to $z \in \mathbb{R}^3 \setminus \partial D$ and deduce that

$$\Delta V + k^2 V = -(1+k^2)E \quad \text{and} \quad \operatorname{div} V = 0 \quad \text{in} \quad D, \tag{3.19}$$

$$\nu \times V_+ = \nu \times V_- \quad \text{and} \quad \nu \times \operatorname{curl} V_+ = \nu \times \operatorname{curl} V_- \quad \text{on} \quad \partial D, \tag{3.20}$$

where the subscripts + and − denote the limits obtained by approaching ∂D from $\mathbb{R}^3 \setminus \overline{D}$ and D, respectively. Using (3.19), (3.20), the equation $\operatorname{curl}\operatorname{curl} E = k^2 E$, and the divergence theorem, we have that

$$\begin{aligned}
&\int_D \left\{ |E|^2 + |\operatorname{curl} E|^2 \right\} dx \\
&= \int_D \left\{ |E|^2 + \overline{E} \cdot \operatorname{curl}\operatorname{curl} E \right\} dx + \int_{\partial D} \nu \cdot \overline{E} \times \operatorname{curl} E \, ds \\
&= (1+k^2) \int_D |E|^2 \, dx + \int_{\partial D} \nu \cdot \overline{E} \times \{\operatorname{curl} V_- - \operatorname{curl} V_+\} \, ds \\
&= \int_D \overline{E} \cdot \left\{ \operatorname{curl}\operatorname{curl} V - k^2 V \right\} dx + \int_{\partial D} \nu \cdot \overline{E} \times \{\operatorname{curl} V_- - \operatorname{curl} V_+\} \, ds \\
&= \int_D \left\{ \operatorname{curl}\operatorname{curl} \overline{E} - k^2 \overline{E} \right\} \cdot V \, dx + \int_{\partial D} \nu \cdot \left\{ V_- \times \operatorname{curl} \overline{E} - \overline{E} \times \operatorname{curl} V_+ \right\} ds \\
&= \int_{\partial D} \left\{ \overline{E}_t \cdot \nu \times \operatorname{curl} V_- - V_t \cdot \nu \times \operatorname{curl} \overline{E} \right\} ds,
\end{aligned}$$

and the proof is complete. □

We are now in position to prove our desired approximation theorem [51].

Theorem 3.4. *The set of Herglotz wave functions E_g for $g \in L^2_t(\Omega)$ is dense in $\overline{M(D)}$ with respect to the $H(\operatorname{curl}, D)$ norm.*

Proof. Assume that $E \in \overline{M(D)}$ is such that

$$(\mathbb{H} g, E)_{H(\operatorname{curl}, D)} = 0$$

for all $g \in L^2_t(\Omega)$. Then by Lemma 3.2 we have that

$$(g, \mathbb{H}^* E)_{L^2_t(\Omega)} = 0$$

for all $g \in L^2_t(\Omega)$, and hence $\mathbb{H}^* E = 0$. But $\mathbb{H}^* E$ is the far field pattern of V defined by (3.17), and hence by Rellich's lemma we have that $V = 0$ in $\mathbb{R}^3 \setminus D$. Hence by Lemma 3.3 we have that $E = 0$, and the proof is complete. □

We now turn our attention to showing that Herglotz wave functions can be used to approximate the solution of a certain mixed boundary value problem with respect to the norm in $X(D, \partial D_I)$. In particular, let $f \in Y(\partial D_P)$, $h \in L^2_t(\partial D_I)$, and consider the problem of finding a solution $E \in X(D, \partial D_I)$ of the interior mixed boundary value problem

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \quad \text{in } D, \tag{3.21}$$

$$\nu \times E = f \quad \text{on } \partial D_P, \tag{3.22}$$

$$\nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu = h \quad \text{on } \partial D_I, \tag{3.23}$$

where $\lambda \in C(\partial D_I)$ and $\lambda(x) \geq \lambda_0 > 0$. We then have the following theorem [27] (in [27] λ was assumed to be constant, but all the results remain valid if $\lambda \in C(\partial D_I)$). Note that if $\partial D_P = \emptyset$, then (3.21)–(3.23) is no longer a mixed boundary value problem but rather an interior impedance boundary value problem.

Theorem 3.5. *Assume that $\partial D_I \neq 0$. Then the interior boundary value problem* (3.21)–(3.23) *has a unique solution $E \in X(D, \partial D_I)$ satisfying*

$$\|E\|_{X(D,\partial D_I)} \leq C\left(\|f\|_{Y(\partial D_P)} + \|h\|_{L^2(\partial D_I)}\right)$$

for some positive constant C.

We now define an operator $\mathcal{H} : L^2_t(\Omega) \to Y(\partial D_P) \times L^2_t(\Gamma_I)$ by

$$\mathcal{H} g := \begin{cases} \nu \times E_g & \text{on } \partial D_P, \\ \nu \times \operatorname{curl} E_g - i\lambda \nu \times (E_g \times \nu) & \text{on } \partial D_I, \end{cases}$$

where E_g is a Herglotz wave function with kernel $g \in L^2_t(\Omega)$. By Theorem 3.5 we see that $\mathcal{H}$ is injective provided $\partial D_I \neq \emptyset$.

Theorem 3.6. *Assume that $\partial D_I \neq \emptyset$. Then the range of $\mathcal{H}$ is dense in $Y(\partial D_P) \times L^2_t(\partial D_I)$.*

Proof. By the change of variables $d \to -d$ and replacing $g(-d)$ by $g(d)$, it suffices to consider the operator $\mathcal{H}$ with E_g written as

$$E_g(x) = \int_\Omega e^{-ikx\cdot d} g(d)\, ds(d).$$

Let $H := Y(\partial D_P) \times L^2_t(\partial D_I)$ with dual space $H^* := Y(\partial D_P)' \times L^2_t(\partial D_I)$ in the componentwise duality pairing. The dual operator $\mathcal{H}^\top : H^* \to L^2_t(\Omega)$ of the operator $\mathcal{H}$ is such that for every $(a_1, a_2) \in H^*$ and $g \in L^2_t(\Omega)$ we have that

$$\langle \mathcal{H} g, (a_1, a_2)\rangle_{H,H^*} = \left\langle g, \mathcal{H}^\top(a_1, a_2)\right\rangle_{L^2_t(\Omega), L^2_t(\Omega)}.$$

It suffices to show that the dual operator $\mathcal{H}^\top$ is injective since [92]

$$\overline{(\text{Range}\,\mathcal{H})} = {}^a\,\text{Kern}\,\mathcal{H}^\top,$$

where

$$
{}^{a}\operatorname{Kern}\mathcal{H}^{\top} := \Big\{(p_1,p_2)\in H : \langle (p_1,p_2),(q_1,q_2)\rangle_{H,H^*} = 0 \quad \forall (q_1,q_2)\in \operatorname{Kern}\mathcal{H}^{\top}\Big\}. \tag{3.24}
$$

In particular, the injectivity of $\mathcal{H}^{\top}$ implies that $\overline{(\operatorname{Range}\mathcal{H})} = H$. Simple computations show that $\mathcal{H}^{\top}$ is defined by

$$
\mathcal{H}^{\top}[a_1,a_2] = d\times\Bigg\{\int_{\partial D_P} e^{-ikx\cdot d}(a_1\times\nu)\,ds(x) + ikd\times\int_{\partial D_I} e^{-ikx\cdot d}(a_2\times\nu)\,ds(x) - i\int_{\partial D_I}\lambda e^{-ikx\cdot d}\left[\nu\times(a_2\times\nu)\right]ds(x)\Bigg\}\times d.
$$

We note that $\mathcal{H}^{\top}[a_1,a_2]$ coincides with the far field pattern of the combined electric and magnetic dipole distributions

$$
P(z) = \frac{1}{k^2}\operatorname{curl}\operatorname{curl}\int_{\partial D_P}\Phi(x,z)(a_1\times\nu)\,ds(x) - \operatorname{curl}\int_{\partial D_I}\Phi(x,z)(a_2\times\nu)\,ds(x) - \frac{i}{k^2}\operatorname{curl}\operatorname{curl}\int_{\partial D_I}\lambda\,\Phi(x,z)[\nu\times(a_2\times\nu)]\,ds(x).
$$

The potential P is well defined and satisfies $\operatorname{curl}\operatorname{curl}P - k^2P = 0$ in $\mathbb{R}^3\setminus\partial D$.

Now assume that $\mathcal{H}^{\top}[a_1,a_2]=0$. Then the far field pattern of P is zero, and hence from Rellich's lemma $P=0$ in $\mathbb{R}^3\setminus D$. Since $a_1\in Y(\partial D_P)'$, there is an extension $(\tilde a_1\times\nu)\in H^{-1/2}(\operatorname{Div},\partial D)$ of $a_1\times\nu$ such that $(\tilde a_1\times\nu)\Big|_{\partial D_I}\in L^2_t(\partial D_I)$. This follows from the fact that if $\nu\times u\in H^{-1/2}(\operatorname{Curl},\partial D)$, then $(\nu\times u)\times\nu$ is in $H^{-1/2}(\operatorname{Div},\partial D)$. Hence we can write

$$
\begin{aligned}
P(z) ={}& \frac{1}{k^2}\operatorname{curl}\operatorname{curl}\int_{\partial D}\Phi(x,z)(\tilde a_1\times\nu)\,ds(x) - \frac{1}{k^2}\operatorname{curl}\operatorname{curl}\int_{\partial D_I}\Phi(x,z)(\tilde a_1\times\nu)\,ds(x)\\
&- \operatorname{curl}\int_{\partial D_I}\Phi(x,z)(a_2\times\nu)\,ds(x) - \frac{i}{k^2}\operatorname{curl}\operatorname{curl}\int_{\partial D_I}\lambda\,\Phi(x,z)[\nu\times(a_2\times\nu)]\,ds(x).
\end{aligned}
$$

Furthermore, as $z\to\partial D$ we have that

$$
\nu\times P_+ - \nu\times P_-\big|_{\partial D_P} = 0, \tag{3.25}
$$

$$
\nu\times P_+ - \nu\times P_-\big|_{\partial D_I} = -(a_2\times\nu), \tag{3.26}
$$

$$
\nu\times\operatorname{curl}P_+ - \nu\times\operatorname{curl}P_-\big|_{\partial D_P} = (\tilde a_1\times\nu), \tag{3.27}
$$

$$
\nu\times\operatorname{curl}P_+ - \nu\times\operatorname{curl}P_-\big|_{\partial D_I} = -i\lambda[\nu\times(a_2\times\nu)], \tag{3.28}
$$

where the subscripts again denote the limits obtained by approaching ∂D from $\mathbb{R}^3 \setminus \overline{D}$ and D, respectively. Therefore combining (3.26) and (3.28) and using the fact that $\nu \times P_+ = \nu \times \operatorname{curl} P_+ = 0$ we obtain

$$\nu \times P_- \Big|_{\partial D_P} = 0, \tag{3.29}$$

$$\big[\nu \times \operatorname{curl} P_- + i\lambda \nu \times (P_- \times \nu)\big]\Big|_{\partial D_I} = 0, \tag{3.30}$$

which are understood in the L^2-limit sense (cf. [50, p. 172]). Thus P is such that $\operatorname{curl}\operatorname{curl} P - k^2 P = 0$ in D and satisfies the boundary conditions (3.29) and (3.30). Using the divergence theorem and a parallel surface argument, one can now conclude (cf. [27, Theorem 2.3]) that $P = 0$ in D. From (3.26), (3.27), and the fact that a_1 and a_2 are tangential fields, we can now conclude that $a_1 = a_2 = 0$. Hence $\mathcal{H}^\top$ is injective and the proof is complete. □

Theorems 3.5 and 3.6 now imply the following theorem [27].

Theorem 3.7. *Assume that $\partial D_I \neq \emptyset$. Then the solution $E \in X(D, \partial D_I)$ of* (3.21)–(3.23) *can be approximated by a Herglotz wave function E_g with kernel $g \in L^2(\Omega)$ with respect to the norm in $X(D, \partial D_I)$.*

Note that if $\partial D_I = \emptyset$, then $X(D, \partial D_I) = H(\operatorname{curl}, D)$ and Theorem 3.7 reduces to Theorem 3.4.

3.3 The Linear Sampling Method

Until a few years ago, essentially all algorithms for solving the inverse scattering problem were based on either a weak scattering approximation, such as the Born or physical optics approximation, or the use of nonlinear optimization techniques. Although nonlinear optimization techniques avoid the restrictive modeling assumptions of weak scattering approximations, for many practical applications such approaches require a priori information that may not be available. Hence in recent years alternative methods for imaging have been developed which avoid incorrect model assumptions but, as opposed to nonlinear optimization techniques, seek only limited information about the scattering object and do not rely on any a priori knowledge of the geometry and physical properties of the scatterer. Examples of such approaches are the LSM [22], the factorization method [77], and the method of singular sources [101]. In this section we will present the LSM for solving the inverse electromagnetic obstacle problem [47], [80] (this method was first introduced for the scalar problem in [49] and [54]). The inverse medium problem will be considered in Chapter 4. For an introduction to the LSM for scalar problems we refer to the recent monograph of Cakoni and Colton [22].

We first consider the case when the scattering object D is a perfect conductor, i.e., E, H is a solution of (2.1)–(2.4), and the incident field is a plane wave, i.e., E^i, H^i is given by (2.20). We assume that we know the electric far field pattern $E_\infty(\hat{x}, d, p)$ for all $d, x \in \Omega$ (we will later consider the case of limited aperture data). Then we can define the *far field operator* $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, ds(d), \quad \hat{x} \in \Omega, \tag{3.31}$$

for $g \in L_t^2(\Omega)$. Since the scattered field depends linearly on the polarization of the incident field, F is a linear operator. Note that by superposition, Fg is the electric far field pattern of the scattered field E^s corresponding to the electric field of an electromagnetic Herglotz pair with kernel ikg as incident field; i.e., E^s is the solution of (2.1)–(2.4) with $\nu \times E^s = -(ik)(\nu \times E_g)$ on ∂D, where E_g is the Herglotz wave function with kernel g. In particular it follows from Theorem 2.5 that F is injective if and only if k is not a Maxwell eigenvalue for D.

We now consider the linear first kind integral equation

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q), \tag{3.32}$$

where

$$E_{e,\infty}(\hat{x}, z, q) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x}\, e^{-ik\hat{x}\cdot z}$$

is the far field pattern of an electric dipole with source at z and polarization q (we could also have considered $E_{e,\infty}$ to be the far field pattern of a magnetic dipole). Equation (3.32) is known as the *far field equation*. If $z \in D$, it is seen that if $g = g_z$ is a solution of the far field equation, then by Theorem 2.5 the scattered field E_g^s due to the incident field ikE_g and the electric dipole $E_e(\cdot, z, q)$ coincide in $\mathbb{R}^3 \setminus \overline{D}$. Hence, by the trace theorem, the tangential traces $\nu \times E_g^s = -ik(\nu \times E_g)$ and $\nu \times E_e(\cdot, z, q)$ coincide on ∂D. As $z \in D$ tends to ∂D we have that $\|\nu \times E_e(\cdot, z, q)\|_{H^{-1/2}(\mathrm{Div}, \partial D)} \to \infty$, and hence $\|\nu \times E_g\|_{H^{-1/2}(\mathrm{Div}, \partial D)} \to \infty$ also. Thus $\|g\|_{L_t^2(\Omega)} \to \infty$ and this behavior determines ∂D. Unfortunately, the above argument is only heuristic since it is based on the assumption that g satisfies the far field equation for $z \in D$, and in general the far field equation has no solution for $z \in D$. This follows from the fact that if g satisfies the far field equation, then the Herglotz wave function ikE_g is the solution of the interior boundary value problem

$$\operatorname{curl}\operatorname{curl} E_z - k^2 E_z = 0 \quad \text{in } D, \tag{3.33}$$

$$\nu \times \left[E_z + E_e(\cdot, z, q)\right] = 0 \quad \text{on } \partial D, \tag{3.34}$$

which in general is not possible. However, it follows from Theorem 3.4 that if k is not a Maxwell eigenvalue for D (i.e., k is such that there exists a nontrivial solution of (3.33), (3.34) for E_e set equal to zero), then the unique solution $E_z \in H(\mathrm{curl}, D)$ of (3.33), (3.34) can be approximated arbitrarily closely in $H(\mathrm{curl}, D)$ by a Herglotz wave function. More generally, if k is not a Maxwell eigenvalue, then the well-posedness of the interior problem (3.33), (3.34) with $-\nu \times E_e(\cdot, z, q)$ replaced by an arbitrary tangential vector field $f \in H^{-1/2}(\mathrm{Div}, \partial D)$ implies that for every $\epsilon > 0$ there exists a $g_\epsilon \in L_t^2(\Omega)$ such that

$$\|\nu \times E_{g_\epsilon} - f\|_{H^{-1/2}(\mathrm{Div}, \partial D)} < \epsilon. \tag{3.35}$$

We now examine the far field equation more closely. To this end, we introduce the bounded linear operator $B : H^{-1/2}(\mathrm{Div}, \partial D) \to L_t^2(\Omega)$, which maps a tangential vector field $f \in H^{-1/2}(\mathrm{Div}, \partial D)$ to the far field pattern E_∞ of the radiating solution E^s of $\operatorname{curl}\operatorname{curl} E^s - k^2 E^s = 0$ in $\mathbb{R}^3 \setminus \overline{D}$ satisfying $\nu \times E^s = f$ on ∂D. Then in terms of the far field operator F we have that

$$Fg = -ikB(\nu \times E_g).$$

B is a compact operator since it is the composition of the bounded linear solution operator mapping the boundary data f onto $(\nu \times E^s, \nu \times H^s) \in (H^{-1/2}(\mathrm{Div}, \Omega_R))^2$, where $\Omega_R :=$

$\{x : |x| = R\}$ with the compact operator which takes this data onto the electric far field pattern (we assume that $\overline{D} \subset B_R := \{x : |x| < R\}$).

Lemma 3.8. *The operator $B : H^{-1/2}(\mathrm{Div}, \partial D) \to L^2_t(\Omega)$ is injective with dense range.*

Proof. Injectivity is a consequence of Theorem 2.5 and the uniqueness of the direct scattering problem. To show that B has dense range, we consider the dual operator $B^\top :\to L^2_t(\Omega) \to H^{-1/2}(\mathrm{Curl}, D)$ given by

$$\langle Bf, g\rangle_{L^2_t(\Omega),L^2_t(\Omega)} = \left\langle f, B^\top g\right\rangle_{H^{-1/2}(\mathrm{Div},\partial D),H^{-1/2}(\mathrm{Curl},\partial D)},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the denoted spaces. By changing the order of integration and integrating by parts it can be shown that (cf. [27])

$$\langle Bf, g\rangle_{L^2_t(\Omega),L^2_t(\Omega)} = \frac{1}{4\pi}\int\limits_{\partial D}\left[f\cdot\left(\mathrm{curl}\, E_g - \mathrm{curl}\,\tilde{E}\right)\right] ds, \tag{3.36}$$

where $\tilde{E} \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{D})$ is the solution of

$$\mathrm{curl}\,\mathrm{curl}\, E^s - k^2 E^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D},$$

$$\nu \times \left(E^s - E_g\right) = 0 \quad \text{on } \partial D$$

and the Herglotz wave function E_g is written in the form (making the change of variables as in the proof of Theorem 3.6)

$$E_g(x) := \int\limits_{\Omega} g(d) e^{-ikx\cdot d}\, ds(d).$$

Hence, noting that the integral in (3.36) is interpreted in the sense of duality between $H^{-1/2}(\mathrm{Div}, \partial D)$ and $H^{-1/2}(\mathrm{Curl}, \partial D)$, we have that

$$(B^\top g)(x) = \nu \times \left(\mathrm{curl}\, E_g(x) - \mathrm{curl}\,\tilde{E}(x)\right) \times \nu, \quad x \in \partial D.$$

To show that B has dense range, it suffices to show that $B^\top$ is injective. To this end, $B^\top g = 0$ implies that $\nu \times \mathrm{curl}\, E_g = \nu \times \mathrm{curl}\,\tilde{E}$ on ∂D and by definition we have that $\nu \times E_g = \nu \times \tilde{E}$ on ∂D. Now let $B_R : \{x : |x| < R\}$ be a ball containing $\overline{D}$ in its interior and consider the solution $\hat{E}$, $\hat{H}$ of Maxwell's equation in B_R defined by

$$\hat{E}(x) := \begin{cases} 0, & x \in D, \\ \tilde{E}(x) - E_g(x), & x \in B_R \setminus \overline{D}, \end{cases}$$

$$\hat{H}(x) := \frac{1}{ik}\mathrm{curl}\,\hat{E}(x).$$

Then using the first Stratton–Chu formula (2.10), (2.11) we see that $\hat{E}(x) = 0$ for $x \in B_R$, and, since R was arbitrary, $\hat{E}(x) = 0$ for $x \in \mathbb{R}^3$, i.e., $E_g(x) = E^s(x)$ for $x \in \mathbb{R}^3 \setminus \overline{D}$. By

Theorem 2.2 this is a contradiction unless $E_g(x) = E^s(x) = 0$ for $x \in \mathbb{R}^3 \setminus \overline{D}$. By Theorem 2.1 this implies that $E_g(x) = 0$ for $x \in \mathbb{R}^3$ and hence $g = 0$, i.e., $B^\top$ is injective. □

Lemma 3.9. *$E_{e,\infty}(\hat{x}, z, q)$ is in the range of B if and only if $z \in D$.*

Proof. If $z \in D$, then $B(-\nu \times E_e(\cdot, z, q)) = E_{e,\infty}(\hat{x}, z, q)$. Now let $z \in \mathbb{R}^3 \setminus D$ and assume that there is a tangential vector field $f \in H^{-1/2}(\mathrm{Div}, D)$ such that $Bf = E_{e,\infty}(\cdot, z, q)$. Then by Theorem 2.5 the scattered field E^s corresponding to the boundary data f and the electric dipole $E_e(\cdot, z, q)$ coincide in $\{x : x \in \mathbb{R}^3 \setminus \overline{D}, x \neq z\}$. But this is a contradiction since $E^s \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{D})$ but $E_\infty(\cdot, z, q)$ is not. □

Next we consider the ill-posed equation

$$Bf = \frac{1}{ik} E_{e,\infty}(\cdot, z, q), \quad z \in \mathbb{R}^3. \tag{3.37}$$

As noted in the proof of Lemma 3.9, for $z \in D$ the tangential vector field $f_z := -\frac{1}{ik}\nu \times E_e(\cdot, z, q)$ is the solution to (3.37). In particular, as $z \to \partial D$ we have that $\|f_z\|_{H^{-1/2}(\mathrm{Div}, \partial D)} \to \infty$. If $z \in \mathbb{R}^3 \setminus \overline{D}$, then from Lemmas 3.8 and 3.9 and using Tikhonov regularization we can construct a regularized solution to (3.37). In particular, there exists $f_z := f_z^\alpha$ corresponding to a parameter $\alpha = \alpha(\delta)$ chosen by a regular regularization strategy (e.g., the Morozov discrepancy principle [50]) such that

$$\left\| Bf_z + \frac{1}{ik} E_{e,\infty}(\cdot, z, q) \right\|_{L_t^2(\Omega)} < \gamma\delta$$

for an arbitrary noise level δ and a constant $\gamma \geq 1$ and

$$\lim_{\alpha \to 0} \|f_z^\alpha\|_{H^{-1/2}(\mathrm{Div}, \partial D)} \to \infty.$$

Noting that $\alpha \to 0$ as $\delta \to 0$ and using (3.35) to approximate f_z^α by $\nu \times E_g$ in $H^{-1/2}(\mathrm{Div}, \partial D)$ now yields the following result [18].

Theorem 3.10. *Assume that k is not a Maxwell eigenvalue for D and that F is the far field operator* (3.31) *corresponding to the scattering problem for a perfect conductor; i.e., E_∞ is the electric far field pattern corresponding to* (2.1)–(2.4), (2.20). *Then the following hold:*

(1) *For $z \in D$ and a given $\epsilon > 0$, there exists a $g_z^\epsilon \in L_t^2(\Omega)$ such that*

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \epsilon,$$

and the corresponding Herglotz wave function $ikE_{g_z^\epsilon}$ converges to the solution of (3.33), (3.34) *in $H(\mathrm{curl}, D)$ as $\epsilon \to 0$.*

(2) *For a fixed $\epsilon > 0$ we have that*

$$\lim_{z \to \partial D} \|E_{g_z^\epsilon}\|_{H(\mathrm{curl}, D)} = \infty \quad \textit{and} \quad \lim_{z \to \partial D} \|g_z^\epsilon\|_{L_t^2(\Omega)} = \infty.$$

(3) *For* $z \in \mathbb{R}^3 \setminus \overline{D}$ *and a given* $\epsilon > 0$, *every* $g_z^\epsilon \in L_t^2(\Omega)$ *that satisfies*

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot,z,q)\|_{L_t^2(\Omega)} < \epsilon$$

is such that

$$\lim_{\epsilon \to 0} \|E_{g_z^\epsilon}\|_{H(\text{curl},D)} = \infty \quad \textit{and} \quad \lim_{\epsilon \to 0} \|g_z^\epsilon\|_{L_t^2(\Omega)} = \infty.$$

A similar result can also be obtained for scattering by a partially coated obstacle (i.e., the scattering problem (3.1)–(3.5)) or a fully coated obstacle (i.e., the scattering problem (3.1)–(3.5) with $\partial D_P = \emptyset$). In this case the analysis follows that for a perfect conductor, except that instead of using Theorem 3.4 we now use Theorem 3.7, and the boundary operator B now maps an ordered pair in $Y(\partial D_P) \times L_t^2(\partial D_I)$ onto the electric far field pattern $E_\infty \in L_t^2(\Omega)$. For details see [27], where the following result is proved.

Theorem 3.11. *Assume that* $\partial D_I \neq \emptyset$ *and let* F *be the far field operator* (3.31) *corresponding to the scattering problem for a coated obstacle; i.e.,* E_∞ *is the electric far field pattern corresponding to* (3.1)–(3.5)*. Then the following hold:*

(1) *For* $z \in D$ *and a given* $\epsilon > 0$, *there exists a* $g_z^\epsilon \in L_t^2(\Omega)$ *such that*

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot,z,q)\|_{L_t^2(\Omega)} < \epsilon,$$

and the corresponding Herglotz wave function $ikE_{g_z^\epsilon}$ *converges to the solution of* (3.21)–(3.23) *in* $X(D,\partial D_I)$ *with* $f = -\nu \times E_e(\cdot,z,q)$ *and* $h = -\nu \times \text{curl}\, E_e(\cdot,z,q) + i\lambda(\nu \times E_e(\cdot,z,q)) \times \nu$ *as* $\epsilon \to 0$.

(2) *For a fixed* $\epsilon > 0$ *we have that*

$$\lim_{z \to \partial D} \|E_{g_z^\epsilon}\|_{X(D,\partial D_I)} = \infty \quad \textit{and} \quad \lim_{z \to \partial D} \|g_z^\epsilon\|_{L_t^2(\Omega)} = \infty.$$

(3) *For* $z \in \mathbb{R}^3 \setminus \overline{D}$ *and a given* $\epsilon > 0$, *every* $g_z^\epsilon \in L_t^2(\Omega)$ *that satisfies*

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot,z,q)\|_{L_t^2(\Omega)} < \epsilon$$

is such that

$$\lim_{\epsilon \to 0} \|E_{g_z^\epsilon}\|_{X(D,\partial D_I)} = \infty \quad \textit{and} \quad \lim_{\epsilon \to 0} \|g_z^\epsilon\|_{L_t^2(\Omega)} = \infty.$$

Theorem 3.11 says that the Herglotz wave function $ikE_{g_z^\epsilon}$ is an approximation in $X(D,\partial D_I)$ to the solution E_z of the interior mixed boundary value problem

$$\text{curl}\,\text{curl}\, E_z - k^2 E_z = 0 \quad \text{in } D, \tag{3.38}$$

$$\nu \times \left[E_z + E_e(\cdot,z,q)\right] = 0 \quad \text{on } \partial D_P, \tag{3.39}$$

$$\nu \times \text{curl}\,(E_z + E_e(\cdot,z,q)) - i\lambda \left[\nu \times (E_z + E_e(\cdot,z,q))\right] \times \nu = 0 \quad \text{on } \partial D_I. \tag{3.40}$$

This fact enables us to obtain estimates for λ [21]. In particular, the following theorem connects the surface impedance λ with E_z (and hence with $ikE_{g_z^\epsilon}$).

Theorem 3.12. *Let $z \in D$, $W_z := E_z + E_e(\cdot, z, q)$ and let $u_T := (\nu \times u) \times \nu$ be the tangential component of a function $u \in H(\text{curl}, D)$. Then*

$$\int_{\partial D_I} (W_z)_T \cdot \lambda(\overline{W}_z)_T \, ds = -\frac{k^2}{6\pi}|q|^2 + k\Re(q \cdot E_z).$$

Proof. By applying the second vector Green's formula and using the boundary conditions for E_z on ∂D we obtain

$$2i \int_{\partial D_I} (W_z)_T \cdot \lambda(\overline{W}_z)_T \, ds = \int_{\partial D} \left(\nu \times W_z \cdot \text{curl}\,\overline{W}_z - \nu \times \overline{W}_z \cdot \text{curl}\, W_z\right) ds$$

$$= \int_{\partial D} \left(\nu \times E_e(\cdot,z,q) \cdot \text{curl}\,\overline{E_e(\cdot,z,q)} - \nu \times \overline{E_e(\cdot,z,q)} \cdot \text{curl}\, E_e(\cdot,z,q)\right) ds$$

$$+ \int_{\partial D} \left(\nu \times E_z \cdot \text{curl}\,\overline{E_e(\cdot,z,q)} - \nu \times \overline{E_e(\cdot,z,q)} \cdot \text{curl}\, E_z\right) ds \tag{3.41}$$

$$+ \int_{\partial D} \left(\nu \times E_e(\cdot,z,q) \cdot \text{curl}\,\overline{E}_z - \nu \times \overline{E}_z \cdot \text{curl}\, E_e(\cdot,z,q)\right) ds.$$

It is easily seen that if $E \in H(\text{curl}, D)$ and $H = \frac{1}{ik}\text{curl}\, E$ is a solution of Maxwell's equations and if $z \in D$, then

$$\begin{aligned} \nu \times E_e(y,z,q) \cdot \text{curl}\,\overline{E(y)} &= -\frac{i}{k}(-ik)\,\text{curl}_z \text{curl}_z q\,\Phi(y,z) \cdot \left(\nu \times \overline{H(y)}\right) \\ &= -q \cdot \text{curl}_z \text{curl}_z \Phi(y,z) \left(\nu \times \overline{H(y)}\right) \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} \nu \times \overline{E(y)} \cdot \text{curl}_y E_e(y,z,q) &= ik\nu \times \overline{E(y)} \cdot H_e(y,z,q) \\ &= ikq \cdot \text{curl}_z \Phi(y,z) \left(\nu \times \overline{E(y)}\right). \end{aligned} \tag{3.43}$$

Hence from the first Stratton–Chu formula (2.10), (2.11) we have that

$$\int_{\partial D} \left(\nu \times E_e(y,z,q) \cdot \text{curl}_y \overline{E(y)} - \nu \times \overline{E(y)} \cdot \text{curl}_y E(y,z,q)\right) ds(y) = ikq \cdot \overline{E(z)}. \tag{3.44}$$

One can also easily derive that

$$
\begin{aligned}
\int_{\partial D} & \left(\nu \times E_e(\cdot,z,q)\cdot \operatorname{curl}\overline{E_e(\cdot,z,q)} - \nu \times \overline{E_e(\cdot,z,q)}\cdot \operatorname{curl} E_e(\cdot,z,q)\right) ds \\
&= -2ik \int_{\Omega} E_{e,\infty}(\cdot,z,q)\cdot \overline{E_{e,\infty}(\cdot,z,q)} ds \\
&= -\frac{ik^3}{8\pi^2}\int_{\Omega} |(\hat{x}\times q)\times \hat{x}|^2 ds = -\frac{2ik^3}{6\pi}.
\end{aligned}
\tag{3.45}
$$

The theorem now follows from (3.41)–(3.45). □

From Theorem 3.12 it now follows that

$$
\max_{x\in\partial D} \lambda(x) \geq \frac{-\frac{k^2}{6\pi}|q|^2 + k\Re(q\cdot E_z)}{\|(W_z)_T\|^2_{L^2(\partial D)}} \tag{3.46}
$$

with equality holding for $\lambda(x) = \text{constant}$. Since E_z can be approximated by $ikE_{g_z^\epsilon}$ where g_z^ϵ is given by Theorem 3.11, (3.46) provides a method for estimating $\max \lambda(x)$ from a knowledge of the electric far field pattern. It is also possible to derive a variational formula for the determination of $\max_{x\in\partial D}\lambda(x)$ [21]. In addition,

$$
\int_{\partial D} \lambda |(\nu\times W_z)\times \nu|^2 ds = -\frac{k^2}{6\pi}|q|^2 + k\Re(q\cdot E_z)
$$

for $z \in B_r \subset D$ can be viewed as an integral equation for λ and thus can be used to compute λ and consequently the support of the coating ∂D_I. However, further analysis of the integral operator on the left-hand side is necessary (see [29] for the scalar case). Numerical examples using (3.46) will be given in Section 3.5.

The above results provide a characterization for the boundary ∂D of the scattering obstacle D. Having found D it is then possible in the case of a coated obstacle to obtain an estimate for $\max_{x\in\partial D}\lambda(x)$ (without knowing ∂D_I). Unfortunately, since the behavior of $E_{g_z^\epsilon}$ is described in terms of a norm depending on the unknown region D, $E_{g_z^\epsilon}$ cannot be used to characterize D. Instead the LSM characterizes the obstacle by the behavior of g_z^ϵ. In particular, given a discrepancy $\epsilon > 0$ and g_z^ϵ, the ϵ-approximate solution of the far field equation, the boundary of the scatterer is reconstructed as the set of points z where the $L_t^2(\Omega)$ norm of g_z^ϵ becomes large. An open question is how to obtain numerically the ϵ-approximate solution of the far field equation given by Theorem 3.10 or Theorem 3.11. In all numerical experiments implemented to date, Tikhonov regularization combined with the Morozov discrepancy principle is used to solve the far field equation.

Although all these experiments indicate that this regularized solution behaves the same way as g_z^ϵ given by Theorem 3.10 or Theorem 3.11, in general there is no mathematical justification for this behavior. However, for the case of the Helmholtz equation, Arens and Lechleiter have shown [7], [8] that, in certain cases, applying Tikhonov regularization to the far field equation leads to a solution g that exhibits the desired behavior.

Obviously, in the context of the above discussion, it would be desirable to modify the far field equation in such a way that it has a solution if and only if $z \in D$. This desire motivated Kirsch to introduce the *factorization method* for solving the inverse scattering problem of shape reconstruction [73], [74]. The applicability of the factorization method is still limited to a restricted class of scattering problems. In particular, to date the method has not been established for the case of Maxwell's equations for a perfect conductor, for coated obstacles, or for limited aperture scattering data. On the other hand, when applicable, the factorization method provides a mathematical justification for using the regularized solution of an approximate far field equation to determine D. A complete discussion of the factorization method for solving the inverse scattering problem can be found in the monograph [77].

3.4 Limited Aperture Data

In many cases of practical interest, the electric far field data $E_\infty(\hat{x}, d, p)$ is restricted to the case when d and $\hat{x}$ are on a subset Ω_0 and Ω_1, respectively, of the unit sphere Ω (possibly $\Omega_0 = \Omega_1$). In the case of limited aperture data the far field equation (3.31) takes the form

$$\int_{\Omega_0} E_\infty(\hat{x}, d, g(d))\, ds(d) = E_{e,\infty}(\hat{x}, z, q), \quad \hat{x} \in \Omega_1.$$

In order to handle this case, we note that the function $g_z^\epsilon \in L_t^2(\Omega)$ in Theorem 3.10 or Theorem 3.11 is the kernel of a Herglotz wave function which approximates the solution of (3.33)–(3.34) or (3.38)–(3.40) with respect to an appropriate norm. Hence, as discussed in [19], to treat the case of limited aperture data, it suffices to show that a Herglotz wave function and its first derivatives can be approximated uniformly on compact subsets of a ball B_R of radius R centered at the origin by a Herglotz wave function with kernel supported in a subset of Ω. This new Herglotz wave function and its kernel can now be used in place of $E_{g_z^\epsilon}$ and g_z^ϵ in Theorem 3.10 or Theorem 3.11. Thus, assuming that k is not a Maxwell eigenvalue for B_R (this is not a restriction since we can always find a ball containing D that has this property), it suffices to show that the set of functions $\nu \times E_g$ with $g \in L_t^2(\Omega)$ having support $\Omega_0 \subset \Omega$ for some subset $\Omega_0 \subset \Omega$ is complete in $H^{-1/2}(\mathrm{Div}, \partial B_R)$.

To this end, let $\varphi \in H^{-1/2}(\mathrm{Curl}, \partial B_R)$ and assume that for a fixed $\Omega_0 \subset \Omega$ we have that

$$\int_{\partial B_R} \varphi(x) \left[\int_{\Omega_0} \overline{g(d)} e^{-ikx\cdot d}\, ds(d) \right] ds(x) = 0 \tag{3.47}$$

for every $g \in L_t^2(\Omega_0)$, where the first integral is interpreted in the sense of duality pairing. We want to show that $\varphi = 0$. By interchanging the order of integration we arrive at

$$\int_{\Omega_0} \overline{g(d)} \left[\int_{\partial B_R} \varphi(x) e^{-ikx\cdot d}\, ds(x) \right] ds(d) = 0$$

for every $g \in L_t^2(\Omega_0)$, which implies that

$$\left(d \times \int_{\partial B_R} \varphi(x) e^{-ikx\cdot d}\, ds(x) \right) \times d = 0, \quad d \in \Omega_0. \tag{3.48}$$

The left-hand side of (3.48) coincides with the far field pattern of the surface potential defined by

$$(A\varphi(x))(y) := \frac{1}{k^2} \operatorname{curl}\operatorname{curl} \int_{\partial B_R} \varphi(x)\Phi(x,y)\, ds(x)$$

for $y \in \mathbb{R}^3 \setminus \partial B_R$. It can be shown [17] that $\nu \times A\varphi$ is continuous across the boundary ∂B_R, and A maps $H^{-1/2}(\mathrm{Curl}, B_R)$ into $H_{\mathrm{loc}}(\mathrm{Curl}, \mathbb{R}^3 \setminus \partial B_R)$. Since $A\varphi$ is a radiating solution to $\operatorname{curl}\operatorname{curl} E - k^2 E = 0$ in $\mathbb{R}^3 \setminus \overline{B}_R$, from (3.48) and Theorem 2.5 we have that $A\varphi(y) = 0$ for $y \in \mathbb{R}^3 \setminus \overline{B}_R$. In particular $\nu \times A\varphi = 0$ on ∂B_R in the sense of the trace theorem and, since k is not a Maxwell eigenvalue, $(A\varphi)(y) = 0$ for $y \in B_R$. Finally, applying the jump relation for $\nu \times (\nabla \times (A\varphi))$ across ∂B_R, we obtain that $\varphi = 0$. This completes the proof. □

Note that "backscattering" data coincides with taking $\Omega_0 = -\Omega_1$; i.e., $d \in \Omega_0$ if and only if $-\hat{x} \in \Omega_0$. Numerical examples of the reconstruction of D using limited aperture electric far field data will be given in subsection 3.5.4. As will be seen, the quality of the reconstruction of D deteriorates as the aperture decreases.

3.5 Numerical Examples in Three Dimensions

In this subsection we collect a few numerical examples for the LSM and related coefficient problems in three dimensions. Unfortunately none of our examples use real data. Indeed we adapt the strategy discussed in Section 1.5. We choose a scatterer, predict the far field (or near field) measurements by a suitable forward solver, and then use the LSM to solve the inverse problem. In every case, in order to avoid inverse crimes, extra noise is added as described in equation (1.13).

The computed far field pattern $E_\infty^{\mathrm{comp}}(\hat{x}, d, p)$ is determined for N measurement points $\{\hat{x}_j\}_{j=1}^N$ on the unit sphere (roughly uniformly distributed; see Figure 3.1). These directions also serve as incident directions, and we use two mutually orthogonal polarizations per incident wave, to be detailed shortly.

The method of discretization of the far field equation is from [93]. In particular, an auxiliary vector $\hat{p}$, $|\hat{p}| = 1$, is chosen such that $\hat{p} \times \hat{x}_j \neq 0$ for any j. Then two polarizations are used, $\hat{p}_j^\theta = (\hat{p} \times \hat{x}_j)/|\hat{p} \times \hat{x}_j|$ and $\hat{p}_j^\phi = \hat{p} \times (\hat{x}_j \times \hat{p})/|\hat{p} \times (\hat{x}_j \times \hat{p})|$.

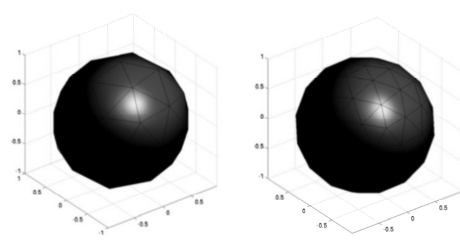


Figure 3.1. *Examples of grids for the unit sphere Ω used in some of the studies in this section. Left: a grid of 42 vertices used for the case of the unit sphere. Right: a grid of 92 points used for the other scatterers.*

The $N \times N$ generalized "matrices" A^θ and A^ϕ are now defined by

$$A^\theta_{i,j} = E^{\text{comp}}_\infty(\hat{x}_i, \hat{x}_j, \hat{p}^\theta_j), \qquad A^\phi_{i,j} = E^{\text{comp}}_\infty(\hat{x}_i, \hat{x}_j, \hat{p}^\phi_j).$$

At each data point $\{\hat{x}_j\}$ on the unit sphere the Herglotz kernel is expanded as

$$g_j(z,q) = g^\theta_j(z,q)\hat{p}^\theta_j + g^\phi_j(z,q)\hat{p}^\phi_j.$$

Then at the discrete level the far field equation (3.32) becomes

$$\sum_{j=1}^{N} \omega_j (g^\theta_j(z,q) A^\theta_{i,j} + g^\phi_j(z,q) A^\phi_{i,j}) = E_{e,\infty}(\hat{x}_i, z, q) \tag{3.49}$$

for $1 \le i \le N$, where the weights $\omega_j > 0$ are chosen to give a consistent quadrature scheme on the unit sphere. The $2N$ unknowns $\{g^\theta_j, g^\phi_j\}$, $1 \le j \le N$, thus satisfy a $2N \times 2N$ system of equations (recall that the entries of the "matrices" $A^\theta_{i,j}$ and $A^\phi_{i,j}$ are themselves tangential vectors). A single linear system can be obtained by taking the dot product of (3.49) with p^θ_i and p^ϕ_i, respectively, for each i. Having obtained a standard linear system, noise is added via (1.13) and, for each z and q, the resulting discrete far field equation is solved by the Morozov/Tikhonov procedure given in Chapter 1. We use a uniform $M \times M \times M$ grid of sampling points uniformly spaced and containing the scatterer. For each z in the sampling grid and for each polarization $q = q_1 = (1,0,0)$, $q_2 = (0,1,0)$, and $q_3 = (0,0,1)$, we can compute an approximation to $\|g(z,q)\|_{L^2(\Omega)}$ by using quadrature at the incident field points (the points $\{\hat{x}_j\}_{j=1}^N$)) and hence compute an approximation to the indicator function

$$G(z) = \frac{1}{3}\left(\|g(z,q_1)\|^{-2}_{L^2(\Omega)} + \|g(z,q_2)\|^{-2}_{L^2(\Omega)} + \|g(z,q_3)\|^{-2}_{L^2(\Omega)}\right)^{1/2}.$$

Isosurfaces of $G(z)$ close to zero give an approximation to the surface of the scatterer, provided the right isosurface value is chosen. By this we mean that we choose C and plot surfaces where $G(z) = C$. As in the case of two-dimensional reconstructions discussed in Section 1.3, the choice of C is still largely an unsolved problem. We refer the reader to the discussion at the end of that section for some references to possible approaches.

A study of G computed using spectral cutoff for a sphere is presented in [43]. In this case, series solutions can be used to obtain explicit formulae for the scattered field and the Herglotz kernel g_z. This study inspired the two-dimensional results presented in subsection 1.5.1, and the qualitative conclusions are similar (although the results in three dimensions are somewhat clearer than those in two dimensions).

3.5.1 A Disconnected Scatterer: Two Balls

Our first example is taken from [52] and is intended to illustrate two advantages of the LSM. First, the method works for one or more scatterers without modification. Second, it is not necessary to know the nature of the scatterer. In Figure 3.2 we show results of reconstructing two penetrable balls, each having a conducting boundary condition with internal parameter $N(x) = 2I$ and conductivity $\eta = 1$ (cf. Section 4.6). The data was computed

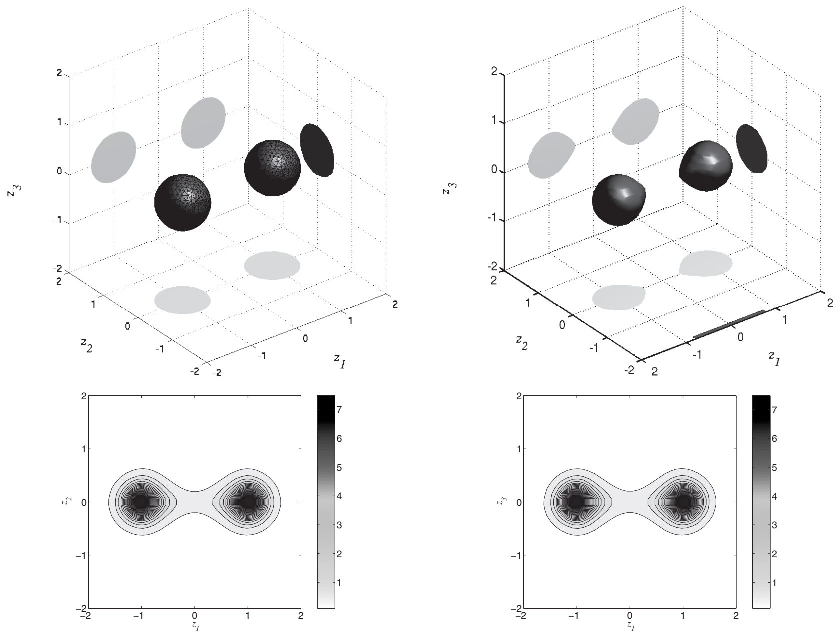


Figure 3.2. *Reconstruction of two balls using the LSM in the case of a conducting boundary condition with a fully coated scatterer and $\eta = 1$. Top left: the exact scatterer. Top right: reconstruction. Bottom left: a contour plot of $G(z)$ in the plane $z_3 = 0$. Bottom right: a contour plot of $G(z)$ in the plane $z_2 = 0$. The solid bar on the top right figure indicates the wavelength of the radiation used. Reprinted from* [52] *with permission.*

using the Ultra Weak Variational Formulation (UWVF) of Maxwell's equations [71] and noise with parameter $\epsilon = 0.01$ was added (see (1.13)). The balls have unit radius and are centered at $(1,0,0)$ and $(-1,0,0)$. The wave number is $k = 6$ (so the wavelength is 1.05) and the conductivity parameter is $\eta = 1$. We use 92 incoming waves and measurements (see Figure 3.1). As can be seen from Figure 3.2, provided a good isosurface of $G(z)$ is chosen, the balls can be reconstructed. The contour plots of $G(z)$ in the same figure show that if the isosurface value is chosen too small, we would predict a dumbbell scatterer. If the value is chosen too large, we predict separated balls that are two small. In Figure 3.3, also from [52], we show a reconstruction of two impenetrable balls, this time both having an impedance boundary condition with impedance (see (3.23)) $\lambda = 1$. No change in the inversion scheme is needed when reconstructing the scatterers in Figure 3.2 or 3.3 despite their having quite different physical properties.

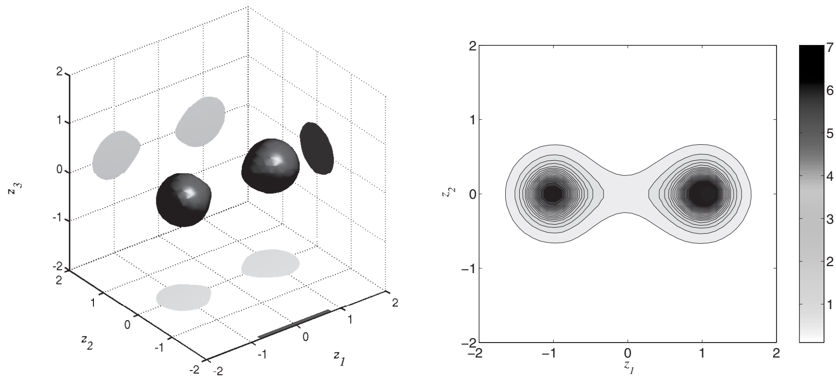


Figure 3.3. *Reconstruction of two balls with impedance boundary conditions where* $\lambda = 1$*. Left: reconstruction. Right: a contour plot of* $1/|\vec{g}_z|$ *in the plane* $z_3 = 0$*. Reprinted from* [52] *with permission.*

3.5.2 The Teapot

Our next example is from [48]. The scatterer is a perfectly conducting teapot[4] shown in Figure 3.4(a). Forward data is computed using an integral equation method via the Electric Field Integral Equation. For the teapot scatterer, the authors use successively $k = 28$, $N = 252$; $k = 56$, $N = 252$; and $k = 96$, $N = 492$ so the wavelength of the radiation ranges from 0.224 down to 0.0654. Results are shown in Figure 3.4. Two comments are in order: (1) the increase of N with k is needed to maintain a good approximation to the far field operator F and is consistent with our discussion of the need to increase N with k in two dimensions (see (1.14)); (2) as k increases (and hence the wavelength decreases) the fidelity of the reconstruction improves. This is to be expected on physical grounds.

The authors of [48] suggest that the roughness observed in the reconstruction at $k = 96$ is due to error in the forward solver. However, we have also observed increased surface roughness whenever k increases (even if care is taken to keep the forward data at a fixed accuracy independent of k).

3.5.3 Impedance Cube

Our next example underlines the need to have sufficiently many measurements to resolve the far field pattern. Here the scatterer is a unit cube with impedance boundary data. The forward problem is approximated by the UWVF [71], and the inverse problem is solved as described at the start of this chapter. We choose the wave number $k = 8$, and λ varies over the surface of the cube. In the left panel of Figure 3.5 we show the reconstruction when the number of incident directions is $N = 42$ (see Figure 3.1) and show in the right

[4]Metal teapots, although poor at keeping tea hot, are quite common in tea drinking regions. The need for the remote detection of a teapot should be obvious to any tea drinker.

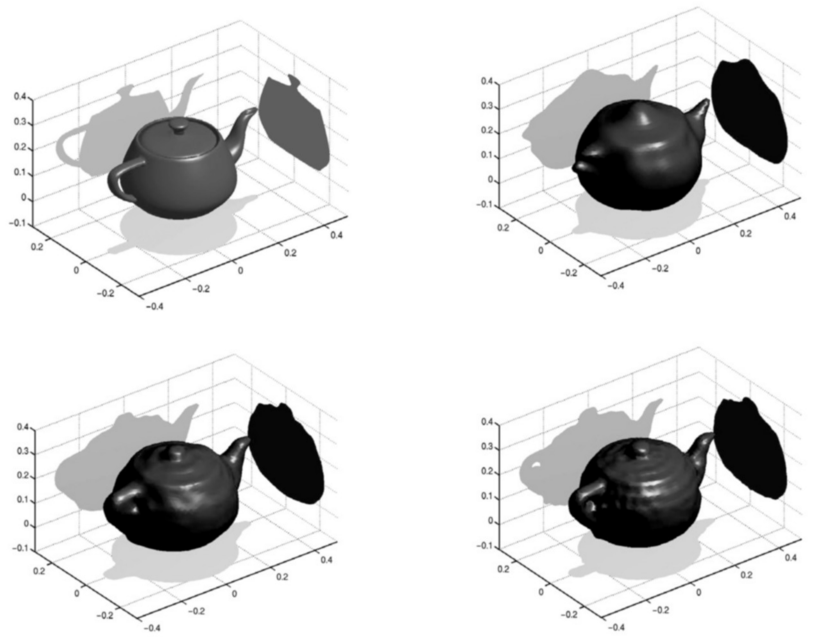


Figure 3.4. *Reconstruction of the teapot. Top left: Original figure. Top right: Reconstruction when* $k = 28$*. Bottom left: Reconstruction when* $k = 56$*. Bottom right: Reconstruction when* $k = 96$*. As expected, smaller features are seen at higher wave numbers. Reprinted from* [48] *with permission.*

panel when $N = 92$. The need for a sufficiently large N is obvious. As yet no estimate for the minimum necessary N like that given in (1.14) has been suggested in the literature, although it is reasonable to conjecture that N should be proportional to $(ka)^2$, where a is a representative radius for the scatterer.

3.5.4 Reconstruction of λ and Limited Aperture

We again present material from [52]. In particular we show how we can reconstruct λ, the impedance parameter, from far field data using equation (3.46) (note that the inequality is an equality if λ is constant). We consider two cases: first, the case when the boundary ∂D also needs to be reconstructed (using the LSM) followed by λ, and second, the case when ∂D is known and only λ needs to be reconstructed. The scatterer is a simple unit sphere, and $k = 3$ with an impedance boundary condition having constant impedance λ. Data is computed for 42 incoming directions and 42 measurement points (see Figure 3.1). In Table 3.1 we show the reconstruction of λ using (3.46) for various choices of λ. Generally, when ∂D is known, the relative error is roughly constant except for $\lambda = 0.1$, whereas when both the LSM (for shape reconstruction) and (3.46) are used, the error is more variable.

Unfortunately, even if the theory appears satisfactory (see Section 3.4), limited aper-

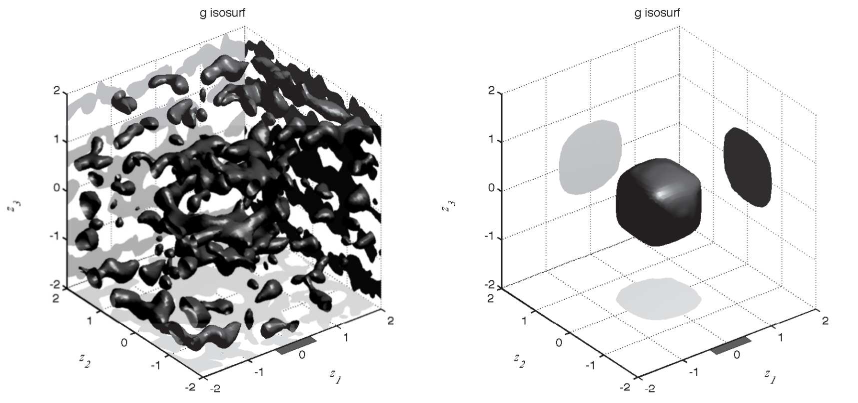


Figure 3.5. *Reconstruction of the unit cube when $k = 8$. An impedance boundary condition with variable impedance is imposed on ∂D. Left panel: results for $N = 42$ incoming waves. Right panel: results for $N = 96$. The left panel shows typical results when too few measurements are used.*

Exact	Exact ∂D	LSM
0.1	0.069	0.072
1	0.96	0.97
1.22	1.17	1.17
2	1.93	1.53

Table 3.1. *Results for the reconstruction of the impedance λ for the unit sphere at $k = 3$. The column "Exact ∂D" shows results using* (5.29) *with the exact boundary ∂D and the column "LSM" shows results using the LSM followed by* (5.29)*. Reproduced from* [52] *with permission.*

ture data can present a severe challenge for the LSM.[5] In Figure 3.6 we show the results of using incident directions chosen to lie in a small spherical cap in Ω. In particular we show results when Ω_0, the set of measurement directions, subtends an angle of 27° in the direction $d = (1,0,0)$ as shown in Figure 3.6(c). The measurement angles (multistatic) are in $\Omega_1 = -\Omega_0$ (i.e., the transmitters and receivers are in the same place—we are using multistatic data in the backscattering direction). We use 39 incoming directions and measurements. The resulting reconstruction of the unit sphere is shown in Figure 3.6(c). We see that in the cross-range direction (i.e., roughly orthogonal to the directions of propagation) the size of the ball is well constructed, but the reconstruction is highly elongated down range. This is typical of limited aperture results, and the elongation increases as the size of the cap decreases. Adding multistatic measurements from a second cap (shown in Figure 3.6(b)) improves the reconstruction on the side of the scatterer closest to the measurement caps (remember that these are opposite the incident direction caps in the figures) as shown in Figure 3.6(d). Contour plots shown in Figures 3.6(e), (f) reinforce these observations.

[5] Alleviating this problem would be an important development in extending the utility of the method.

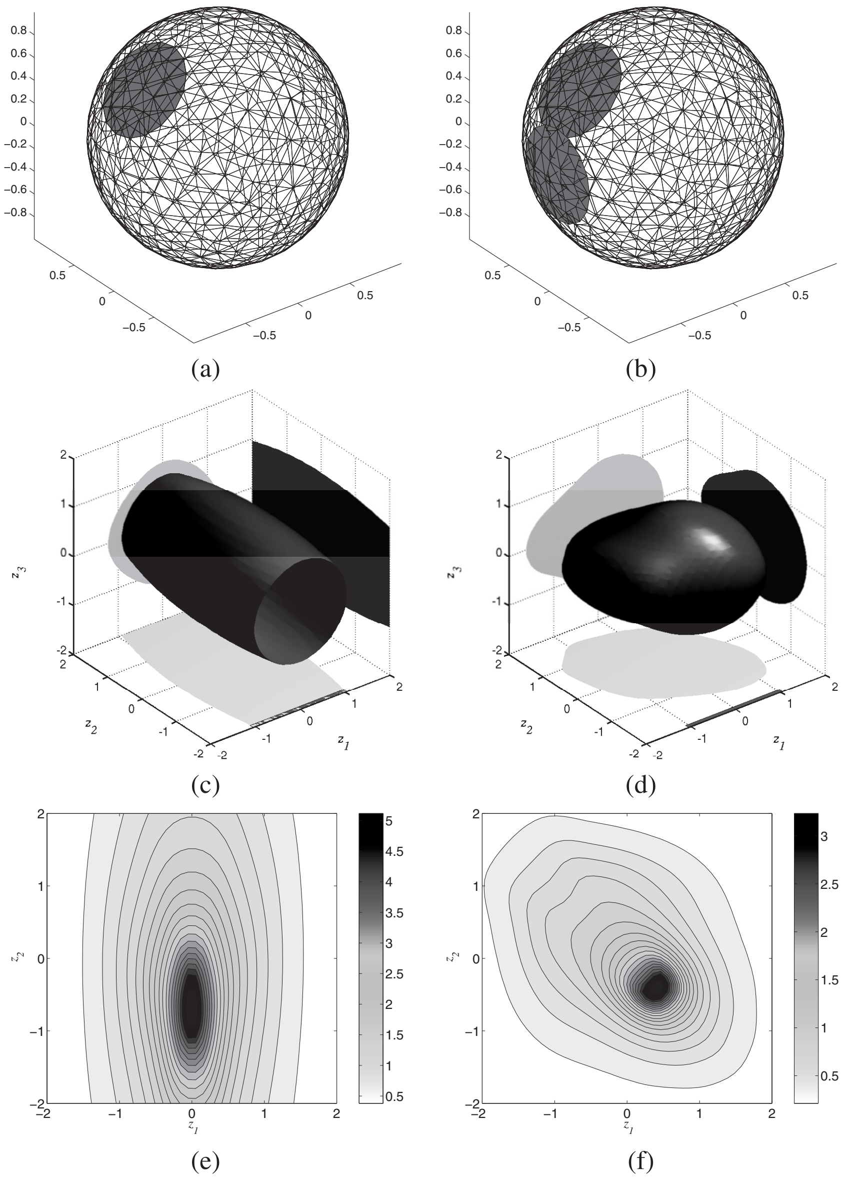


Figure 3.6. *Results of reconstructing the unit sphere using limited aperture data, where the sphere has an impedance boundary condition with* $\lambda = 0.1$. *Top row: the domain* Ω_1. *Middle row: isosurface. Bottom row: a contour plot of* $G(z)$ *in the plane* $z_3 = 0$ *corresponding to the middle row. Reprinted from* [52] *with permission.*

Chapter 4

The Inverse Scattering Problem for Anisotropic Media

In this chapter we consider the inverse scattering problem for a penetrable scatterer. In the context of the LSM, this was first considered by Haddar and Monk [66], who assumed that the coefficient $N(x)$ (see (4.2)) is smooth. This is obviously not generally a practical assumption. So in this chapter we describe more recent results, including uniqueness theory, where $N(x)$ is allowed to be a discontinuous matrix function of position. Obviously, all the results of this chapter hold true in the particular case of isotropic media, i.e., for $N(x) = n(x)I$ where $n(x)$ is a piecewise smooth function in $\overline{D}$. A related optimization method, the dual space method, is considered in [50].

More specifically, we assume that the electric permittivity ϵ and conductivity σ of the medium are real 3×3 matrix valued functions, whereas the magnetic permeability of the medium is a constant μ_0. The positive constants ϵ_0 and μ_0 are the electric permittivity and magnetic permeability of the dielectric background medium (i.e., the conductivity is zero). Let the bounded region $D \subset \mathbb{R}^3$ again be the support of the anisotropic inhomogeneity with piecewise smooth boundary ∂D such that $\mathbb{R}^3 \setminus \overline{D}$ is connected. We denote by $N(x)$ the matrix index of refraction of the anisotropic medium defined by $N(x) = (\epsilon(x) + i\sigma(x)/\omega)/\epsilon_0$ for $x \in \overline{D}$. The scattering of time-harmonic electromagnetic incident fields E^i, H^i by the anisotropic medium leads to the following set of equations for the interior electric and magnetic fields E, H and the scattered electric and magnetic fields E^s, H^s:

$$\left.\begin{aligned} \operatorname{curl} E^s - ikH^s = 0 \\ \operatorname{curl} H^s + ikE^s = 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{4.1}$$

$$\left.\begin{aligned} \operatorname{curl} E - ikH = 0 \\ \operatorname{curl} H + ikN(x)E = 0 \end{aligned}\right\} \quad \text{in } D, \tag{4.2}$$

$$\left.\begin{aligned} \nu \times (E^s + E^i) - \nu \times E = 0 \\ \nu \times (H^s + H^i) - \nu \times H = 0 \end{aligned}\right\} \quad \text{on } \partial D, \tag{4.3}$$

where E^i, H^i are taken to be plane waves given by

$$
\begin{aligned}
E^i(x,d,p) &= \frac{i}{k}\operatorname{curl}\operatorname{curl} p\, e^{ikx\cdot d} \\
&= ik(d\times p)\times d\, e^{ikx\cdot d}, \\
H^i(x,d,p) &= \operatorname{curl} p\, e^{ikx\cdot d}, \\
&= ikd\times p\, e^{ikx\cdot d}
\end{aligned}
\tag{4.4}
$$

and the Silver–Müller radiation condition

$$\lim_{r\to\infty}(H^s\times x - rE^s) = 0. \tag{4.5}$$

To give our assumptions on the data for this problem, we first need to make precise the definition of bounded positive definite matrix fields.

Definition 4.1. *A matrix field K is said to be bounded positive definite on D if $K \in L^\infty(D,C)^{3\times 3}$ and if there exists a constant $\gamma > 0$ such that*

$$\Re(K\xi,\xi) \geq \gamma\,|\xi|^2 \ \ \forall\xi\in C \quad \textit{and a.e. in } D. \tag{4.6}$$

We assume that N and N^{-1} are symmetric positive definite matrix fields whose entries are piecewise smooth functions in $\overline{D}$ and $\bar{\xi}\cdot\Im(N(x))\xi \geq 0$ for all $\xi\in\mathbb{C}^3$ and all $x\in\overline{D}$. Note that in (4.1)–(4.5) the continuity of the tangential component of the electric and magnetic fields is assumed where $N(x)$ is discontinuous. We assume that $N(x)$ is discontinuous across nonintersecting smooth interfaces.

The scattering problem (4.1)–(4.5) can be seen as a particular case of the following transmission problem if we set $f := \nu\times E^i$ and $h := \nu\times H^i$:

$$\left.\begin{aligned}\operatorname{curl} E^s - ikH^s &= 0\\ \operatorname{curl} H^s + ikE^s &= 0\end{aligned}\right\}\quad \text{in } \mathbb{R}^3\setminus\overline{D}, \tag{4.7}$$

$$\left.\begin{aligned}\operatorname{curl} E - ikH &= 0\\ \operatorname{curl} H + ikN(x)E &= 0\end{aligned}\right\}\quad \text{in } D, \tag{4.8}$$

$$\left.\begin{aligned}\nu\times E - \nu\times E^s &= f\\ \nu\times H - \nu\times H^s &= h\end{aligned}\right\}\quad \text{on } \partial D \tag{4.9}$$

together with the Silver–Müller radiation condition (2.3). The following well-posedness of the direct scattering problem is well known (see [78], [93]).

Theorem 4.2. *Given $f,h\in H^{-1/2}(\mathrm{Div},\partial D)$ and the above assumptions on D and N, the transmission problem (4.7)–(4.9) has a unique solution $E^s,H^s\in H_{loc}(\mathrm{curl},\mathbb{R}^3\setminus\overline{D})$ and $E,H\in H(\mathrm{curl},D)$. Moreover, this solution satisfies*

$$
\begin{aligned}
&\|E\|_{H(\mathrm{curl},D)} + \|E^s\|_{H_{loc}(\mathrm{curl},B_R\setminus\overline{D})} + \|H\|_{H(\mathrm{curl},D)} + \|H^s\|_{H_{loc}(\mathrm{curl},B_R\setminus\overline{D})} \\
&\leq C\left(\|f\|_{H^{-1/2}(\mathrm{Div},\partial D)} + \|h\|_{H^{-1/2}(\mathrm{Div},\partial D)}\right)
\end{aligned}
\tag{4.10}
$$

for some positive constant C depending on R but not on f and h.

The inverse problem we consider here is to determine D and some information about N from the measured far field data at a fixed frequency. It is already known (cf. [64]) that in the case of an anisotropic medium the (matrix) index of refraction N is not uniquely determined by the electric far field pattern $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d \in \Omega$ and $p \in \mathbb{R}^3$ even if it is known for an interval of frequencies. However, in the next section we show that the support D of the inhomogeneity can be uniquely determined from a knowledge of $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d \in \Omega$ and $p \in \mathbb{R}^3$ at a fixed frequency. An important ingredient for the uniqueness theorem and for solving the inverse problem is the analysis of the corresponding interior transmission problem which will be discussed in sections 4.1 and 4.2. We then derive the LSM for the reconstruction of D and establish a lower bound for the Euclidean norm of the matrix N that is computable from a knowledge of the far field data. We conclude the chapter by outlining some results on the inverse scattering problem for an anisotropic medium that is partially coated by a very thin layer of highly conductive material.

4.1 Uniqueness Theorems

To prove the uniqueness theorem for the determination of the support D of an anisotropic medium, we follow the approach in [20], which is based on the ideas of [67]. In particular, we first need to study the interior transmission problem corresponding to the scattering problem (4.1)–(4.5). In this case, the interior transmission problem is formulated as the problem of finding functions E, E_0, H, and H_0 defined in D that satisfy

$$\left.\begin{aligned} \operatorname{curl} E - ikH &= 0 \\ \operatorname{curl} H + ikN(x)E &= 0 \end{aligned}\right\} \quad \text{in } D, \tag{4.11}$$

$$\left.\begin{aligned} \operatorname{curl} E_0 - ikH_0 &= 0 \\ \operatorname{curl} H_0 + ikE_0 &= 0 \end{aligned}\right\} \quad \text{in } D, \tag{4.12}$$

$$\left.\begin{aligned} \nu \times E - \nu \times E_0 &= \phi \\ \nu \times H - \nu \times H_0 &= \psi \end{aligned}\right\} \quad \text{on } \partial D. \tag{4.13}$$

The well-posedness of this interior transmission problem is nontrivial and is the subject of the following section. Our goal here is to show that a slightly modified interior transmission problem has a unique solution in $H(\operatorname{curl}, D) \times H(\operatorname{curl}, D)$. It turns out that this modified interior transmission problem provides a fundamental tool in the proof of the uniqueness theorem.

The modified interior transmission problem. Without loss of generality we assume that D is simply connected. This is not a restriction since, as will be seen in the following sections, one need only consider the interior transmission problem in each connected component of D separately. To take advantage of the jumps in the boundary conditions, we formulate the modified interior transmission problem in terms of magnetic fields. To this end let m be a positive number, $F_1, F_2 \in (L^2(D))^3$, and $f, h \in H^{-1/2}(\operatorname{Div}, \partial D)$. We want to find $H_0, H \in H(\text{curl}, D)$ satisfying

$$\text{curl curl } H_0 + H_0 = F_1 \quad \text{in } D, \tag{4.14}$$
$$\text{curl } N^{-1}\text{curl } H + mH = F_2 \quad \text{in } D, \tag{4.15}$$
$$\nu \times \text{curl } H_0 - \nu \times N^{-1} \text{curl } H = f \quad \text{on } \partial D, \tag{4.16}$$
$$\nu \times H_0 - \nu \times H = h \quad \text{on } \partial D. \tag{4.17}$$

Our aim is to reformulate (4.14)–(4.17) as a variational problem. To this end we introduce the sesquilinear form $\mathcal{A}$ defined on $H(\text{curl}, D) \times H(\text{curl}, D)$ by

$$\begin{aligned}\mathcal{A}(H_0, V; \Phi, \Psi) = &\int_D [(\text{curl } H_0) \cdot (\text{curl } \Phi) + H_0 \cdot \Phi] dx + \int_{\partial D} H_{0T} \cdot (\nu \times \Psi) ds \\ &+ \int_D \left[\frac{1}{m}(\text{curl } V) \cdot (\text{curl } \Psi) + nV \cdot \Psi\right] dx + \int_{\partial D} (\nu \times V) \cdot \Phi_T ds,\end{aligned} \tag{4.18}$$

where (H_0, V) and (Φ, Ψ) are in $H(\text{curl}, D) \times H(\text{curl}, D)$ and $U_T := \nu \times (\nu \times U)$. We also introduce the antilinear form

$$L(\Phi, \Psi) = \int_D \left[F_1 \cdot \Phi + \frac{1}{m} F_2 \cdot (\text{curl } \Psi)\right] dx + \int_{\partial D} (h \times \nu) \cdot (\nu \times \Psi) ds - \int_{\partial D} f \cdot \Phi_T ds \tag{4.19}$$

for $(\Phi, \Psi) \in H(\text{curl}, D) \times H(\text{curl}, D)$. Note that the integrals over ∂D are interpreted as the duality between $H^{-1/2}(\text{Div}, \partial D)$ and $H^{-1/2}(\text{Curl}, \partial D)$.

The variational formulation of problem (4.14)–(4.17) is as follows: Find $(H_0, V) \in H(\text{curl}, D) \times H(\text{curl}, D)$ such that

$$\mathcal{A}(H_0, V; \Phi, \Psi) = L(\Phi, \Psi) \qquad \forall (\Phi, \Psi) \in H(\text{curl}, D) \times H(\text{curl}, D). \tag{4.20}$$

The following theorem proves the equivalence of problems (4.16) and (4.20).

Theorem 4.3. (a) *If* (H_0, H) *is a solution to* (4.16)*, then* (H_0, V) *with* $V := N^{-1}(\text{curl } H)$ *is a solution to* (4.20).

(b) *Conversely, if* (H_0, V) *is a solution to* (4.20)*, then* (H_0, H) *with* $H := -\frac{1}{m}(\text{curl } V) + \frac{1}{m} F_2$ *is a solution to* (4.16).

Proof. (a) Let (H_0, H) be a solution of problem (4.16), and set $V := N^{-1}\text{curl } H$. Since $\text{curl } V = F_2 - mH$, then $V \in H(\text{curl}, D)$. Moreover, V satisfies

$$\text{curl curl } V + mN(x)V = \text{curl } F_2 \tag{4.21}$$

interpreted in the sense of distributions. Now taking the L^2 scalar product of the first equation of (4.14) and (4.15) with a function $\Phi \in H(\text{curl}, D)$, integrating by parts, and using the boundary condition (4.16), which now takes the form

$$\nu \times \text{curl } H_0 - \nu \times V = f \quad \text{on } \partial D,$$

we obtain

$$\int_D [(\mathrm{curl}\, H_0)\cdot(\mathrm{curl}\,\Phi) + H_0\cdot\Psi]dx + \int_{\partial D} (\nu\times V)\cdot\Phi_T ds$$
$$= \int_D F_1\cdot\Phi dx - \int_{\partial D} f\cdot\Phi_T ds. \tag{4.22}$$

We now take the L^2 scalar product of (4.21) with a function $\Psi \in H(\mathrm{curl}, D)$ and integrate by parts to obtain

$$\int_D \left[\frac{1}{m}(\mathrm{curl}\, V)\cdot(\mathrm{curl}\,\Phi) + nV\cdot\Psi\right]dx - \frac{1}{m}\int_D F_2\cdot(\mathrm{curl}\,\Psi)dx$$
$$-\frac{1}{m}\int_{\partial D}(\mathrm{curl}\, V)_T\cdot(\nu\times\Psi)ds + \frac{1}{m}\int_{\partial D} F_{2T}\cdot(\nu\times\Psi)ds = 0. \tag{4.23}$$

From the fact that curl $V = F_2 - mH$ and the boundary condition (4.17), we have

$$\frac{1}{m}\int_{\partial D}[-(\mathrm{curl}\, V)_T + F_{2T}]\cdot(\nu\times\Psi)ds = \int_{\partial D} H_{0T}\cdot(\nu\times\Psi)ds - \int_{\partial D}(h\times\nu)\cdot(\nu\times\Psi)ds.$$

Combining (4.24) with (4.23) and using (4.16) we finally obtain

$$\int_D \left[\frac{1}{m}(\mathrm{curl}\, V)\cdot(\mathrm{curl}\,\Psi) + nV\cdot\Psi\right]dx \tag{4.24}$$
$$+\int_{\partial D} H_{0T}\cdot(\nu\times\Psi)ds = \frac{1}{m}\int_D F_2\cdot(\mathrm{curl}\,\Psi)dx + \int_{\partial D}(h\times\nu)\cdot(\nu\times\Psi)ds.$$

Adding (4.22) and (4.24) shows that (H_0, V) is a solution of (4.20).

(b) Let (H_0, V) be a solution of (4.20). It is obvious that H_0 and V satisfy (4.22) and (4.24), respectively. Set $H := -\frac{1}{m}\mathrm{curl}\, V + \frac{F_2}{m}$. By taking sufficiently smooth test functions Ψ in (4.24), one sees that V satisfies

$$\frac{1}{m}\mathrm{curl}\,\mathrm{curl}\, V + N(x)V = \mathrm{curl}\,\frac{F_2}{m} \quad \text{in } D,$$

which means that curl $H + N(x)V = 0$ in D. Therefore H is in $H(\mathrm{curl}, D)$ and satisfies (4.15). Now by taking smooth functions Φ, the variational expression (4.22) yields the first equation of (4.14). It is easy to verify that the boundary conditions (4.16), (4.17) for H_0 and H are also satisfied. This ends the proof. □

Theorem 4.4. *Assume that there exists a constant $\gamma > 1$ such that for $x \in D$*

$$\Re\left(\bar{\xi}\cdot N(x)\xi\right) \ge \gamma|\xi|^2 \quad \forall\, \xi\in\mathbb{C}^3 \quad \textit{and} \quad \frac{1}{m} \ge \gamma. \tag{4.25}$$

Then problem (4.20) *has a unique solution* $(H_0, V) \in H(\text{curl}, D) \times H(\text{curl}, D)$. *This solution satisfies the a priori estimate*

$$\|H_0\|_{H(\text{curl},D)} + \|V\|_{H(\text{curl},D)} \le C\big(\|F_1\|_{L^2(D)} + \|F_2\|_{L^2(D)} + \|f\|_{H^{-1/2}(\text{Div},\partial D)} + \|h\|_{H^{-1/2}(\text{Div},\partial D)}\big), \tag{4.26}$$

where the constant $C > 0$ *is independent of* F_1, F_2, f, h, *and* ∂D.

Proof. Classical trace theorems and Schwarz's inequality ensure the continuity of the sesquilinear form $\mathcal{A}$ and of the antilinear form L on $H(\text{curl}, D) \times H(\text{curl}, D)$ as well as the existence of a positive constant c independent of F_1, F_2, f, and h such that

$$\|L\| \le c\big(\|F_1\|_{L^2} + \|F_2\|_{L^2} + \|f\|_{H^{-1/2}} + \|h\|_{H^{-1/2}}\big). \tag{4.27}$$

Next we take the real part of $\mathcal{A}$ for $(H_0, V) \in H(\text{curl}, D) \times H(\text{curl}, D)$ and use the assumption (4.25) to obtain

$$\Re\big(\mathcal{A}(H_0, V; \overline{H_0}, \overline{V})\big) \ge \gamma\, \|H_0\|^2_{H(\text{curl},D)} + \|V\|^2_{H(\text{curl},D)} + 2\Re\big(\langle \bar{H}_0, V\rangle\big),$$

where $\langle \overline{H}_0, V\rangle$ denotes the duality between $H^{-1/2}(\text{Div}, \partial D)$ and $H^{-1/2}(\text{Curl}, \partial D)$ defined by

$$\langle \overline{H}_0, V\rangle := \int_{\partial D} (\nu \times V) \cdot \overline{H}_{0T}\, ds = \int_D [(\text{curl}\, V) \cdot \overline{H}_0 - (\text{curl}\, \overline{H_0}) \cdot V]\, dx.$$

By Schwarz's inequality we have that

$$\big|\langle \overline{H_0}, V\rangle\big| \le \|H_0\|_{H(\text{curl},D)} \|V\|_{H(\text{curl},D)},$$

and therefore

$$\Re\big(\mathcal{A}(H_0, V; \overline{H_0}, \overline{V})\big) \ge \gamma\, \|H_0\|^2_{H(\text{curl},D)} + \|V\|^2_{H(\text{curl},D)} - 2\, \|H_0\|_{H(\text{curl},D)} \|V\|_{H(\text{curl},D)}.$$

Using the identity $\gamma x^2 + y^2 - 2xy = \frac{\gamma+1}{2}(x - \frac{2}{\gamma+1}\, y)^2 + \frac{\gamma-1}{2} x^2 + \frac{\gamma-1}{\gamma+1} y^2$, we conclude that

$$\Re\big(\mathcal{A}(H_0, V; \overline{H_0}, \overline{V})\big) \ge \frac{\gamma - 1}{\gamma + 1}\Big(\|H_0\|^2_{H(\text{curl},D)} + \|V\|^2_{H(\text{curl},D)}\Big).$$

Now taking the imaginary part of $\mathcal{A}$ and using the fact that $\Im(N) \ge 0$ implies that there exists a positive constant c such that

$$\Im\big(\mathcal{A}(H_0, V; \overline{H_0}, \overline{V})\big) \ge c\|\nu \times V\|_{L^2(\partial D_2)}.$$

Hence we have that

$$\big|\mathcal{A}(H_0, V; \overline{H_0}, \overline{V})\big| \ge C_1\Big(\|H_0\|^2_{H(\text{curl},D)} + \|V\|^2_{H(\text{curl},D)}\Big)$$

for some $C_1 > 0$, and thus $\mathcal{A}$ is coercive. The unique determination of (H, V) and the a priori estimate are therefore a direct consequence of the Lax–Milgram lemma applied to $\mathcal{A}$ in $H(\text{curl}, D) \times H(\text{curl}, D)$ and (4.27). This proves the theorem. $\square$

Theorem 4.5. *Under the assumptions of Theorem* 4.4*, problem* (4.16) *has a unique solution* $(H_0, H) \in H(\text{curl}, D) \times H(\text{curl}, D)$*. This solution satisfies the a priori estimate*

$$\|H_0\|_{H(\text{curl},D)} + \|H\|_{H(\text{curl},D)}$$
$$\leq 2C \left(\|F_1\|_{L^2(D)} + \|F_2\|_{L^2(D)} + \|f\|_{H^{-1/2}(\text{Div},\partial D)} + \|h\|_{H^{-1/2}(\text{Div},\partial D)}\right),$$

where the constant $C > 0$ *is independent of* F_1*,* F_2*,* f*,* h*, and* ∂D*.*

Proof. It only remains to prove that the uniqueness of the variational problem (4.20) implies the uniqueness of the modified interior transmission problem (4.16). Then the theorem is a consequence of Theorems 4.3 and 4.4.

Consider two solutions (H_{01}, H_1) and (H_{02}, H_2) to (4.16). Then from Theorem 4.3, $(H_{01}, N\text{curl } H_1)$ and $(H_{02}, N\text{curl } H_2)$ are two solutions to (4.20), whence $H_{01} = H_{02}$ and $N\text{curl } H_1 = N\text{curl } H_2$. Since N^{-1} is bounded and D is simply connected, the latter means that there exists a function $P \in H^1(D)$, uniquely determined up to a real constant, such that $H_1 - H_2 = \text{curl } P$. Equation (4.15) yields $m\,\text{curl } P = 0$ in D and hence $H_1 = H_2$. □

The extra condition $\Re\left(\bar{\xi} \cdot N(x)\xi\right) \geq \gamma|\xi|^2$ for some $\gamma > 1$, $x \in D$, and $\xi \in \mathbb{C}^3$ is not an essential restriction. In particular, it is possible to prove that if $\Re\left(\bar{\xi} \cdot N^{-1}(x)\xi\right) \geq \gamma|\xi|^2$ for some $\gamma > 1$, then there exists a unique solution of the modified transmission problem (4.16). In this case one writes a variational formulation for $V := \text{curl } H_0$ and H in $H(\text{curl}, D)$ and $H(\text{curl}, D)$, respectively, and follows a similar procedure as above (see [34] for the corresponding scalar case). Note also that, since $N(x)$ is a symmetric matrix, $\Re\left(\bar{\xi} \cdot N(x)\xi\right) = \bar{\xi} \cdot \Re(N(x))\xi$.

Unique determination of the support. We now turn our attention to proving a uniqueness result for the determination of the support of an anisotropic inhomogeneity. As we remarked at the beginning of this chapter, this is the best that one can do in this direction since N is not uniquely determined. Throughout this section, we will assume that the matrix of the index of refraction N has $C^1(\overline{D})$ entries and that the boundary ∂D is smooth, assumptions which are needed for technical reasons only (see Remark 4.1). To this end, we consider a slightly different inverse problem; namely, given the scattered fields $E^s|_{S_R}$ and $H^s|_{S_R}$ on a large sphere S_R of radius R surrounding $\overline{D}$ for all incident plane waves $E^i(x) = \frac{i}{k}\text{curl}\,\text{curl}\, p\, e^{ikx\cdot d}$ and $H^i(x) := \text{curl}\, p\, e^{ikx\cdot d}$, $x \in \mathbb{R}^3$, with polarization $p \in \mathbb{R}^3$ and incident direction $d \in \Omega$, find the support D of N (note that H can be computed from E and conversely). Then the main result of this section states that D can be uniquely determined by the above data. Note that from Rellich's lemma the scattered fields $E^s|_{S_R}$ and $H^s|_{S_R}$ on S_R can be uniquely found from the electric far field pattern and conversely [50]. Hence the result proved here is equivalent to the result of the unique determination of D from the knowledge of $E_\infty(\hat{x}, d, p_i)$ for $\hat{x}, d \in \Omega$ and $p \in \mathbb{R}^3$. As usual, we assume that the frequency is fixed.

We need the following regularity result for the solution of the transmission problem (4.7)–(4.9) (see [20] for the proof). Let $H_t^{1/2}(\partial D)$ denote the space of tangential vector valued functions in $H^{1/2}(\partial D)$.

Lemma 4.6. *Suppose that* $\mathrm{Div}_{\partial D} f \in H^{1/2}(\partial D)$, $h \in H_t^{1/2}(\partial D)$, *and that* D *and* N *satisfy the assumptions outlined at the beginning of this chapter and, in addition,* N *has* $C^1(\overline{D})$ *entries and* ∂D *is smooth. Then the magnetic fields of the solution to the transmission problem* (4.7)–(4.9) *satisfy* $H \in (H^1(D))^3$ *and* $H^s \in (H^1_{loc}(D_e))^3$. *The following norm estimate holds:*

$$\|H\|_{H^1(D)} + \|H^s\|_{H^1(B_R\setminus\overline{D})} \le C\big(\|H\|_{H(\mathrm{curl},D)} + \|H^s\|_{H(\mathrm{curl},B_R\setminus\overline{D})} + \|h\|_{H_t^{1/2}(\partial D)} + \|\mathrm{Div}_{\partial D} f\|_{H^{1/2}(\partial D)}\big) \quad (4.28)$$

with C *a positive constant depending on* R *but not on* H, f, *and* h.

The same type of regularity can also be obtained for the electric fields E, E^s.

Lemma 4.7. *Suppose that* $f \in H_t^{1/2}(\partial D)$, $\mathrm{Div}_{\partial D} h \in H^{1/2}(\partial D)$, *and that* D *and* N *satisfy the assumptions outlined at the beginning of this chapter, and, in addition,* N *has* $C^1(\overline{D})$ *entries and* ∂D *is smooth. Then the electric fields of the solution to the transmission problem* (4.7)–(4.9) *satisfy* $E \in (H^1(D))^3$ *and* $E^s \in (H^1_{loc}(D_e))$. *The following norm estimate holds:*

$$\|E\|_{H^1(D)} + \|E^s\|_{H^1(B_R\setminus\overline{D})} \le C\Big(\|E\|_{H(\mathrm{curl},D)} + \|E^s\|_{H(\mathrm{curl},B_R\setminus\overline{D})} + \|f\|_{H_t^{1/2}(\partial D)} + \|\mathrm{Div}_{\partial D} h\|_{H^{1/2}(\partial D)}\Big) \quad (4.29)$$

with C *a positive constant depending on* R *but not on* E, f, *and* h.

Theorem 4.8. *Let the domains* D_1 *and* D_2 *with the boundaries* ∂D_1 *and* ∂D_2 *and the index of refraction* N_1 *and* N_2, *respectively, satisfy the assumptions outlined at the beginning of this chapter. In addition, we assume that either* $\bar{\xi}\cdot\Re(N_1)\xi \ge \gamma|\xi|^2$ *or* $\bar{\xi}\cdot\Re(N_1^{-1})\xi \ge \gamma|\xi|^2$, *and either* $\bar{\xi}\cdot\Re(N_2)\xi \ge \gamma|\xi|^2$ *or* $\bar{\xi}\cdot\Re(N_2^{-1})\xi \ge \gamma|\xi|^2$ *for some* $\gamma > 1$. *If the scattered fields* (E_1, H_1) *corresponding to the data* D_1, N_1 *and* (E_2, H_2) *corresponding to the data* D_2, N_2 *coincide on a large sphere* S_R *of radius* R *for all incident plane waves with arbitrary direction* d *and polarization* p, *then* $D_1 \equiv D_2$.

Proof. We consider the electromagnetic field generated by an electric dipole located at z given by

$$E_e^i(x;z,p) = \frac{i}{k}\mathrm{curl}_x\mathrm{curl}_x\, p\,\Phi(x,z),$$
$$H_e^i(x;z,p) = \mathrm{curl}_x\, p\,\Phi(x,z), \quad (4.30)$$

where $\Phi(x,z)$ is the fundamental solution to the Helmholtz equation given by (2.9).

Let G denote the unbounded connected component of $\mathbb{R}^3 \setminus (\overline{D}_1 \cap \overline{D}_2)$ and $E_e^{s,1(2)}(\cdot, z, p)$ and $H_e^{s,1(2)}(\cdot, z, p)$ be the scattered fields corresponding to $D_{1(2)}$ and the incident field $E^i := E_e^i(x;z,p)$ and $H^i := H_e^i(x;z,p)$. Since the scattered fields coincide on S_R for all plane waves, then from the mixed reciprocity relation (3.6), Theorem 2.5, and the well-posedness of the transmission problem we have that the scattered fields $E_e^{s,1}(\cdot,z,p), H_e^{s,1}(\cdot,z,p)$ and $E_e^{s,2}(\cdot,z,p), H_e^{s,2}(\cdot,z,p)$ coincide in S_R for all z.

Now let us assume that D_1 is not included in D_2. Then there exists a point z such that $z \in \partial D_1$ and $z \notin \partial D_2$. In particular, we have that the points $z_n = z + \frac{\epsilon}{n}\nu(z)$ lie in G for all natural numbers n and ϵ sufficiently small, where $\nu(z)$ is the unit outward normal vector to ∂D_1 at z. Due to the singular behavior of $\Phi(x,z)$, it is obvious that $\|H_e^i(\cdot, z_n, \nu(z))\|_{H(\mathrm{curl},D_1)} \to \infty$ as $n \to \infty$, where $H_e^i(\cdot, z_n, \nu(z))$ is the magnetic field of the electric dipole (4.30) with polarization $\nu(z)$. We now consider the incident fields

$$H_n(x) = \frac{H_e^i(x\,; z_n, \nu(z))}{\|H_e^i(\cdot\,; z_n, \nu(z))\|_{H(\mathrm{curl},D_1)}},$$
$$E_n(x) = -\frac{1}{ik}\mathrm{curl}\; H_n(x) \tag{4.31}$$

for $x \in \overline{D}_1 \cup \overline{D}_2$ and denote by $E_n^{j,s}, H_n^{j,s}$ and $E_n^{j,int}, H_n^{j,int}$ the corresponding solutions of (4.1)–(4.5) for the domains $D_j,\ j = 1,2$. Note that $E_n(x)$ and $H_n(x)$ are uniformly bounded in $H(\mathrm{curl}, D_1)$. The trace theorem for $H(\mathrm{curl})$ and Theorem 4.2 then show that the corresponding scattered fields and interior fields are uniformly bounded in their respective norms. For later use we need to show that the sequence H_n is uniformly bounded in $H^1(D_1)$. To this end, with the help of a cutoff function χ supported in $B_{2\epsilon}(z)$ and $\chi = 1$ in $B_\epsilon(z)$, we first write

$$\|(1-\chi)H_n + \chi H_n\|_{H^1(D_1)} \le C + \frac{\|H_e^i(\cdot\,; z_n, \nu(z))\|_{H^1(D_1\cap B_\epsilon(z))}}{\|H_e^i(\cdot\,; z_n, \nu(z))\|_{H(\mathrm{curl},D_1\cap B_\epsilon(z))}}. \tag{4.32}$$

Simple computations show that

$$\|H_e^i(\cdot\,; z_n, \nu(z))\|^2_{H^1(D_1\cap B_\epsilon(z))} = \|\nabla_x \Phi(x,z_n) \times \nu(z)\|^2_{L^2(D_1\cap B_\epsilon(z))}$$
$$+ \|\nabla_x \nabla_x \Phi(x,z_n) \times \nu(z)\|^2_{L^2(D_1\cap B_\epsilon(z))} = \frac{1}{|z_n - z|^3}\,[A_1 + O(|z_n - z|)]$$

and

$$\|H_e^i(\cdot\,; z_n, \nu(z))\|^2_{H(\mathrm{curl},D_1\cap B_\epsilon(z))} = \|\nabla_x \Phi(x,z_n) \times \nu(z)\|^2_{L^2(D_1\cap B_\epsilon(z))}$$
$$+ \|k^2\Phi(x,z_n)\nu(z) + \nabla_x \nabla_x \Phi(x,z_n) \cdot \nu(z)\|^2_{L^2(D_1\cap B_\epsilon(z))} = \frac{1}{|z_n - z|^3}\,[A_2 + O(|z_n - z|)].$$

Furthermore a straightforward but long computation shows that

$$A_2 = 2\pi \int_{\pi/2}^{\pi} \int_0^{\infty} \frac{t^2(3\cos^2\theta + 1)\sin\theta}{(t^2 + 1 - 2t\cos\theta)^3}\,dt\,d\theta > 0,$$

whence (4.32) is uniformly bounded for $n \in N$.

Now let $B_\epsilon(z)$ be a ball of radius $\epsilon > 0$ centered at z. Since $E_e(\cdot, z_n, \nu(z))$ and $H_e(\cdot, z_n, \nu(z))$ together with their derivatives are uniformly bounded in every compact subset of $\mathbb{R}^3 \setminus B_{2\epsilon}(z)$, from the estimates (4.10), (4.28), (4.29) applied to the scattered field corresponding to D_2, we have that

$$\lim_{n\to\infty} \|E_n\|_{H^1(D_2)} = \lim_{n\to\infty} \|H_n\|_{H^1(D_2)} = 0,$$

whence

$$\lim_{n\to\infty} \|E_n^{2,s}\|_{H^1(B_R\cap G)} = \lim_{n\to\infty} \|H_n^{2,s}\|_{H^1(B_R\cap G)} = 0.$$

But $\hat{x}\cdot H_n^{1,s}|_{S_R} = \hat{x}\cdot H_n^{2,s}|_{S_R}$, and therefore by the uniqueness of the exterior Maxwell problem outside B_R and unique continuation (cf. Theorem 2.1) we conclude that

$$\lim_{n\to\infty} \|E_n^{1,s}\|_{H^1(B_R\cap G)} = \lim_{n\to\infty} \|E_n^{2,s}\|_{H^1(B_R\cap G)} = 0,$$
$$\lim_{n\to\infty} \|H_n^{1,s}\|_{H^1(B_R\cap G)} = \lim_{n\to\infty} \|H_n^{2,s}\|_{H^1(B_R\cap G)} = 0.$$

Hence from trace theorems and with the help of a cutoff function $\chi \in C_0^\infty(B_{\epsilon'}(z))$, where $\epsilon' > 0$ is small enough to ensure that $B_{\epsilon'}(z)\cap D_1 = B_{\epsilon'}(z)\cap Z$, we conclude that

$$\begin{aligned}&\lim_{n\to\infty} \|\nu\times\chi H_n^{1,s}\|_{H^{-1/2}(\mathrm{Div},\partial D_1)}\\ &\quad= \lim_{n\to\infty} \|\nu\times(\mathrm{curl}\,\chi H_n^{1,s})\|_{H^{-1/2}(\mathrm{Div},\partial D_1)} = 0.\end{aligned} \tag{4.33}$$

Let D_{je} denote the exterior of D_j, $j = 1,2$. In the exterior of $B_{2\epsilon}(z)$ the $H^1(B_R\setminus B_{2\epsilon}(z))$ norm of E_n and H_n remain uniformly bounded, and therefore from regularity results $(1-\chi)E_n^{1,s}, (1-\chi)H_n^{1,s}$ are also uniformly bounded in $H^1((B_R\cap D_{1e})\setminus B_{2\epsilon}(z))$. Using the compact embedding of $H^1(B_R\cap D_{1e})$ in $H^{(1-\tau)}(B_R\cap D_{1e})$ for $0<\tau\le 1$, we can now select an $H(\mathrm{curl}, B_R\cap D_{1e})$ convergent subsequence $(1-\chi)H_{n_j}^{1,s}$ from $(1-\chi)H_n^{1,s}$. Hence, $\nu\times(1-\chi)H_{n_j}^{1,s}$ and $\nu\times[\mathrm{curl}\,(1-\chi)H_{n_j}^{1,s}]$ are convergent in $H^{-1/2}(\mathrm{Div},\partial D_1)$ as well. Combining this fact with (4.33) we have that the sequences

$$\nu\times H_{n_j}^{1,s} \text{ and } \nu\times\mathrm{curl}\, H_{n_j}^{1,s}$$

converge in the trace space $H^{-1/2}(\mathrm{Div},\partial D_1)$.

Estimate (4.28) shows that $H_n^{1,int}$ is uniformly bounded in $H^1(D_1)$ because H_n is uniformly bounded in $H^1(D_1)$, and consequently the boundary data is uniformly bounded in the required trace spaces. Obviously, H_{n_j} and $H_{n_j}^{1,int}$ in D_1 solve the modified interior transmission problem (4.16) with $F_1 := H_{n_j}$, $F_2 := H_{n_j}^{1,int}$, and boundary data $f := \nu\times(\mathrm{curl}\,\times H_{n_j}^{1,s})$, $h := \nu\times H_{n_j}^{1,s}$. By using the compact embedding of $H^1(D_1)$ in $L^2(D_1)$ we can select from H_{n_j} and $H_{n_j}^{1,s}$ convergent subsequences in $L^2(D_1)$, which we again denote by H_{n_j} and $H_{n_j}^{1,s}$. Theorem 4.5 now gives that H_{n_j} converges with respect to the norm $H(\mathrm{curl}, D_1)$ to a function $H_0\in H(\mathrm{curl}, D_1)$. Obviously H_0 satisfies curl curl $H_0 - k^2H_0 = 0$ in the weak sense. But $H_0|_{D_1\setminus B_{2\epsilon}(z)} = 0$ since the function H_{n_j} converges uniformly to zero outside the ball $B_{2\epsilon}$. Therefore $H_0 = 0$ in all of D_1. But this contradicts the fact that $\|H_n\|_{H(\mathrm{curl},D_1)} = 1$ for all $n\in N$.

Since one can derive the same contradiction for the assumption that D_2 is not included in D_1, we have proved that $D_1 = D_2$. □

Remark 4.1. The assumption that the matrix index of refraction has entries in $C^1(\overline{D})$ can be relaxed to the assumption that the entries are piecewise smooth, as we have assumed throughout this book. In this case, the regularity results stated in Lemmas 4.6 and 4.7 do not hold any longer. However, for a piecewise smooth index of refraction it is possible to obtain [55] H^s, $s > 0$, regularity for the electric and magnetic fields, which suffices to carry through the proof of Theorem 4.8.

For a general anisotropic medium we cannot hope to determine ϵ and μ. However, if $N(x) = n(x)I$, where $n(x)$ is a smooth scalar function of position, it is possible to prove the uniqueness of $n(x)$ as we shall now show using the arguments of [53]. In particular, in the remainder of this section we shall suppose that $m(x) := 1 - n(x) \in C_0^3(\mathbb{R}^3)$. We begin with a few observations. Due to the assumption on the regularity of m, we have that $E, H \in C^2(\mathbb{R}^3)$ and that the electric field E satisfies the vector Helmholtz equation

$$\operatorname{curl}\operatorname{curl} E - k^2 n E = 0, \tag{4.34}$$

where

$$E = E^i + E^s \text{ together with } \operatorname{div} nE = 0 \tag{4.35}$$

in $\mathbb{R}^3$ and E^s satisfies the Silver–Müller radiation condition. Furthermore, let S denote the set

$$S = \{E(x, d_j, p_i) : E \text{ is a solution of (4.34)–(4.35)}\},$$

where k is fixed, $\{d_j\}$ is a countable dense set in Ω, and p_i, $i = 1,2,3$, are three linearly independent polarizations. Then if X denotes the closure in $L^2(B_R)$ of the set of all solutions to (4.34), (4.35) in $B_R := \{x : |x| < R\}$, S is complete in X [68]. (For the scalar case see Lemma 10.4 of [50]). We can now prove the following basic result [53].

Theorem 4.9. *Suppose the solutions of* (4.34)–(4.35) *corresponding to the refractive indices* $\sqrt{n_1}$ *and* $\sqrt{n_2}$ *have the same electric far field patterns for three linearly independent polarizations* p *and all* $d \in \Omega$. *Let* B_R *be a ball containing the supports of* $m_1 = 1 - n_1$ *and* $m_2 = 1 - n_2$, *and let* B *be a ball in* $\mathbb{R}^3$ *such that* $\overline{B}_R \subset B$. *Then if* $E_j \in C^2(B)$ *is any solution of* (4.34), (4.35) *in* B *with* $n = n_j$, $j = 1,2$, *we have that*

$$\int_{B_R} E_1(x) \cdot (n_1(x) - n_2(x))\, E_2(x)\, dx = 0.$$

Proof. Let E_1 and E_2 be the solutions of the scattering problem (4.34)–(4.35) corresponding to n_1 and n_2, respectively, which have the same electric far field patterns for three linearly independent polarizations p and all $d \in \Omega$. Then by Theorem 2.5 we have that $E_1(x) = E_2(x)$ for $x \in \mathbb{R}^3 \setminus B_R$. Let $E = E_1 - E_2$, and let $\mathcal{L}_j$ denote the differential operator defined by (4.34) with $n = n_j$, $j = 1,2$. Then, since

$$\mathcal{L}_2(E) = \mathcal{L}_2(E_1) = k^2(n_1 - n_2)E_1 \tag{4.36}$$

and $E(x) = 0$ for $x \in \mathbb{R}^3 \setminus B_R$, we have from Green's theorem that

$$\int_{B_R} E_2 \cdot \mathcal{L}_2(E)\, dx = \int_{B_R} \mathcal{L}_2(E_2) \cdot E\, dx = 0 \tag{4.37}$$

for all solutions E_2 of $\mathscr{L}_2(E_2) = 0$ in B_R. But (4.36) and (4.37) now imply that

$$\int_{B_R} E_2(x) \cdot (n_1(x) - n_2(x)) \cdot E_1(x)\,dx = 0,$$

and the theorem follows from the fact that S is complete in X. □

Our next step is to construct a solution of (4.34), (4.35) of the form

$$E(x) = e^{i\zeta\cdot x}(\eta + R_\zeta(x)), \tag{4.38}$$

where $\zeta \in \mathbb{C}^3 \setminus \mathbb{R}^3, \eta \in \mathbb{C}^3$ are constant vectors such that

$$\zeta \cdot \zeta = k^2, \quad \zeta \cdot \eta = 0.$$

Substituting (4.38) into (4.34), (4.35) gives

$$\tilde{\nabla} \times \tilde{\nabla} \times R_\zeta = k^2(n-1)\eta + k^2 n R_\zeta, \tag{4.39}$$

$$\tilde{\nabla} \cdot R_\zeta = -\alpha \cdot (\eta + R_\zeta), \tag{4.40}$$

where $\tilde{\nabla} := \nabla + i\zeta$ and $\alpha(x) := \nabla n(x)/n(x)$. If we further define $\tilde{\Delta} := \nabla + 2i\zeta \cdot \nabla - k^2$ and use the formula

$$\tilde{\nabla} \times \tilde{\nabla} \times R_\zeta = -\tilde{\Delta} R_\zeta + \tilde{\nabla}\tilde{\nabla} \cdot R_\zeta,$$

we have from (4.39) and (4.40) that

$$(\Delta + 2i\zeta \cdot \nabla)R_\zeta = -\tilde{\nabla}(\alpha \cdot (\eta + R_\zeta)) + k^2 m(\eta + R_\zeta), \tag{4.41}$$

where again $m = 1 - n$. Hence we must construct a solution to (4.41). To this end, for $1/2 < \delta < 1$, we define the Hilbert space $L^2_\delta(\mathbb{R}^3)$ by

$$L^2_\delta(\mathbb{R}^3) = \left\{ f \in L^2(\mathbb{R}^3) : \|f\|_\delta := \left(\int_{\mathbb{R}^3} \left(1 + |x|^2\right)^\delta |f(x)|^2 dx \right)^{1/2} < \infty \right\}$$

and denote by $H^2_\delta(\mathbb{R}^3)$ the Sobolev space of functions having derivatives up to second order in $L^2_\delta(\mathbb{R}^3)$. Define the Fourier transform F by

$$F(f) = \hat{f}(\xi) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\,dx$$

and the integral operator $G_\zeta : L^2_\delta(\mathbb{R}^3) \to L^2_{-\delta}(\mathbb{R}^3)$ by

$$G_\zeta(f) := F^{-1}\left(\frac{\hat{f}(\xi)}{\xi^2 + 2\zeta \cdot \xi} \right).$$

From Proposition 3.6 of [104] and Proposition 2.1 of [96] we have that G_ζ is bounded and that there exists a positive constant C independent of ζ such that

$$\|G_\zeta\| \leq \frac{C}{|\zeta|}. \tag{4.42}$$

Applying G_ζ to (4.41) now yields the integral equation

$$R_\zeta = G_\zeta\left[\tilde{\nabla}\left(\alpha\cdot(\eta+R_\zeta)\right)\right] - k^2 G_\zeta\left[m(\eta+R_\zeta)\right]. \tag{4.43}$$

We shall now show that for $|\zeta|$ sufficiently large there exists a unique solution to (4.43). We first need to prove the following lemma, where $n^{1/2}$ denotes the principal value of the square root of n.

Lemma 4.10. *For any $v \in L^2_\delta(\mathbb{R}^3)$ and $|\zeta|$ sufficiently large, the equation*

$$\left(\Delta + 2i\zeta\cdot\nabla + \alpha\cdot\tilde{\nabla}\right)u = v$$

has a unique solution $u \in H^2_{-\delta}(\mathbb{R}^3)$ satisfying

$$u = -n^{-1/2}G_\zeta(n^{1/2}v) + f_\zeta,$$

where

$$\|f_\zeta\| \leq \frac{C}{|\zeta|^2}$$

for some positive constant C independent of ζ.

Proof. From the identity

$$n^{-1/2}(\Delta + 2i\zeta\cdot\nabla)n^{1/2}u = \left(\Delta + 2i\zeta\cdot\nabla + \alpha\cdot\tilde{\nabla}\right)u + qu,$$

where $q := \Delta n^{1/2}/n^{1/2} \in C^1_0(\mathbb{R}^3)$, we see that solving

$$\left(\Delta + 2i\zeta\cdot\nabla + \alpha\cdot\tilde{\nabla}\right)u = v$$

is equivalent to solving

$$(\Delta + 2i\zeta\cdot\nabla - q)f = g, \tag{4.44}$$

where $f := n^{1/2}u$ and $g := n^{1/2}v$. But (4.44) can be rewritten as the integral equation

$$f + G_\zeta(gf) = -G_\zeta g,$$

which by (4.42) can be solved by successive approximations for $|\zeta|$ sufficiently large (noting that a function in $L^2_{-\delta}(\mathbb{R}^3)$ multiplied by a continuous function of compact support is in $L^2_{-\delta}(\mathbb{R}^3)$). By Lemma 2.11 of [96] we see that f, and hence u, is in $H^2_{-\delta}(\mathbb{R}^3)$, and the lemma is proved. □

We can now prove the following theorem, which is the key ingredient of our promised uniqueness theorem.

Theorem 4.11. *For $|\zeta|$ sufficiently large the integral equation* (4.43) *has a unique solution $R_\zeta \in C^2(\mathbb{R}^3)$ that satisfies* (4.39) *and* (4.40)*; i.e., E as defined by* (4.38) *satisfies* (4.34) *and* (4.35).

Proof. Straightforward calculations show that any solution R_ζ of (4.43) in $L^2_{-\delta}(\mathbb{R}^3)$ is in $C^2(\mathbb{R}^3)$ and satisfies (4.39) and (4.40) [53]. Hence to prove the theorem we must show that for $|\zeta|$ sufficiently large there exists a solution $R_\zeta \in L^2_{-\delta}(\mathbb{R}^3)$ of the integral equation (4.43). To this end, we must examine the term $\tilde{\nabla}\left(\alpha \cdot \left(\eta + R_\zeta\right)\right)$, which appears in the first term of (4.43) and appears to go to infinity as $|\zeta|$ tends to infinity. From the vector identity

$$\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla) B + (B \cdot \nabla) A$$

we see that

$$\tilde{\nabla}\left(\alpha \cdot \left(\eta + R_\zeta\right)\right) = \alpha \times \left(\tilde{\nabla} \times R_\zeta\right) + \left(\alpha \cdot \tilde{\nabla}\right) R_\zeta + \left(R_\zeta \cdot \nabla\right)\alpha + \tilde{\nabla}(\alpha \cdot \eta),$$

and hence the terms which are potentially troublesome are $\alpha \times \left(\tilde{\nabla} \times R_\zeta\right)$ and $\left(\alpha \cdot \tilde{\nabla}\right) R_\zeta$. However, by Lemma 4.10, the term $\left(\alpha \cdot \tilde{\nabla}\right) R_\zeta$ can be easily handled. Hence we must examine $Q := \tilde{\nabla} \times R_\zeta$. Since any solution of (4.43) satisfies (4.39), we see that

$$\tilde{\nabla} \times Q = k^2(n-1)\eta + k^2 n R_\zeta,$$

and hence

$$\tilde{\nabla} \times \tilde{\nabla} \times Q = k^2 \nabla n + \left(\eta + R_\zeta\right) + k^2(n-1)\zeta \times \eta + k^2 n Q.$$

Since $\tilde{\nabla} \cdot Q = 0$, this now implies that

$$\Delta Q + 2i\zeta \cdot \nabla Q = k^2 \nabla m \times \left(\eta + R_\zeta\right) + k^2 m (i\zeta \times \eta + Q),$$

and since $I + k^2 G_\zeta m$ is invertible in $L^2_{-\delta}(\mathbb{R}^3)$ for large $|\zeta|$ we have that

$$Q = -\left(I + k^2 G_\zeta m\right)^{-1} G_\zeta \left(k^2 \nabla m \times \left(\eta + R_\zeta\right) + k^2 m (i\zeta \times \eta)\right) \tag{4.45}$$

for $|\zeta|$ sufficiently large. We can now conclude that if $R_\zeta \in L^2_{-\delta}(\mathbb{R}^3)$ is a solution of (4.43), then R_ζ satisfies the integral equation

$$\begin{aligned} R_\zeta = {} & G_\zeta[\alpha \times Q] + G_\zeta\left[\left(\alpha \cdot \tilde{\nabla}\right) R_\zeta\right] + G_\zeta\left[\left(R_\zeta \cdot \nabla\right)\alpha\right] \\ & + G_\zeta\left[\tilde{\nabla}\alpha \cdot \eta\right] - k^2 G_\zeta\left[m\left(\eta + R_\zeta\right)\right], \end{aligned} \tag{4.46}$$

where Q is given by (4.45). Furthermore, for $|\zeta|$ sufficiently large, the integral equation (4.45), (4.46) has a unique solution in $L^2_{-\delta}(\mathbb{R}^3)$ due to (4.42) and Lemma 4.10.

We shall now use the unique solvability of (4.45), (4.46) to deduce that (4.43) also has a unique solution in $L^2_{-\delta}(\mathbb{R}^3)$. To do this we first note that (4.43) is of Fredholm type in $L^2(B_R)$, where $B_R := \{x : |x| < R\}$ contains the support of $m = 1 - n$. This follows from the compact embedding of $H^q(B_R)$ in $L^2(B_R)$ for $q = 1, 2$ and the facts that $G_\zeta m : L^2(B_R) \to H^2(B_R)$ is bounded and $R_\zeta \to G_\zeta\left[\tilde{\nabla}\left(\alpha \cdot R_\zeta\right)\right]$ is bounded from $L^2(B_R)$ into $H^1(B_R)$ since

$$G_\zeta\left(\tilde{\nabla} f\right) = F^{-1}\left(\frac{i\xi + i\zeta}{\xi^2 + 2\zeta \cdot \zeta}\hat{f}(\xi)\right).$$

Suppose now that R^h_ζ is a solution of the homogeneous equation

$$R^h_\zeta = G_\zeta\left[\tilde{\nabla}\left(\alpha \cdot R^h_\zeta\right)\right] - k^2 G_\zeta\left[m R^h_\zeta\right] \tag{4.47}$$

in $L^2(B_R)$. Then R^h_ζ also satisfies the homogeneous equation corresponding to (4.45), (4.46) (viewed as an integral equation in $L^2(B_R)$). Note that, since α and m have compact support, R^h_ζ can be continued as a solution of (4.47) in $L^2(\mathbb{R}^3)$. Since this homogeneous equation is invertible in $L^2(B_R)$ as well as $L^2_{-\delta}(\mathbb{R}^3)$, we see that $R^h_\zeta = 0$. Hence by the Fredholm alternative we can conclude that there exists a unique solution of (4.43) in $L^2(B_R)$. Defining $R_\zeta(x)$ for $x \in \mathbb{R}^3$ by the right-hand side of (4.43) and recalling that $D \subset B_R$ now yields a solution of (4.43) that is defined in all of $\mathbb{R}^3$. From (4.43) and the compact support of α and m we see that $R_\zeta \in L^2_{-\delta}(\mathbb{R}^3)$. The theorem is now proved. □

We shall now use the above results to establish our desired uniqueness theorem. We need the following simple lemma.

Lemma 4.12. *Suppose* $\zeta \in \mathbb{C}^3 \setminus \mathbb{R}^3$, $\eta \in \mathbb{C}^3$, *satisfy* $\zeta \cdot \zeta = k^2$ *and* $\zeta \cdot \eta = 0$ *such that as* $|\zeta| \to \infty$, *the limits*

$$\lim_{|\zeta|\to\infty} \frac{\zeta}{|\zeta|} = \zeta_0, \qquad \lim_{|\eta|\to\infty} \eta = \eta_0$$

exist. If R_ζ *is the solution of* (4.39), (4.40) *given by Theorem* 4.11, *then*

$$R_\zeta = i|\zeta| n^{-1/2} G_\zeta\left[n^{-1/2}\alpha \cdot \eta_0\right]\zeta_0 + f_\zeta,$$

where

$$\lim_{|\zeta|\to\infty} \|f_\zeta\|_{-\delta} = 0.$$

Proof. From (4.45), (4.46) we see that $\|Q\|_{-\delta} \le C$, where the positive constant C is independent of ζ. Since

$$\tilde{\nabla}(\alpha \cdot \eta) = i\zeta_0(\alpha \cdot \eta)|\zeta| + O(1),$$

the lemma follows from (4.46) and Lemma 4.10. □

We now choose two specific sets of vectors ζ, η, corresponding to n_1 and n_2, respectively, satisfying the hypothesis of Lemma 4.12. In particular, we choose an arbitrary vector $\xi \in \mathbb{R}^3$ and assume that the coordinate axes have been rotated such that in the new coordinate system $\xi = (a,0,0)$. In this coordinate system we shall define vectors ζ_1, ζ_2 and η_1, η_2, with the understanding that the corresponding vectors for an arbitrary ξ are obtained by rotation. More specifically, for $\xi = (a,0,0)$ we define ζ_1, ζ_2 and η_1, η_2 in terms of a real parameter c by

$$\begin{aligned} \zeta_1 &= \left(\frac{a}{2}, i\sqrt{c^2+\frac{a^2}{4}-k^2}, c\right), \\ \zeta_2 &= \left(\frac{a}{2}, -i\sqrt{c^2+\frac{a^2}{4}-k^2}, -c\right), \\ \eta_1 &= \frac{1}{\sqrt{c^2+a^2}}\left(c, 0, -\frac{a}{2}\right), \\ \eta_2 &= \frac{1}{\sqrt{c^2+a^2}}\left(c, 0, \frac{a}{2}\right) \end{aligned} \tag{4.48}$$

and note that

$$\begin{aligned} \lim_{c\to\infty} \eta_j &= \eta_0 := (1,0,0), \quad j=1,2, \\ \lim_{c\to\infty} \frac{\zeta_1}{|\zeta_1|} &= \zeta_0 := \frac{1}{\sqrt{2}}(0,i,1), \\ \lim_{c\to\infty} \frac{\zeta_2}{|\zeta_2|} &= -\zeta_0, \end{aligned}$$

and

$$\zeta_1+\zeta_2 = \xi, \quad \zeta_0\cdot\zeta_0 = 0, \quad \eta_0\cdot\zeta_0 = 0. \tag{4.49}$$

Lemma 4.12 now implies that

$$\left(\eta_1 + R_{\zeta_1}\right)\cdot\left(\eta_2 + R_{\zeta_2}\right) = 1 + o(1) \tag{4.50}$$

in the L^1 norm over compact subsets of $\mathbb{R}^3$ as $c \to \infty$ since $\zeta_0\cdot\zeta_0 = \zeta_0\cdot\eta_0 = 0$. We can now prove our desired uniqueness theorem [53] (see also [98]).

Theorem 4.13. *Let $m = 1-n \in C_0^3(\mathbb{R}^3)$ and let $p_i, i = 1,2,3$, be three linearly independent polarizations. Then n is uniquely determined by the electric far field patterns $E_\infty(\hat{x}, d, p_i)$ corresponding to the incident fields* (4.4) *(for $p = p_i$) for a fixed wave number $k > 0$, $d, \hat{x} \in \Omega$ and $i = 1,2$.*

Proof. Suppose the electric far field patterns corresponding to n_1 and n_2 are the same. Let R_{ζ_j}, $j = 1,2$, be the solution of (4.39), (4.40) corresponding to ζ_j, η_j and the refractive index $\sqrt{n_j}$ with ζ_j, η_j given by an appropriate rotation of (4.48). Then from (4.50) we have that

$$\lim_{c\to\infty} \left(\eta_1 + R_{\zeta_1}\right)\cdot\left(\eta_2 + R_{\zeta_2}\right) = 1,$$

and from (4.38), (4.49) and Theorem 4.9 we have that

$$\int_{B_R} e^{i\xi\cdot x}\,(n_1(x) - n_2(x))\,dx = 0$$

for all $\xi \in \mathbb{R}^3$. The theorem now follows by the Fourier Integral Theorem. $\square$

For an alternate proof of Theorem 4.13 based on the use of Fourier series see [68].

4.2 The Interior Transmission Problem

The interior transmission problem plays an important role in the solution of the inverse scattering problem for penetrable scatterers. In Section 4.1 we investigated the solvability in $H(\text{curl}, D)$ of a modified interior transmission problem, which differs from the interior transmission problem by lower order terms in the equations. However, the approach there cannot be used to prove that the interior transmission problem (4.11)–(4.13) is well posed because it is not a compact perturbation of the modified one, due to the lack of compactness of the embedding mapping from $H(\text{curl}, D)$ into $L^2(D)$. In this section, we use a different approach to study the well-posedness of the interior transmission problem, which provides only the existence of $L^2(D)$ solutions and is based on a variational approach for a fourth order differential equation equivalent to the interior transmission problem for the electric field, which is developed in [65] and [36]. We refer the reader to [75] for an alternative approach based on a combined integral equation and variational formulation in the isotropic case.

To motivate the analysis of the interior transmission problem, we recall the definition of the far field operator $F : L^2_t(\Omega) \to L^2_t(\Omega)$:

$$(Fg)(\hat{x}) := \int_{\Omega} E_\infty(\hat{x}, d, g(d))ds(d). \tag{4.51}$$

The following important property of the far field operator is essential for the solution of the inverse problem.

Theorem 4.14. *The far field operator $F : L^2_t(\Omega) \to L^2_t(\Omega)$ corresponding to the scattering problem* (4.1)–(4.5) *is injective with dense range if and only if there does not exist a nontrivial solution to the homogeneous interior transmission problem*

$$\left.\begin{aligned} \text{curl}\,\text{curl}\, E - k^2 N(x)E = 0 \\ \text{curl}\,\text{curl}\, E_0 - k^2 E_0 = 0 \end{aligned}\right\} \quad \text{in } D, \tag{4.52}$$

$$\left.\begin{aligned} \nu \times E = \nu \times E_0 \\ \nu \times \text{curl}\, E = \nu \times \text{curl}\, E_0 \end{aligned}\right\} \quad \text{on } \partial D \tag{4.53}$$

such that $E_0 := E_g$ and $H_0 := H_g$ are an electromagnetic Herglotz pair with kernel ikg.

Proof. The injectivity of F is proved as follows. The equation $Fg = 0$ holds if and only if the scattered field corresponding to the incident field (E_g, H_g) is identically zero, i.e., (4.52), (4.53) is satisfied. Since $g = 0$ if and only if $E_g = 0$, the statement on injectivity follows. On the other hand, F has dense range if and only the L^2 adjoint $F^* : L^2_t(\Omega) \to L^2_t(\Omega)$ of F is injective. But from the reciprocity relation

$$q \cdot E_\infty(\hat{x}, d, p) = p \cdot E_\infty(-d, -\hat{x}, q)$$

we easily derive that

$$(F^*h)(d) = \overline{(Fg)(-d)}, \qquad d \in \Omega,$$

where $g(\hat{x}) = \overline{h(-\hat{x})}$, whence F^* is injective if and only if F is injective. This proves the theorem. □

Definition 4.15. *The values of $k > 0$ for which the homogeneous interior transmission problem* (4.52)–(4.53) *has nontrivial solutions are called* transmission eigenvalues.

Now we turn our attention to the main goal of this section, the study of the interior transmission problem. Recalling that $(\cdot,\cdot)_D$ denotes the $L^2(D)$ scalar product and

$$H_0(\mathrm{curl}, D) := \{u \in H(\mathrm{curl}, D) : \; u \times \nu = 0 \text{ on } \partial D\},$$

where $H(\mathrm{curl}, D)$ is defined by (2.5), we define

$$\mathcal{U}(D) := \{u \in H(\mathrm{curl}, D) : \; \mathrm{curl}\, u \in H(\mathrm{curl}, D)\}, \tag{4.54}$$

$$\mathcal{U}_0(D) := \{u \in H_0(\mathrm{curl}, D) : \; \mathrm{curl}\, u \in H_0(\mathrm{curl}, D)\}, \tag{4.55}$$

equipped with the scalar product $(u, v)_{\mathcal{U}} = (u, v)_{\mathrm{curl}} + (\mathrm{curl}\, u, \mathrm{curl}\, v)_{\mathrm{curl}}$ and the corresponding norm $\|\cdot\|_{\mathcal{U}}$. We note that $C_0^\infty(D)$ is dense in $\mathcal{U}_0(D)$ (see the appendix of [65]).

Let F and F_0 be two vector valued functions on D, and let φ, ψ be two tangential vector fields on ∂D. After expressing the magnetic fields in terms of the electric fields, the interior transmission problem (4.11)–(4.13) is formulated as the problem of finding two vector valued functions E and E_0 such that

$$\begin{aligned} \mathrm{curl}\,\mathrm{curl}\, E - k^2 N E = F \quad & \text{in } D, \\ \mathrm{curl}\,\mathrm{curl}\, E_0 - k^2 E_0 = F_0 \quad & \text{in } D, \end{aligned} \tag{4.56}$$

$$\begin{aligned} (E - E_0) \times \nu = \varphi \quad & \text{on } \partial D, \\ \mathrm{curl}(E - E_0) \times \nu = \psi \quad & \text{on } \partial D. \end{aligned} \tag{4.57}$$

The existence of solutions to this problem will be studied for data that satisfies the following assumption.

Assumption 4.1. The data F, F_0, φ, and ψ is such that

(a) F and F_0 are in $L^2(D)^3$.

(b) φ and ψ are tangential functions defined on ∂D such that there exists a function w in $\mathcal{U}(D)$ satisfying

$$w \times \nu = \varphi \text{ and } (\mathrm{curl}\, w) \times \nu = \psi \text{ on } \partial D.$$

Let us denote by $Y(\partial D)$ the set of (φ, ψ) satisfying (b) equipped with the norm

$$\|(\varphi, \psi)\|_{Y(\partial D)} := \inf_{w \text{ as in (b)}} \|w\|_{\mathcal{U}(D)}.$$

It is proved in [65] that if ∂D is a C^3 boundary, then $H_t^{3/2}(\partial D) \times H_t^{1/2}(\partial D)$ is continuously embedded in $Y(\partial D)$, where $H_t^{3/2}(\partial D)$ and $H_t^{1/2}(\partial D)$ are the spaces of tangential vectors that componentwise are in $H^{3/2}(\partial D)$ and $H^{1/2}(\partial D)$, respectively. In the applications to inverse problems w can be easily constructed from the fundamental solution E_e and a suitable cutoff function.

Definition 4.16. *A strong solution to* (4.56)–(4.57) *is a pair* $(E, E_0) \in L^2(D)^3$ *that satisfies* (4.56) *in the sense of distributions such that* $E - E_0 \in \mathcal{U}(D)$ *and* $E - E_0$ *satisfies* (4.57).

We remark that, in general, the solutions to this problem do not belong to $H(\text{curl}, D)$. Examples of such solutions can be easily constructed by taking

$$E = E_0 = h,$$

where h is a function in $L^2(D)^3$ such that $\text{curl}\,\text{curl}\,h = 0$ in D and $\text{curl}\,h \notin L^2(D)^3$. In cylindrical coordinates (r, θ, z) and for D a bounded domain, where the z axis is tangent to ∂D and does not intersect D, one can take

$$h(r, \theta, z) = r^{-\alpha} \cos(\alpha\theta) e_z,$$

where $0 < \alpha < 1$, and e_z denotes a vector in the z direction.

To study the existence and uniqueness of solutions to (4.56)–(4.57) we rewrite it as a fourth order boundary value problem. For that purpose we need to assume that $N - I$ is invertible almost everywhere in D.

Setting

$$u = E - E_0 \quad \text{and} \quad v = NE - E_0, \tag{4.58}$$

we obtain that

$$E = (N - I)^{-1}(v - u), \quad E_0 = (N - I)^{-1}(Nu - v). \tag{4.59}$$

Taking the difference between two equations in (4.56) we get

$$\text{curl}\,\text{curl}\,u = k^2 v + (F - F_0) \text{ in } D. \tag{4.60}$$

In particular,

$$E = (N - I)^{-1}(k^{-2}(\text{curl}\,\text{curl}\,u - (F - F_0)) - u). \tag{4.61}$$

Substituting for E in (4.56) one now obtains the following fourth order partial differential equation satisfied by u:

$$\begin{aligned} &(\text{curl}\,\text{curl} - k^2 N)(N - I)^{-1}(\text{curl}\,\text{curl}\,u - k^2 u) \\ &\quad = \text{curl}\,\text{curl}(N - I)^{-1}(F - F_0) + k^2 (N - I)^{-1}(NF_0 - F) \qquad \text{in } D. \end{aligned} \tag{4.62}$$

In addition, from (4.57) one obtains that

$$u \times \nu = \varphi, \quad (\text{curl}\,u) \times \nu = \psi \text{ on } \partial D. \tag{4.63}$$

Hence, based on (4.58)–(4.60) we can state the following result.

Theorem 4.17. *Assume that* $(N - I)^{-1}$ *is a bounded matrix field in* D *and that the data satisfies Assumption* 4.1. *Then the existence and uniqueness of a strong solution to* (4.56)–(4.57) *is equivalent to the existence and uniqueness of* $u \in \mathcal{U}(D)$ *and* $v \in L^2(D)^3$ *satisfying* (4.60) *and* (4.62)–(4.63).

Variational formulations. The study of (4.62)–(4.63) will be done using a variational framework. Using the denseness in $\mathcal{U}_0(D)$ of regular functions with compact support in D [65], one can easily see that $u \in \mathcal{U}(D)$ satisfies (4.62) if and only if

$$\begin{aligned} &\big((N - I)^{-1}(\text{curl}\,\text{curl}\,u - k^2 u), (\text{curl}\,\text{curl}\,u' - k^2 \bar{N} u')\big)_D \\ &\quad = \big((N - I)^{-1}(F - F_0), (\text{curl}\,\text{curl}\,u' - k^2 u')\big)_D + k^2 \left(F_0, u'\right)_D \end{aligned} \tag{4.64}$$

for all $u' \in \mathcal{U}_0(D)$. Now set

$$\ell(u') = \Big((N-I)^{-1}(F-F_0), (\operatorname{curl}\operatorname{curl} u' - k^2 u')\Big)_D + k^2 \left(F_0, u'\right)_D,$$

which defines an antilinear form on $\mathcal{U}(D)$. Using the identity $N(N-I)^{-1} = I + (N-I)^{-1}$, one can rewrite (4.64) in one of the following equivalent forms:

$$\mathcal{A}_k(u,u') - k^2 \mathcal{B}(u,u') = \ell(u') \ \forall u' \in \mathcal{U}_0(D) \tag{4.65}$$

or

$$-\tilde{\mathcal{A}}_k(u,u') + k^2 \mathcal{B}(u,u') = \ell(u') \ \forall u' \in \mathcal{U}_0(D), \tag{4.66}$$

where $\mathcal{A}_k$, $\tilde{\mathcal{A}}_k$, and $\mathcal{B}$ are sesquilinear forms on $\mathcal{U}(D) \times \mathcal{U}(D)$ defined by

$$\mathcal{A}_k(u,u') = \Big((N-I)^{-1}(\operatorname{curl}\operatorname{curl} u - k^2 u), (\operatorname{curl}\operatorname{curl} u' - k^2 u')\Big)_D + k^4 \left(u, u'\right)_D,$$

$$\tilde{\mathcal{A}}_k(u,u') = \Big((I-N)^{-1}(\operatorname{curl}\operatorname{curl} u - k^2 N u), (\operatorname{curl}\operatorname{curl} u' - k^2 \bar{N} u')\Big)_D + k^4 \left(Nu, u'\right)_D,$$

and

$$\mathcal{B}(u,u') = \left(\operatorname{curl} u, \operatorname{curl} u'\right)_D, \tag{4.67}$$

where the expression for $\mathcal{B}$ is obtained after using the identity

$$\left(\operatorname{curl}\operatorname{curl} u, u'\right)_D = \left(\operatorname{curl} u, \operatorname{curl} u'\right)_D$$

for all $(u,u') \in \mathcal{U}(D) \times \mathcal{U}_0(D)$.

Our goal now is to establish the existence and uniqueness of $u \in \mathcal{U}(D)$ that satisfies (4.64) and (4.63) by proving that (4.65) and (4.66) form a Fredholm set of equations given suitable assumptions on N. For the study of (4.66) it is more convenient to use the following equivalent expression of $\tilde{\mathcal{A}}_k$:

$$\begin{aligned} \tilde{\mathcal{A}}_k(u,u') &= \left(N(I-N)^{-1}(\operatorname{curl}\operatorname{curl} u - k^2 u), (\operatorname{curl}\operatorname{curl} u' - k^2 u')\right)_D \\ &\quad + \left(\operatorname{curl}\operatorname{curl} u, \operatorname{curl}\operatorname{curl} u'\right)_D. \end{aligned} \tag{4.68}$$

Lemma 4.18. *Assume that there exists a constant $\gamma > 0$ such that*

$$\Re((N-I)^{-1}\xi, \xi) \geq \gamma\, |\xi|^2 \ \ \forall \xi \in C \ \ \textit{and a.e. in } D, \tag{4.69}$$

$$(\textit{respectively, } \Re(N(I-N)^{-1}\xi, \xi) \geq \gamma\, |\xi|^2 \ \ \forall \xi \in C \ \ \textit{and a.e. in } D). \tag{4.70}$$

Then $\mathcal{A}_k$ (respectively, $\tilde{\mathcal{A}}_k$) is a coercive sesquilinear form on $\mathcal{U}_0(D) \times \mathcal{U}_0(D)$.

Proof. Let us prove first the result for $\mathcal{A}_k$. Using (4.69) yields

$$\Re(\mathcal{A}_k(u_0,u_0)) \geq \gamma \|\operatorname{curl}\operatorname{curl} u_0 - k^2 u_0\|^2_{L^2(D)} + \|u_0\|^2_{L^2(D)}.$$

Setting $X = \|\operatorname{curl}\operatorname{curl} u_0\|_{L^2(D)}$ and $Y = k^2 \|u_0\|_{L^2(D)}$, one has

$$\|\operatorname{curl}\operatorname{curl} u_0 - k^2 u_0\|^2_{L^2(D)} \geq X^2 - 2XY + Y^2,$$

and therefore

$$\Re(\mathcal{A}_k(u_0,u_0)) \geq \gamma X^2 - 2\gamma XY + (1+\gamma)Y^2. \tag{4.71}$$

Using the identity

$$\gamma X^2 - 2\gamma XY + (1+\gamma)Y^2 = \left(\gamma + \frac{1}{2}\right)\left(Y - \frac{\gamma}{\gamma+\frac{1}{2}}X\right)^2 + \frac{1}{2}Y^2 + \frac{\gamma}{1+2\gamma}X^2,$$

one concludes that

$$\Re(\mathcal{A}_k(u_0,u_0)) \geq \frac{\gamma}{1+2\gamma}\left(X^2+Y^2\right). \tag{4.72}$$

Integrating by parts, one has the following equality, valid for $u_0 \in \mathcal{U}_0(D)$:

$$\|\mathrm{curl}\,\mathrm{curl}\,u_0 - k^2u_0\|^2_{L^2(D)} = \|\mathrm{curl}\,\mathrm{curl}\,u_0\|^2_{L^2(D)} - 2k^2\|\mathrm{curl}\,u_0\|^2_{L^2(D)} + k^4\|u_0\|^2_{L^2(D)}.$$

Therefore

$$2k^2\|\mathrm{curl}\,u_0\|^2_{L^2(D)} \leq X^2+Y^2,$$

which combined with (4.72) yields the existence of a constant c_k (independent of u_0 and γ) such that

$$|\mathcal{A}_k(u_0,u_0)| \geq c_k\,\frac{\gamma}{1+2\gamma}\|u_0\|^2_{\mathcal{U}}. \tag{4.73}$$

The sesquilinear form $\tilde{\mathcal{A}}_k$ also satisfies (4.73) under condition (4.70) since (as one can easily check)

$$\Re(\tilde{\mathcal{A}}_k(u_0,u_0)) \geq (\gamma+1)X^2 - 2\gamma XY + \gamma Y^2.$$

Hence we can conclude a similar estimate to (4.73) for $\tilde{\mathcal{A}}_k(u_0,u_0)$. □

Based on the Riesz representation theorem we now define the operator $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ by

$$\left(Bu_0,\,u'\right)_{\mathcal{U}} = \mathcal{B}(u_0,u') \;\; \forall\, u' \in \mathcal{U}_0(D).$$

Lemma 4.19. *The operator $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is compact.*

Proof. Let $\{u_n\}$ be a bounded sequence in $\mathcal{U}_0(D)$. We can extract a subsequence, denoted again by $\{u_n\}$, that converges weakly to some u_0 in $\mathcal{U}_0(D)$. Now we recall, provided the boundary of D is sufficiently smooth, that the space of functions

$$\{u \in H_0(\mathrm{curl},D) : \mathrm{div}\,u = 0 \text{ in } D\}$$

is continuously embedded into $H^1(D)$. We deduce that the sequence $\{\mathrm{curl}\,u_n\}$ is bounded in $H^1(D)$. By the Rellich compact embedding theorem, we deduce that $\{\mathrm{curl}\,u_n\}$ converges strongly to $\mathrm{curl}\,u_0$ in $L^2(D)$. From the definition of B and using the Schwarz inequality we obtain

$$\|B(u_n-u_0)\|_{\mathcal{U}(D)} \leq \|\mathrm{curl}(u_n-u_0)\|_{L^2(D)}.$$

Hence $\{Bu_n\}$ converges strongly to Bu_0 in $\mathcal{U}_0(D)$. □

Based on Lemmas 4.19 and 4.18, we are in position to prove the first main theorem of this section.

Theorem 4.20. *Assume that $(N-I)^{-1}$ or $N(I-N)^{-1}$ is a bounded positive definite matrix field on D and that k is not a transmission eigenvalue. Then for all data (F, F_0, φ, ψ) satisfying Assumption* 4.1 *there exists a unique solution $u \in \mathcal{U}(D)$ to* (4.63)–(4.64) *such that*

$$\|u\|_{\mathcal{U}(D)} \leq C\left(\|F\|_{L^2(D)} + \|F_0\|_{L^2(D)} + \|(\varphi, \psi)\|_{Y(\partial D)}\right),$$

where $C > 0$ is a constant independent of u and (F, F_0, φ, ψ).

Proof. Let us first prove this theorem in the case where $N(I-N)^{-1}$ is a bounded positive definite matrix field on D. In this case, one can easily see that $\tilde{\mathcal{A}}_k$ is a continuous sesquilinear form on $\mathcal{U}(D) \times \mathcal{U}(D)$. Based on the Riesz representation theorem, one can therefore define a continuous operator $\tilde{A}_k : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ such that

$$\left(\tilde{A}_k u_0, u'\right)_{\mathcal{U}} = \tilde{\mathcal{A}}_k(u_0, u') \;\; \forall\, u' \in \mathcal{U}_0(D).$$

Lemma 4.18 and the Lax–Milgram theorem prove that $\tilde{A}_k : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is a bijective operator. The identity $(N-I)^{-1} = N(N-I)^{-1} - I$ implies that the antilinear form ℓ is continuous on $\mathcal{U}_0(D)$. We denote by $\ell \in \mathcal{U}_0(D)$ the Riesz representative of ℓ in $\mathcal{U}_0(D)$. Let w be as in Assumption 4.9 and define m such that

$$-\tilde{\mathcal{A}}_k(w, u') + k^2 \mathcal{B}(u_0, u') = \left(m, u'\right)_{\mathcal{U}} \;\; \forall\, u' \in \mathcal{U}_0(D).$$

Then (4.63)–(4.64) is equivalent to $u = w + u_0$, where $u_0 \in \mathcal{U}_0(D)$ is the solution of

$$-\tilde{A}_k u_0 + k^2 B u_0 = m + \ell \quad \text{in } \mathcal{U}_0(D). \tag{4.74}$$

Since $\tilde{A}_k$ is an isomorphism and B is compact, the Fredholm alternative can be applied to (4.74). Hence, assuming that k is not a transmission eigenvalue we have the existence and uniqueness of a solution u_0 to (4.74) satisfying the a priori estimate.

The proof in the case where $(N-I)^{-1}$ is a bounded positive definite matrix can be done exactly the same way by replacing $-\tilde{A}_k + k^2 B$ by $A_k + k^2 B$, where $A_k : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is defined by

$$\left(A_k u_0, u'\right)_{\mathcal{U}} = \mathcal{A}_k(u_0, u') \;\; \forall\, u' \in \mathcal{U}_0(D). \tag{4.75}$$

This proves the theorem. ☐

Theorem 4.21. *Assume that $(N-I)^{-1}$ or $N(I-N)^{-1}$ is a bounded positive definite matrix field on D. Then the following hold:*

(a) *The set of interior transmission eigenvalues, if they exist, is discrete, with $+\infty$ as the only possible accumulation point.*

(b) *If $\Im(N\xi, \xi) > 0$ for all $\xi \in C \setminus \{0\}$ and almost everywhere in D, then the set of eigenvalues is empty.*

Remark 4.2. *We study the existence of transmission eigenvalues in Section* 4.5.

Proof. The proof of part (a) is based on the use of the analytic Fredholm theory. For the sake of presentation we consider only the case when $(N-I)^{-1}$ is bounded and positive definite. Case (b) can be proved in the same way by replacing $\mathcal{A}(\cdot,\cdot)$ with $\tilde{\mathcal{A}}(\cdot,\cdot)$. We first prove that A_k^{-1} is analytic for $k \in \mathbb{C}$ in a neighborhood of the positive real axis, where A_k is defined by (4.75). Let $k_1 > 0$. Then there exists a positive constant C independent of k such that

$$\|(A_k - A_{k_1})u_0\| \le C(|k^2 - k_1^2|\|\operatorname{curl}\operatorname{curl} u_0\|_{L^2(D)}\|u_0\|_{L^2(D)} + |k^4 - k_1^4|\|u_0\|^2_{L^2(D)}.$$

Hence, A_k is a bijective operator for $|k - k_1|$ sufficiently small. Moreover, since $k \mapsto A_k$ is analytic, then $k \mapsto A_k^{-1}$ is analytic in a neighborhood of k_1.

It suffices to show that for $k > 0$ small enough, the operator $A_k - B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is an isomorphism; in other words, sufficiently small positive k are not transmission eigenvalues. To this end, let $u_0 \in \mathcal{U}_0(D)$ be such that

$$\mathcal{A}_k(u_0, u') - k^2\mathcal{B}(u_0, u') = 0 \;\forall u' \in \mathcal{U}_0(D).$$

First we observe that since $u_0 \times \nu = 0$ on ∂D, we have

$$\operatorname{curl} u_0 \cdot \nu = 0 \text{ on } \partial D.$$

On the other hand, the continuous embedding of

$$\{u \in H_0(\operatorname{curl}, D) : \ \operatorname{div} u = 0 \text{ in } D\}$$

into $H^1(D)^3$ implies that $\operatorname{curl} u_0 \in H_0^1(D)^3$. The Poincaré inequality now implies the existence of a constant $C > 0$ such that

$$\|\operatorname{curl} u_0\|^2_{L^2(D)} \le C\|\nabla \operatorname{curl} u_0\|^2_{L^2(D)}.$$

Let $\tilde{v}_0$ be the extension of $\operatorname{curl} u_0$ by zero outside D. Then

$$\begin{aligned}\|\nabla \operatorname{curl} u_0\|^2_{L^2(D)} = \|\nabla \tilde{v}_0\|^2_{L^2(\mathbb{R})} &= \|\operatorname{curl} \tilde{v}_0\|^2_{L^2(\mathbb{R})} + \|\operatorname{div} \tilde{v}_0\|^2_{L^2(\mathbb{R})} \\ &= \|\operatorname{curl} \tilde{v}_0\|^2_{L^2(D)} + \|\operatorname{div} \tilde{v}_0\|^2_{L^2(D)}.\end{aligned}$$

We therefore obtain that

$$\|\operatorname{curl} u_0\|^2_{L^2(D)} \le C\|\operatorname{curl}\operatorname{curl} u_0\|^2_{L^2(D)}.$$

From inequality (4.72) (satisfied here by $\mathcal{A}_k$) we now obtain that

$$\begin{aligned}\Re(\mathcal{A}_k(u_0, u_0) - k^2\mathcal{B}(u_0, u_0)) \quad &\ge \quad \frac{\gamma}{1+2\gamma}\Big(\|\operatorname{curl}\operatorname{curl} u_0\|^2_{L^2(D)} + k^4\|u_0\|^2_{L^2(D)}\Big) \\ &\quad -Ck^2\|\operatorname{curl}\operatorname{curl} u_0\|^2_{L^2(D)}.\end{aligned}$$

Therefore there are no eigenvalues such that $k^2 \le \gamma/(C(1+2\gamma))$.

Part (b) does not require the assumption on the positive definite property of the corresponding matrices. Note that $\Im(N\xi, \xi) > 0$ implies $\Im((N-I)^{-1}\xi, \xi) < 0$. Now assume that u_0 is a solution of

$$\mathcal{A}_k(u_0, u') - k^2\mathcal{B}(u_0, u') = 0 \;\forall u' \in \mathcal{U}_0(D).$$

Taking the imaginary part, one deduces that

$$\operatorname{curl}\operatorname{curl} u_0 - k^2 u_0 = 0 \in D.$$

Since $u_0 \times \nu = 0$ and $\operatorname{curl} u_0 \times \nu = 0$ on ∂D, the extension of u_0 outside D by zero gives an outgoing solution to Maxwell's equation in $\mathbb{R}^3$ with vanishing far field. This implies that this function vanishes on $\mathbb{R}^3$, and therefore $u_0 = 0$. □

4.3 Determination of the Support

This section is dedicated to the solution of the inverse problem for an anisotropic inhomogeneous medium. The *inverse scattering problem* we consider here is to determine the support D and information on the index of refraction N from knowledge of the electric far field pattern $E_\infty(\hat{x}, d, p)$ for all $\hat{x} \in \Omega_1 \subseteq \Omega$, $d \in \Omega_2 \subseteq \Omega$, $p \in \mathbb{R}^3$, and possibly for an interval of frequencies. From Theorem 2.8 and Section 3.4, in the following we can assume that we know the far field pattern for all d and $\hat{x}$ in Ω.

We start with the determination of the support D using the LSM. Similarly to the discussion of Section 3.3, the LSM for determining the support of the inhomogeneity is based on the study of the far field equation

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q), \qquad z, q \in \mathbb{R}^3,$$

where F is the far field operator defined by (4.51), with $E_\infty(\hat{x}, d, p)$ being the far field pattern corresponding to the scattering problem (4.1)–(4.5) with incident field $E^i := E^i(\cdot, d, p)$, $H^i := H^i(\cdot, d, p)$ being a time-harmonic plane wave given by (2.20), and $E_{e,\infty}(\hat{x}, z, q)$ being the electric far field pattern of the electric dipole given by (2.15). Consider now the following interior transmission problem:

$$\left.\begin{array}{l} \operatorname{curl}\operatorname{curl} E^z - k^2 N(x) E^z = 0 \\ \operatorname{curl}\operatorname{curl} E_0^z - k^2 E_0^z = 0 \end{array}\right\} \text{ in } D, \tag{4.76}$$

$$\left.\begin{array}{l} \nu \times E^z - \nu \times E_0^z = \nu \times E_e(\cdot, d, q) \\ \nu \times \operatorname{curl} E^z - \nu \times \operatorname{curl} E_0^z = \nu \times \operatorname{curl} E_e(\cdot, d, q) \end{array}\right\} \text{ on } \partial D. \tag{4.77}$$

At the beginning of Section 3.3 we showed that the far field equation (3.32) corresponding to the scattering of a plane wave by a perfect conductor was solvable if and only if there exists a solution E_z of the interior problem (3.33), (3.34) such that E_z is the electric field of an electromagnetic Herglotz pair with kernel ikg. In the same way one can prove the following theorem.

Theorem 4.22. *There exists a solution $g \in L_t^2(\Omega)$ of the far field equation* (4.51) *for an inhomogeneous anisotropic medium if and only if there exists a solution E_0^z of the interior transmission problem* (4.76)–(4.77) *and E_0^z is the electric field of an electromagnetic Herglotz pair with kernel ikg.*

In order to connect the interior transmission problem (4.76)–(4.77) with the forward scattering problem we need to consider the scattering problem (4.1)–(4.5) with more general incident fields. To this end we define the space of incident fields

$$H_{inc}(D) := \left\{ E_0 \in L^2(D) : \text{ such that } \operatorname{curl}\operatorname{curl} E_0 - k^2 E_0 = 0 \quad \text{in } D \right\}, \tag{4.78}$$

where the equation is satisfied in the distributional sense. For an $E_0 \in H_{inc}(D)$ we consider the scattering problem for the scattered field E^s corresponding to E_0 as an incident field,

$$\begin{array}{rcll} \operatorname{curl}\operatorname{curl} E^s - k^2 E^s &=& 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \operatorname{curl}\operatorname{curl} E^s - k^2 N E^s &=& k^2(N-1)E_0 & \text{in } D, \\ \nu \times E^s_+ &=& \nu \times E^s_- & \text{on } \partial D, \\ \nu \times \operatorname{curl} E^s_+ &=& \nu \times \operatorname{curl} E^s_- & \text{on } \partial D, \\ \lim\limits_{r\to\infty} (\operatorname{curl} E^s \times x - ikrE^s) &=& 0, & \end{array} \tag{4.79}$$

where E^s_+ and E^s_- denote the limits of E^s approaching ∂D from $\mathbb{R}^3 \setminus \overline{D}$ and D, respectively. It can be shown that (4.79) has a unique solution [78]. In particular, this solution E^s satisfies the a priori estimate

$$\|E^s\|_{H(\operatorname{curl}, B_R)} \le C \|E_0\|_{L^2(D)}, \tag{4.80}$$

where $C > 0$ is a positive constant independent of E_0, and B_R is a ball of radius R. We also note that the solution E^s of (4.79) has the integral representation [65]

$$E^s(x) = \operatorname{curl}_x \operatorname{curl}_x \int_D \Phi(x,y)(N(y)-I)\big(E^s(y) + E_0(y)\big)\, dy \tag{4.81}$$

for $x \in \mathbb{R}^3 \setminus \overline{D}$, where $\Phi(x,y)$ is given by (2.9).

We now show that $H_{inc}(D)$ is the closure in $L^2(D)$ of the space of entire solutions to Maxwell's equations. To this end, consider $M_n^m(x) := \operatorname{curl}(x u_n^m(x))$ and $u_n^m(x) := j_n(k_b|x|)Y_n^m(x/|x|)$, where $\{Y_n^m, m = -n, \dots, n, n = 0, 1, \dots\}$ is the set of orthonormal spherical harmonics and j_n denotes the spherical Bessel function of order n (see Section 2.3).

Lemma 4.23. *The space*

$$H := \operatorname{span}\{M_n^m, \operatorname{curl} M_n^m : n = 1, 2, \dots, m = -n, \dots, n\}$$

is dense in $H_{inc}(D)$.

Proof. The proof is taken from [65]. Let $\overline{H}$ be the closure of H in $H_{inc}(D)$ and let $E_0 \in H_{inc}(D)$ be in the orthogonal complement of $\overline{H}$. We define

$$E(x) = \int_D \Phi(x,y)E_0(y)\,dy + \frac{1}{k^2}\operatorname{grad}\operatorname{div} \int_D \Phi(x,y)E_0(y)\,dy, \quad x \in \mathbb{R}^3,$$

where $\Phi(x,y)$ is defined by (2.9). Using the regularity properties of the volume potential (see [50, Theorem 8.2]) we have that $E \in L^2_{loc}(\mathbb{R}^3)$ and $\operatorname{curl} E \in H^1_{loc}(\mathbb{R}^3)$, whence $E \in \mathcal{U}_{loc}(\mathbb{R}^3)$. By definition we have that

$$\operatorname{curl}\operatorname{curl} E - k^2 E = E_0 \quad \text{in } D, \tag{4.82}$$

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}. \tag{4.83}$$

Now let B be an open ball that contains $\overline{D}$. Then (4.82) implies that in $\mathbb{R}^3 \setminus B$

$$E(x) = \frac{1}{k^2}\operatorname{curl}\operatorname{curl} \int_D \Phi(x,y)E_0(y)dy,$$

and from Theorem 2.9 it follows that for all $x \in \mathbb{R}^3 \setminus B$

$$E(x) = \sum_{n=1}^{\infty} \frac{i}{n(n+1)} \sum_{m=-n}^{n} \Big[k \left(E_0, M_n^m\right)_{L^2(D)} N_n^m(x) + \frac{1}{k} \left(E_0, \operatorname{curl} M_n^m\right)_{L^2(D)} \operatorname{curl} N_n^m(x) \Big],$$

where $(\cdot,\cdot)_{L^2(D)}$ denotes the $L^2(D)$-inner product, $N_n^m(x) := \operatorname{curl}(x h_n^{(1)}(k|x|) Y_n^m(\hat{x}))$, and $h_n^{(1)}$ denotes the spherical Hankel function of the first kind of order n. From the fact that E_0 is orthogonal to H with respect to the $L^2(D)$-inner product, we conclude that $E = 0$ in $\mathbb{R}^3 \setminus \overline{D}$. Hence, taking the $L^2(D)$-inner product of (4.82) with E_0, we obtain that

$$\|E_0\|^2_{L^2(D)} = \Big(\operatorname{curl}\operatorname{curl} E - k^2 E,\, E_0\Big)_{L^2(D)}. \tag{4.84}$$

Finally, in view of zero boundary traces of E, and since the test functions are dense in $\mathcal{U}_0(D)$, we obtain after integrating by parts that the right-hand side of (4.84) is zero since $\operatorname{curl}\operatorname{curl} E_0 - k^2 E_0 = 0$ in the distribution sense. Hence $E_0 = 0$, which ends the proof. □

Remark 4.3. As a consequence of Lemma 4.23 we have that the set of Herglotz electric wave functions E_g for $g \in L^2_t(\Omega)$ is dense in $H_{inc}(D)$ with respect to the $L^2(D)$ norm.

We now turn our attention to the justification of the LSM for the determination of the support D of the inhomogeneity. To this end let $\mathcal{F} : H_{inc}(D) \to L^2(\Omega)$ be the operator that takes $E_0 \in H_{inc}(D)$ to the far field of the corresponding radiating solution E^s to (4.79). By superposition we have that

$$(Fg)(\hat{x}) = \mathcal{F}(E_g)(\hat{x}), \qquad \hat{x} \in \Omega. \tag{4.85}$$

The integral representation (4.81) of E^s and the far field asymptotic behavior of $\Phi(x,y)$ imply that

$$\mathcal{F}(E_0)(\hat{x}) = k^2 \int_D e^{-ik\hat{x}\cdot y} \left[\hat{x} \times (N(y) - I)\left(E^s(y) + E_0(y)\right) \times \hat{x}\right] ds(y). \tag{4.86}$$

The well-posedness of (4.79), Theorem 4.14, and (4.85) imply the following result.

Lemma 4.24. *Assume that k is not a transmission eigenvalue. Then the operator $\mathcal{F}$: $H_{inc}(D) \to L^2(\Omega)$ is compact, injective, and has dense range.*

Lemma 4.25. *Assume that k is not a transmission eigenvalue. Then $E_{e,\infty}(\hat{x},z,q)$ is in the range of $\mathcal{F}$ if and only if $z \in D$. For $z \in D$, the unique solution $E_0^z \in H_{inc}(D)$ of $\mathcal{F}(E_0^z) = E_{e,\infty}(\hat{x},z,q)$ satisfies $\lim_{z\to\partial D} \|E_0^z\|_{L^2(D)} = \infty$.*

Proof. Let $z \in D$ and let $E_0^z \in L^2(D), E^z \in L^2(D)$ be the unique solution of the interior transmission problem (4.76)–(4.77). Then $E^s := E_z(\cdot, z, q)$ in $\mathbb{R}^3 \setminus \overline{D}$, and $E^s := E^z - E_0^z$

in D is a solution to (4.79) with $E_0 := E_0^z$ as the incident field. Furthermore, since the far field of E^s is $E_{e,\infty}(\cdot,z,q)$, we conclude that $\mathcal{F}(E_0^z) = E_{e,\infty}(\cdot,x,z,q)$. The a priori estimate (4.80) implies that for $z \in D$, $\|E_e(\cdot,z,q)\|_{H(\text{curl},B_R\setminus\overline{D})} \le C\|E_0^z\|_{L^2(D)}$, and since $\|E_e(\cdot,z,q)\|_{H(\text{curl},B_R\setminus\overline{D})} \to \infty$ as $z \to \partial D$, so does $\|E_0^z\|_{L^2(D)}$.

Now assume that $z \in \mathbb{R}^3 \setminus \overline{D}$ and $E_0^z \in H_{inc}(D)$ is such that $\mathcal{F}(E_0^z) = E_{e,\infty}(\cdot,z,q)$. Let $E^s \in H_{loc}(\text{curl},\mathbb{R}^3)$ be the scattered field corresponding to E_0^z. Then from Rellich's lemma (Theorem 2.3) and the unique continuation principle we conclude that $E^s = E_e(\cdot,z,q)$ in $\mathbb{R}^3 \setminus (\overline{D} \cup \{z\})$. This is a contradiction since E^s is in $H_{loc}(\mathbb{R}^3 \setminus \overline{D})$, whereas $E_e(\cdot,z,q)$ is not due to the singularity at z. □

Now we are ready to prove the main theorem of this section, which is the basis of the LSM.

Theorem 4.26. *Assume that k is not a transmission eigenvalue, and let F be the far field operator corresponding to the scattering problem* (4.1)–(4.5)*. Then*

1. *for $z \in D$ and a given $\epsilon > 0$ there exists a $g_z^\epsilon \in L_t^2(\Omega)$ such that*

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot,z,q)\|_{L_t^2(\Omega)} < \epsilon,$$

and the corresponding Herglotz function $E_{g_z^\epsilon}$ converges to E_0^z in the $L^2(D)$ norm as $\epsilon \to 0$, where E_z^0, E_z is the solution of (4.76)–(4.77).

2. *for a fixed $\epsilon > 0$, we have that*

$$\lim_{z\to\partial D} \|E_{g_z^\epsilon}\|_{L^2(D)} = \infty \quad \textit{and} \quad \lim_{z\to\partial D} \|g_z^\epsilon\|_{L_t^2(\Omega)} \to \infty.$$

3. *for $z \in \mathbb{R}^3 \setminus D$ and a given $\epsilon > 0$, there exists $g_z^\epsilon \in L_t^2(\Omega)$ satisfying*

$$\|Fg_z^\epsilon - E_{e,\infty}(\hat{x},z,q)\|_{L_t^2(\Omega)} < \epsilon$$

such that

$$\lim_{\epsilon\to 0} \|E_{g_z^\epsilon}\|_{L^2(D)} = \infty \quad \textit{and} \quad \lim_{\epsilon\to 0} \|g_z^\epsilon\|_{L_t^2(\Omega)} \to \infty.$$

Proof. Parts 1 and 2 of the theorem are a direct consequence of Lemmas 4.23 and 4.25. The approximate solution g_z^ϵ of the far field equation is the kernel of the Herglotz wave function that approximates E_0^z in the $L^2(D)$ norm with discrepancy ϵ, where E_z^0, E_z is the solution of (4.76)–(4.77). Thus $E_{g_z^\epsilon}$, and consequently g_ϵ^z, satisfies the stated properties due to the behavior of E_0^z.

For $z \in \mathbb{R}^3 \setminus D$, from Lemmas 4.24 and 4.25, by using Tikhonov regularization we can construct a regularized solution to $\mathcal{F}(E_0) = E_{e,\infty}(\cdot,z,q)$. In particular, there exists $E_0^z := E_0^{z,\alpha}$ corresponding to a parameter $\alpha = \alpha(\delta)$ chosen by a regular regularization strategy (e.g., the Morozov discrepancy principle) such that

$$\|\mathcal{F}(E_0^z) - E_{e,\infty}(\hat{x},z,q)\|_{L_t^2(\Omega)} < \delta$$

for an arbitrary noise level δ and

$$\lim_{\alpha\to 0}\|E_0^z\|_{L^2(D)} \to \infty.$$

Note that $\alpha \to 0$ as $\delta \to 0$. Part 3 of the theorem follows by approximating E_0^z arbitrarily closely by E_g. ☐

This approximate (regularized) solution g_z^ϵ given by Theorem 4.26 can now be used to reconstruct D.

4.4 A Lower Bound for $\|N\|_2$

Having found D by the LSM, we are now concerned with finding some information on the (matrix) index of refraction N from knowledge of $E_\infty(\hat{x},d,p)$ for $\hat{x},d \in \Omega$ and $p \in \mathbb{R}^3$. In the case of anisotropic media, as we mentioned before, $E_\infty(d,\hat{x},p)$ for all $\hat{x},d \in \Omega$ and $p \in \mathbb{R}^3$ does not uniquely determine the matrix N even if this data is known for an interval of values of k. Our aim is to provide inequalities that are satisfied by all dielectric anisotropic media that give rise to the same far field data. The information needed to do this is obtained from the smallest transmission eigenvalue, which can be determined from the far field data. In particular, instead of avoiding transmission eigenvalues, as in the LSM, we will now have them play a central role. Following the ideas in [25] and [28], we will show that in certain circumstances a lower bound for the Euclidean norm of $N(x)$ can be obtained from knowledge of the smallest transmission eigenvalue. The existence of transmission eigenvalues is proved in the following section.

To indicate why transmission eigenvalues can be computed from the far field data, we remind the reader that the LSM fails when k is a transmission eigenvalue. In particular, the norm of the (regularized) solution to the far field equation

$$(Fg)(\hat{x}) = \frac{ik}{4\pi}(\hat{x}\times q)\times \hat{x}e^{-ik\hat{x}\cdot z_0}, \qquad z_0 \in D, \tag{4.87}$$

can be expected to be large for such values of k. This provides us with a method for determining the smallest transmission eigenvalue from the far field data (see Section 3.5).

We now derive a relationship between $N(x)$ and the smallest transmission eigenvalue. Since transmission eigenvalues do not occur when the anisotropic medium is absorbing (Theorem 4.21), in this section we assume that $\Im(N) = 0$.

Theorem 4.27. *Assume that $\bar{\xi}\cdot(N-I)^{-1}\xi \ge \gamma|\xi|^2$ in D for all $\xi \in \mathbb{C}^2$ and $\gamma > 0$. Then all transmission eigenvalues satisfy $k^2 \ge \frac{\gamma}{1+\gamma}\lambda_1(D)$, where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.*

Proof. We use the notation from Section 4.2. Since N is a real valued matrix, it follows from the proof of Lemma 4.18 that

$$\mathcal{A}_k(u,u) \ge \gamma X^2 - 2\gamma XY + (\gamma+1)Y^2, \tag{4.88}$$

where $X = \|\mathrm{curl}\,\mathrm{curl}\,u\|_{L^2(D)}$ and $Y = k^2\|u\|_{L^2(D)}$. From the identity

$$\gamma X^2 - 2\gamma XY + (\gamma+1)Y^2 = \epsilon\left(Y - \frac{\gamma}{\epsilon}X\right)^2 + \left(\gamma - \frac{\gamma^2}{\epsilon}\right)X^2 + (1+\gamma-\epsilon)Y^2 \tag{4.89}$$

for $\gamma < \epsilon < \gamma + 1$, we now obtain that

$$\mathcal{A}_k(u,u) - k^2\mathcal{B}(u,u) \geq \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\operatorname{curl}\operatorname{curl} u\|^2_{L^2(D)} + (1+\gamma-\epsilon)k^2\|u\|^2_{L^2(D)} - k^2\|\operatorname{curl} u\|^2_{L^2(D)}. \tag{4.90}$$

From the proof of Theorem 4.21 and the Poincaré inequality we have that

$$\|\operatorname{curl} u\|_{L^2(D)} \leq \frac{1}{\lambda_1(D)}\|\operatorname{curl}\operatorname{curl} u\|_{L^2(D)}, \tag{4.91}$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D. Hence from (4.90) and assuming that $\gamma < \epsilon < \gamma + 1$ we have that

$$\mathcal{A}_k(u,u) - k^2\mathcal{B}(u,u) \geq \left(\gamma - \frac{\gamma^2}{\epsilon} - \frac{k^2}{\lambda_1(D)}\right)\|\operatorname{curl}\operatorname{curl} u\|^2_{L^2(D)} + (1+\gamma-\epsilon)k^2\|u\|^2_{L^2(D)}.$$

Therefore there are no transmission eigenvalues for $k^2 < \left(\gamma - \gamma^2/\epsilon\right)\lambda_1(D)$ for $\gamma < \epsilon < \gamma + 1$. In particular, taking ϵ arbitrarily close to $\gamma + 1$, we have that if $k^2 < \frac{\gamma}{1+\gamma}\lambda_1(D)$, then k is not a transmission eigenvalue. This proves the theorem. $\square$

Theorem 4.28. *Assume that $\bar{\xi} \cdot N(I-N)^{-1}\xi \geq \gamma|\xi|^2$ in D for all $\xi \in \mathbb{C}^2$ and $\gamma > 0$. Then all transmission eigenvalues satisfy $k^2 \geq \lambda_1(D)$, where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.*

Proof. The proof is similar to the proof of Theorem 4.27. Here we need to use the sesquilinear form $\tilde{\mathcal{A}}_k(u,u)$ which, since N is symmetric, can be rewritten as

$$\tilde{\mathcal{A}}_k(u,\psi) := \left(N(I-N)^{-1}\left(\operatorname{curl}\operatorname{curl} u + k^2 u\right), \left(\operatorname{curl}\operatorname{curl}\psi + k^2\psi\right)\right)_{L^2(D)} + (\operatorname{curl}\operatorname{curl} u, \operatorname{curl}\operatorname{curl}\psi)_{L^2(D)}.$$

Similarly as in the proof of Theorem 4.27, for $\gamma < \epsilon < \gamma + 1$ we have that

$$\tilde{\mathcal{A}}_k(u,u) - k^2\mathcal{C}(u,u) \geq (1+\gamma-\epsilon)\|\operatorname{curl}\operatorname{curl} u\|^2_{L^2(D)} + \left(\gamma - \frac{\gamma^2}{\epsilon}\right)k^2\|u\|^2_{L^2(D)} - k^2\frac{1}{\lambda_1(D)}\|\operatorname{curl}\operatorname{curl} u\|^2_{L^2(D)},$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D. In particular, $\mathcal{B}_k(u,u) - k^2\mathcal{C}(u,u)$ is coercive as long as $k^2 < (\gamma + 1 - \epsilon)\lambda_1(D)$ for $\gamma < \epsilon < \gamma + 1$. In particular, by taking $\epsilon > 0$ arbitrarily close to α we have that k such that $k^2 < \lambda(D)$ are not transmission eigenvalues. $\square$

We are now ready to formulate the main result of this section, which provide estimates on the matrix index of refraction N under the assumption that the anisotropic material is a dielectric. Note that in this case the symmetric matrices N and N^{-1} are bounded below, i.e., $\bar{\xi} \cdot N\xi \geq \beta|\xi|^2$ and $\bar{\xi} \cdot N^{-1}\xi \geq \alpha|\xi|^2$, for all $\xi \in C^2 \setminus \{0\}$ and all $x \in D$ for some constants $\alpha > 0$ and $\beta > 0$.

Let $\|N\|_2$ denote the Euclidean norm of N, which is the largest eigenvalue of N since the matrix is positive definite. We denote by $\lambda_1(x) \leq \lambda_2(x) \leq \lambda_3(x)$ the eigenvalues of N for $x \in D$. The above assumptions guarantee that $\beta < \lambda_1(x)$ and $\alpha < 1/\lambda_3(x)$ for $x \in D$, since λ_3 is the reciprocal of the smallest eigenvalue of N^{-1}, which by assumption is bigger than α. We recall that

$$\|N\|_2 = \lambda_3 = \sup_{\|\xi\|=1} (\bar{\xi} \cdot N\xi) \text{ and } \lambda_1 = \inf_{\|\xi\|=1} (\bar{\xi} \cdot N\xi).$$

Theorem 4.29.

1. *Assume that* $\Im(N(x)) = 0$ *and* $\|N(x)\|_2 \geq \delta > 1$ *for all* $x \in D$ *and some constant* δ. *Then,*
$$\sup_D \|N\|_2 \geq \frac{\lambda_1(D)}{k^2}, \tag{4.92}$$
where k *is a transmission eigenvalue and* $\lambda_1(D)$ *is the first Dirichlet eigenvalue of* $-\Delta$ *on* D.

2. *Assume that* $\Im(N(x)) = 0$ *and* $0 < \beta \leq \|N(x)\|_2 \leq \delta < 1$ *for all* $x \in D$ *and some constants* β *and* δ. *Then, if* k *is a transmission eigenvalue,*
$$k^2 \geq \lambda_1(D),$$
where $\lambda_1(D)$ *is the first Dirichlet eigenvalue of* $-\Delta$ *on* D.

Proof. To prove the first part of the theorem, let k be a transmission eigenvalue. The assumptions on N imply that there exists a constant $\gamma > 0$ such that $\bar{\xi} \cdot (N-I)^{-1}\xi \geq \gamma|\xi|^2$. Indeed,

$$\inf_{\xi \in C^2} \bar{\xi} \cdot (N-I)^{-1}\xi = \frac{1}{\lambda_3 - 1}|\xi|^2 \geq \frac{1}{1/\alpha - 1}|\xi|^2 = \gamma|\xi|^2, \quad x \in D,$$

since $1 < \lambda_3 \leq 1/\alpha$. Now, without loss of generality, we take

$$\gamma = \inf_{|\xi|=1} \bar{\xi} \cdot (N(x_0) - I)^{-1}\xi \qquad \text{for an appropriate } x_0 \in D.$$

From Theorem 4.27 we have that $\frac{\gamma}{\gamma+1} < k^2/\lambda_1(D)$. Using the fact that γ is the reciprocal of the largest eigenvalue of $N(x_0) - I$, we have that

$$\gamma = \frac{1}{\lambda_3(x_0) - 1} = \frac{1}{\|N(x_0)\|_2 - 1} \geq \frac{1}{\sup_D \|N(x)\|_2 - 1}.$$

Now since $\sup_D \|N\|_2 \geq 1$ by assumption, we conclude that

$$\sup_D \|N\|_2 \geq \frac{1}{\gamma} + 1 > \frac{\lambda_1(D)}{k^2},$$

which proves the first part of the theorem.

In order to show the second part of the theorem, it suffices to show that the assumptions on N imply that there exists a constant $\gamma > 0$ such that $\bar{\xi} \cdot N(I-N)^{-1}\xi \geq \gamma|\xi|^2$. Then the result follows from Theorem 4.28. To this end, we have that $N(I-N)^{-1} = (I-N)^{-1} - I$. Hence

$$\inf_{\xi \in C^2} \bar{\xi} \cdot N(I-N)^{-1}\xi = \inf_{\xi \in C^2} \bar{\xi} \cdot (I-N)^{-1}\xi - |\xi|^2 = \left(\frac{1}{1-\lambda_3} - 1\right)|\xi|^2$$
$$\geq \left(\frac{1}{1-\beta} - 1\right)|\xi|^2 = \gamma|\xi|^2, \quad x \in D.$$

This ends the proof. □

The above theorem provides a lower bound for $\|N\|_2$ in terms of the first transmission eigenvalue only in the case when it is known a priori that $\|N(x)\|_2 \geq \delta > 1$ for $x \in D$. In particular, if k_1 is the smallest transmission eigenvalue, then

$$\sup_D \|N\|_2 \geq \frac{\lambda_1(D)}{k_1^2}, \tag{4.93}$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D. We remind the reader that D (and hence $\lambda_1(D)$) can be determined from the far field data using the LSM, and the first transmission eigenvalue can also be computed from the far field data.

4.5 The Existence of Transmission Eigenvalues

As discussed above, transmission eigenvalues play an important role in the solution of the inverse scattering problem for inhomogeneous media. On one hand, in the context of sampling methods for reconstructing the support of the scatterer discussed in Section 4.3, one needs to avoid those frequencies that correspond to transmission eigenvalues, and hence it is important to know that the transmission eigenvalues form a discrete set. This was shown in Theorem 4.21. On the other hand, one can use transmission eigenvalues to obtain information about physical properties of the scattering object as discussed in Section 4.4, and therefore it is important to know whether they exist and to understand their connection with the index of refraction. This application is based on the results in [26], which justify the numerical observation that transmission eigenvalues can be computed from the far field data. Either way, the investigation of the spectral properties of the interior transmission problem has become an active area of research in inverse scattering theory. Note that the LSM is based on keeping the wave number k fixed and determining the support D of the scatterer by "sampling" a region containing D by the point z. On the other hand, if $z \in D$ is kept fixed and k is varied, we can use the far field equation to determine the smallest transmission eigenvalue; i.e., the regularized solution of the far field equation will have a large norm when k is a transmission eigenvalue. Then from knowledge of the first transmission eigenvalue it is possible to obtain information about the index of refraction of the inhomogeneous medium. Our goal in this section is to prove that there exists an infinite discrete set of transmission eigenvalues defined by Definition 4.15 and provide lower and upper bounds for the first transmission eigenvalue, thus improving the results presented in Section 4.4. We refer the reader to [30], [33], [38], [76] for more detailed discussions on transmission eigenvalues in electromagnetic scattering theory. The existence of transmission eigenvalues in the scalar case was first shown by Päivärinta and Sylvester [99].

The interior transmission eigenvalue problem (4.52)–(4.53) can be set in the following abstract analytic framework, which is introduced in [36]. In particular, let U be a separable Hilbert space with scalar product $(\cdot,\cdot)$, let A be a bounded, positive definite, and self-adjoint operator on U, and let B be a nonnegative, self-adjoint, and compact bounded linear operator on U. Then there exists an increasing sequence of positive real numbers $(\lambda_j)_{j\geq 1}$ and a sequence $(u_j)_{j\geq 1}$ of elements of U such that $Au_j = \lambda_j Bu_j$. The sequence $(u_j)_{j\geq 1}$ forms a basis of $(A\ker(B))^\perp$, and if $\ker(B)^\perp$ has infinite dimension, then $\lambda_j \to +\infty$ as $j \to \infty$. Furthermore, these eigenvalues satisfy a min-max principle (see Corollary 2.1 in [37]); namely,

$$\lambda_j = \min_{W\subset \mathcal{U}_j}\left(\max_{u\in W\setminus\{0\}} \frac{(Au,u)}{(Bu,u)}\right), \tag{4.94}$$

where $\mathcal{U}_j$ denotes the set of all j-dimensional subspaces W of U such that $W\cap\ker(B) = \{0\}$. These eigenvalues can be arranged in increasing order.

Let $\tau \longmapsto A_\tau$ be a continuous mapping from $]0,\infty[$ to the set of self-adjoint and positive definite bounded linear operators on U, and consider the generalized eigenvalue problem

$$A_\tau u - \lambda_j(\tau)Bu = 0, \qquad u\in U. \tag{4.95}$$

Obviously from (4.94) we have that λ_j for every $j\in\mathbb{N}$ is a continuous function of τ in $]0,\infty[$. The following theorem provides the fundamental tool for proving the existence of transmission eigenvalues.

Theorem 4.30. *Let $\tau \longmapsto A_\tau$ be a continuous mapping from $]0,\infty[$ to the set of self-adjoint and positive definite bounded linear operators on U, and let B be a self-adjoint and nonnegative compact bounded linear operator on U. We assume that there exists two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. *$A_{\tau_0} - \tau_0 B$ is positive on U;*

2. *$A_{\tau_1} - \tau_1 B$ is nonpositive on an m-dimensional subspace of U.*

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1,\dots,m$ has at least one solution in $[\tau_0,\tau_1]$, where $\lambda_j(\tau)$ is the jth eigenvalue (counting multiplicity) of A_τ with respect to B, i.e., $\ker(A_\tau - \lambda_j(\tau)B) \neq \{0\}$.

Proof. First, we can deduce from (4.94) that for all $j\geq 1$, $\lambda_j(\tau)$ is a continuous function of τ. Assumption 1 shows that $\lambda_j(\tau_0) > \tau_0$ for all $j\geq 1$. Assumption 2 implies in particular that $W_k\cap\ker(B) = \{0\}$. Hence, another application of (4.94) implies that $\lambda_j(\tau_1)\leq \tau_1$ for $1\leq j\leq k$. The desired result is then obtained by applying the intermediate value theorem. □

Returning to the homogeneous interior transmission problem (4.52)–(4.53) we recall that N, N^{-1}, and either $(N-I)^{-1}$ or $(I-N)^{-1}$ are bounded positive definite real matrix valued functions on D, and a solution of (4.52)–(4.53) is such that $E\in (L^2(D))^3$, $E_0 \in$

$(L^2(D))^3$, and $E - E_0 \in \mathcal{U}_0(D)$, where $\mathcal{U}_0(D)$ is defined by (4.55). As shown in Section 4.2, (4.52)–(4.53) is equivalent to finding $u = E - E_0 \in \mathcal{U}_0(D)$ such that

$$(\mathrm{curl}\,\mathrm{curl} - k^2 N)(N - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, u - k^2 u) = 0, \tag{4.96}$$

which in variational form can be written as

$$\int_D (N - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, u - k^2 u) \cdot (\mathrm{curl}\,\mathrm{curl}\, v - k^2 N v)\, dx = 0 \quad \forall\, v \in \mathcal{U}_0(D). \tag{4.97}$$

Letting $\tau := k^2$, we notice that (4.52)–(4.53) can be written as an operator equation

$$A_\tau u - \tau B u = 0 \quad \text{and} \quad \tilde{A}_\tau u - \tau B u = 0, \quad \text{for} \quad u \in \mathcal{U}_0(D). \tag{4.98}$$

Here the bounded linear operators $A_\tau : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$, $\tilde{A}_\tau : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$, and $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ are the operators defined using the Riesz representation theorem associated with the sesquilinear forms $\mathcal{A}_\tau$, $\tilde{\mathcal{A}}$, and $\mathcal{B}$, which are given by

$$\mathcal{A}_\tau(u,v) := \Big((N - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, u - \tau u), (\mathrm{curl}\,\mathrm{curl}\, v - \tau v)\Big)_D + \tau^2 (u, v)_D\,, \tag{4.99}$$

$$\begin{aligned}\tilde{\mathcal{A}}_\tau(u,v) :=& \Big(N(I - N)^{-1}(\mathrm{curl}\,\mathrm{curl}\, u - \tau u), (\mathrm{curl}\,\mathrm{curl}\, v - \tau v)\Big)_D \\ &+ (\mathrm{curl}\,\mathrm{curl}\, u, \mathrm{curl}\,\mathrm{curl}\, v)_D\,,\end{aligned} \tag{4.100}$$

and

$$\mathcal{B}(u,v) := (\mathrm{curl}\, u, \mathrm{curl}\, v)_D\,, \tag{4.101}$$

respectively, where $(\cdot,\cdot)_D$ denotes the $L^2(D)$-inner product (see Section 4.2).

The properties of these operators are studied in Sections 4.2 and 4.4. Let $\sigma_*(x) > 0$ and $\sigma^*(x) > 0$ be the smallest and the largest eigenvalue, respectively, of the positive definite symmetric 3×3 matrix N. Recall that the largest eigenvalue $\sigma^*(x)$, which coincides with the Euclidean norm $\|N(x)\|_2$, is given by $\sigma^*(x) = \sup_{\|\xi\|=1}(\bar{\xi} \cdot N(x)\xi)$, and the smallest eigenvalue $\sigma_*(x)$ is given by $\sigma_*(x) = \inf_{\|\xi\|=1}(\bar{\xi} \cdot N(x)\xi)$. In the following we define $n^* := \sup_D \sigma^*(x)$ and $n_* := \inf_D \sigma_*(x)$. Let $\lambda_1(D)$ again be the first Dirichlet eigenvalue for $-\Delta$ in D. The following lemma follows from the results of sections 4.2 and 4.4.

Lemma 4.31. *The operators $A_\tau : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$, $\tilde{A}_\tau : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$, $\tau > 0$, and $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ are self-adjoint. Furthermore, B is a positive compact operator. If $(N - I)^{-1}$ is a bounded positive definite matrix function on D, then A_τ is a positive definite operator and*

$$(A_\tau u - \tau B u, u)_{\mathcal{U}_0(D)} \geq \alpha \|u\|^2_{\mathcal{U}_0(D)} > 0 \quad \forall \quad 0 < \tau < \frac{\lambda_1(D)}{n^*}.$$

If $N(I - N)^{-1}$ is a bounded positive definite matrix function on D, then $\tilde{A}_\tau$ is a positive definite operator and

$$\big(\tilde{A}_\tau u - \tau B u, u\big)_{\mathcal{U}_0(D)} \geq \alpha \|u\|^2_{\mathcal{U}_0(D)} > 0 \quad \forall \quad 0 < \tau < \lambda_1(D).$$

Note that the kernel of $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is given by

$$\text{Kernel}(B) = \Big\{ u \in \mathcal{U}_0(D) \quad \text{such that } u := \nabla\varphi,\ \varphi \in H^1(D) \Big\}.$$

To prove the existence of transmission eigenvalues we use Theorem 4.30. In particular, we need to ensure that assumption 2 of this theorem is satisfied for the operators $A_\tau : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$, $\tilde{A}_\tau : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$, $\tau > 0$, and $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$. To this end, we need to consider the corresponding interior transmission eigenvalue problems for a ball with a constant index of refraction. Let $B_R \subset \mathbb{R}^3$ be a ball of radius R centered at the origin and let $n_0 > 0$ be a constant different from one. In [50] it is shown, by using separation of variables, that

$$\left.\begin{aligned} \operatorname{curl}\operatorname{curl} w - k^2 n_0 w = 0 \\ \operatorname{curl}\operatorname{curl} v - k^2 v = 0 \end{aligned}\right\} \quad \text{in} \quad B_R, \tag{4.102}$$

$$\left.\begin{aligned} w \times \nu = v \times \nu \\ \operatorname{curl} w \times \nu = \operatorname{curl} v \times \nu \end{aligned}\right\} \quad \text{on} \quad \partial B_R \tag{4.103}$$

has a countable discrete set of eigenvalues. Denote by k_{R,n_0} the first transmission eigenvalue, which is the smallest zero of the determinants

$$W_p(k) = \det\begin{pmatrix} j_p(kR) & j_p(k\sqrt{n_0}R) \\ -j_p'(kR) & -\sqrt{n_0}\, j_p'(k\sqrt{n_0}R) \end{pmatrix} \quad \text{for} \quad p \geq 1, \tag{4.104}$$

where j_p is the spherical Bessel function of order p. We call $u^{B_R,n_0} := w^{B_R,n_0} - v^{B_R,n_0}$ the eigenfunction corresponding to k_{R,n_0}. We have that $u^{B_R,n_0} \in \mathcal{U}_0(B_R)$ and

$$\int_{B_R} \frac{1}{n_0 - 1} (\operatorname{curl}\operatorname{curl} u^{B_R,n_0} - k_{R,n_0}^2 u^{B_R,n_0}) \cdot (\operatorname{curl}\operatorname{curl} \overline{u}^{B_R,n_0} - k_{R,n_0}^2 n_0 \overline{u}^{B_R,n_0})\, dx = 0. \tag{4.105}$$

By definition, the eigenvectors u^{B_R,n_0} for (4.102)–(4.103) are not in the kernel of $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$.

Remark 4.4. The multiplicity of transmission eigenvalues is finite since if k_1 is a transmission eigenvalue, then the kernel of $I - \tau_0 A_{\tau_0}^{-1/2} B A_{\tau_0}^{-1/2}$ or $I - \tau_0 \tilde{A}_{\tau_0}^{-1/2} B \tilde{A}_{\tau_0}^{-1/2}$, where $\tau_0 := k_0^2$, is finite since the operators $\tau_0 A_{\tau_0}^{-1/2} B A_{\tau_0}^{-1/2}$ (if $1/(n-1) > \gamma > 0$) and $\tau_0 \tilde{A}_{\tau_0}^{-1/2} B \tilde{A}_{\tau_0}^{-1/2}$ (if $n/(1-n) > \gamma > 0$) are compact and self-adjoint.

The above discussion provides all the necessary ingredients to apply Theorem 4.30 to (4.98) to prove the existence of an infinite discrete set of transmission eigenvalues [33].

Theorem 4.32. *Assume that $N \in L^\infty(D, \mathbb{R}^{3\times 3})$, satisfies either of the following assumptions for every $\xi \in \mathbb{C}^3$ such that $\|\xi\| = 1$ and some constants $\alpha > 0$ and $\beta > 0$:*

(1) $1 + \alpha \leq n_* \leq (\bar{\xi} \cdot N(x)\xi) \leq n^* < \infty$;

(2) $0 < n_* \leq (\bar{\xi} \cdot N(x)\xi) \leq n^* < 1 - \beta$.

Then there exists an infinite countable set of transmission eigenvalues corresponding to (4.52)–(4.53) *with* $+\infty$ *as the only accumulation point.*

Proof. First let us suppose that assumption (1) holds. This assumption also implies that

$$0 < \frac{1}{n^* - 1} \leq (\bar{\xi} \cdot (N(x) - I)^{-1} \xi) \leq \frac{1}{n_* - 1} < \infty.$$

Therefore, from Lemma 4.31, A_τ and B satisfy the requirement of Theorem 4.30, with $U = \mathcal{U}_0(D)$, and also assumption 1 of Theorem 4.30, with $\tau_0 \leq \lambda_1(D)/n^*$. Next let k_{1,n_*} be the first transmission eigenvalue for the ball B of radius $R = 1$ and $n_0 := n_*$. This transmission eigenvalue is the smallest zero of (4.104) for $R := 1$ and $n_0 := n_*$. By a scaling argument, it is obvious that $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue corresponding to the ball of radius $\epsilon > 0$ with an index of refraction n_*. Now take $\epsilon > 0$ small enough such that D contains $m := m(\epsilon) \geq 1$ disjoint balls $B_\epsilon^1, B_\epsilon^2, \ldots, B_\epsilon^m$ of radius ϵ, i.e., $\overline{B_\epsilon^j} \subset D$, $j = 1, \ldots, m$, and $\overline{B_\epsilon^j} \cap \overline{B_\epsilon^i} = \emptyset$ for $j \neq i$. Then $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue for each of these balls with index of refraction n_*, and we let $u^{B_\epsilon^j, n_*} \in \mathcal{U}_0(B_\epsilon^j)$, $j = 1, \ldots, m$, be the corresponding eigenfunction. The extension by zero $\tilde{u}^j$ of $u^{B_\epsilon^j, n_*}$ to the whole of D is obviously in $\mathcal{U}_0(D)$ due to the boundary conditions on $\partial B_{\epsilon,n_*}^j$. Furthermore, the vectors $\{\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^m\}$ are linearly independent and orthogonal in $\mathcal{U}_0(D)$ since they have disjoint supports, and from (4.105)

$$0 = \int_D \frac{1}{n_0 - 1} (\mathrm{curl}\,\mathrm{curl}\,\tilde{u}^j - k_{\epsilon,n_*}^2 \tilde{u}^j) \cdot (\mathrm{curl}\,\mathrm{curl}\,\overline{\tilde{u}^j} - k_{\epsilon,n_*}^2 n_0 \overline{\tilde{u}^j})\,dx \tag{4.106}$$

for $j = 1, \ldots, m$.

Let $\mathcal{U}$ be the m-dimensional subspace of $\mathcal{U}_0(D)$ spanned by $\{\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^m\}$. Since each $\tilde{u}^j$, $j = 1, \ldots, m$, satisfies (4.106) and each has disjoint supports, we have that for $\tau_1 := k_{\epsilon,n_*}^2$ and for every $\tilde{u} \in \mathcal{U}$

$$\left(A_{\tau_1} \tilde{u}^j - \tau_1 B \tilde{u}^j, \tilde{u}^j \right)_{\mathcal{U}_0(D)} \tag{4.107}$$

$$\leq \int_D \frac{1}{n_0 - 1} \left(\mathrm{curl}\,\mathrm{curl}\,\tilde{u}^j - k_{\epsilon,n_*}^2 \tilde{u}^j \right) \cdot \left(\mathrm{curl}\,\mathrm{curl}\,\overline{\tilde{u}^j} - k_{\epsilon,n_*}^2 n_0 \overline{\tilde{u}^j} \right) dx = 0. \tag{4.108}$$

This means that assumption 2 of Theorem 4.30 is also satisfied, and therefore we can conclude that there are $m(\epsilon)$ transmission eigenvalues (counting multiplicity) inside $[\tau_0, k_{\epsilon,n_*}]$. Note that $m(\epsilon)$ and k_{ϵ,n_*} both go to $+\infty$ as $\epsilon \to 0$. Since the multiplicity of each eigenvalue is finite we have shown, by letting $\epsilon \to 0$, that there exists an infinite countable set of transmission eigenvalues that accumulate at ∞.

If the index of refraction is such that assumption (2) of this theorem holds, then we have that

$$0 < \frac{n_*}{1 - n_*} \leq N(x)(I - N(x))^{-1} \leq \frac{n^*}{1 - n^*} < \infty$$

and therefore according to Lemma 4.31, $\tilde{A}_\tau$ and B, $\tau > 0$, satisfy the requirements and assumption 1 of Theorem 4.30 with $U = \mathcal{U}_0(D)$ for $\tau_0 \leq \lambda_1(D)$. In this case, based on

(4.100) the rest of the proof for checking the validity of assumption 2 of Theorem 4.30 goes exactly in the same way as for the previous case if one replaces n_* by n^*. ☐

The following theorem provides lower and upper bounds for the first transmission eigenvalue.

Theorem 4.33. *Let $k_{1,D,N(x)}$ be the first transmission eigenvalue for* (4.52)–(4.53) *and let α and β be positive constants. Denote by k_{1,D,n_*} and k_{1,D,n^*} the first transmission eigenvalue of* (4.52)–(4.53) *for $N = n_* I$ and $N = n^* I$, respectively.*

1. *If $||N(x)||_2 \geq \alpha > 1$, then $0 < k_{1,D,n^*} \leq k_{1,D,N(x)} \leq k_{1,D,n_*}$.*
2. *If $0 < ||N(x)||_2 \leq 1-\beta$, then $0 < k_{1,D,n_*} \leq k_{1,D,N(x)} \leq k_{1,D,n^*}$.*

Proof. We sketch the proof for the case of $||N(x)||_2 \geq \alpha > 1$. Obviously, for any $u \in \mathcal{U}_0(D)$ we have

$$\begin{aligned}
&\frac{\|\nabla\times\nabla\times u - \tau u\|_D^2 + \tau^2\|u\|_D^2}{(n^*-1)\|\nabla\times u\|_D^2}\\
&\quad\leq \frac{\big((N-I)^{-1}(\nabla\times\nabla\times u-\tau u),(\nabla\times\nabla\times u-\tau u)\big)_D+\tau^2\|u\|_D^2}{\|\nabla\times u\|_D^2}\\
&\quad\leq \frac{\|\nabla\times\nabla\times u - \tau u\|_D^2 + \tau^2\|u\|_D^2}{(n_*-1)\|\nabla\times u\|_D^2}.
\end{aligned} \tag{4.109}$$

Therefore we have that for an arbitrary $\tau > 0$

$$\lambda_1(\tau,D,n^*)-\tau \leq \lambda(\tau,D,N(x))-\tau \leq \lambda_1(\tau,D,n_*)-\tau, \tag{4.110}$$

where $\lambda_1(\tau,D,n^*)$, $\lambda(\tau,D,N(x))$, and $\lambda_1(\tau,D,n_*)$ are given by

$$\lambda_1(\tau,D) = \inf_{u\in\mathcal{W}_0(D)} \frac{(A_\tau u,u)_{\mathcal{U}}}{(Bu,u)_{\mathcal{U}}} \tag{4.111}$$

corresponding to the index of refraction n^*, $N(x)$, and n_*, respectively. Now for $\tau_1 := k_{1,D,n^*}^2$ we have that $\lambda(\tau_1,D,N(x))-\tau_1 \geq 0$. Again using (4.110) for $\tau_2 := k_{1,D,n_*}^2$ we have that $\lambda(\tau_2,D,N(x))-\tau_2 \leq 0$. Then by continuity of the mapping $\tau \to \lambda_1(\tau,D,N(x))$ there is an eigenvalue corresponding to $D, N(x)$ between k_{1,D,n^*} and k_{1,D,n_*}. To complete the proof we need to show that this is the first eigenvalue for $D, N(x)$. Indeed, if $k_{1,D,N(x)} < k_{1,D,n^*}$, then from (4.110) $\lambda_1(\tau_3,D,n^*)-\tau_3 \leq 0$ for $\tau_3 := k_{1,D,N(x)}^2$. On the other hand, for $\tau_0 > 0$ sufficiently small we have $\lambda_1(\tau_0,D,n^*)-\tau_0 \geq 0$, which means that there is a transmission eigenvalue for D, n^* less than the first one, which is a contradiction. The theorem now follows. ☐

Recalling that $k_{1,D,N(x)}$ can be computed from the far field measurements, our approach to estimating n_* and n^* is based on computing a constant n such that $k_{1,D,N(x)}$ is the first transmission eigenvalue corresponding to (4.52)–(4.53) with $N := nI$ for this n. From the above theorem, which shows that transmission eigenvalues for n constant are monotonically decreasing with respect to n, we have that $n_* \leq n \leq n^*$. To fully justify this idea one needs to show that for a constant index of refraction the first transmission eigenvalue depends continuously on n. This result is proved in [30].

4.6 Partially Coated Objects

In certain applications the scattering object is (possibly) partially coated by a thin layer of a highly conductive material. Such problems arise, for example, in the detection of decoys and the testing of the integrity of coatings. In this section we formulate the inverse problem for a coated anisotropic medium, state the main theorem of the LSM, and show that the same solution of the far field equation that is used to determine the support of the inhomogeneity can also be used to determine the surface conductivity, which gives information about the thickness and physical properties of the coating. Let D be the support of the anisotropic dielectric having the (matrix) index of refraction N. We assume that D and the 3×3 matrix valued function $N(x)$ satisfy the assumptions stated at the beginning of this chapter. In addition we assume that the boundary $\partial D = \overline{\partial D_u} \cup \overline{\partial D_c}$ is split into two open disjoint parts ∂D_u and ∂D_c, where ∂D_c is the portion of the boundary coated by a thin layer of highly conductive material. The physical properties of the thin coating layer are described by the positive function $\eta > 0$, called the surface conductivity, which is defined and bounded on ∂D_c [5]. We assume that the surface conductivity satisfies $\eta(x) \geq \eta_0 > 0$ on ∂D_c. Note that the case when $\partial D_c = \emptyset$ becomes the problem considered previously in this chapter, whereas the case when $\partial D_u = \emptyset$ corresponds to a fully coated obstacle.

The scattering of time-harmonic electromagnetic plane waves E^i, H^i, given by (3.5), by the (possibly) partially coated anisotropic medium leads to the same set of equations given at the beginning of this chapter for the interior electromagnetic field E, H in D and the scattered electromagnetic field E^s, H^s in $\mathbb{R}^3 \setminus \overline{D}$, namely,

$$\left.\begin{array}{l} \operatorname{curl} E^s - ikH^s = 0 \\ \operatorname{curl} H^s + ikE^s = 0 \end{array}\right\} \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{4.112}$$

$$\left.\begin{array}{l} \operatorname{curl} E - ikH = 0 \\ \operatorname{curl} H + ikN(x)E = 0 \end{array}\right\} \quad \text{in } D, \tag{4.113}$$

together with Silver–Müller radiation condition (4.5). On the boundary ∂D the tangential component of the total electric field is continuous, whereas the tangential component of the total magnetic field is continuous across the uncoated part ∂D_u and satisfies the so-called conducting boundary condition [5] on the coated part ∂D_c, i.e.,

$$\nu \times (E^s + E^i) - \nu \times E = 0 \qquad \text{on } \partial D, \tag{4.114}$$

$$\nu \times (H^s + H^i) - \nu \times H = 0 \qquad \text{on } \partial D_u, \tag{4.115}$$

$$\nu \times (H^s + H^i) - \nu \times H = \eta(x)\, \nu \times \Big[(E^s + E^i) \times \nu\Big] \qquad \text{on } \partial D_c. \tag{4.116}$$

The well-posedness of this problem is established in [20]. In particular it is shown that the transmission problem (4.112)–(4.116) has a unique solution $E \in X(D, \partial D_c)$, $E^s \in X(\mathbb{R}^3 \setminus \overline{D}, \partial D_c)$, $H \in H(\operatorname{curl}, D)$, $H^s \in H(\operatorname{curl}, D_e)$, and this solution depends continuously on the incident field E^i, H^i in the respective norms (see Chapter 3 for the definition of these spaces).

The support D of the (partially) coated anisotropic obstacle can be determined by the LSM from knowledge of the electric far field patterns $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d \in \Omega$ and $p \in \mathbb{R}^3$. A uniqueness result for the support D is proved in [20] by using ideas similar to those in the proof of Theorem 4.8. Indeed, all the results of Sections 4.1–4.3 can be proved for the current problem with the necessary modifications arising from the use of a different solution space for the corresponding interior transmission problem

$$\operatorname{curl}\operatorname{curl} E^z - k^2 N(x) E^z = 0 \quad \text{in } D, \tag{4.117}$$

$$\operatorname{curl}\operatorname{curl} E_0^z - k^2 E_0^z = 0 \quad \text{in } D, \tag{4.118}$$

$$\nu \times E^z - \nu \times E_0^z = \nu \times E_e(\cdot, d, q) \quad \text{on } \partial D, \tag{4.119}$$

$$\nu \times \operatorname{curl} E^z - \nu \times \operatorname{curl} E_0^z = \nu \times \operatorname{curl} E_e(\cdot, d, q) \quad \text{on } \partial D_u, \tag{4.120}$$

$$\nu \times \operatorname{curl} E^z - \nu \times \operatorname{curl} E_0^z = \nu \times \operatorname{curl} E_e(\cdot, d, q) + ik\eta(x)\nu \times \left[(E_0^z + E_e(\cdot, z, q)) \times \nu\right] \quad \text{on } \partial D_c. \tag{4.121}$$

The interior transmission problem (4.117)–(4.121) is studied in [36] by the same method as the one presented in Section 4.2. In particular, it is shown that (4.117)–(4.121) has a unique solution $E_0^z \in L^2(D)$, $E^z \in L^2(D)$ such that $E^z - E_0^z \in \mathcal{U}(D)$ and $\nu \times E_0^z|_{\partial D_c} \in L^2(\partial D_c)$ provided that $z \in D$ and k is not an η-transmission eigenvalue, which are the values of $k > 0$ for which the uniqueness of interior transmission problem (4.117)–(4.121) fails. In addition in [36] it is shown that η-transmission eigenvalues form a subset of the transmission eigenvalues defined in Section 4.2 and therefore are at most discrete.

As the reader already knows, the LSM looks for a solution $g \in L_t^2(\Omega)$ to the far field equation

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))ds(d) = E_{e,\infty}(\hat{x}, z, q), \qquad z, q \in \mathbb{R}^3,$$

where now $E_\infty(\hat{x}, d, p)$ is the far field corresponding to the scattering problem for a partially coated anisotropic media (4.112)–(4.116). The following theorem can be proved in exactly the same way as Theorem 4.26.

Theorem 4.34. *Assume that k is not an η-transmission eigenvalue and let F be the far field operator corresponding to the scattering problem* (4.112)–(4.116). *Then*

1. *for $z \in D$ and a given $\epsilon > 0$ there exists a $g_z^\epsilon \in L_t^2(\Omega)$ such that*

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \epsilon,$$

and the corresponding Herglotz function $E_{g_z^\epsilon}$ converges to E_0^z in $L^2(D) \cap L^2(\partial D_c)$ as $\epsilon \to 0$, where E_z^0, E_z satisfies the interior transmission problem (4.117)–(4.121).

2. *for a fixed $\epsilon > 0$, we have that*

$$\lim_{z \to \partial D} \|E_{g_z^\epsilon}\|_{L^2(D) \cap L^2(\partial D_c)} = \infty \quad \text{and} \quad \lim_{z \to \partial D} \|g_z^\epsilon\|_{L_t^2(\Omega)} \to \infty.$$

3. *for $z \in \mathbb{R}^3 \setminus D$ and a given $\epsilon > 0$, there exists $g_z^\epsilon \in L_t^2(\Omega)$ satisfying*

$$\|Fg_z^\epsilon - E_{e,\infty}(\hat{x}, z, q)\|_{L_t^2(\Omega)} < \epsilon$$

and such that

$$\lim_{\epsilon \to 0} \|E_{g_z^\epsilon}\|_{L^2(D) \cap L^2(\partial D_c)} = \infty \quad \text{and} \quad \lim_{\epsilon \to 0} \|g_z^\epsilon\|_{L_t^2(\Omega)} \to \infty.$$

We emphasize that the LSM for the determination of the support of the scattering object is independent of the physical properties of the scattering object, as it is based on the solution of the far field equation which has the same form for all of our scattering problems (remember that the right-hand side can be the electric far field pattern of either a magnetic dipole or an electric dipole).

Having determined D by the LSM, it is possible to use the approximate solution g_z^ϵ to the far field equation to estimate η. In particular, this is possible since $E_{g_z^\epsilon}$ approximates E_0^z, where E_0^z, E_z is the unique solution of the interior transmission problem (4.117)–(4.121). Before doing this we remark that a uniqueness result for η can be found in [22], [39] (another uniqueness theorem can be shown in a way similar to that in [29] for the scalar case).

In the following we formally derive an equation for η. The procedure is based on the application of Green's formulas to E_z^0, E_z, which are less smooth than needed. This technical difficulty can be avoided by connecting the solution of the interior transmission problem with a direct scattering problem. We omit the details here and direct the reader to [22], [39] and to the end of Section 6.3 for a similar situation.

Theorem 4.35. *Assume that k is neither a Maxwell eigenvalue nor an η-transmission eigenvalue and that $\Im(N) = 0$. Then, for any point z in D we have that*

$$\int_{\partial D_c} \eta\,|\nu \times (E_0^z + E_e(\cdot, z, q))|^2\, ds = -\frac{k^2}{6\pi}\|q\|^2 + \Re\left(E_0^z(z)\right), \tag{4.122}$$

where E^z, E_0^z is a solution to the interior transmission problem (4.117)–(4.121).

Proof. Let E^z and E_0^z be the unique solution of the interior transmission problem (4.117)–(4.121). Applying the vector Green's formula we have that

$$\int_{\partial D} \left(\nu \times E^z \cdot \operatorname{curl} \overline{E^z} - \nu \times \overline{E^z} \cdot \operatorname{curl} E^z\right) ds = 2i \int_D \overline{E}^z \cdot \Im(N) E^z\, dx = 0. \tag{4.123}$$

On the other hand, using the transmission conditions across ∂D and defining $W^z := E_0^z + E_e(\cdot, z, q)$, we have that

$$\begin{aligned} &\int_{\partial D} \left(\nu \times E^z \cdot \operatorname{curl} \overline{E^z} - \nu \times \overline{E^z} \cdot \operatorname{curl} E^z\right) ds \\ &= \int_{\partial D} \left(\nu \times W^z \cdot \operatorname{curl} \overline{W^z} - \nu \times \overline{W^z} \cdot \operatorname{curl} W^z\right) ds \\ &\quad -2ik \int_{\partial D_c} \eta\,|(\nu \times W^z) \times \nu|^2\, ds. \end{aligned} \tag{4.124}$$

Again using the vector Green's formula, along with the integral representation formula, and connecting the radiating solution $E_e(\cdot, z, q)$ to its far field pattern in a way similar to that in the proof of Theorem 3.12, we obtain

$$\int_{\partial D} \left(\nu \times W^z \cdot \operatorname{curl} \overline{W^z} - \nu \times \overline{W^z} \cdot \operatorname{curl} W^z\right) ds \tag{4.125}$$

$$= -\frac{ik^3}{3\pi} \|q\|^2 + ikq \cdot \left[E_0^z(z) + \overline{E_0^z}(z)\right].$$

Hence, combining (4.123), (4.124), and (4.125) we have that

$$2ik \int_{\partial D_c} \eta \, |(\nu \times W^z) \times \nu|^2 \, ds = -\frac{ik^3}{3\pi} \|q\|^2 + ikq \cdot \left[E_0^z(z) + \overline{E_0^z}(z)\right],$$

which proves the result. ◻

Equation (4.122) immediately provides a lower bound for η:

$$\sup_{x \in \partial D_c} \eta(x) \geq \frac{-\frac{k^2}{6\pi}\|q\|^2 + \Re\left(E_0^z(z)\right)}{\int_{\partial D} |\nu \times (E_0^z + E_e(\cdot, z, q))|^2 \, ds}.$$

In the particular case when the obstacle is fully coated (i.e., $\partial D_c = \partial D$ and the coating is homogeneous (i.e., η is a constant)), (4.122) provides an estimate for the constant η. Finally, let $\tilde{\eta}$ denote the extension of η by zero to the whole boundary ∂D. Then we can rewrite (4.122) as

$$\int_{\partial D} \tilde{\eta} \, |\nu \times (E_0^z + E_e(\cdot, z, q))|^2 \, ds = -\frac{k^2}{6\pi} \|q\|^2 + \Re\left(E_0^z(z)\right), \quad z \in B_r \subset D. \tag{4.126}$$

Viewing (4.126) as an integral equation for $\tilde{\eta}$, it is possible to solve it to determine η as well as the coated part ∂D_c (see [29] in the scalar case).

We end with the important remark that E_0^z can be approximated by the electric Herglotz wave function $E_{g_z^\epsilon}$, with the kernel being the approximate solution to the far field equation provided by Theorem 4.34.

Chapter 5
The Inverse Scattering Problem for Thin Objects

In many wave scattering problems occurring in practice one encounters thin objects, where the thickness of the object is small compared to the wavelength and other characteristic dimensions. Such problems can be mathematically modeled by a boundary value problem for an open surface in $\mathbb{R}^3$. In this chapter we consider the inverse electromagnetic scattering problem of determining the shape of a thin object, referred to as a screen, from knowledge of the incident time-harmonic electromagnetic plane wave and the electric far field pattern of the scattered wave at a fixed frequency. We will consider two types of thin objects, namely, a perfect conductor and a thin object which is perfectly conducting on one side and on the other side behaves as an imperfect conductor (e.g., a metallic thin object coated on one side). The latter is called a mixed screen. The main goal of this chapter is to establish the validity of the LSM for solving the inverse scattering problem for this class of problems following [24] and [31].

5.1 Scattering by Thin Objects

Before studying the inverse scattering problem for screens we need to set up the analytical framework for the direct scattering problem. To this end let Γ denote the screen. In the following, Γ is assumed to be a bounded, simply connected, oriented, piecewise smooth open surface in $\mathbb{R}^3$ bounded by a piecewise smooth boundary curve l. We consider Γ as part of a piecewise smooth boundary ∂D of some bounded domain $D \subset \mathbb{R}^3$. Let ν denote the normal vector to Γ that coincides with the outward normal vector defined almost everywhere on ∂D. For a vector field u, we denote by $\nu \times u^+|_\Gamma$, $\gamma_T^+ u|_\Gamma$, and $\nu \cdot u^+|_\Gamma$ ($\nu \times u^-|_\Gamma$, $\gamma_T^- u|_\Gamma$, and $\nu \cdot u^-|_\Gamma$) the restriction to Γ of the traces $\nu \times u|_{\partial D}$, $\gamma_T u|_{\partial D}$, and $\nu \cdot u|_{\partial D}$, respectively, from the outside (from the inside) of ∂D, where $\gamma_T u := \nu \times (u \times \nu)$ is the tangential component of u.

The perfectly conducting screen. After eliminating the magnetic field, the scattering of electromagnetic incident waves by the open surface Γ with a perfectly conducting boundary condition is modeled as the problem of finding a scattered field E^s that satisfies

$$\operatorname{curl}\operatorname{curl} E^s - k^2 E^s = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Gamma}, \tag{5.1}$$

$$\gamma_T^{\pm} E^s = f \quad \text{on} \quad \Gamma, \tag{5.2}$$

$$\lim_{r\to\infty} (\operatorname{curl} E^s \times x - ikrE^s) = 0, \tag{5.3}$$

where $f := -\gamma_T^{\pm} E^i$.

Mixed screen. The scattering of electromagnetic waves by the open surface Γ, which is on one side a perfect conductor and on the other side an imperfect conductor, leads to the boundary value problem

$$\operatorname{curl}\operatorname{curl} E^s - k^2 E^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Gamma}, \tag{5.4}$$

$$\gamma_T^{-} E^s = f \quad \text{on } \Gamma^-, \tag{5.5}$$

$$\nu \times \operatorname{curl} E^s - i\lambda\gamma_T^{+} E^s = h \quad \text{on } \Gamma^+, \tag{5.6}$$

$$\lim_{r\to\infty} (\operatorname{curl} E^s \times x - ikrE^s) = 0, \tag{5.7}$$

where $f := -\gamma_T^- E^i$ and $h := -\left(\nu \times \operatorname{curl} E^{i+} - i\lambda\gamma_T^+ E^i\right)$, and E^s is the scattered electric field. Here $\lambda := \lambda(x)$ is a piecewise continuous function on Γ such that $\lambda(x) \geq \lambda_0 > 0$ and depends on the thickness and material properties of the coating as well as on the wave number k.

For both of the above direct scattering problems we assume that the incident field E^i is the electric field of time-harmonic electromagnetic plane waves given by (4.4).

The direct scattering problem for perfectly conducting screens is studied in [1], [14], and [24], whereas the direct scattering problem for mixed screens is studied in [31]. We summarize here the main results concerning the direct problems, which play an essential role in the study of the inverse problem. To this end, we need to properly define the trace spaces on the open surface Γ. In particular, we first introduce the spaces

$$H^{1/2}(\Gamma) := \left\{ u|_\Gamma : u \in H^{1/2}(\partial D) \right\},$$

$$\tilde{H}^{1/2}(\Gamma) := \left\{ u \in H^{1/2}(\Gamma) : \operatorname{supp} u \subseteq \overline{\Gamma} \right\}.$$

Now we denote by $H^{-1/2}(\Gamma)$ the dual space of $\tilde{H}^{1/2}(\Gamma)$ and by $\tilde{H}^{-1/2}(\Gamma)$ the dual space of $H^{1/2}(\Gamma)$, with $L^2(\Gamma)$ as the pivot space. Note that $\tilde{H}^{-1/2}(\Gamma)$ can also be identified with

$$\tilde{H}^{-1/2}(\Gamma) := \left\{ u \in H^{-1/2}(\Gamma) : \operatorname{supp} u \subseteq \overline{\Gamma} \right\}.$$

Moreover, we define

$$H^{-1/2}(\operatorname{Div}, \Gamma) := \left\{ u|_\Gamma : u \in H^{-1/2}(\operatorname{Div}, \partial D) \right\},$$

$$H^{-1/2}(\operatorname{Curl}, \Gamma) := \left\{ u|_\Gamma : u \in H^{-1/2}(\operatorname{Curl}, \partial D) \right\},$$

where $H^{-1/2}(\operatorname{Div}, \partial D)$ and $H^{-1/2}(\operatorname{Curl}, \partial D)$ are defined by (3.10) and (3.15), respectively. We denote by $\tilde{H}^{-1/2}(\operatorname{Div}, \Gamma)$ the dual space of $H^{-1/2}(\operatorname{Curl}, \Gamma)$ in the duality pairing $\left\langle H^{-1/2}(\operatorname{Div}, \partial D), H^{-1/2}(\operatorname{Curl}, \partial D)\right\rangle$. This space contains tangential fields u such that

$u \in (\tilde{H}^{-1/2}(\Gamma))^3$, $\operatorname{Div} u \in \tilde{H}^{-1/2}(\Gamma)$, and

$$\int_{\partial D} u \cdot \operatorname{Grad} v \, ds + \int_{\partial D} \operatorname{Div} u \, v \, ds = 0$$

for every $v \in H^{\frac{3}{2}}(\Gamma)$. The latter means that the normal trace of u at the edge l is well defined and is zero, that is, $\nu_l \cdot u|_l = 0$, where ν_l is the exterior normal vector at the boundary l of Γ (for smooth screens see [1] and [41], and for piecewise smooth screens see [15], [16], and [14]). Note also that a function $u \in \tilde{H}^{-1/2}(\operatorname{Div}, \Gamma)$ can be extended by zero to a function in $H^{-1/2}(\operatorname{Div}, \partial D)$. It is known that the trace operators $\nu \times u^{\pm}|_\Gamma$ and $\gamma_T^{\pm}|_\Gamma$ map $H(\operatorname{curl}, B_R \setminus \overline{\Gamma})$ into $H^{-1/2}(\operatorname{Div}, \Gamma)$ and $H^{-1/2}(\operatorname{Curl}, \Gamma)$, respectively.

Theorem 5.1. *Given $f \in H^{-1/2}(\operatorname{Curl}, \Gamma)$, the scattering problem* (5.1)–(5.3) *has a unique solution $E \in H_{loc}(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$, which depends continuously on f with respect to the norms in the respective spaces.*

For later use we note that the unique solution of (5.1)–(5.3) is given by

$$\begin{aligned} E(x) &= \frac{1}{k^2} \operatorname{curl}_x \operatorname{curl}_x \int_\Gamma \phi(y) \Phi(x,y) \, ds(y), \\ &= \int_\Gamma \phi(y)\Phi(x,y)\,ds(y) + \frac{1}{k^2} \operatorname{grad}_x \int_\Gamma \operatorname{div}_\Gamma \phi(y) \Phi(x,y) \, ds(y) \end{aligned} \tag{5.8}$$

for $x \in \mathbb{R}^3 \setminus \overline{\Gamma}$ with $\phi \in \tilde{H}^{-1/2}(\operatorname{Div}, \Gamma)$ being the unique solution of the integral equation

$$A\phi = f, \tag{5.9}$$

where the integral operator $A : \tilde{H}^{-1/2}(\operatorname{Div}, \Gamma) \to H^{-1/2}(\operatorname{Curl}, \Gamma)$ is an isomorphism and is given by

$$(A\phi)(x) := \gamma_T \left(\int_\Gamma \phi(y)\Phi(x,y)\,ds(y) + \frac{1}{k^2} \operatorname{grad}_x \int_\Gamma \operatorname{div}_\Gamma \phi(y) \Phi(x,y) \, ds(y) \right).$$

We next consider the direct scattering problem for a mixed screen (5.4)–(5.7). Due to the impedance condition the natural space for the solution of this problem is $X(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$, which we recall is defined by

$$X(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) := \{u \in H_{loc}(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) : \nu \times u^+|_\Gamma \in L^2_t(\Gamma)\},$$

equipped with the natural norm

$$\|u\|^2_{X(\operatorname{curl}, B_R \setminus \overline{\Gamma})} := \|u\|^2_{H(\operatorname{curl}, B_R \setminus \overline{\Gamma})} + \|\nu \times u^+\|^2_{L^2(\Gamma)}. \tag{5.10}$$

We need to specify the space of $\gamma_T^- E$ for $E \in X(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$, which is obviously a closed subspace of $H^{-1/2}(\operatorname{Curl}, \Gamma)$ since $X(\operatorname{curl}, B_R \setminus \overline{\Gamma})$ is a closed subspace of $H(\operatorname{curl}, B_R \setminus \overline{\Gamma})$.

(Note that $\gamma_T^+ E \in H^{-1/2}(\mathrm{Curl},\Gamma)\cap L_t^2(\Gamma)$.) To this end we introduce

$$Y(\Gamma) := \Big\{ f \in (H^{-1/2}(\Gamma))^3 \,:\, \exists u \in H(\mathrm{curl}, B_R \setminus \overline{\Gamma}), \\ \gamma_T^+ u|_\Gamma \in L_t^2(\Gamma) \quad \text{and} \quad f = \gamma_T^- u|_\Gamma \Big\},$$

which is a Banach space with respect to the norm

$$\|f\|^2_{Y(\Gamma)} := \inf\{\|u\|^2_{H(\mathrm{curl},B_R\setminus\overline{\Gamma})} + \|\nu \times u\|^2_{L^2(\Gamma_I)}\}, \tag{5.11}$$

where the infimum is taken over all functions $u \in H(\mathrm{curl}, B_R \setminus \overline{\Gamma})$ such that $\gamma_T^+ u|_\Gamma \in L_t^2(\Gamma)$ and $f = \gamma_T^- u|_\Gamma$. Again let ∂D be a closed surface containing Γ, let B_R be a large ball containing D, and let $u \in H(\mathrm{curl}, B_R \setminus \overline{\Gamma})$ be such that $\nu \times u|_{\partial B_R} = 0$, $\gamma_T^+ u|_\Gamma \in L_t^2(\Gamma)$, and $f = \gamma_T^- u|_\Gamma$. Applying integration by parts in D and $B_R \setminus \overline{D}$ and using the fact that the tangential components of functions in $H(\mathrm{curl}, B_R \setminus \overline{\Gamma})$ are continuous across $\partial D \setminus \overline{\Gamma}$, we obtain

$$\begin{aligned} \langle f,\phi\rangle &:= \int_\Gamma (\nu \times u^-)\cdot(\gamma_T^-\phi)\,ds \\ &= -\int_{B_R} (\mathrm{curl}\, u\cdot\phi - u\cdot\mathrm{curl}\,\phi)\,dv + \int_\Gamma (\nu\times u^+)\cdot(\gamma_T^+\phi)\,ds. \end{aligned} \tag{5.12}$$

Here $\phi \in X(\mathrm{curl}, B_R \setminus \overline{\Gamma})$ is such that $\nu \times \phi|_{\partial B_R} = 0$. In particular (5.12) defines a duality relation and characterizes the dual space $Y'(\Gamma)$ of $Y(\Gamma)$ (see also (3.11), (5.12)).

Theorem 5.2. *For any boundary data $f \in Y(\Gamma)$ and $h \in L_t^2(\Gamma)$ there exists a unique solution $E \in X(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ of* (5.4)–(5.7) *which depends continuously on (f,h) with respect to the norm in the respective spaces.*

Let $E_\infty(\hat{x},d,p)$ denote the far field pattern of the scattered field E^s corresponding to either problem (5.1)–(5.3) or (5.4)–(5.7). The *inverse scattering problem* we consider in this chapter is to determine Γ from the knowledge of the electric far field $E_\infty(\hat{x},d,p)$ for $\hat{x},d \in \Omega$ and $p \in \mathbb{R}^3$. (Note that we do not assume a priori any knowledge of λ.)

We end this section with a sketch of the proof of *a uniqueness theorem* for the inverse problem. The proof presented here follows the general framework for the uniqueness of electromagnetic inverse obstacle problems given in [81]. The main ingredients are the well-posedness of the corresponding direct problem and the mixed reciprocity relation (3.6).

Theorem 5.3. *Let B denote the boundary condition of either type* (5.2) *(perfectly conducting screen) or type* (5.5), (5.6) *(mixed screen). Assume that Γ_1 and Γ_2 are two open surfaces satisfying the assumptions in the beginning of this section, with the boundary conditions B_1 and B_2, respectively, such that the far field patterns coincide for all incident directions d and $p \in \mathbb{R}^3$. Then $\Gamma_1 = \Gamma_2$ and $B_1 = B_2$.*

Proof. The proof proceeds along the same lines as the proof of Theorem 3.1. First, by Rellich's lemma (Theorem 2.5) from the coincidence of the far field pattern it follows that the corresponding scattered fields E_1^s and E_2^s coincide in the unbounded component G of $\mathbb{R}^3 \setminus (\overline{\Gamma}_1 \cup \overline{\Gamma}_2)$. Using the mixed reciprocity relation exactly in the same way as in the proof of the first part of Theorem 3.1, we have that

$$E_{1,e}^s(x,z,p) = E_{2,e}^s(x,z,p), \qquad x,z \in G,$$

where $E_{1,e}^s(\cdot,z,p)$ and $E_{2,e}^s(\cdot,z,p)$ are the scattered electric fields due to the incident field being an electric dipole located at z with polarization p for Γ_1 and Γ_2, respectively.

Now assume that $\Gamma_1 \neq \Gamma_2$. Then we can find points $x^* \in \Gamma_1$ and $x^* \notin \Gamma_2$, such that $\nu(x^*)$ is defined, and consider $z_n = x^* + \frac{1}{n}\nu(x^*) \in G$. Then in view of the well-posedness of the direct scattering problem for Γ_2 with boundary condition B_2, on one hand we obtain that

$$\lim_{n\to\infty} \|B_1(E_{2,e}^s(x,z_n,p)) - B_1(E_{2,e}^s(x,x*,p))\|_{X_1} = 0,$$

where X_1 is the boundary data space corresponding to Γ_1 with boundary condition B_1. On the other hand we find that

$$\lim_{n\to\infty} \|B_1(E_{2,e}^s(x,z_n,p))\|_{X_1} = \lim_{n\to\infty} \|B_1(E_{1,e}^s(x,z_n,p))\|_{X_1} = \infty$$

because the boundary condition for $E_{1,e}^s(x,z_n,p)$ is given in terms of the electric dipole which does not belong to the boundary data space due to the singularity at $z = x^*$. We have arrived at a contradiction, and hence $\Gamma_1 = \Gamma_2$.

Next, denoting $\Gamma = \Gamma_1 = \Gamma_2$, $E^s = E_1^s = E_2^s$, and the total field $E = E_1 = E_2$, we assume that we have a different boundary condition $B_1 \neq B_2$. If there is an open part Γ_0 of Γ where $\lambda_1 \neq \lambda_2$ (on the same side of Γ), then from $(\lambda_1 - \lambda_2)\gamma_T^+ E = 0$ we deduce that $\gamma_T^+ E = 0$, and from the impedance condition we deduce $\nu \times E^+ = 0$ on Γ_0 as well. Then from the Holmgren's theorem (Theorem 2.4 in [80]; see also the second part of the proof of Theorem 3.1) we conclude that $E = 0$ in $\mathbb{R}^3 \setminus \overline{D}$, which is a contradiction. For the same reason, it is not possible to have, on an open part $\Gamma_0 \subset \Gamma$, different types of boundary conditions since this also leads to zero Cauchy data for the total field on Γ_0. This proves that $B_1 = B_2$, which ends the proof of the theorem. □

5.2 Approximation Theorems

As we have seen, approximations properties of electromagnetic Herglotz wave functions are fundamental in the justification of the LSM. We show that appropriate traces of the scattered field can be approximated by the corresponding traces of the electric Herglotz wave functions given by (3.27). First, we proceed with the operator $\mathcal{H}_c : L_t^2(\Omega) \to H^{-1/2}(\mathrm{Curl},\Gamma)$ defined by

$$\mathcal{H}_c g := \gamma_T\, E_g, \tag{5.13}$$

where we recall that $\gamma_T u := (\nu \times u) \times \nu$.

Theorem 5.4. *The range of $\mathcal{H}_c$ is dense in $H^{-1/2}(\mathrm{Curl},\Gamma)$.*

Proof. As in Theorem 3.6 it suffices to consider the operator $\mathcal{H}_c$ with E_g written as

$$E_g(x) = \int_\Omega \mathrm{e}^{-ikx\cdot d} g(d)\, ds(d). \tag{5.14}$$

The dual operator $\mathcal{H}_c^\top : \tilde{H}^{-1/2}(\mathrm{Div},\Gamma) \to L_t^2(\Omega)$ of the operator $\mathcal{H}_c$ is such that for every $\alpha \in \tilde{H}^{-1/2}(\mathrm{Div},\Gamma)$ and $g \in L_t^2(\Omega)$ we have

$$\langle \mathcal{H}_c g, \alpha\rangle_{H_{\mathrm{curl}}^{-1/2}, \tilde{H}_{\div}^{-1/2}} = \left\langle g, \mathcal{H}_c^\top \alpha\right\rangle_{L_t^2, L_t^2},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the denoted spaces. It is enough to show that the dual operator $\mathcal{H}_c^\top$ is injective. Then similarly to the proof of Theorem 3.6, the result follows from the fact that the range of $\mathcal{H}_c$ can be characterized as $\overline{(\mathrm{Range}\,\mathcal{H}_c)} = {}^a\mathrm{Kern}\,\mathcal{H}_c^\top$, where the annihilator ${}^a(\ )$ is defined by (3.24). In particular, the injectivity of $\mathcal{H}_c^\top$ implies that $\overline{(\mathrm{Range}\,\mathcal{H}_c)} = H^{-1/2}(\mathrm{Curl},\Gamma)$. Simple computations shows that the dual operator $\mathcal{H}_c^\top$ is defined by

$$\mathcal{H}_c^\top(\alpha) = d \times \left\{ \int_\Gamma \alpha(x) e^{-ikx\cdot d} \alpha\, ds \right\} \times d.$$

One sees that $\mathcal{H}_c^\top(\alpha)$ coincides with the far field pattern of the electric single layer potential

$$P(z) = \frac{1}{k^2} \mathrm{curl\ curl} \int_\Gamma \alpha(x)\Phi(x,z)\, ds(x), \qquad z \notin \overline{\Gamma},$$

with $\Phi(x,z)$ given by (2.9). The potential $P(z)$ is well defined for $z \in \mathbb{R}^3 \setminus \overline{\Gamma}$ and satisfies $\mathrm{curl\,curl}\, P - k^2 P = 0$. In addition $P : \tilde{H}^{-1/2}(\mathrm{Div},\Gamma) \to H(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$. Now let us assume that $\mathcal{H}_c^\top(\alpha) = 0$. This means that the far field pattern of P is zero, and from Rellich's lemma $P = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$. As $z \to \Gamma$ we have from the jump relations of the electric single layer potential [41], [50] that

$$\nu \times \mathrm{curl}\, P^+ - \nu \times \mathrm{curl}\, P^-|_\Gamma = \alpha. \tag{5.15}$$

Hence from (5.15) we conclude that $\alpha = 0$. Thus $\mathcal{H}^\top$ is injective, which proves the theorem. □

We remark that Theorem 5.4 implies that any function $f \in H^{-1/2}(\mathrm{Curl},\Gamma)$ can be approximated arbitrarily closely by the tangential trace of a Herglotz function E_g.

Next, we prove a similar result for the mixed trace of the Herglotz wave function, which will be used to study the inverse scattering problem for mixed screens. We first define the operator $\mathcal{H}_m : L_t^2(\Omega) \to Y(\Gamma) \times L_t^2(\Gamma)$ by

$$\mathcal{H}_m g := \begin{cases} \gamma_T E_g & \text{on } \Gamma^-, \\ \nu \times \mathrm{curl}\, E_g - i\lambda\gamma_T E_g & \text{on } \Gamma^+, \end{cases} \tag{5.16}$$

with the Herglotz function E_g written as in (5.14).

Theorem 5.5. *The range of $\mathcal{H}_m$ is dense in $Y(\Gamma) \times L_t^2(\Gamma)$.*

Proof. Let $H := Y(\Gamma) \times L_t^2(\Gamma)$ with dual $H^* := Y'(\Gamma) \times L_t^2(\Gamma)$ in the componentwise duality pairing. By the same reasoning as in the proof of Theorem 5.4 we need to prove the dual operator $\mathcal{H}_m^\top : H^* \to L_t^2(\Gamma)$ is injective. Straightforward calculations show that

$$\mathcal{H}_m^\top[\alpha,\beta] = d \times \left\{ \int_\Gamma e^{-ikx\cdot d} \alpha \, ds - ik\, d \times \int_\Gamma e^{-ikx\cdot d}(\nu \times \beta)\, ds - i\lambda \int_\Gamma e^{-ikx\cdot d} \beta \, ds \right\} \times d.$$

Note that α and β are tangential fields defined on Γ. Obviously, $\mathcal{H}^\top[\alpha,\beta]$ coincides with the far field pattern of the combined electric and magnetic potentials

$$Q(z) = \frac{1}{k^2} \text{curl curl} \int_\Gamma \Phi(x,z)\alpha(x)\, ds(x) + \text{curl} \int_\Gamma \Phi(x,z)(\nu \times \beta(x))\, ds(x) - i\lambda \frac{1}{k^2} \text{curl curl} \int_\Gamma \Phi(x,z)\beta(x) ds(x), \qquad z \notin \overline{\Gamma}.$$

The potential $Q(z)$ again is well defined for $z \in \mathbb{R}^3 \setminus \overline{\Gamma}$ and satisfies $\text{curl}\,\text{curl}\, Q - k^2 Q = 0$. Assume now that $\mathcal{H}_m^\top[\alpha,\beta] = 0$. This means that the far field pattern of Q is zero, and from Rellich's lemma $Q = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$. As $z \to \Gamma$ we have that

$$\nu \times Q^+ - \nu \times Q^-|_\Gamma = \nu \times \beta, \tag{5.17}$$

$$\nu \times \text{curl}\, Q^+ - \nu \times \text{curl}\, Q^-|_\Gamma = \alpha - i\lambda\beta, \tag{5.18}$$

where the jump relations are well defined in the sense of the L^2 limit (see [50, p. 172]) due to the relation (5.12) and the fact that β is a square integrable tangential field. Hence from (5.17) and (5.18) we conclude that $\alpha = \beta = 0$. Thus $\mathcal{H}^\top$ is injective, which proves the theorem. □

We remark that Theorem 5.5 implies that any pair $(f,g) \in Y(\Gamma) \times L_t^2(\Gamma)$ can be approximated arbitrarily closely by the mixed trace of the same Herglotz wave function E_g.

5.3 Solution of the Inverse Problem

In this section we employ the above analysis to justify the LSM for determining the shape of an open surface Γ. The justification of the LSM for an obstacle with an empty interior differs from the one previously discussed for the cases of obstacles with nonempty interior due to the fact that there are no interior sampling points.

Let $E_\infty(\hat{x},d,p)$, $\hat{x},d \in \Omega$, and $p \in \mathbb{R}^3$ be the far field pattern corresponding to either the scattering problem (5.1)–(5.3) or (5.4)–(5.7), and consider the far field operator $F : L_t^2(\Omega) \to L_t^2(\Omega)$, which is again defined by (3.31). The LSM is based on the *far field equation*

$$(Fg)(\hat{x}) = E_\infty(\hat{x}), \qquad \hat{x} \in \Omega, \tag{5.19}$$

where the right-hand side E_∞ is the far field pattern of a suitable (to be defined later) radiating solution to Maxwell's equations.

Consider the solution operator $\mathcal{S}_c$, which maps the data $f \in H^{-1/2}(\mathrm{Curl},\Gamma)$ to the far field pattern of the solution to (5.1)–(5.3). Similarly, we consider the operator $\mathcal{S}_m$, which maps the boundary data $(f,h) \in Y(\Gamma) \times L^2_t(\Gamma)$ to the far field pattern of the radiating solution to (5.4)–(5.7). Hence F and $\mathcal{S}_c$ are related through the relation

$$(Fg) = -\mathcal{S}_c(\mathcal{H}_c g), \tag{5.20}$$

where $\mathcal{H}_c$ is given by (5.13) for the case of a perfectly conducting screen, and similarly F and $\mathcal{S}_m$ are related through the relation

$$(Fg) = -\mathcal{S}_m(\mathcal{H}_m g), \tag{5.21}$$

where $\mathcal{H}_m$ is given by (5.16) for the case of a mixed screen.

Lemma 5.6. *Suppose that there does not exist a Herglotz wave function such that its tangential component vanishes on Γ. Then the linear operator $\mathcal{S}_c : H^{-1/2}_{\mathrm{curl}}(\Gamma) \to L^2_t(\Omega)$ is injective, compact, and has dense range.*

Proof. The injectivity follows from the uniqueness of the scattering problem and Rellich's lemma. Since $\mathcal{S}_c$ is the composition of the bounded operator that takes the boundary data to the scattered solution on a large sphere ∂B_R of radius R and the compact operator (see [50, Theorem 6.8]) that maps data on ∂B_R to the corresponding far field, $\mathcal{S}_c$ is compact.

Next, we prove that the range of $\mathcal{S}_c$ is dense by showing that the dual operator $\mathcal{S}_c^\top : \tilde{H}^{-1/2}(\mathrm{Div},\Gamma) \times L^2_t(\Gamma)$ is injective. To this end it is easy to see that

$$\langle \mathcal{S}_c(c), g\rangle = \frac{1}{4\pi}\int_\Gamma c \cdot \left[\nu \times \mathrm{curl}\, \tilde{E}^- - \nu \times \mathrm{curl}\, E_g\right] ds,$$

where now $\tilde{E} \in H(\mathrm{curl}, D_e)$ is the solution of (5.1)–(5.3) with boundary data

$$\gamma_T^\pm \tilde{E} = \gamma_T^\pm E_g \quad \text{on} \quad \Gamma \tag{5.22}$$

and denotes the jump of the denoted function across Γ. Hence

$$4\pi\,\mathcal{S}_c^\top g = \left[\nu \times \mathrm{curl}\,\tilde{E} - \nu \times \mathrm{curl}\, E_g\right] \in \tilde{H}^{-1/2}(\mathrm{Div},\Gamma). \tag{5.23}$$

Now let $\mathcal{S}_c^\top g \equiv 0$. Then from (5.22), (5.23), and the fact that the tangential components of E_g and its kernel are continuous across Γ, we obtain that $[\nu \times \tilde{E}] = 0$ and $[\nu \times \mathrm{curl}\,\tilde{E}] = 0$ across Γ. Since $\tilde{E} \in H_{loc}(\mathrm{curl}, \mathbb{R}^3)$ is a radiating solution to Maxwell's equations, we have that $\tilde{E} = 0$. Hence (5.22) implies that $\nu \times E_g = 0$ on Γ. But this is not possible by assumption, and therefore E_g must be identically zero, which can happen only if the kernel $g \equiv 0$. Hence, $\mathcal{S}_c^\top$ is injective, which implies that $\mathcal{S}_c$ has dense range. The proof is now complete. We remark that there are special geometries of Γ that allow for a nontrivial electric Herglotz wave function E_g to satisfy $\nu \times E_g = 0$ on Γ. ◻

Lemma 5.7. *The linear operator $\mathcal{S}_m : Y(\Gamma) \times L^2_t(\Gamma) \to L^2_t(\Omega)$ is injective, compact, and has dense range.*

Proof. The injectivity and compactness of $\mathcal{S}_m$ can be proved as in the first part of Lemma 5.6. To prove that $\mathcal{S}_m$ has dense range we will again show that the dual operator $\mathcal{S}_m^\top : L_t^2(\Omega) \to Y'(\Gamma) \times L_t^2(\Gamma)$ is injective. Long, but straightforward, calculations [31] show that

$$\begin{aligned}\langle \mathcal{S}_m(f,h), g\rangle &= \frac{1}{4\pi}\int_\Gamma f\cdot(\nu\times\operatorname{curl}\tilde{E}^- - \nu\times\operatorname{curl}E_g^-)\,ds \\ &+ \frac{1}{4\pi}\int_\Gamma h\cdot(\gamma_T^+ E_g - \tilde{\gamma}_T^+ E)\,ds,\end{aligned}$$

where $\tilde{E} \in X(D_e,\Gamma)$ is the solution of the (5.4)–(5.7) with boundary data

$$\begin{aligned}&\gamma_T^-\tilde{E} = \gamma_T E_g \quad \text{on} \quad \Gamma, \\ &\nu\times\operatorname{curl}\tilde{E}^+ - i\lambda\gamma_T^+\tilde{E} = \nu\times\operatorname{curl}E_g - i\lambda\gamma_T E_g \quad \text{on} \quad \Gamma.\end{aligned} \tag{5.24}$$

Hence

$$4\pi\,\mathcal{S}_m^\top g = \begin{cases} (\nu\times\operatorname{curl}\tilde{E}^- - \nu\times\operatorname{curl}E_g^-) \in Y(\Gamma)', \\ (\gamma_T^+ E_g - \gamma_T^+\tilde{E}) \in L_t^2(\Gamma). \end{cases} \tag{5.25}$$

Now let $\mathcal{S}_m^\top g \equiv 0$. Then (5.25) and (5.24) imply that $\nu\times(\tilde{E}-E_g)^\pm = 0$ and $\nu\times(\operatorname{curl}\tilde{E} - \operatorname{curl}E_g)^\pm = 0$. But since $\tilde{E}$ is a radiating solution while E_g is an entire solution, we now see that E_g must be identically zero, which can happen only if the kernel $g\equiv 0$. Hence, $\mathcal{S}_m^\top$ is injective, which implies that $\mathcal{S}_m$ has dense range. This ends the proof of the lemma. □

The following lemmas will help us to choose the right-hand side of the far field equation (5.19) appropriately. We denote by $C_0^\infty(L)$ the space of C^∞ functions with compact support in $\overline{L}$.

Lemma 5.8. *For any open surface L and a tangential field $\alpha_L \in \left(C_0^\infty(L)\right)^3$ we define $E_\infty^L \in L_t^2(\Omega)$ by*

$$E_\infty^L := \left(\hat{x}\times\int_L \alpha_L(y)e^{-ik\hat{x}\cdot y}ds(y)\right)\times\hat{x}. \tag{5.26}$$

Then, $E_\infty^L \in \operatorname{Range}(\mathcal{S}_c)$ if and only if $L\subset\Gamma$.

Lemma 5.9. *For any open surface L and two tangential fields $\alpha_L, \beta_L \in \left(C_0^\infty(L)\right)^3$ we define $E_\infty^L \in L_t^2(\Omega)$ by*

$$E_\infty^L := \hat{x}\times\left(\int_L \alpha_L(y)e^{-ik\hat{x}\cdot y}ds(y) + \hat{x}\times\int_L \beta_L(y)e^{-ik\hat{x}\cdot y}ds(y)\right)\times\hat{x}. \tag{5.27}$$

Then, $E_\infty^L \in \operatorname{Range}(\mathcal{S}_m)$ if and only if $L\subset\Gamma$.

In the following we give the proof of Lemma 5.9. The proof of Lemma 5.8 is a particular case of the proof of Lemma 5.9.

Proof. First, assume that $L \subset \Gamma$ and let $\alpha_L, \beta_L \in \left(C_0^\infty(L)\right)^3$ be tangential fields. We again consider a closed boundary ∂D that contains Γ. We notice that (5.27) is the far field pattern of the potential V defined by

$$V(x) := \frac{1}{k^2} \operatorname{curl}\operatorname{curl} \int_L \alpha_L(y)\Phi(x,y)ds(y) + \frac{i}{k}\operatorname{curl} \int_L \beta_L(y)\Phi(x,y)ds(y).$$

Since the extensions $\tilde{\alpha}_L$ and $\tilde{\beta}_L$ of α_L and β_L, respectively, by zero to the whole boundary ∂D are C^∞ functions, we have that V is smooth enough and satisfies $\operatorname{curl}\operatorname{curl} V - k^2 V = 0$. Moreover, using the jump relations of the vector potentials across ∂D [41], [50], we have that V satisfies the following mixed boundary conditions on Γ:

$$\begin{aligned}
f :&= \gamma_T^- V = -\frac{i}{2k}\tilde{\beta}_L \times \nu + \frac{1}{k^2}(A\tilde{\alpha}_L)^- + \frac{i}{k}(B\tilde{\beta}_L)^-,\\
h :&= \left(\nu \times \operatorname{curl} V^+ - i\lambda\gamma_T^+ V\right) = \frac{1}{2}\tilde{\alpha}_L + \frac{\lambda}{2k}\tilde{\beta}_L \times \nu + \nu \times (B\tilde{\alpha}_L)^+ \\
&+ \frac{i}{k}\nu \times (A\tilde{\beta}_L)^+ - \frac{i\lambda}{k^2}(A\tilde{\alpha}_L)^+ + \frac{\lambda}{k}(B\tilde{\beta}_L)^+,
\end{aligned}$$

where the boundary operators A and B are given by

$$\begin{aligned}
(A\phi)^\pm(x) &= \gamma_T^\pm \operatorname{curl}\operatorname{curl} \int_{\partial D} \phi(y)\Phi(x,y)ds(y),\\
(B\phi)^\pm(x) &= \gamma_T^\pm \operatorname{curl} \int_{\partial D} \phi(y)\Phi(x,y)ds(y), \qquad x \in \partial D.
\end{aligned}$$

Since $f \in Y(\Gamma)$ and $h \in L_t^2(\Gamma)$ we have that E_∞^L is in the range of $\mathcal{S}_m$.

Now let $L \not\subset \Gamma$ and assume, on the contrary, that $E_\infty^S \in \operatorname{Range}(\mathcal{S}_m)$; i.e., there exist $f \in Y(\Gamma)$ and $h \in L_t^2(\Gamma)$ such that $E_\infty^L = E_\infty^s$, where E_∞^s is the far field pattern of the radiating solution E^s to (5.4)–(5.7) corresponding to this boundary data f, h. Hence by Rellich's lemma and the unique continuation principle, we have that $E^s(x)$ and

$$V(x) := \frac{1}{k^2} \operatorname{curl}\operatorname{curl} \int_L \alpha_L(y)\Phi(x,y)ds(y) + \frac{i}{k}\operatorname{curl} \int_L \beta_L(y)\Phi(x,y)ds(y)$$

coincide for $x \in \mathbb{R}^3 \setminus (\overline{\Gamma} \cup \overline{L})$. Now let $x_0 \in L$, $x_0 \notin \Gamma$, and let $B_\epsilon(x_0)$ be a small ball with its center at x_0 such that $B_\epsilon(x_0) \cap \Gamma = \emptyset$. Then E^s is analytic in $B_\epsilon(x_0)$ while V has a singularity at x_0, which is a contradiction. This proves the lemma. $\square$

Since, if $L \not\subset \Gamma$, $E_\infty^L \notin \operatorname{Range}\mathcal{S}_c$ with E_∞^L given by (5.26), or $E_\infty^L \notin \operatorname{Range}\mathcal{S}_m$ with E_∞^L given by (5.27), applying regularization techniques to the compact operators $\mathcal{S}_c$ or $\mathcal{S}_m$, respectively, we have the following results.

Lemma 5.10. *Suppose that there does not exist a Herglotz wave function such that its tangential component vanishes on* Γ. *Consider the equation*

$$\mathcal{S}_c(f) = E_\infty^L, \qquad f \in H^{-1/2}(\mathrm{Curl}, \Gamma),$$

where E_∞^L *is given by* (5.26)*, and let* $L \not\subset \Gamma$. *Then for every* $\delta > 0$ *there exists* f_α *depending on the regularization parameter* $\alpha > 0$ *such that*

$$\|\mathcal{S}_c(f_\alpha) - E_\infty^L\|_{L_t^2(\Omega)} < \delta \quad \text{and} \quad \lim_{\alpha\to 0} \|f_\alpha\|_{H^{-1/2}(\mathrm{Curl},\Gamma)} = \infty.$$

Lemma 5.11. *Consider the equation*

$$\mathcal{S}_m(f,h) = E_\infty^L, \qquad (f,h) \in Y(\Gamma) \times L_t^2(\Gamma),$$

where E_∞^L *is given by* (5.27)*, and let* $L \not\subset \Gamma$. *Then for every* $\delta > 0$ *there exists* (f_α, h_α) *depending on the regularization parameter* $\alpha > 0$ *such that*

$$\|\mathcal{S}_m(f_\alpha, h_\alpha) - E_\infty^L\|_{L_t^2(\Omega)} < \delta \quad \text{and} \quad \lim_{\alpha\to 0} \|(f_\alpha, h_\alpha)\|_{Y(\Gamma)\times L_t^2(\Gamma)} = \infty.$$

Note that in the above lemmas $\alpha \to 0$ as $\delta \to 0$.

We now have all the ingredients to derive the LSM for screens. Denote by $\mathcal{W}$ the set of piecewise smooth open surfaces L and consider the far field equation

$$(Fg)(\hat{x}) = E_\infty^L(\hat{x}), \qquad L \in \mathcal{W}, \tag{5.28}$$

where E_∞^L is given by (5.26) if F is the far field operator corresponding to (5.1)–(5.3), and E_∞^L is given by (5.27) if F is the far field operator corresponding to (5.4)–(5.7). Combining Lemmas 5.8 and 5.10, using the factorization (5.20) of the far field operator F together with the fact that any boundary function $f \in H^{-1/2}(\mathrm{Curl}, \Gamma)$ can be approximated arbitrarily close by $\mathcal{H}_c g$ for a $g \in L_t^2(\Omega)$ (Theorem 5.4), and finally, making use of the continuity of the operator $\mathcal{S}_c$, we can prove the following theorem.

Theorem 5.12. *Assume that* Γ *is a bounded, oriented, piecewise smooth open surface and that there does not exist a Herglotz wave function such that its tangential component vanishes on* Γ. *Then if* F *is the far field operator corresponding to* (5.1)–(5.3) *and* E_∞^L *is given by* (5.26)*, we have the following:*

1. *If* $L \subset \Gamma$, *then for any arbitrary* $\epsilon > 0$ *there exists a solution* $g_\epsilon^L \in L_t^2(\Omega)$ *of the inequality*

   $$\|Fg_\epsilon^L - E_\infty^L\|_{L_t^2(\Omega)} < \epsilon,$$

   and the corresponding trace of the Herglotz wave function $\mathcal{H}_c g_\epsilon^L$ *converges in* $H^{-1/2}(\mathrm{Curl}, \Gamma)$ *as* $\epsilon \to 0$.

2. *If* $L \not\subset \Gamma$, *then for any arbitrary* $\epsilon > 0$*, every solution* $g_\epsilon^L \in L_t^2(\Omega)$ *of the inequality*

   $$\|Fg_\epsilon^L - E_\infty^L\|_{L_t^2(\Omega)} < \epsilon$$

   is such that

   $$\lim_{\epsilon\to 0} \|g_\epsilon^L\|_{L_t^2(\Omega)} = \infty \quad \text{and} \quad \lim_{\epsilon\to 0} \|\mathcal{H}_c g_\epsilon^L\|_{H(curl, B_R)} = \infty,$$

   where $\mathcal{H}_c g_\epsilon^L$ *is as defined by* (5.13).

Note that, unlike the case of an obstacle with nonempty interior, the LSM fails for special geometries instead of special frequencies.

Similarly, combining Lemmas 5.9 and 5.11, using the factorization (5.21) of the far field operator F together with the fact that any boundary data $(f,h) \in Y(\Gamma) \times L_t^2(\Gamma)$ can be approximated arbitrarily close by $\mathcal{H}_m g$ for a $g \in L_t^2(\Omega)$ (Theorem 5.5), and finally, making use of the continuity of the operator $\mathcal{S}_m$, we can prove the following theorem.

Theorem 5.13. *Assume that Γ is a bounded, oriented, and piecewise smooth open surface. Then if F is the far field operator corresponding to* (5.4)–(5.7) *and E_∞^L is given by* (5.27), *we have the following:*

1. *If $L \subset \Gamma$, then for any arbitrary $\epsilon > 0$ there exists a solution $g_\epsilon^L \in L_t^2(\Omega)$ of the inequality*
$$\|Fg_\epsilon^L - E_\infty^L\|_{L_t^2(\Omega)} < \epsilon,$$
and the corresponding trace of the Herglotz wave function $\mathcal{H}_m g_\epsilon^L$ converges in $Y(\Gamma) \times L_t^2(\Gamma)$ as $\epsilon \to 0$.

2. *If $L \not\subset \Gamma$, then for any arbitrary $\epsilon > 0$, every solution $g_\epsilon^L \in L_t^2(\Omega)$ of the inequality*
$$\|Fg_\epsilon^L - E_\infty^L\|_{L_t^2(\Omega)} < \epsilon$$
is such that
$$\lim_{\epsilon \to 0} \|g_\epsilon^L\|_{L_t^2(\Omega)} = \infty \quad \textit{and} \quad \lim_{\epsilon \to 0} \|\mathcal{H}_m g_\epsilon^L\|_{Y(\Gamma)\times L_t^2(\Gamma)} = \infty,$$
where $\mathcal{H}_m g_\epsilon^L$ is as defined by (5.16).

There are several ways to implement the LSM. One way is to replace E_∞^L in the far field equation (5.28) by an expression independent of L and test by sampling points. To this end, we note that as L degenerates to a point z with α_L, β_L an appropriate delta sequence, we have that the integral in (5.27) approaches

$$\frac{ik}{4\pi}\left[(\hat{x} \times q_1) \times \hat{x} e^{-ik\hat{x}\cdot z} + (\hat{x} \times q_2) e^{-ik\hat{x}\cdot z}\right],$$

where q_1, q_2 are two constant vectors, and this equation can be used for both types of screens considered here. Note that the first term is the electric far field of an electric dipole and the second term is the magnetic far field of an electric dipole. In particular, one can take $q_1 = q_2$ (which corresponds to choosing $\alpha_L = \beta_L$) and obtain the following limiting expression for E_∞^L:

$$\frac{ik}{4\pi}\left[(\hat{x} \times q) \times \hat{x} e^{-ik\hat{x}\cdot z} + (\hat{x} \times q) e^{-ik\hat{x}\cdot z}\right],$$

where q is a constant vector. These expressions for E_∞^L are valid for both types of screens. Hence, roughly speaking, independent of the physical properties of the screen, the shape of the screen Γ can be characterized as the set of points where the $L^2_t(\Omega)$ norm of an approximate (regularized) solution of the far field equation

$$(Fg)(\hat{x}) = \frac{ik}{4\pi}\left[(\hat{x}\times q_1)\times\hat{x}e^{-ik\hat{x}\cdot z} + (\hat{x}\times q_2)e^{-ik\hat{x}\cdot z}\right] \tag{5.29}$$

becomes large.

We end this chapter by noting that it is possible to use the approximate solution to the far field equation to determine λ as well. We refer the reader to [107] for an approach to doing this for a related problem in the scalar case.

5.4 Numerical Reconstruction of Screens

The majority of the theory and numerical results in this book relates to scatterers D with a nonzero volume. Both heuristically and mathematically we make use of properties of the far field equation for $z \in D$ and $z \notin D$. Remarkably, as shown in this chapter, the LSM can also be used to image objects with zero volume. In particular, we have described a method, culminating in equation (5.29), that allows us to use the LSM to approximate the shape of screens [31].

We show two representative results from [24] that demonstrate the power of the method. For a given screen, the authors solve the forward mixed screen problem using the Ultra Weak Variational Formulation (UWVF) of Maxwell's equations modified slightly to allow for the mixed boundary conditions. After adding noise to the far field pattern computed at the 42 points in Figure 3.1(a) using 42 incoming plane waves, this data is then used to solve the inverse problem using a regularized version of (5.29). In the first example, which again demonstrates how the LSM can handle disconnected scatterers with mixed boundary conditions, two square screens are reconstructed. The upper square is a perfect conductor on both upper and lower surfaces, while the lower square is a perfect conductor on the lower surface and has an impedance boundary condition with $\lambda = 2$ on the upper side. The results using $k = 2$ are shown in Figure 5.1.

The second example is an L-shaped scatterer, which is a perfect conductor on all surfaces except the inner side (i.e., the side facing the viewer in Figure 5.2) of the vertical square where there is an impedance boundary condition with $\lambda = 2$. The reconstruction with $k = 2$ (so the wavelength is 3.14) is shown in Figure 5.2. Despite the long wavelength in this case (compared to the scatterer) the method reconstructs the object. The perfectly conducting surface is sharper, but even the impedance surface is reconstructed well.

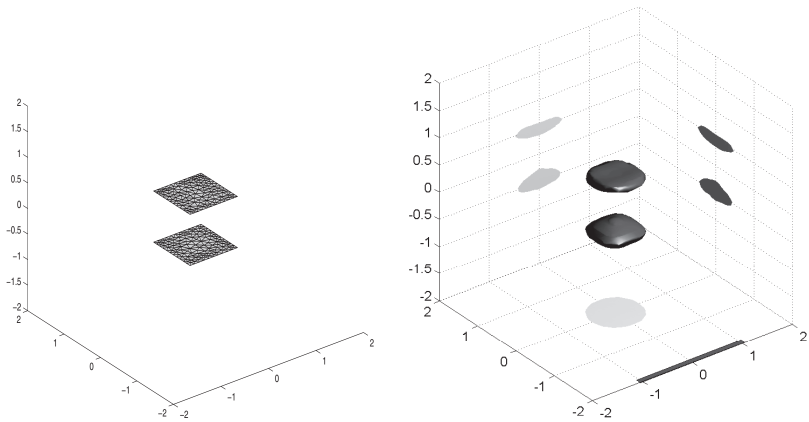


Figure 5.1. *Here, the right panel shows the reconstruction of the disconnected screens shown in the left panel. The upper screen is perfectly conducting, while the lower screen is perfectly conducting on the lower surface and imperfectly conducting on the upper surface. This figure is reproduced from* [31] *with permission.*

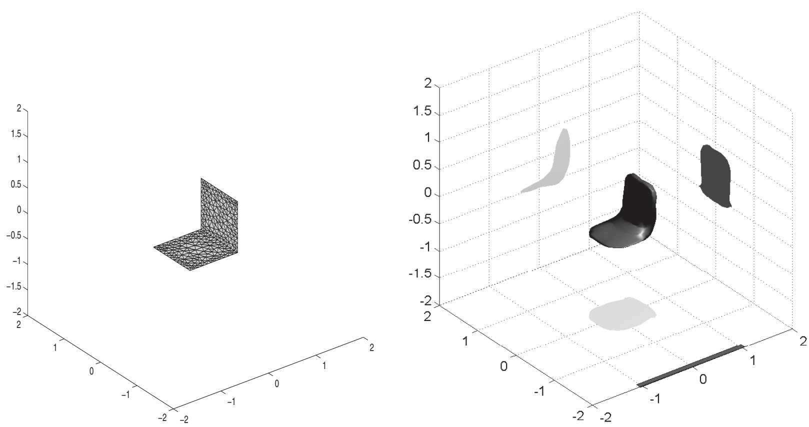


Figure 5.2. *Here, the right panel shows the reconstruction of the L-shaped screen shown in the left panel. All surfaces are perfectly conducting except the facing surface of the vertical screen, which is imperfectly conducting. This figure is reproduced from* [31] *with permission.*

Chapter 6
The Inverse Scattering Problem for Buried Objects

Up to now we have only discussed the inverse scattering problem for obstacles situated in a homogeneous background. However, in most applications the unknown target is embedded in an inhomogeneous background. The use of electromagnetic fields to detect buried objects has a long history and continues to be an active area of research [3], [4], [9], [32], [61], [62]. Of particular interest is the use of such methods to detect chemical waste deposits, examine urban infrastructure, and locate land mines. However, from a practical point of view, there are two main reasons why such imaging problems are challenging. The first is the difficulty of distinguishing the scattered field due to the target from the scattered fields due to the earth, the antenna, and, in particular, the air-earth interface. The second reason is that the material properties of the target are in general unknown. For example, a land mine can be made of metal or plastic, whereas a rusted barrel of chemical waste deposits is typically modeled by a complicated mixed boundary value problem involving a dielectric of unknown permittivity. Because of these reasons, traditional methods of imaging, such as the use of weak scattering approximations and nonlinear optimization techniques, remain problematic.

The linear sampling method (LSM) has a number of features which make it attractive for the imaging of buried objects. In particular, it is a linear method that does not ignore multiple scattering effects and determines the shape of a target without requiring any a priori knowledge of the target's physical properties. However, until recently, the implementation of the LSM for a nonhomogeneous background media required knowledge of the Green's function for the background media. This is obviously an unattractive feature if it is desired to use this method for the detection of buried objects, particularly if the scattering effects due to the antenna play a significant role. In order to overcome the problem of needing to compute the Green's function for the background media, a new version of the LSM, based on the reciprocity gap functional, was introduced by Colton and Haddar [46] for the scalar case and by Cakoni, Fares, and Haddar [32] for the vector case. However, in imaging nothing is free, and the price paid for avoiding the need to compute the Green's function is that one now needs to measure both the electric and magnetic fields corresponding to time-harmonic electric dipoles as incident fields.

We begin this chapter with a brief discussion of the LSM for objects buried in a known inhomogeneous background using near field measurements. However, the main focus here is the analysis of the reciprocity gap functional method for solving the inverse scattering problem for buried objects based on [32] and [46] (see also [63]). In order to make the presentation more friendly to the reader, we assume that the background is piecewise homogeneous and limit ourselves to the case of perfectly conducting objects and inhomogeneous media. The approach is extendable to more general backgrounds [95] as well as to more complicated objects, such as partially coated perfect conductors and inhomogeneous media. We refer the reader to [35] and [23] for discussions of these cases.

6.1 Scattering by Buried Objects

We consider the scattering of a time-harmonic electromagnetic field of frequency ω by a scattering object embedded in a piecewise homogeneous background medium in $\mathbb{R}^3$. Throughout this chapter we assume that the magnetic permeability $\mu_0 > 0$ of the background medium is a positive constant, whereas the electric permittivity $\epsilon(x)$ and conductivity $\sigma(x)$ are piecewise constant. Moreover, we assume that for $|x| = r > R$, for R sufficiently large, $\sigma = 0$ and $\epsilon(x) = \epsilon_0$. Then the electric field $\mathcal{E}$ and magnetic field $\mathcal{H}$ in the background medium satisfy the time-harmonic Maxwell's equations

$$\nabla \times \mathcal{E} - i\omega\mu_0\mathcal{H} = 0, \qquad \nabla \times \tilde{\mathcal{H}} + (i\omega\epsilon(x) - \sigma(x))\mathcal{E} = 0.$$

After an appropriate scaling [50] and elimination of the magnetic field, we now have that in the background medium $\mathcal{E}$ satisfies

$$\operatorname{curl}\operatorname{curl} E - k^2 n(x) E = 0,$$

where $\mathcal{E} = \frac{1}{\sqrt{\epsilon_0}} E$, $k^2 = \epsilon_0\mu_0\omega^2$, and $n(x) = \frac{1}{\epsilon_0}(\epsilon(x) + i\frac{\sigma(x)}{\omega})$. Note that the piecewise constant function $n(x)$ satisfies $n(x) = 1$ for $r > R$, $\Re(n) > 0$, and $\Im(n) \geq 0$. The surfaces across which $n(x)$ is discontinuous are assumed to be piecewise smooth and closed.

The incident field is considered to be the electric field of an electric dipole located at $x_0 \in \Lambda$ with polarization $p \in \mathbb{R}^3$, situated in a layer with constant index of refraction n_s, and is given by

$$E_e(x, x_0, p, k_s) := \frac{i}{k_s} \operatorname{curl}_x \operatorname{curl}_x p \frac{e^{ik_s|x-x_0|}}{4\pi|x - x_0|}, \tag{6.1}$$

where $k_s^2 = k^2 n_s$. We denote by $\mathbb{G}(x, x_0)$ the free space Green's tensor of the background medium and define $E^i(x) := E^i(x, x_0, p) = \mathbb{G}(x, x_0)p$, which satisfies

$$\operatorname{curl}\operatorname{curl} E^i(x) - k^2 n(x) E^i(x) = p\,\delta(x - x_0) \quad \text{in } \mathbb{R}^3, \tag{6.2}$$

where δ denotes the Dirac distribution. Note that E^i can be written as

$$E^i(x) = E_e(x, x_0, p, k_s) + E_b^s(x), \tag{6.3}$$

where $E_b^s = E_b^s(\cdot, x_0, p)$ is the electric scattered field due to the background medium.

The Green's function $E^i := E^i(x, x_0, p)$ satisfies the Silver–Müller radiation condition

$$\lim_{r\to\infty} (\operatorname{curl} E^i \times x - ikr E^i) = 0 \tag{6.4}$$

uniformly in $\hat{x} = x/|x|$, $r = |x|$.

Remark 6.1. The analysis of the inverse problems considered in this chapter allows for more complicated backgrounds. Possible cases are the following:

1. The index of refraction n of the background medium can be a piecewise continuous function in a bounded region of $\mathbb{R}^3$ (not necessarily piecewise constant) but n must still be equal to 1 outside a big ball.

2. It is also possible to consider the problem of objects buried in an unbounded multilayer medium. In this case, the radiation condition and mathematical analysis of the forward problem become more complicated (see [56] for the case of a two-layered medium).

In the above two cases, the following analysis of the inverse scattering problems remains essentially the same.

Let D denote the support of the scattering object embedded in the piecewise homogeneous background described above. We suppose that $\mathbb{R}^3 \setminus \overline{D}$ is connected and that the boundary ∂D is piecewise smooth. We denote by ν the outward unit normal defined almost everywhere on ∂D. We consider the cases when D is a perfect conductor and D is an inhomogeneous anisotropic media.

The scattering problem for a perfect conductor. The scattering problem for a buried perfect conductor D, given $E^i = E^i(\cdot, x_0, p) = \mathbb{G}(\cdot, x_0)p$, is to find the total field $E \in H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D} \cup \{x_0\})$ such that the scattered field $E^s := (E - E^i) \in H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D})$ satisfies

$$\text{curl curl } E - k^2 n(x) E = p\,\delta(x - x_0) \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \tag{6.5}$$

$$\nu \times E = 0 \quad \text{on} \quad \partial D, \tag{6.6}$$

$$\lim_{r \to \infty} \left(\text{curl } E^s \times x - ikr E^s \right) = 0. \tag{6.7}$$

The scattering problem for anisotropic media. We assume that the index of refraction of the scattering object is a symmetric matrix-valued function denoted by $N(x)$, $x \in \overline{D}$, whose entries are bounded, complex-valued, piecewise continuous functions such that

$$\bar{\xi} \cdot \Im(N)\xi \ge 0 \text{ and } \bar{\xi} \cdot \Re(N)\xi \ge \gamma |\xi|^2 \, \forall \xi \in \mathbb{C}^3 \text{ and all } x \in \overline{D}, \tag{6.8}$$

where γ is a positive constant. Note that we assume that the magnetic permittivity of the scattering object is the same as that of the background medium. Then the scattering problem for a buried anisotropic medium, given $E^i = E^i(\cdot, x_0, p) = \mathbb{G}(\cdot, x_0)p$, is to find $E^{int} \in H(\text{curl}, D)$ and $E^s = E - E^i \in H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D})$ such that (E^{int}, E) satisfies

$$\text{curl curl } E - k^2 n(x) E = p\,\delta(x - x_0) \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \tag{6.9}$$

$$\text{curl curl } E^{int} - k^2 N(x) E^{int} = 0 \quad \text{in} \quad D, \tag{6.10}$$

$$\nu \times E - \nu \times E^{int} = 0 \quad \text{on} \quad \partial D, \tag{6.11}$$

$$\nu \times \text{curl } E - \nu \times \text{curl } E^{int} = 0 \quad \text{on} \quad \partial D, \tag{6.12}$$

$$\lim_{r \to \infty} \left(\text{curl } E^s \times x - ikr E^s \right) = 0. \tag{6.13}$$

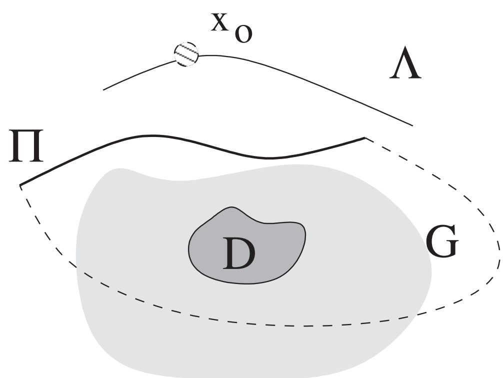


Figure 6.1. *Example of a geometry for the LSM.*

Note that (6.5) and (6.9), (6.10) are assumed to hold in the distributional sense, whereas the boundary conditions are valid in the sense of traces. Of course the continuity of E and curl E across the interfaces is assumed [93]. As already mentioned in Chapters 3 and 4 (see [20], [27], [78], [93]) both direct scattering problems are well posed.

6.2 Near Field Data

In this section we briefly describe the LSM for near field measurements. We consider a bounded domain G such that $\overline{D}$ is contained in G and an open surface Λ is contained in $\mathbb{R}^3 \setminus \overline{G}$. In this configuration (see Figure 6.1) G is the interrogation region and Λ is the surface, where the incident sources are placed. Let ∂G denote the piecewise smooth boundary of G. Let $\Pi \subseteq \partial G$ be a part of the boundary ∂G, where the measurements are made. Note that Λ may be a subset of ∂G and possibly the same as Π. The *inverse scattering problem* we are interested in is to determine D from knowledge of the tangential components $\nu \times E^s$ of the scattered electric field $E^s = E^s(\cdot, x_0, p)$ measured on Π for all point sources $x_0 \in \Lambda$ and two linearly independent polarizations p tangent to Λ at x_0. Here and in what follows, ν is always the outward unit normal to the surface under consideration unless otherwise stated.

Adapting the uniqueness proofs in Sections 2.1 and 4.1 (see also [81] and [20]) to the case of near field data, one can prove that, under appropriate assumptions on N, D is uniquely determined from a knowledge of the tangential component of the scattered electric field on Π corresponding to all $x_0 \in \Lambda$ and $p \in \mathbb{R}^3$. In the proofs one makes use of the unique continuation principle for the equation in the background (cf. [58] and [59] for the scalar case). For closely related uniqueness results for objects in a piecewise homogeneous medium, see [89], [90].

The main goal here is to reformulate the LSM in terms of the near field data. To this end we consider the *near field operator* $\mathcal{F} : L^2_t(\Lambda) \to L^2_t(\Pi)$ and the *near field equation*

$$(\mathcal{F}\varphi_z)(x) := \int_\Lambda \nu(x) \times E^s(x, y, \varphi_z(y))\, ds(y) = \nu(x) \times \mathbb{G}(x,z)q \tag{6.14}$$

for all $x \in \Pi$ and $z \in \mathbb{R}^3$, where $q \in \mathbb{R}^3$ and $\mathbb{G}(x,z)$ is the Green's tensor for the background medium. The LSM is based on finding a tangential field $\varphi_z \in L^2_t(\Lambda)$ that satisfies (6.14). We remind the reader that E^s is the scattered field due to the incident wave being $E^i(x) = E_e(x, x_0, p, k_s) + E^s_b(x)$ by a perfect conductor or an anisotropic medium. By superposition, $\mathcal{F}\varphi$ is the rotated tangential component on Π of the scattered electric field corresponding to the potential

$$(\mathcal{S}\varphi)(x) := \int_\Lambda \varphi(y)\mathbb{G}(x,y)\,ds(y) \tag{6.15}$$

as the incident wave. The analysis of the LSM with far field data done in Section 3.2 for the scattering problem (6.5)–(6.7), and in Sections 4.3 and 4.5 for the scattering problem (6.9)–(6.13), can be carried through by merely replacing the electric Herglotz function with the potential given by (6.15). To avoid repetition we state in the following the main theorem that provides the theoretical basis of the LSM for buried objects with near field data. The approximation properties of the potentials (6.15) that are needed for this analysis will be proved in the next section in the context of the reciprocity gap function.

In order to formulate our main result we need to consider the interior problem corresponding to (6.5)–(6.7),

$$\operatorname{curl}\operatorname{curl} E_z - k^2 n(x) E_z = 0 \quad \text{in} \quad D, \tag{6.16}$$

$$\nu \times E_z = \nu \times \mathbb{G}(\cdot, z)q \quad \text{on} \quad \partial D, \tag{6.17}$$

which for $z \in D$ has a solution $E_z \in H(\operatorname{curl}, D)$ provided that k is not a Maxwell's eigenvalue [27]. We call $k > 0$ a *Maxwell eigenvalue for D* if the homogeneous problem (6.16)–(6.17) (i.e., with $\mathbb{G}(\cdot, z) = 0$) has a nontrivial solution. We also consider the interior transmission problem corresponding to (6.9)–(6.13),

$$\operatorname{curl}\operatorname{curl} E_0^z - k^2 n(x) E_0^z = 0 \quad \text{in} \quad D, \tag{6.18}$$

$$\operatorname{curl}\operatorname{curl} E^z - k^2 N(x) E^z = 0 \quad \text{in} \quad D, \tag{6.19}$$

$$\nu \times E^z - \nu \times E_0^z = \nu \times \mathbb{G}(\cdot, z)q \quad \text{on} \quad \partial D, \tag{6.20}$$

$$\nu \times \operatorname{curl} E^z - \nu \times \operatorname{curl} E_0^z = \nu \times \operatorname{curl} \mathbb{G}(\cdot, z)q \quad \text{on} \quad \partial D, \tag{6.21}$$

which for $z \in D$ and appropriate assumptions (see the next section and [35]) has a unique solution $E^z \in L^2(D)$ and $E_0^z \in L^2(D)$. We call $k > 0$ a *transmission eigenvalue for D* if the homogeneous problem (6.18)–(6.21) (i.e., with $\mathbb{G}(\cdot, z) = 0$) has a nontrivial solution.

Theorem 6.1. *Assume that either*

(a) $\mathcal{F}$ *is the near field operator corresponding to* (6.5)–(6.7)*, and k is not a Maxwell eigenvalue, or*

(b) $\mathcal{F}$ *is the near field operator corresponding to* (6.9)–(6.13)*, and k is not a transmission eigenvalue.*

Then we have the following:

1. *For $z \in D$ and a given $\epsilon > 0$, there exists a $\varphi_z^\epsilon \in L^2_t(\Lambda)$ such that*

$$\|\mathcal{F}\varphi_z^\epsilon - \nu \times \mathbb{G}(\cdot, z)q\|_{L^2_t(\Pi)} < \epsilon,$$

and the corresponding potential $\mathcal{S}\varphi_z^\epsilon$ *converges to the solution* E^z *of* (6.16)–(6.17) *in* $H(\text{curl}, D)$ *in case* (a), *or to* E_0^z *where* E_0^z, E^z *is the solution of* (6.18)–(6.21) *in* $L^2(D)$ *in case* (b) *as* $\epsilon \to 0$.

2. *For a fixed* $\epsilon > 0$, *we have that*

$$\lim_{z\to\partial D} \|\mathcal{S}\varphi_z^\epsilon\|_X = \infty \quad \textit{and} \quad \lim_{z\to\partial D} \|\varphi_z^\epsilon\|_{L_t^2(\Lambda)} = \infty,$$

where $X := H(\text{curl}, D)$ *in case* (a) *and* $X := L^2(D)$ *in case* (b).

3. *For* $z \in \mathbb{R}^3 \setminus \overline{\Gamma}$ *and a given* $\epsilon > 0$, *every* $\varphi_z^\epsilon \in L_t^2(\Lambda)$ *that satisfies*

$$\|\mathcal{F}\varphi_z^\epsilon - \nu \times \mathbb{G}(\cdot, z)q\|_{L_t^2(\Pi)} < \epsilon$$

is such that

$$\lim_{\epsilon\to 0} \|\mathcal{S}\varphi_z^\epsilon\|_X = \infty \quad \textit{and} \quad \lim_{\epsilon\to 0} \|\varphi_z^\epsilon\|_{L_t^2(\Lambda)} = \infty,$$

where $X := H(\text{curl}, D)$ *in case* (a) *and* $X := L^2(D)$ *in case* (b).

The solution provided by Theorem 6.1 can be used to determine the shape D of the buried target independently of its physical properties (see Section 6.4 for numerical examples) provided that an analytic expression of the Green's function for the background media is available.

6.3 The Reciprocity Gap Functional Method

The LSM for buried objects works for any background medium for which the Green's function is available. However, since the expression of the background Green's function is part of the far field equation, it is crucial for the LSM that it be accurately computed. Often this task is difficult and computationally expensive especially for complicated geometries. A new version of the LSM based on the reciprocity gap function, referred to in the following as the *reciprocity gap functional method*, avoids this problem at the expense of needing more data. In particular, if the tangential component of both the electric and magnetic field is measured on ∂G (see Figure 6.2) only the Green's tensor for G is needed. In order to clearly present the main ideas of the method, throughout this section we assume that the medium inside G is homogeneous, and in this case we can choose the Green's tensor to be an electric dipole.

More precisely, let G be a bounded region in $\mathbb{R}^3$ with piecewise smooth boundary ∂G containing the scatterer $\overline{D}$, and suppose that the medium inside G is homogeneous with constant index of refraction n_b and wave number $k_b^2 = k^2 n_b$. We assume that the open surface Λ, which is the location of the sources, is part of a closed analytic surface situated in a homogeneous layer of $\mathbb{R}^3 \setminus \overline{G}$ with constant index of refraction n_s and wave number $k_s^2 = k^2 n_s$ (see Figure 6.2). In order to formulate the inverse problem we assume that the tangential components of *both* the total electric field $E := E(\cdot, x_0, p)$ and total magnetic field $H = \frac{1}{ik}\text{curl}\, E$ are known on ∂G. Furthermore, without loss of generality, for the following analysis we assume that Λ is the entire closed analytic surface surrounding G

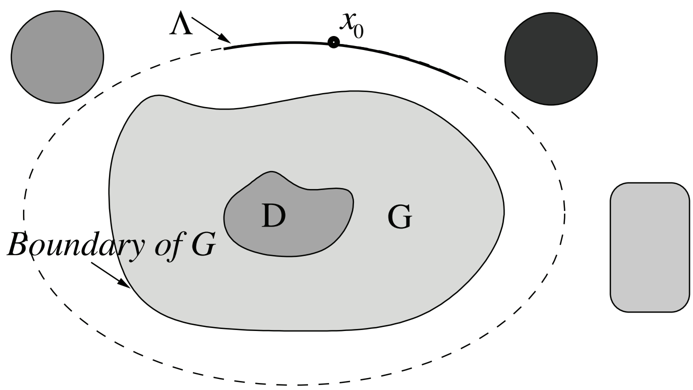


Figure 6.2. *Example of a geometry for the reciprocity gap functional method.*

situated in the layer with index of refraction n_s. By an analyticity argument our analysis remains valid if Λ is only a portion of this surface.

The *inverse scattering problem* we are interested in is to determine the support D of the scattering object from knowledge of the tangential components $\nu \times E$ and $\nu \times \operatorname{curl} E$ measured on ∂G for all points $x_0 \in \Lambda$ and $p \in \mathbb{R}^3$. For later use, we define

$$\mathcal{U} := \left\{ E(\cdot, x_0, p),\ x_0 \in \Lambda,\ p \in \mathbb{R}^3 \right\}, \tag{6.22}$$

which represents the set of electric fields corresponding to these measurements.

The reciprocity gap functional method is based on an integral equation of the first kind constructed from the *reciprocity gap operator*. To this end, define

$$\mathbb{H}(G) := \left\{ W \in H(\operatorname{curl}, G),\ \text{ such that } \operatorname{curl}\operatorname{curl} W - k_b^2 W = 0 \ \text{ in } G \right\}.$$

The reciprocity gap operator is obtained from the *reciprocity gap functional* $\mathcal{R}$ defined on $\mathcal{U} \times \mathbb{H}(G)$ by

$$\mathcal{R}(E, W) := \int_{\partial G} \{(\nu \times E) \cdot \operatorname{curl} W - (\nu \times W) \cdot \operatorname{curl} E\}\, ds, \tag{6.23}$$

where the integrals are interpreted in the sense of the duality between $H^{-1/2}(\mathrm{Div}, \partial G)$ and $H^{-1/2}(\mathrm{Curl}, \partial G)$. Notice that in the absence of a scattering object D, the right-hand side of (6.23) is zero for all point sources, whereas if D is present, this right-hand side defines a nonzero function of the source location x_0 and of the source polarization p. This observation motivates using (6.23) to set up an integral equation whose solution is an indicator function for D. To this end, we define the *reciprocity gap operator* $R : \mathbb{H}(G) \to L_t^2(\Lambda)$ by

$$R(W)(x_0) \cdot p = \mathcal{R}(E(\cdot, x_0, p), W) \tag{6.24}$$

for all $x_0 \in \Lambda$ and $p \in \mathbb{R}^3$. Notice that this definition makes sense since E depends linearly on the polarization p and so does $\mathcal{R}$.

Our inversion scheme is based on the construction of an integral equation using the reciprocity gap operator and a family of solutions in $\mathbb{H}(G)$, which satisfy certain properties to be made precise later. To fix our ideas we consider the single layer potential $A\varphi$ defined by

$$(A\varphi)(x) := \operatorname{curl}\operatorname{curl}\int_{\tilde{\Lambda}} \varphi(y)\Phi(x,y,k_b)\,ds, \qquad \varphi \in L^2_t(\tilde{\Lambda}), \tag{6.25}$$

where $\tilde{\Lambda}$ is the boundary of a ball containing G and Λ in its interior, and $\Phi(x,y,k_b)$ is given by (2.9) with $k = k_b$. The *sampling operator* $S : L^2_t(\tilde{\Lambda}) \to L^2_t(\Lambda)$ is defined by

$$S\varphi := R\,A\varphi \tag{6.26}$$

for $\varphi \in L^2_t(\tilde{\Lambda})$. Using the definition of R and interchanging the order of integration, it is readily seen that S is an integral operator whose (matrix) kernel $s(x_0,y)$ is defined by

$$(s(x_0,y)\cdot q)\cdot p = \mathcal{R}(E(\cdot,x_0,p), \operatorname{curl}\operatorname{curl}(q\,\Phi(\cdot,y,k_b)))$$

for $(x_0,y) \in \Lambda \times \tilde{\Lambda}$ and $p,q \in \mathbb{R}^3$.

The key ingredients of the reciprocity gap functional method are the properties of the reciprocity gap operator and the fact that set $\left\{A\varphi :\ \varphi \in L^2_t(\tilde{\Lambda})\right\}$ is dense in appropriate solution spaces.

Lemma 6.2. *Let B_R be a ball containing D and contained in the domain bounded by $\tilde{\Lambda}$. Then the set $\left\{A\varphi, \varphi \in L^2_t(\tilde{\Lambda})\right\}$ is dense in $L^2_t(\partial B_R)$.*

Proof. Without loss of generality we assume that k_b is not a Maxwell eigenvalue for B_R (which is not a restriction since we can always find such a ball). Noting that

$$\operatorname{curl}_x\operatorname{curl}_x\int_{\tilde{\Lambda}} \varphi(y)\Phi(x,y,k_b)\,ds(y) = -ik_b\int_{\tilde{\Lambda}} \mathcal{G}(x,y)^\top\varphi(y)\,ds(y),$$

where $\mathcal{G}$ is given by

$$\mathcal{G}(x,y) = \Phi(x,y,k_b)I + \frac{1}{k_b^2}\operatorname{grad}_x\operatorname{div}_x\Phi(x,y,k_b)I \tag{6.27}$$

and $\top$ denotes the transposed matrix, assume there exists $a \in L^2_t(\partial B_{\mathbb{R}})$ such that

$$\int_{\partial B_R} \overline{a}(x)\cdot\int_{\tilde{\Lambda}} \mathcal{G}(x,y)^\top\varphi(y)\,ds(y)\,ds(x) = 0 \tag{6.28}$$

for every $\varphi \in L^2_t(\tilde{\Lambda})$. We want to show that $a = 0$. By interchanging the order of integration we arrive at

$$\int_{\tilde{\Lambda}} \varphi(y)\cdot\int_{\partial B_R} \mathcal{G}(x,y)\overline{a}(x)\,ds(x)\,ds(y) = 0$$

for every $\varphi \in L^2_t(\tilde{\Lambda})$. This implies that

$$\nu \times \int_{\partial B_R} \mathcal{G}(x,y)\overline{a}(x)\,ds(x) = 0 \quad \text{on } \tilde{\Lambda}.$$

Hence, using the uniqueness of the exterior Maxwell problem and analytic continuation [93], we have that the surface potential

$$(Va)(y) := \int_{\partial B_R} \mathcal{G}(x,y)\overline{a}(x)\,ds(x), \qquad y \in \mathbb{R}^3 \setminus \partial B_R, \qquad a \in L^2_t(\partial B_R),$$

is zero outside ∂B_R. By continuity of the tangential component of Va across ∂B_R and the fact that k_b is not a Maxwell eigenvalue for B_R, we conclude that $Va = 0$ in B_R as well. Finally, by applying the jump relation for $\nu \times \nabla \times (Va)$ across ∂B_R [50], we obtain that $a \equiv 0$. This ends the proof. □

Remark 6.2. Alternatively one can use, instead of the single layer potential, the electric Herglotz wave function Eg defined by (2.24). Then the sampling operator is given by

$$\tilde{S} : L^2_t(\Omega) \to L^2_t(\Lambda) \text{ such that } \tilde{S}g = REg,$$

where Ω is the unit sphere. Thus the analysis that follows also holds, with S replaced by $\tilde{S}$.

Now let $z \in G$ be a sampling point, let $q \in \mathbb{R}^3 \setminus \{0\}$ be an arbitrary vector, and let

$$E_e(x,z,q,k_b) := \frac{i}{k_b}\operatorname{curl}_x \operatorname{curl}_x q\, \Phi(x,z,k_b) \tag{6.29}$$

be the electric field of the electric dipole corresponding to k_b. We associate with this dipole the function $\ell_z \in L^2_t(\Lambda)$ defined by

$$\ell_z(x_0)\cdot p = \mathcal{R}(E(\cdot,x_0,p),\, E_e(\cdot,z,q,k_b)) \tag{6.30}$$

for $x_0 \in \Lambda$ and $p \in \mathbb{R}^3$. The reciprocity gap functional method consists in seeking for each sampling point z an approximate solution to the ill-posed integral equation

$$S\varphi_z = \ell_z, \qquad \varphi_z \in L^2_t(\Lambda), \tag{6.31}$$

which can be equivalently written, using the definition of S and ℓ_z, in the form

$$\mathcal{R}(E, A\varphi_z) = \mathcal{R}(E, E_e(\cdot,z,q,k_b)), \qquad \varphi_z \in L^2_t(\tilde{\Lambda}), \quad E \in \mathcal{U}. \tag{6.32}$$

We emphasize that, as opposed to the LSM, the background Green's function $\mathbb{G}(\cdot,x_0)p$ does not appear in the integral equation (6.31).

In the following we develop the analytical framework for studying (6.31) for both scattering problems (6.5)–(6.7) and (6.9)–(6.13).

The reciprocity gap functional method for perfect conductors. Let $E = E(\cdot, x_0, p) = E^s(\cdot, x_0, p) + \mathbb{G}(\cdot, x_0)p$ and $H = \frac{1}{ik}\operatorname{curl} E$ be the total electric and magnetic fields, respectively, corresponding to the scattering problem (6.5)–(6.7).

Lemma 6.3. *Assume that k is not a Maxwell eigenvalue. Then the operator $R : \mathbb{H}(G) \to L^2_t(\Lambda)$ defined by* (6.24) *is injective.*

Proof. $RW = 0$ means $\mathcal{R}(E(\cdot, x_0, p), W) = 0$ for all $x_0 \in \Lambda$ and $p \in \mathbb{R}^3$. Since both E and W satisfy Maxwell's equations in $G \setminus \overline{D}$, we have, using the boundary condition for E on ∂D, that

$$0 = \int_{\partial D} (\nu \times E) \cdot \operatorname{curl} W - (\nu \times W) \cdot \operatorname{curl} E \, ds = -\int_{\partial D} (\nu \times W) \cdot \operatorname{curl} E \, ds,$$

where the first integral is interpreted in the sense of duality between $H^{-1/2}(\mathrm{Div}, \partial D)$ and $H^{-1/2}(\mathrm{Curl}, \partial D)$. Now let $\tilde{E}$ be the unique solution to (see [27])

$$\operatorname{curl}\operatorname{curl} \tilde{E} - k^2 n(x)\tilde{E} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \tag{6.33}$$

$$\nu \times (\tilde{E} - W) = 0 \quad \text{on} \quad \partial D, \tag{6.34}$$

$$\lim_{r\to\infty} \left(\operatorname{curl} \tilde{E} \times x - ikr\tilde{E}\right) = 0. \tag{6.35}$$

Then from the above problem, the boundary conditions for the total field $E = E^s + \mathbb{G}(\cdot, x_0)p$, and (6.34), (6.35), we have that

$$\begin{aligned} 0 = -\int_{\partial D_u} (\nu \times \tilde{E}) \cdot \operatorname{curl} E \, ds &= \int_{\partial D} (\nu \times E) \cdot \operatorname{curl} \tilde{E} - (\nu \times \tilde{E}) \cdot \operatorname{curl} E \, ds \\ &= \int_{\partial D} [\nu \times (E^s + \mathbb{G}(\cdot, x_0)p)] \cdot \operatorname{curl} \tilde{E} - (\nu \times \tilde{E}) \cdot \operatorname{curl} (E^s + \mathbb{G}(\cdot, x_0)p) \, ds. \end{aligned}$$

Since E^s and $\tilde{E}$ are both radiating solutions to the same equation, the above equation simplifies to

$$\begin{aligned} 0 &= \int_{\partial D} (\nu \times \mathbb{G}(\cdot, x_0)p) \cdot \operatorname{curl} \tilde{E} - (\nu \times \tilde{E}) \cdot \operatorname{curl} \mathbb{G}(\cdot, x_0)p \, ds \\ &= -p \cdot \tilde{E}(x_0). \end{aligned} \tag{6.36}$$

Since p is an arbitrary polarization, we obtain $\nu \times \tilde{E}(x_0) = 0$ for $x_0 \in \Lambda$. Furthermore, since $\tilde{E}$ is a radiating solution to Maxwell's equations outside the domain bounded by Λ, we conclude by the uniqueness of the scattering problem for a perfect conductor (cf. [50], [93]) that $\tilde{E} = 0$ outside the domain bounded by Λ. Then the unique continuation principle [93] implies that $\tilde{E} = 0$ outside D, whence $\nu \times W = 0$ on ∂D. Finally, since k is not a Maxwell eigenvalue, from the uniqueness of the interior boundary value problem for W we conclude that $W = 0$, which proves the lemma. ☐

Lemma 6.4. *Assume that k is not a Maxwell eigenvalue. Then the operator $R : \mathbb{H}(G) \to L^2_t(\Lambda)$ defined by* (6.24) *has dense range.*

Proof. Consider $\beta \in L^2_t(\Lambda)$ and assume that

$$(RW,\beta)_{L^2_t(\Lambda)} = 0 \quad \forall W \in \mathbb{H}(G).$$

From (6.24) and the bilinearity of $\mathcal{R}$, one has that

$$(RW,\beta)_{L^2_t(\Lambda)} = \int_\Lambda \mathcal{R}(E(\cdot,x_0,\alpha(x_0)),W)\,ds(x_0) = \mathcal{R}(\mathcal{E},W),$$

where

$$\mathcal{E}(x) = \int_\Lambda E(x,x_0,\alpha(x_0))\,ds(x_0) \tag{6.37}$$

and $\alpha = (\beta \cdot p)\,p$. Using the second vector Green's formula and the boundary conditions for E, one concludes that

$$0 = \mathcal{R}(\mathcal{E},W) = -\int_{\partial D} (\nu \times W) \cdot \operatorname{curl} \mathcal{E}\,ds \tag{6.38}$$

for all $W \in \mathbb{H}(G)$, where again the first integral is interpreted in the sense of duality between $H^{-1/2}(\mathrm{Div},\partial D)$ and $H^{-1/2}(\mathrm{Curl},\partial D)$. Since $\mathbb{H}(G)$ contains the electric Herglotz wave functions given by (2.24), from Theorem 2.8 in [27] and the well-posedness of the interior boundary value problem for a perfect conductor, one has that the set

$$\{\nu \times W|_{\partial D} \text{ for all } W \in \mathbb{H}\}$$

is dense in $H^{-1/2}(\mathrm{Curl},\partial D)$. Therefore

$$\nu \times \operatorname{curl} \mathcal{E} = 0 \quad \text{on } \partial D.$$

The boundary conditions for $\mathcal{E}$ imply that both $\nu \times \mathcal{E} = 0$ and $\nu \times \operatorname{curl} \mathcal{E} = 0$ on ∂D. This means that the extension of $\mathcal{E}$ by zero inside D satisfies Maxwell's equations inside the domain bounded by Λ with the index n set equal to n_b inside D. From the unique continuation principle [93] one has that $\mathcal{E}$ is zero inside the domain bounded by Λ and outside D. Noting that

$$\mathcal{E}(x) = \int_\Lambda (E^s(x,x_0,\alpha(x_0)) + \mathbb{G}(x,x_0)\alpha(x_0))\,ds(x_0)$$

one concludes that $\mathcal{E} \times \nu$ is continuous across Λ. The uniqueness theorem for the exterior problem for Maxwell's equations with boundary data $\nu \times \mathcal{E} = 0$ on Λ implies that $\mathcal{E} = 0$ outside the domain bounded by Λ. Finally, from the jump relations of the vector potential across Λ [50], we have that

$$0 = \operatorname{curl} \mathcal{E}|_{\Lambda^+} - \operatorname{curl} \mathcal{E}|_{\Lambda^-} = -\alpha \quad \text{on } \Lambda.$$

Hence $(\beta \cdot p)\,p = 0$ for all $p \in \mathbb{R}^3$, which implies that $\beta = 0$. This ends the proof. ☐

Remark 6.3. It is easy to prove (cf. Theorem 4.8 in [22]) that the operator $R : \mathbb{H}(G) \to L^2_t(\Lambda)$ is compact.

From Lemma 6.2, Theorem 7.9 in [50], and the fact that the electric Herglotz functions are dense in $H(\mathrm{curl}, D)$, we have the following lemma.

Lemma 6.5. *The set* $\left\{A\varphi, \varphi \in L^2_t(\tilde{\Lambda})\right\}$ *is dense in* $H(\mathrm{curl}, D)$.

Combining the above results, we have the following theorem.

Theorem 6.6. *The sampling operator* $S : L^2_t(\tilde{\Lambda}) \to L^2_t(\Lambda)$ *defined by* (6.26) *is compact and injective and has dense range provided that k is not a Maxwell eigenvalue.*

Our main goal now is to study the solvability of (6.31) (or equivalently (6.32)). An important observation is that for $z \in D$ it is easy to see from (6.32) that (6.31) has a solution $\varphi_z \in L^2_t(\tilde{\Lambda})$ if and only if $E_z := A\varphi_z$ is a solution of

$$\mathrm{curl}\,\mathrm{curl}\, E_z - k_b^2 E_z = 0 \quad \text{in} \quad D, \tag{6.39}$$

$$\nu \times [E_z - E_e(\cdot, z, q, k_b)] = 0 \quad \text{on} \quad \partial D. \tag{6.40}$$

This is generally not possible. The best we can hope for is to find an approximate solution to (6.31) by approximating the unique solution to (6.39)–(6.40) by $A\varphi_z$.

The following theorem is the basis of the reciprocity gap functional method for perfectly conducting scatterers.

Theorem 6.7. *Assume that k is not a Maxwell eigenvalue for D. Then the following hold:*

1. *For* $z \in D$ *and a given* $\epsilon > 0$, *there exists a* $\varphi_z^\epsilon \in L^2_t(\tilde{\Lambda})$ *such that*

$$\|S\varphi_z^\epsilon - \ell_z\|_{L^2_t(\Lambda)} < \epsilon,$$

and the corresponding single layer potential $A\varphi_z^\epsilon$ *converges to* E_z *in* $H(\mathrm{curl}, D)$ *as* $\epsilon \to 0$, *where* E_z *is the unique solution of* (6.39)–(6.40).

2. *For a fixed* $\epsilon > 0$, *we have that*

$$\lim_{z \to \partial D} \|A\varphi_z^\epsilon\|_{H(\mathrm{curl}, D)} = \infty \quad \textit{and} \quad \lim_{z \to \partial D} \|\varphi_z^\epsilon\|_{L^2_t(\tilde{\Lambda})} = \infty.$$

3. *For* $z \in G \setminus \overline{D}$ *and a given* $\epsilon > 0$, *every* $\varphi_z^\epsilon \in L^2_t(\tilde{\Lambda})$ *that satisfies*

$$\|S\varphi_z^\epsilon - \ell_z\|_{L^2_t(\Lambda)} < \epsilon$$

is such that

$$\lim_{\epsilon \to 0} \|A\varphi_z^\epsilon\|_{H(\mathrm{curl}, D)} = \infty \quad \textit{and} \quad \|\varphi_z^\epsilon\|_{L^2_t(\tilde{\Lambda})} = \infty.$$

Proof. Let $z \in D$. Since $W \in \mathbb{H}(G)$ and $E_e(\cdot, z, q, k_b)$ satisfy $\mathrm{curl}\,\mathrm{curl}\, W - k_b W = 0$ in $G \setminus \overline{D}$, by integrating by parts and using the boundary condition for the total field, we have that

$$\begin{aligned} S\varphi_z^\epsilon - \ell_z &:= \mathcal{R}(E, W) - \mathcal{R}(E, E_e(\cdot, z, q, k_b)) \\ &= -\int_{\partial D} (\nu \times W - \nu \times E_e(\cdot, z, q, k_b)) \cdot \mathrm{curl}\, E\, ds. \end{aligned}$$

From the proof of Lemma 6.3 we see that $\mathcal{R}(E,W) = \mathcal{R}(E, E_e(\cdot, z, q, k_b))$ has a unique solution W if and only if there exists a $W \in \mathbb{H}(G)$ satisfying (6.39)–(6.40), which is generally not true. However, from Lemma 6.5, for every $\epsilon > 0$ there exists a single layer potential $A\varphi_z^\epsilon$ that approximates the unique solution of (6.39)–(6.40) with discrepancy ϵ. In particular, φ_z^ϵ is an approximate solution to (6.31), and $A\varphi_z^\epsilon$ converges to the solution of (6.39)–(6.40) in $H(\mathrm{curl}, D)$ as $\epsilon \to 0$. Next, since $\nu \times E_e(\cdot, z, q) \to \infty$ in $H^{-1/2}(\mathrm{Curl}, \partial D)$ as z approaches the boundary, we obtain from the well-posedness of the interior boundary value problem that, for a fixed $\epsilon > 0$, $\lim_{z \to \partial D} \|A\varphi_z^\epsilon\|_{H(\mathrm{curl},D)} = \infty$ and $\lim_{z \to \partial D} \|\varphi_z^\epsilon\|_{L_t^2(S^2)} = \infty$.

Now consider $z \in G \setminus \overline{D}$, and let φ_z^ϵ and $A\varphi_z^\epsilon$ be such that

$$\|S\varphi_z^\epsilon - \ell_z\|_{L_t^2(\Lambda)} := \|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot, z, q, k_b))\|_{L_t^2(\Lambda)} < \epsilon. \tag{6.41}$$

Note that from Theorem 6.6 we can always find such an $A\varphi_z^\epsilon$. Assume, contrary to the theorem, that $\|A\varphi_z^\epsilon\|_{H(\mathrm{curl},D)} < C$, where the positive constant C is independent of ϵ. From the trace theorems, we have that the mixed trace of $A\varphi_z^\epsilon$ is also bounded in the corresponding norms. Noting that the total field can be written as $E(\cdot, x_0, p) = E^s(\cdot, x_0, p) + \mathbb{G}(\cdot, x_0)p$ and integrating by parts, we obtain that

$$\begin{aligned}
\mathcal{R}(E, E_e(x,z,q,k_b)) = &\int_{\partial G} (\nu \times E^s(x,x_0,p)) \cdot \mathrm{curl}\, E_e(x,z,q,k_b)\, ds(x) \\
&- \int_{\partial G} (\nu \times E_e(x,z,q,k_b)) \cdot \mathrm{curl}\, E^s(x,x_0,p)\, ds(x) \\
&+ \int_{\partial G} (\nu \times \mathbb{G}(x,x_0)p) \cdot \mathrm{curl}\, E_e(x,z,q,k_b)\, ds(x) \\
&- \int_{\partial G} (\nu \times E_e(x,z,q,k_b)) \cdot \mathrm{curl}\, \mathbb{G}(x,x_0)p\, ds(x).
\end{aligned}$$

Due to the symmetry of the background Green's function, $E^s(x,x_0,p)$ as a function of x_0 satisfies $\mathrm{curl}_{x_0} \mathrm{curl}_{x_0} E^s(x,x_0,p) - k^2 n(x_0) E^s(x,x_0,p) = 0$ in the domain bounded by Λ and ∂D. Hence the first two integrals in the above equation give a solution $W(x_0)$ to the same equation as the one satisfied by $E^s(\cdot, x_0, p)$, whereas the last two integrals add up to $-\mathbb{G}(z,x_0)p$ by the Stratton–Chu formula and the fact that $E_e(x,z,q,k_b)$ is the fundamental solution of $\mathrm{curl}\,\mathrm{curl}\, E - k_b^2 E = 0$. On the other hand, we have that

$$\mathcal{R}(E, A\varphi_z^\epsilon) = -\int_{\partial D} (\nu \times A\varphi_z^\epsilon) \cdot \mathrm{curl}\, E\, ds. \tag{6.42}$$

Combining the above equalities, we obtain that

$$\begin{aligned}
\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot, z, q, k_b)) = &-\int_{\partial D} (\nu \times A\varphi_z^\epsilon) \cdot \mathrm{curl}\, E\, ds \\
&- W(x_0) + \mathbb{G}(z, x_0)p.
\end{aligned} \tag{6.43}$$

Now since $\|A\varphi_z^\epsilon\|_{H(\mathrm{curl},D)} < C$ there exists a subset, still denoted by $A\varphi_z^\epsilon$, that converges weakly to a $V \in H(\mathrm{curl}, D)$ as $\epsilon \to 0$, and therefore $\nu \times A\varphi_z^\epsilon$ converges weakly to $\nu \times V$ in the duality pairing $H^{-1/2}(\mathrm{Div}, \partial D)$, $H^{-1/2}(\mathrm{Curl}, \partial D)$. Now set

$$\tilde{W}(x_0) = \lim_{\epsilon \to 0} \mathcal{R}(E, A\varphi_z^\epsilon) = -\int_{\partial D} (\nu \times V) \cdot \mathrm{curl}\, E(\cdot, x_0, p)\, ds.$$

From (6.41) we have that

$$\tilde{W}(x_0) = W(x_0) + \mathbb{G}(z, x_0)p, \qquad x_0 \in \Lambda. \tag{6.44}$$

Since $\tilde{W}(x_0)$ and $W(x_0)$ can be continued as radiating solutions to

$$\mathrm{curl}_{x_0}\mathrm{curl}_{x_0} E^s(x, x_0, p) - k^2 n(x_0) E^s(x, x_0, p) = 0$$

outside the domain bounded by Λ, we deduce by uniqueness and the unique continuation principle [93] that (6.44) holds true for x_0 in $\mathbb{R}^3 \setminus (\overline{D} \cup \{z_0\})$. We now arrive at a contradiction by letting $x_0 \to z$. Hence $A\varphi_z^\epsilon$ is unbounded in the $H(\mathrm{curl}, D)$ norm as $\epsilon \to 0$, which proves the theorem. $\square$

The support D can now be determined by the behavior of the solution to (6.31) according to Theorem 6.6.

It is possible to develop the reciprocity gap functional method for obstacles with mixed boundary conditions such as those considered in Chapter 3. In this case the support of the scattering object can be determined by the solution of the same equation as above. Furthermore this solution can be used to obtain information about the surface impedance. For more details see [23].

The reciprocity gap functional method for anisotropic media. Now, let $E = E(\cdot, x_0, p) = E^s(\cdot, x_0, p) + \mathbb{G}(\cdot, x_0)p$ and $H = \frac{1}{ik}\mathrm{curl}\, E$ be the total electric and magnetic fields, respectively, corresponding to the scattering problem (6.9)–(6.13). In our analysis, the following interior transmission problem will play the role of (6.39)–(6.40):

$$\left.\begin{array}{r} \mathrm{curl\,curl}\, E_0^z - k^2 n_b E_0^z = 0 \\ \mathrm{curl\,curl}\, E^z - k^2 N(x) E^z = 0 \end{array}\right\} \quad \text{in} \quad D, \tag{6.45}$$

$$\left.\begin{array}{r} \nu \times E_0^z - \nu \times E^z = \nu \times E_e(\cdot, z, q, k_b) \\ \nu \times \mathrm{curl}\, E_0^z - \nu \times \mathrm{curl}\, E^z = \nu \times \mathrm{curl}\, E_e(\cdot, z, q, k_b) \end{array}\right\} \quad \text{on} \quad \partial D, \tag{6.46}$$

where $E^z \in L^2(D)$, $E_0^z \in L^2(D)$ such that $E^z - E_0^z \in H(\mathrm{curl}, D)$, $\mathrm{curl}\,(E^z - E_0^z) \in H(\mathrm{curl}, D)$. It is easy to see in [35] that $E^s := E_e(\cdot, z, q, k_b)$ in $\mathbb{R}^3 \setminus \overline{D}$ and that $E^s := E^z - E_0^z$ in D is the scattered field corresponding to E_0^z in $H_{inc}(D) := \{u \in L^2(D) : \mathrm{curl\,curl}\, u - k^2 n_b u = 0\}$ as the incident wave, satisfying

$$\begin{array}{rll} \mathrm{curl\,curl}\, E^s - k^2 n_b E^s = 0 & \text{in} & \mathbb{R}^3 \setminus \overline{D}, \\ \mathrm{curl\,curl}\, E^s - k^2 N E^s = k^2 (N - n_b I) E_0^z & \text{in} & D, \\ \nu \times E_+^s - \nu \times E_-^s = 0 & \text{on} & \partial D, \\ \nu \times \mathrm{curl}\, E_+^s - \nu \times \mathrm{curl}\, E_-^s = 0 & \text{on} & \partial D, \\ \lim_{r \to \infty} (\mathrm{curl}\, E^s \times x - i k_b r E^s) = 0, & & \end{array} \tag{6.47}$$

where E^s_+ and E^s_- denote the limit of E^s approaching ∂D from $\mathbb{R}^3 \setminus \overline{D}$ and D, respectively.

Modifying the approach in Section 4.2 to account for n_b [35] one can show that the Fredholm alternative applies to (6.45)–(6.46) under either of the following three conditions:

Condition 1:

$$M := (n_b I - N)^{-1} \text{ is a bounded matrix,} \tag{6.48}$$

Condition 2:

$$\Im(M) \text{ is nonnegative on } D, \text{ and} \tag{6.49}$$

$$\Im(M) - \{\Im(n_b M)\}^2 \{\Im(M)\}^{-1} \text{ is nonnegative on } D. \tag{6.50}$$

Condition 3: Letting $\tilde{M} := n_b N M$, either

$$\Re(M) \text{ and } \Re(\tilde{M}) \text{ are nonnegative on } D \tag{6.51}$$

and both matrices

$$\Re(\tilde{M}) - \{\Re(NM)\}^2 \{\Re(M)\}^{-1} \quad \text{and} \quad \Re(M) - \{\Re(NM)\}^2 \{\Re(\tilde{M})\}^{-1} \tag{6.52}$$

are uniformly positive definite on D, or

$$-\Re(M) \text{ and } -\Re(\tilde{M}) \text{ are nonnegative on } D \tag{6.53}$$

and both matrices

$$\{\Re(n_b M)\}^2 \{\Re(M)\}^{-1} - \Re(\tilde{M}) \quad \text{and} \quad \{\Re(n_b M)\}^2 \{\Re(\tilde{M})\}^{-1} - \Re(M) \tag{6.54}$$

are uniformly positive definite on D.

We remark that these conditions also apply to a more general case, where n_b is a matrix valued function that commutes with N. When n_b and N are real scalars, one can easily verify that the first condition is equivalent to $0 < N < n_b$ and that the third set of conditions is equivalent to $0 < n_b < N$. If n_b is a real scalar, the first set of conditions is equivalent to $\Im(N) > 0$ on D. One can also show that if the second matrix in (6.49) is uniformly positive definite on D, then the uniqueness of solutions holds true. In general, we need to exclude the corresponding transmission eigenvalues, which are known to be at most discrete [36].

Lemma 6.8. *Assume that k is not a transmission eigenvalue for D as defined in Section* 6.2 *(note that $n(x) = n_b$ in* (6.18)*). Then the operator $R : \mathbb{H}(G) \to L^2_t(\Lambda)$ defined by* (6.24) *is injective.*

Proof. From (6.24), $RW = 0$ means $\mathcal{R}(E(\cdot, x_0, p), W) = 0$ for all $x_0 \in \Lambda$ and $p \in \mathbb{R}^3$. Using the second vector Green's formula and the transmission conditions (6.11)–(6.12), we have that

$$\begin{aligned} 0 &= \int_{\partial D} (\nu \times E) \cdot \operatorname{curl} W - (\nu \times W) \cdot \operatorname{curl} E \, ds \qquad (6.55) \\ &= \int_{\partial D} (\nu \times E) \cdot \operatorname{curl} W - (\nu \times W) \cdot \operatorname{curl} E \, ds \\ &= \int_{\partial D} (\nu \times E^{int}) \cdot \operatorname{curl} W - (\nu \times W) \cdot \operatorname{curl} E^{int} \, ds. \end{aligned}$$

Now (see [20]) let $\tilde{E}^{int} \in H(\text{curl}, D)$ and $\tilde{E} \in H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D})$ be the unique solution to

$$\text{curl curl}\, \tilde{E}^{int} - k^2 N(x)\tilde{E}^{int} = 0 \quad \text{in } D, \tag{6.56}$$

$$\text{curl curl}\, \tilde{E} - k^2 n(x)\tilde{E} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{6.57}$$

$$\left.\begin{array}{r} \nu \times (\tilde{E} + W) - \nu \times \tilde{E}^{int} = 0 \\ \nu \times \text{curl}\,(\tilde{E} + W) - \nu \times \text{curl}\, \tilde{E}^{int} = 0 \end{array}\right\} \text{ on } \partial D, \tag{6.58}$$

$$\lim_{r\to\infty} \left(\text{curl}\, \tilde{E} \times x - ikr\tilde{E}\right) = 0. \tag{6.59}$$

Expressing W in the last equation of (6.55) in terms of $\tilde{E}$ and $\tilde{E}^{int}$ using (6.58) and the fact that E^{int} and $\tilde{E}^{int}$ satisfy the same equation in D, we obtain that

$$0 = \int_{\partial D} (\nu \times \tilde{E}) \cdot \text{curl}\, E^{int} - (\nu \times E^{int}) \cdot \text{curl}\, \tilde{E}\, ds - ik\eta \int_{\partial D_c} (\nu \times \tilde{E}) \cdot (\nu \times E^{int})\, ds. \tag{6.60}$$

Next, expressing E^{int} in terms of the total exterior field $E = E^s + \mathbb{G}(\cdot, x_0)p$ using the transmission conditions (6.11), (6.12), the fact that E^s and $\tilde{E}$ are radiating solutions to the same equation outside D and, finally, the Stratton–Chu representation formula outside D [50], [93], we can rewrite (6.60) as

$$\begin{aligned} 0 &= \int_{\partial D} (\nu \times \tilde{E}) \cdot \text{curl}\,(E^s + \mathbb{G}(\cdot, x_0)p) - [\nu \times (E^s + \mathbb{G}(\cdot, x_0)p)] \cdot \text{curl}\, \tilde{E}\, ds \\ &= \int_{\partial D} (\nu \times \tilde{E}) \cdot \text{curl}\, \mathbb{G}(\cdot, x_0)p - (\nu \times \mathbb{G}(\cdot, x_0)p) \cdot \text{curl}\, \tilde{E}\, ds = -p \cdot \tilde{E}(x_0). \end{aligned}$$

Since p is an arbitrary polarization, we obtain that $\nu \times \tilde{E}(x_0) = 0$ for all $x_0 \in \Lambda$. Furthermore, since $\tilde{E}$ is a radiating solution to curl curl $\tilde{E} - k^2 n(x)\tilde{E} = 0$ outside the domain bounded by Λ and satisfies $\nu \times \tilde{E} = 0$ on Λ, we can conclude by the uniqueness theorem for scattering by a perfect conductor that $\tilde{E} = 0$ outside the domain bounded by Λ. Finally, from the unique continuation principle [93], we have that $\tilde{E} = 0$ outside D as well. Therefore, $E_0 := W$ and $E^{int} := \tilde{E}^{int}$ satisfy the homogeneous interior transmission problem (6.45)–(6.46) (i.e., with $E_e = 0$), hence from the assumption that k is not a transmission eigenvalue we finally obtain that $W = 0$ in D. This proves the lemma. □

Lemma 6.9. *Assume that k is not a transmission eigenvalue for D. Then the operator* $R : \mathbb{H}(G) \to L^2_t(\Lambda)$ *defined by* (6.24) *has dense range.*

Proof. Consider $\beta \in L^2_t(\Lambda)$ and assume that

$$(RW, \beta)_{L^2_t(\Lambda)} = 0 \quad \forall W \in \mathbb{H}(G).$$

From (6.24) and the bilinearity of $\mathcal{R}$ one has that

$$(RW, \beta)_{L^2_t(\Lambda)} = \int_{\Lambda} \mathcal{R}(E(\cdot, x_0, \alpha(x_0)), W)\, ds(x_0) = \mathcal{R}(\mathcal{E}, W),$$

where

$$\mathcal{E}(x) = \int_\Lambda E(x, x_0, \alpha(x_0))\, ds(x_0) \tag{6.61}$$

and $\alpha = (\beta \cdot p)\, p$. Letting

$$\mathcal{E}^{int}(x) = \int_\Lambda E(x, x_0, \alpha(x_0))\, ds(x_0), \tag{6.62}$$

by linearity we have that $\mathcal{E}$ and $\mathcal{E}^{int}$ satisfy the scattering problem (6.9)–(6.13). Using the second vector Green's formula and the transmission conditions for $\mathcal{E}$ and $\mathcal{E}^{int}$, one concludes that

$$0 = \mathcal{R}(\mathcal{E}, W) = k^2 \int_D (N - n_b I)\, \mathcal{E}^{int} \cdot W\, dx \tag{6.63}$$

for all $W \in \mathbb{H}(G)$. Since $\mathbb{H}(G)$ contains the space H of Lemma 4.23, we conclude from this lemma and (6.63) that $\mathcal{E}^{int} = 0$ in D and that $\nu \times \mathcal{E}|_{\partial D_c} = 0$. The transmission conditions now imply that both $\nu \times \mathcal{E} = 0$ and $\nu \times \operatorname{curl} \mathcal{E} = 0$ on ∂D. This means that the extension of $\mathcal{E}$ by 0 inside D satisfies Maxwell's equations inside the domain bounded by Λ, with the index n set equal to n_b inside D. From the unique continuation principle [93] one has that $\mathcal{E}$ is zero inside the domain bounded by Λ and outside D. Noting that

$$\mathcal{E}(x) = \int_\Lambda (E^s(x, x_0, \alpha(x_0)) + \mathbb{G}(x, x_0)\alpha(x_0))\, ds(x_0),$$

one concludes that $\mathcal{E} \times \nu$ is continuous across Λ. The uniqueness theorem for the exterior problem for Maxwell's equations with boundary data $\nu \times \mathcal{E} = 0$ on Λ implies that $\mathcal{E} = 0$ outside the domain bounded by Λ as well. Finally, from the jump relations of the vector potential across Λ [50], we have that

$$0 = \operatorname{curl} \mathcal{E}|_{\Lambda^+} - \operatorname{curl} \mathcal{E}|_{\Lambda^-} = -\alpha \quad \text{on} \quad \Lambda.$$

Hence $(\beta \cdot p)\, p = 0$ for all $p \in \mathbb{R}^3$, which implies that $\beta = 0$. This ends the proof. □

Using Lemmas 4.23 and 6.2 we can prove the following.

Lemma 6.10. *The set* $\left\{A\varphi, \varphi \in L^2_t(\tilde{\Lambda})\right\}$ *is dense in* $H_{inc}(D)$.

Consequently we also have the next lemma.

Lemma 6.11. *The sampling operator* $S : L^2_t(\tilde{\Lambda}) \to L^2_t(\Lambda)$ *is compact. It is also injective with dense range provided that k is not a transmission eigenvalue for D.*

Now we are ready to prove the main theorem for the reciprocity gap functional method for buried (partially) coated anisotropic media.

Theorem 6.12. *Assume that k, N, and n_b are such that the interior transmission problem* (6.45)–(6.46) *is well posed (see conditions* (6.48)–(6.49)*). Then the following hold:*

1. *For $z \in D$ and a given $\epsilon > 0$, there exists a $\varphi_z^\epsilon \in L_t^2(\tilde{\Lambda})$ such that*

$$\|S\varphi_z^\epsilon - \ell_z\|_{L_t^2(\Lambda)} < \epsilon,$$

and the corresponding single layer potential $A\varphi_z^\epsilon$ converges to E_0^z in $L^2(D)$ as $\epsilon \to 0$, where (E_0^z, E^z) is the solution of (6.45)–(6.46).
Moreover, for a fixed $\epsilon > 0$, we have that

$$\lim_{z\to\partial D} \|A\varphi_z^\epsilon\|_{L^2(D)} = \infty \quad \textit{and} \quad \lim_{z\to\partial D} \|\varphi_z^\epsilon\|_{L_t^2(\tilde{\Lambda})} = \infty.$$

2. *For $z \in G \setminus \overline{D}$ and a given $\epsilon > 0$, every $\varphi_z^\epsilon \in L_t^2(\tilde{\Lambda})$ that satisfies*

$$\|S\varphi_z^\epsilon - \ell_z\|_{L_t^2(\Lambda)} < \epsilon$$

is such that

$$\lim_{\epsilon\to 0} \|A\varphi_z^\epsilon\|_{L^2(D)} = \infty \quad \textit{and} \quad \|\varphi_z^\epsilon\|_{L_t^2(\tilde{\Lambda})} = \infty.$$

Proof. Consider $z \in D$ and let E_0^z and E^z be the solution to the interior transmission problem (6.45)–(6.46). Since both $W \in \mathbb{H}(G)$ and $E_e(\cdot, z, q, k_b)$ satisfy curl curl $U - k_b U = 0$ in $G \setminus \overline{D}$, by integrating by parts and using the equations satisfied by the total electric field, we have that for $W \in \mathbb{H}(G)$

$$\mathcal{R}(E, W) = k^2 \int_D (N - n_b I) E W \, dx. \tag{6.64}$$

From Lemma 6.3 we see that $\mathcal{R}(E, W) = \mathcal{R}(E, E_e(\cdot, z, q, k_b))$ has a unique solution W if and only if W coincides with E_0^z in D. But this is generally not possible. However, from Lemma 6.2, for every $\epsilon > 0$ we can find a $\varphi_z^\epsilon \in L_t^2(\tilde{\Lambda})$ such that

$$\|E_0^z - A\varphi_z^\epsilon\|_{H_{inc}(D, \partial D_c)} < \epsilon,$$

which implies that

$$\left\| \mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_0^z) \right\|_{L_t^2(\Lambda)} < C\epsilon$$

for some positive constant $C > 0$, whence

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot, z, q, k_b))\|_{L_t^2(\Lambda)} < C\epsilon.$$

By construction, $A\varphi_z^\epsilon$ converges to E_0^z in the $L^2(D)$ norm as $\epsilon \to 0$. We now observe that $E^s := -E_e(\cdot, z, q, k_b)$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$ and $E^s := E^z - E_0^z$ in D satisfy the scattering problem (6.47). From the well-posedness of (6.47) we have

$$\|E_e(\cdot, z, q, k_b)\|_{X(B_R \setminus \overline{D})} \le C \|E_z^0\|_{L^2(D)}.$$

Hence, due to the singularity of the electric dipole, we have that $\|E_z^0\|_{L^2(D)} \to \infty$ as $z \to \partial D$ and hence so does $\|A\varphi_z^\epsilon\|_{L^2(D)}$ and $\|\varphi_z^\epsilon\|_{L_t^2(\tilde{\Lambda})}$.

Now we consider $z \in G \setminus \overline{D}$ and let φ_z^ϵ and its corresponding single layer potential $A\varphi_z^\epsilon$ be such that

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot, z, q, k_b))\|_{L_t^2(\Lambda)} < \epsilon. \tag{6.65}$$

Note that from Lemma 6.4 we can always find such an $A\varphi_z^\epsilon$. Assume to the contrary that $\|A\varphi_z^\epsilon\|_{H_{inc}(D,\partial D_c)} < C$, where the positive constant C is independent of ϵ. Noting that the total field can be written as $E(\cdot, x_0, p)E^s(\cdot, x_0, p) + \mathbb{G}(\cdot, x_0)p$ and integrating by parts, we obtain that

$$\begin{aligned}\mathcal{R}(E, E_e(x,z,q,k_b)) = &\int_{\partial G} (\nu \times E^s(x,x_0,p)) \cdot \operatorname{curl} E_e(x,z,q,k_b)\,ds(x)\\ &- \int_{\partial G} (\nu \times E_e(x,z,q,k_b)) \cdot \operatorname{curl} E^s(x,x_0,p)\,ds(x)\\ &+ \int_{\partial G} (\nu \times \mathbb{G}(x,x_0)p) \cdot \operatorname{curl} E_e(x,z,q,k_b)\,ds(x)\\ &- \int_{\partial G} (\nu \times E_e(x,z,q,k_b)) \cdot \operatorname{curl} \mathbb{G}(x,x_0)p\,ds(x).\end{aligned}$$

Due to the symmetry of the background Green's function, $E^s(x,x_0,p)$ as a function of x_0 satisfies $\operatorname{curl}_{x_0}\operatorname{curl}_{x_0} E^s(x,x_0,p) - k^2 n(x_0)E^s(x,x_0,p) = 0$ in the domain bounded by Λ and ∂D. Hence the first two integrals in the above equation give a solution $W(x_0)$ to the same equation satisfied by $E^s(\cdot, x_0, p)$, whereas the last two integrals add up to $-\mathbb{G}(z,x_0)p$ by the Stratton–Chu formula and the fact that $E_e(x,z,q,k_b)$ is the fundamental solution of $\operatorname{curl}\operatorname{curl} E - k_b^2 E = 0$. On the other hand, we have that

$$\mathcal{R}(E, A\varphi_z^\epsilon) = k^2 \int_D (N - n_b I)\,E^{int} \cdot A\varphi_z^\epsilon\,dx.$$

Combining the above equalities we obtain that

$$\begin{aligned}\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot, z, q, k_b)) = &-W(x_0) + \mathbb{G}(z,x_0)p\\ &+k^2 \int_D (N - n_b I)\,E^{int} \cdot A\varphi_z^\epsilon\,dx.\end{aligned} \tag{6.66}$$

Now, since $\|A\varphi_z^\epsilon\|_{L^2(D)} < C$ there exists a subsequence, still denoted by $A\varphi_z^\epsilon$, that converges weakly to a $V \in L^2(D)$ as $\epsilon \to 0$. For $x_0 \in \Lambda$ we set

$$\tilde{W}(x_0) = \lim_{\epsilon\to 0} \mathcal{R}(E, A\varphi_z^\epsilon) = k^2 \int_D (N - n_b I)\,E^{int} \cdot V\,dx.$$

Then from (6.41) we have that

$$\tilde{W}(x_0) = W(x_0) + \mathbb{G}(z,x_0)p, \qquad x_0 \in \Lambda. \tag{6.67}$$

Since $\tilde{W}(x_0)$ and $W(x_0)$ can be continued as radiating solutions to

$$\operatorname{curl}_{x_0}\operatorname{curl}_{x_0} E^s(x,x_0,p) - k^2 n(x_0) E^s(x,x_0,p) = 0$$

outside the domain bounded by Λ, we deduce by uniqueness and the unique continuation principle that (6.67) holds true in $\mathbb{R}^3 \setminus (\overline{D} \cup \{z_0\})$. We now arrive at a contradiction by letting $x_0 \to z$. Hence $A\varphi_z^\epsilon$ is unbounded in $L^2(D)$ as $\epsilon \to 0$, which proves the theorem. □

We refer the reader to [36] for discussion of the reciprocity gap functional method for an anisotropic medium partially coated with a thin layer of highly conducting material. There it is also shown how to obtain information on the surface conductivity from the approximate regularized solution provided by Theorem 6.12.

6.4 Numerical Reconstruction of Buried Objects

Despite the rather gloomy conclusion of subsection 3.5.4 that limited aperture causes difficulties, it is possible to image objects using limited aperture data even in a nonconstant background, provided the measurement patches are large enough. In particular, we show results from a model problem intended to simulate buried objects [32]. The known background medium is assumed to consist of two regions. For $x_3 > 0$ the region models air with $N(x) = 1$, whereas for $x_3 < 0$ the domain is conducting with $N(x) = n_b I$ for some complex constant n_b. Some of the geometry for this problem, together with the particular target used in this study, is shown in Figure 6.3. In this figure the lower parallelepiped is the region below ground used to search for the scatterer (i.e., where sampling points z are placed). The upper rectangle marks the air-earth interface.

Near field point sources are used in the air region $x_3 > 0$. In particular the incident field is due to a dipole point source and can be computed using Sommerfeld integrals [93]. Let us temporarily denote by λ the wavelength of the radiation (not to be confused with λ, the impedance). Recalling that $\lambda = 2\pi/k$ is the wavelength in the air, the authors of [32] choose sources on a $3\lambda \times 3\lambda$ domain at $x_3 = \lambda/2$. In this grid 25×25 sources are used, each with two horizontal polarizations resulting in 1,250 data measurements. Data is computed by an integral equation code. For the LSM the same points are used for measurements, and the results are shown in Figure 6.4.

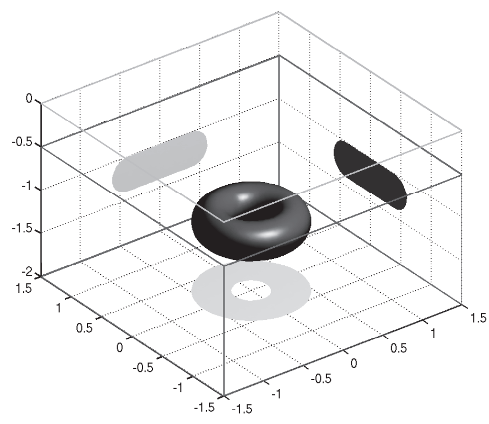


Figure 6.3. *The exact scatter for the buried torus. This figure is reproduced from* [32] *with permission.*

For the reciprocity gap scheme, it is argued that the field in the earth drops rapidly to zero due to the conductivity of the earth, and hence contributions to the reciprocity gap functional from reciprocity gap measurement surfaces in the earth can be ignored if these are far enough from the scatterer. Hence for the reciprocity gap, measurements are taken on a series of 40×40 grids on the surface of the earth (i.e., at $z_3 = 0$)

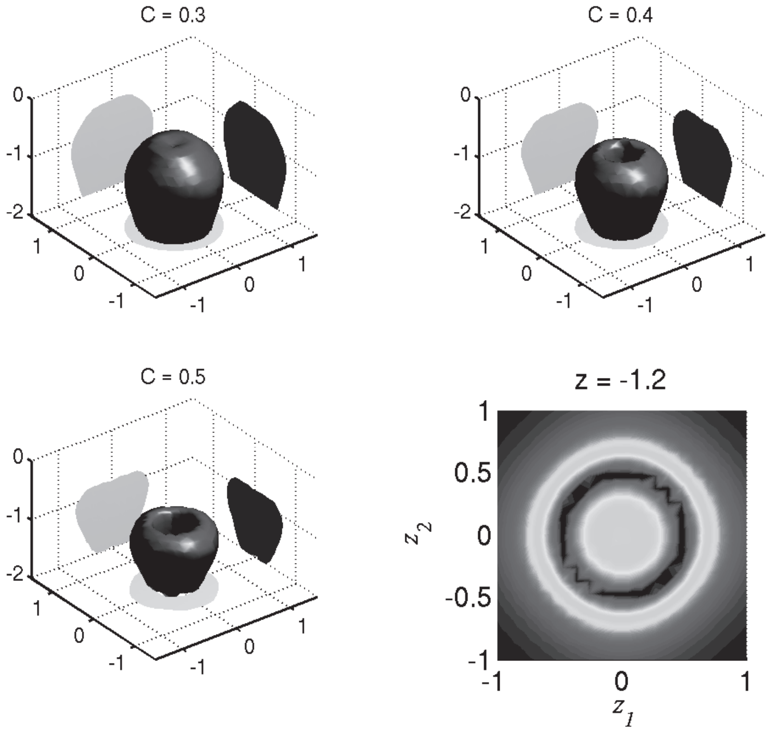


Figure 6.4. *Reconstruction of the buried torus using the LSM. The upper pair of panels and the bottom left panel show the isosurface $G(z) = C$ for various choices of C. The bottom right panel shows a contour plot of $G(z)$ at $z_3 = -1.2$ and shows that the hole in the torus is quite evident. The exact figure is shown in Figure* 6.3. *This figure is reproduced from* [32] *with permission.*

in the square $[s_{i,z} - 2\lambda, s_{i,x} + 2\lambda] \times [s_{iy} - 2\lambda, s_{iy} + 2\lambda]$, where (s_{ix}, s_{iy}) is the (x_i, x_y) position of the ith source, and the field is assumed to vanish away from the measurement grid. The surface $\tilde{\Lambda} = [-\lambda, \lambda] \times [-\lambda, \lambda]$ at $x_3 = \lambda/2$ is used to parameterize the single layer potential appearing in the reciprocity gap scheme, again using a 25×25 grid. The reconstructions for the reciprocity gap method are shown in Figure 6.5 using data with noise added according to (1.13) with $\epsilon = 0.01$. Provided the cutoff C for the isosurface is chosen appropriately, a good reconstruction can be observed.

The results of both the LSM and the reciprocity gap method results show that, with a suitable choice of the isosurface constant, these methods can accurately reconstruct an object in a nontrivial background medium (the layered medium). In addition, the data is limited aperture even if the collection area is quite large.

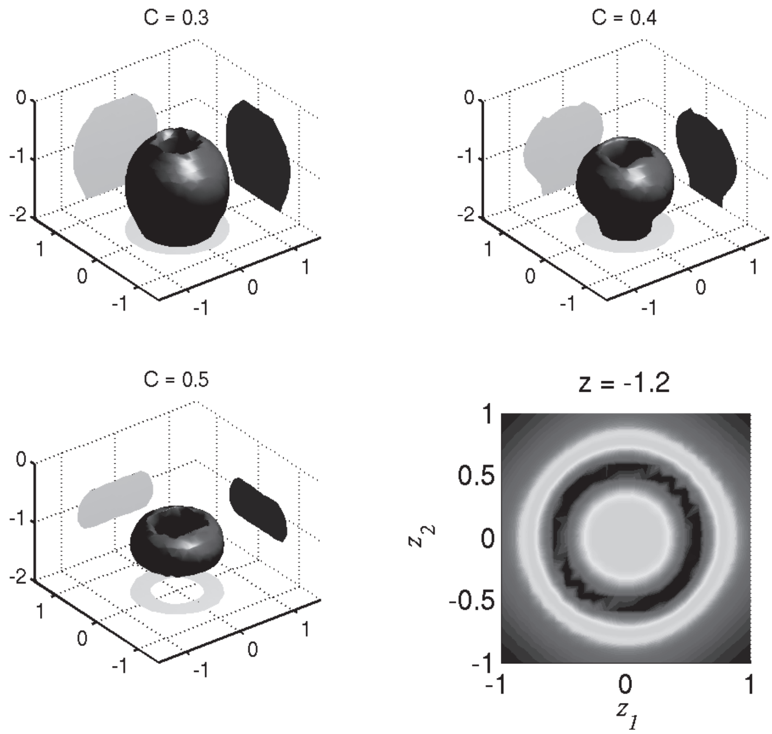


Figure 6.5. *Reconstruction of the buried torus using the reciprocity gap method. The upper pair of panels and the bottom left panel show the isosurface* $G(z) = C$ *for various choices of* C. *The bottom right panel shows a contour plot of* $G(z)$ *at* $z_3 = -1.2$ *and shows that the hole in the torus is quite evident. The exact figure is shown in Figure* 6.3. *This figure is reproduced from* [32] *with permission.*

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