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Qualitative Methods in Inverse Scattering Theory

An Introduction

With 14 Figures

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Preface

The field of inverse scattering theory has been a particularly active field in applied mathematics for the past twenty five years. The aim of research in this field has been to not only detect but also to identify unknown objects throught the use of acoustic, electromagnetic or elastic waves. Although the success of such techniques as ultrasound and x-ray tomography in medical imaging has been truly spectacular, progress has lagged in other areas of application which are forced to rely on different modalities using limited data in complex environments. Indeed, as pointed out in [58] concerning the problem of locating unexploded ordinance, "Target identification is the great unsolved problem. We detect almost everything, we identify nothing."

Until a few years ago, essentially all existing algorithms for target identification were based on either a weak scattering approximation or on the use of nonlinear optimization techniques. A survey of the state of the art for acoustic and electromagnetic waves as of 1998 can be found in [33]. However, as the demands of imaging increased, it became clear that incorrect model assumptions inherent in weak scattering approximations impose severe limitations on when reliable reconstructions are possible. On the other hand, it was also realized that for many practical applications nonlinear optimization techniques require a priori information that is in general not available. Hence in recent years alternative methods for imaging have been developed which avoid incorrect model assumptions but, as opposed to nonlinear optimization techniques, only seek limited information about the scattering object. Such methods come under the general title of *qualitative methods in inverse scattering theory*. Examples of such an approach are the linear sampling method, [29, 37], the factorization method [66, 67] and the method of singular sources [96, 98] which seek to determine an approximation to the shape of the scattering obstacle but in general provide only limited information about the material properties of the scatterer.

This book is designed to be an introduction to qualitative methods in inverse scattering theory, focusing on the basic ideas of the linear sampling method and its close relative the factorization method. The obvious question is an introduction for whom? One of the problems in making these new ideas in inverse scattering theory available to the wider scientific and engineering community is that the research papers in this area make use of mathematics that may be beyond the training of a reader who is not a professional mathematician. This book is an effort to overcome this problem and to write a monograph that is accessible to anyone having a mathematical background only in advanced calculus and linear algebra. In particular, the necessary material on functional analysis, Sobolev spaces and the theory of ill-posed problems will be given in the first two chapters. Of course, in order to do this in a short book such as this one, some proofs will not be given nor will all theorems be proven in complete generality. In particular, we will use the mapping and discontinuity properties of double and single layer potentials with densities in the Sobolev spaces $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$ respectively but will not prove any of these results, referring for their proofs to the monographs [75] and [85]. We will furthermore restrict ourselves to a simple model problem, the scattering of time harmonic electromagnetic waves by an infinite cylinder. This choice means that we can avoid the technical difficulties of three dimensional inverse scattering theory for different modalities and instead restrict our attention to the simpler case of two dimensional problems governed by the Helmholtz equation. For a glimpse of the problems arising in the three dimensional "real world", we conclude our book with a brief discussion of the qualitative approach to the inverse scattering problem for electromagnetic waves in $\mathbb{R}^3$ (see also [12]).

Although, for the above reasons, we do not discuss the qualitative approach to the inverse scattering problem for modalities other than electromagnetic waves, the reader should not assume that such approaches to not exist! Indeed, having mastered the material in this book, the reader will be fully prepared to understand the literature on qualitative methods for inverse scattering problems arising in other areas of application such as in acoustics and elasticity. In particular, for qualitative methods in the inverse scattering problem for acoustic waves and underwater sound see [6, 92, 112, 113], and [114] whereas for elasticity we refer the reader to [4, 20, 21, 48, 91, 93] and [105].

In closing, we would like to acknowledge the scientific and financial support of the Air Force Office of Scientific Research and in particular Dr. Arje Nachman of AFOSR and Dr. Richard Albanese of Brooks Air Force Base. Finally, a special thanks to our colleague Peter Monk who has been a participant with us in developing the qualitative approach to inverse scattering theory and whose advice and insights have been indispensable to our research efforts.

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Functional Analysis and Sobolev Spaces

Much of the recent work on inverse scattering theory is based on the use of special topics in functional analysis and the theory of Sobolev spaces. The results that we plan to present in this book are no exception. Hence we begin our book by providing a short introduction to the basic ideas of functional analysis and Sobolev spaces that will be needed to understand the material that follows. Since these two topics are the subject matter of numerous books at various levels of difficulty, we can only hope to present the bare rudiments of each of these fields. Nevertheless, armed with the material presented in this chapter, the reader will be well prepared to follow the arguments presented in subsequent chapters of this book.

We begin our presentation with the definition and basic properties of normed spaces and in particular Hilbert spaces. This is followed by a short introduction to the elementary properties of bounded linear operators and in particular compact operators. Included here is a proof of the Riesz theorem for compact operators on a normed space and the spectral properties of compact operators. We then proceed to a discussion of the adjoint operator in a Hilbert space and a proof of the Hilbert-Schmidt theorem. We conclude our chapter with an elementary introduction to Sobolev spaces. Here, following [75], we base our presentation on Fourier series rather than the Fourier transform and prove special cases of Rellich's theorem, the Sobolev imbedding theorem and the trace theorem.

1.1 Normed Spaces

We begin with the basic definition of a normed space X. We will always assume that $X \neq \{0\}$.

Definition 1.1. *Let X be a vector space over the field $\mathbb{C}$ of complex numbers. A function $||\cdot|| : X \to \mathbb{R}$ such that*

1. $||\varphi|| \geq 0$,

2. $||\varphi|| = 0$, *if and only if* $\varphi = 0$,
3. $||\alpha\varphi|| = |\alpha| \, ||\varphi||$ *for all* $\alpha \in \mathbb{C}$,
4. $||\varphi + \psi|| \leq ||\varphi|| + ||\psi||$

for all $\varphi, \psi \in X$ *is called a* norm *on* X. *A vector space* X *equipped with a norm is called a* normed space.

Example 1.2. The vector space $\mathbb{C}^n$ of ordered n-tuples of complex numbers $(\xi_1, \xi_2, \cdots, \xi_n)$ with the usual definitions of addition and scalar multiplication is a normed space with norm

$$||x|| := \left(\sum_1^n |\xi_i|^2 \right)^{\frac{1}{2}}$$

where $x = (\xi_1, \xi_2, \cdots, \xi_n)$. Note that the *triangle inequality* $||x + y|| \leq ||x|| + ||y||$ is simply a restatement of Minkowski's inequality for sums [3].

Example 1.3. Consider the vector space X of continuous complex valued functions defined on the interval $[a, b]$ with the obvious definitions of addition and scalar multiplication. Then

$$||\varphi|| := \max_{a \leq x \leq b} |\varphi(x)|$$

defines a norm on X and we refer to the resulting normed space as $C\,[a, b]$.

Example 1.4. Let X be the vector space of square integrable functions on $[a, b]$ in the sense of Lebesgue. Then it is easily seen that

$$||\varphi|| := \left[\int_a^b |\varphi(x)|^2 \, dx \right]^{\frac{1}{2}}$$

defines a norm on X. We refer to the resulting normed space as $\mathrm{L}^2[a, b]$.

Given a normed space X, we now introduce a topological structure on X. A sequence $\{\varphi_n\}, \varphi_n \in X$, *converges* to $\varphi \in X$ if $||\varphi_n - \varphi|| \to 0$ as $n \to \infty$ and we write $\varphi_n \to \varphi$. If Y is another normed space, a function $A : X \to Y$ is *continuous* at $\varphi \in X$ if $\varphi_n \to \varphi$ implies that $A\varphi_n \to A\varphi$. In particular, it is an easy exercise to show that $||\cdot||$ is continuous. A subset $U \subset X$ is *closed* if it contains all limits of convergent sequences of U. The *closure* $\overline{U}$ of U is the set of all limits of convergent sequences of U. A set U is called *dense* in X if $\overline{U} = X$.

In applications we are usually only interested in normed spaces that have the property of *completeness*. To define this property, we first note that a sequence $\{\varphi_n\}, \varphi_n \in X$, is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists an integer $N = N(\epsilon)$ such that $||\varphi_n - \varphi_m|| < \epsilon$ for all $m, n \geq N$. We then call a subset U of X *complete* if every Cauchy sequence in U converges to an element of U.

Definition 1.5. *A complete normed space X is called a* Banach space*.*

It can be shown that for each normed space X there exists a Banach space $\hat{X}$ such that X is isomorphic and isometric to a dense subspace of $\hat{X}$, i.e. there is a linear bijective mapping I from X onto a dense subspace of $\hat{X}$ such that $||I\varphi||_{\hat{X}} = ||\varphi||_X$ for all $\varphi \in X$ [79]. $\hat{X}$ is said to be the *completion* of X. For example, $[a, b]$ with the norm $||x|| = |x|$ for $x \in [a, b]$ is the completion of the set of rational numbers in $[a, b]$ with respect to this norm. It can be shown that the completion of the space of continuous complex valued functions on the interval $[a, b]$ with respect to the norm $||\cdot||$ defined by

$$||\varphi|| := \left[\int_a^b |\varphi(x)|^2 \, dx\right]^{\frac{1}{2}}$$

is the space $\mathrm{L}^2[a, b]$ defined above.

We now introduce vector spaces which have an *inner product* defined on them.

Definition 1.6. *Let X be a vector space over the field $\mathbb{C}$ of complex numbers. A function $(\cdot, \cdot) : X \times X \to \mathbb{C}$ such that*

1. $(\varphi, \varphi) \geq 0$,
2. $(\varphi, \varphi) = 0$ *if and only if* $\varphi = 0$,
3. $(\varphi, \psi) = \overline{(\psi, \varphi)}$,
4. $(\alpha\varphi + \beta\psi, \chi) = \alpha(\varphi, \chi) + \beta(\psi, \chi)$ *for all* $\alpha, \beta \in \mathbb{C}$

for all $\varphi, \psi, \chi \in X$ is called an inner product *on X.*

Example 1.7. For $x = (\xi_1, \xi_2, \cdots, \xi_n), y = (\eta_1, \eta_2, \cdots, \eta_n)$ in $\mathbb{C}^n$,

$$(x, y) := \sum_1^n \xi_i \overline{\eta_i}$$

is an inner product on $\mathbb{C}^n$.

Example 1.8. An inner product on $\mathrm{L}^2[a, b]$ is given by

$$(\varphi, \psi) := \int_a^b \varphi\overline{\psi} \, dx.$$

Theorem 1.9. *An inner product satisfies the* Cauchy-Schwarz inequality

$$|(\varphi, \psi)|^2 \leq (\varphi, \varphi)(\psi, \psi)$$

for all $\varphi, \psi \in X$ with equality if and only if φ and ψ are linearly dependent.

Proof. The inequality is trivial for $\varphi = 0$. For $\varphi \neq 0$ and

$$\alpha = -\frac{\overline{(\varphi, \psi)}}{(\varphi, \psi)} \quad , \quad \beta = (\varphi, \varphi)$$

we have that

$$\begin{aligned} 0 \le (\alpha\varphi + \beta\psi, \alpha\varphi + \beta\psi) &= |\alpha|^2(\varphi, \varphi) + 2\mathrm{Re}\left\{\alpha\overline{\beta}(\varphi, \psi)\right\} + |\beta|^2(\psi, \psi) \\ &= (\varphi, \varphi)(\psi, \psi) - |(\varphi, \psi)|^2 \end{aligned}$$

from which the inequality of the theorem follows. Equality holds if and only if $\alpha\varphi + \beta\psi = 0$ which implies that φ and ψ are linearly dependent since $\beta \neq 0$. □

A vector space with an inner product defined on it is called an *inner product space*. If X is an inner product space, then $||\varphi|| := (\varphi, \varphi)^{\frac{1}{2}}$ defines a norm on X. If X is complete with respect to this norm, X is called a *Hilbert space*. A *subspace* U of an inner product space X is a vector subspace of X taken with the inner product on X restricted to $U \times U$.

Example 1.10. With the inner product of the previous example, $\mathrm{L}^2[a, b]$ is a Hilbert space.

Two elements φ and ψ of a Hilbert space are called *orthogonal* if $(\varphi, \psi) = 0$ and we write $\varphi \perp \psi$. A subset $U \subset X$ is called an *orthogonal system* if $(\varphi, \psi) = 0$ for all $\varphi, \psi \in U$ with $\varphi \neq \psi$. An orthogonal system U is called an *orthonormal system* if $||\varphi|| = 1$ for every $\varphi \in U$. The set

$$U^\perp := \{\psi \in X : \psi \perp U\}$$

is called the *orthogonal complement* of the subset U.

Now let $U \subset X$ be a subset of a normed space X and let $\varphi \in X$. An element $v \in U$ is called a *best approximation* to φ with respect to U if

$$||\varphi - v|| = \inf_{u \in U} ||\varphi - u|| \ .$$

Theorem 1.11. *Let U be a subspace of a Hilbert space X. Then v is a best approximation to $\varphi \in X$ with respect to U if and only if $\varphi - v \perp U$. To each $\varphi \in X$ there exists at most one best approximation with respect to U.*

Proof. The theorem follows from

$$||(\varphi - v) + \alpha u||^2 = ||\varphi - v||^2 + 2\alpha\mathrm{Re}(\varphi - v, u) + \alpha^2||u||^2 \tag{1.1}$$

which is valid for all $v, u \in U$ and $\alpha \in \mathbb{R}$. In particular, if $u \neq 0$ then the minimum of the right hand side of (1.1) occurs when

$$\alpha = -\frac{\mathrm{Re}(\varphi - v, u)}{||u||^2}$$

and hence $||(\varphi - v) + \alpha u||^2 > ||\varphi - v||^2$ unless $\varphi - v \perp U$. On the other hand, if $\varphi - v \perp U$ then $||(\varphi - v) + \alpha u||^2 \geq ||\varphi - v||^2$ for all α and u which implies that v is a best approximation to φ. Finally, if there were two best approximations v_1 and v_2, then $(\varphi - v_1, u) = (\varphi - v_2, u) = 0$ and hence $(\varphi, u) = (v_1, u) = (v_2, u)$ for every $u \in U$. Thus $(v_1 - v_2, u) = 0$ for every $u \in U$ and, setting $u = v_1 - v_2$, we see that $v_1 = v_2$. □

Theorem 1.12. *Let U be a complete subspace of a Hilbert space X. Then to every element of X there exists a unique best approximation with respect to U.*

Proof. Let $\varphi \in X$ and choose $\{u_n\}, u_n \in U$, such that

$$||\varphi - u_n||^2 \leq d^2 + \frac{1}{n} \tag{1.2}$$

where $d := \inf_{u \in U} ||\varphi - u||$. Then, from the easily verifiable *parallelogram equality*

$$||\varphi + \psi||^2 + ||\varphi - \psi||^2 = 2\left(||\varphi||^2 + ||\psi||^2\right),$$

we have that

$$\begin{aligned} ||(\varphi - u_n) + (\varphi - u_m)||^2 + ||u_n + u_m||^2 &= 2\,||\varphi - u_n||^2 + 2\,||\varphi - u_m||^2 \\ &\leq 4d^2 + \frac{2}{n} + \frac{2}{m} \end{aligned}$$

and, since $\frac{1}{2}(u_n + u_m) \in U$, we have that

$$\begin{aligned} ||u_n - u_m||^2 &\leq 4d^2 + \frac{2}{n} + \frac{2}{m} - 4\left\|\varphi - \frac{1}{2}(u_n + u_m)\right\|^2 \\ &\leq \frac{2}{n} + \frac{2}{m}. \end{aligned}$$

Hence $\{u_n\}$ is a Cauchy sequence and, since U is complete, u_n converges to an element $v \in U$. Passing to the limit in (1.2) implies that v is a best approximation to φ with respect to U. Uniqueness follows from Theorem 1.11. □

We note that if U is a closed (and hence complete) subspace of a Hilbert space X then we can write $\varphi = v + \varphi - v$ where $\varphi - v \perp U$, i.e. U is the *direct sum* of U and its orthogonal complement which we write as

$$X = U \oplus U^\perp.$$

If U is a subset of a vector space X, the set spanned by all finite linear combinations of elements of U is denoted by $\operatorname{span} U$. A set $\{\varphi_n\}$ in a Hilbert space X such that $\operatorname{span}\{\varphi_n\}$ is dense in X is called a *complete set*.

Theorem 1.13. *Let $\{\varphi_n\}_1^\infty$ be an orthonormal system in a Hilbert space X. Then the following are equivalent:*

a. *$\{\varphi_n\}_1^\infty$ is complete.*
b. *Each $\varphi \in X$ can be expanded in a* Fourier series

$$\varphi = \sum_1^\infty (\varphi, \varphi_n)\, \varphi_n \, .$$

c. *For every $\varphi \in X$ we have* Parseval's equality

$$||\varphi||^2 = \sum_1^\infty |(\varphi, \varphi_n)|^2 \, .$$

d. *$\varphi = 0$ is the only element in X with $(\varphi, \varphi_n) = 0$ for every integer n.*

Proof. a $\Rightarrow$ b: Theorems 1.11 and 1.12 imply that

$$u_n = \sum_1^n (\varphi, \varphi_k)\, \varphi_k$$

is the best approximation to φ with respect to $\mathrm{span}\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$. Since $\{\varphi_n\}_1^\infty$ is complete, there exists $\hat{u}_n \in \mathrm{span}\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ such that $||\hat{u}_n - \varphi|| \to 0$ as $n \to \infty$ and since $||\hat{u}_n - \varphi|| \geq ||u_n - \varphi||$ we have that $u_n \to \varphi$ as $n \to \infty$.
b $\Rightarrow$ c: We have that

$$||u_n||^2 = (u_n, u_n) = \sum_1^n |(\varphi, \varphi_k)|^2 \, .$$

Now let $n \to \infty$ and use the continuity of $||\cdot||$.
c $\Rightarrow$ d: This is trivial.
d $\Rightarrow$ a: Set $U := \overline{\mathrm{span}\{\varphi_n\}}$ and assume $X \neq U$. Then there exists $\varphi \in X$ with $\varphi \notin U$. Since U is a closed subspace of X, U is complete. Hence, by Theorem 1.12, the best approximation v to φ with respect to U exists and satisfies $(v - \varphi, \varphi_n) = 0$ for every integer n. By assumption this implies $v = \varphi$ which is a contradiction. Hence $X = U$. $\square$

As a consequence of part b of the above theorem, a complete orthonormal system in a Hilbert space X is called an *orthonormal basis* for X.

1.2 Bounded Linear Operators

An operator $A : X \to Y$ mapping a vector space X into a vector space Y is called *linear* if

$$A\,(\alpha\varphi + \beta\psi) = \alpha A\varphi + \beta A\psi$$

for all $\varphi, \psi \in X$ and $\alpha, \beta \in \mathbb{C}$.

Theorem 1.14. *Let X and Y be normed spaces and $A : X \to Y$ a linear operator. Then A is continuous if it is continuous at one point.*

Proof. Suppose A is continuous at $\varphi_0 \in X$. Then for every $\varphi \in X$ and $\varphi_n \to \varphi$ we have that

$$A\varphi_n = A\,(\varphi_n - \varphi + \varphi_0) + A\,(\varphi - \varphi_0) \to A\varphi_0 + A\,(\varphi - \varphi_0) = A\varphi$$

since $\varphi_n - \varphi + \varphi_0 \to \varphi_0$. □

A linear operator $A : X \to Y$ from a normed space X into a normed space Y is called *bounded* if there exists a positive constant C such that

$$||A\varphi|| \leq C\,||\varphi||$$

for every $\varphi \in X$. The *norm* of A is the smallest such C, i.e. (dividing by $||\varphi||$ and using the linearity of A)

$$||A|| := \sup_{||\varphi||=1} ||A\varphi|| \quad , \quad \varphi \in X.$$

If $Y = \mathbb{C}$, A is called a bounded linear *functional*. The space X^* of bounded linear functionals on a normed space X is called the *dual space* of X.

Theorem 1.15. *Let X and Y be normed spaces and $A : X \to Y$ a linear operator. Then A is continuous if and only if it is bounded.*

Proof. Let $A : X \to Y$ be bounded and let $\{\varphi_n\}$ be a sequence in X such that $\varphi_n \to 0$ as $n \to \infty$. Then $||A\varphi_n|| \leq C\,||\varphi_n||$ implies that $A\varphi_n \to 0$ as $n \to \infty$, i.e. A is continuous at $\varphi = 0$. By Theorem 1.14 A is continuous for all $\varphi \in X$.

Conversely, let A be continuous and assume that there is no C such that $||A\varphi|| \leq C\,||\varphi||$ for all $\varphi \in X$. Then there exists a sequence $\{\varphi_n\}$ with $||\varphi_n|| = 1$ such that $||A\varphi_n|| \geq n$. Let $\psi_n := ||A\varphi_n||^{-1}\,\varphi_n$. Then $\psi_n \to 0$ as $n \to \infty$ and hence by the continuity of A we have that $A\psi_n \to A0 = 0$ which is a contradiction since $||A\psi_n|| = 1$ for every integer n. Hence A must be bounded. □

Example 1.16. Let $K(x,y)$ be continuous on $[a,b] \times [a,b]$ and define $A : \mathrm{L}^2[a,b] \to \mathrm{L}^2[a,b]$ by

$$(A\varphi)(x) := \int_a^b K(x,y)\varphi(y)\,dy\,.$$

Then

$$\begin{aligned}
||A\varphi||^2 &= \int_a^b |(A\varphi)(x)|^2 \, dx \\
&= \int_a^b \left| \int_a^b K(x,y)\varphi(y)\, dy \right|^2 \, dx \\
&\leq \int_a^b \int_a^b |K(x,y)|^2 \, dy \int_a^b |\varphi(y)|^2 \, dy\, dx \\
&= ||\varphi||^2 \int_a^b \int_a^b |K(x,y)|^2 \, dx\, dy \,.
\end{aligned}$$

Hence A is bounded and

$$||A|| \leq \left[\int_a^b \int_a^b |K(x,y)|^2 \, dx\, dy \right]^{\frac{1}{2}} .$$

Let X be a Hilbert space and $U \subset X$ a nontrivial subspace. A bounded linear operator $P : X \to U$ with the property that $P\varphi = \varphi$ for every $\varphi \in U$ is called a *projection operator* from X onto U. Suppose U is a nontrivial closed subspace of X. Then $X = U \oplus U^\perp$ and we define the *orthogonal projection* $P : X \to U$ by $P\varphi = v$ where v is the best approximation to φ. Then clearly $P\varphi = \varphi$ for $\varphi \in U$ and P is bounded since $||\varphi||^2 = ||P\varphi + (\varphi - P\varphi)||^2 = ||P\varphi||^2 + ||\varphi - P\varphi||^2 \geq ||P\varphi||^2$ by the orthogonality property of v (Theorem 1.11). Since $||P\varphi|| \leq ||\varphi||$ and $P\varphi = \varphi$ for $\varphi \in U$, we in fact have that $||P|| = 1$.

Our next step is to introduce the central idea of *compactness* into our discussion. A subset U of a normed space X is called *compact* if every sequence of elements in U contains a subsequence that converges to an element in U. U is called *relatively compact* if its closure is compact. A linear operator $A : X \to Y$ from a normed space X into a normed space Y is a *compact operator* if it maps each bounded set in X into a relatively compact set in Y. This is equivalent to requiring that for each bounded sequence $\{\varphi_n\}$ in X the sequence $\{A\varphi_n\}$ has a convergent subsequence in Y. Note that, since compact sets are bounded, compact operators are clearly bounded. It is also easy to see that linear combinations of compact operators are compact and the product of a bounded operator and a compact operator is a compact operator.

Theorem 1.17. *Let X be a normed space and Y a Banach space. Suppose $A_n : X \to Y$ is a compact operator for each integer n and there exists a linear operator A such that $||A - A_n|| \to 0$ as $n \to \infty$. Then A is a compact operator.*

Proof. Let $\{\varphi_m\}$ be a bounded sequence in X. We will use a diagonalization procedure to show that $\{A\varphi_m\}$ has a convergent subsequence in Y. Since A_1 is a compact operator, $\{\varphi_m\}$ has a subsequence $\{\varphi_{1,m}\}$ such that $\{A_1\varphi_{1,m}\}$ is convergent. Similarly, $\{\varphi_{1,m}\}$ has a subsequence $\{\varphi_{2,m}\}$ such that $\{A_2\varphi_{2,m}\}$

is convergent. Continuing in this manner, we see that the diagonal sequence $\{\varphi_{m,m}\}$ is a subsequence of $\{\varphi_m\}$ such that, for every fixed positive integer n, the sequence $\{A_n\varphi_{m,m}\}$ is convergent. Since $\{\varphi_m\}$ is bounded, say $||\varphi_m|| \le C$ for all m, $||\varphi_{m,m}|| \le C$ for all m. We now use the fact that $||A - A_n|| \to 0$ as $n \to \infty$ to conclude that for each $\epsilon > 0$ there exists an integer $n_0 = n_0(\epsilon)$ such that

$$||A - A_{n_0}|| < \frac{\epsilon}{3C}$$

and, since $\{A_{n_0}\varphi_{m,m}\}$ is convergent, there exists an integer $N = N(\epsilon)$ such that

$$||A_{n_0}\varphi_{j,j} - A_{n_0}\varphi_{k,k}|| < \frac{\epsilon}{3}$$

for $j, k > N$. Hence, for $j, k > N$, we have that

$$\begin{aligned}||A\varphi_{j,j} - A\varphi_{k,k}|| &\le ||A\varphi_{j,j} - A_{n_0}\varphi_{j,j}|| + ||A_{n_0}\varphi_{j,j} - A_{n_0}\varphi_{k,k}|| \\ &\quad + ||A_{n_0}\varphi_{k,k} - A\varphi_{k,k}|| \\ &\le ||A - A_{n_0}||\,||\varphi_{j,j}|| + \frac{\epsilon}{3} + ||A_{n_0} - A||\,||\varphi_{k,k}|| \\ &< \epsilon\,.\end{aligned}$$

Thus $\{A\varphi_{m,m}\}$ is a Cauchy sequence and therefore convergent in the Banach space Y. □

Example 1.18. Consider the operator $A : \mathrm{L}^2[a,b] \to \mathrm{L}^2[a,b]$ defined as in the previous example by

$$(A\varphi)(x) := \int_a^b K(x,y)\varphi(y)\,dy$$

where $K(x,y)$ is continuous on $[a,b] \times [a,b]$. Let $\{\varphi_n\}$ be a complete orthonormal set in $\mathrm{L}^2[a,b]$. Then it is easy to show that $\{\varphi_n(x)\varphi_m(y)\}$ is a complete orthonormal set in $\mathrm{L}^2\left([a,b] \times [a,b]\right)$. Hence

$$K(x,y) = \sum_{i,j=1}^{\infty} a_{ij}\varphi_i(x)\varphi_j(y)$$

in the mean square sense and by Parseval's equality

$$\int_a^b \int_a^b |K(x,y)|^2 \, dx\,dy = \sum_{i,j=1}^{\infty} |a_{ij}|^2 \,.$$

Furthermore,

$$\int_a^b \int_a^b \left| K(x,y) - \sum_{i,j=1}^{n} a_{ij}\varphi_j(x)\varphi_j(y) \right|^2 dx\,dy = \sum_{i,j=n+1}^{\infty} |a_{ij}|^2$$

which can be made as small as we please for n sufficiently large. Hence A can be approximated in norm by A_n where

$$(A_n\varphi)(x) := \int_a^b \left[\sum_{i,j=1}^{n} a_{ij}\varphi_i(x)\varphi_j(y)\right] \varphi(y)\, dy\,.$$

But $A_n : \mathrm{L}^2[a,b] \to \mathrm{L}^2[a,b]$ has finite-dimensional range. Hence if $U \subset X$ is bounded, $A_n(U)$ is a set in a finite-dimensional space $A_n(X)$. By the Bolzano-Weierstrass theorem, $A_n(U)$ is relatively compact, i.e. A_n is a compact operator. Theorem 1.17 now implies that A is a compact operator.

Lemma 1.19 (Riesz Lemma). *Let X be a normed space, $U \subset X$ a closed subspace such that $U \neq X$ and $\alpha \in (0,1)$. Then there exists $\psi \in X$, $||\psi|| = 1$, such that $||\psi - \varphi|| \geq \alpha$ for every $\varphi \in U$.*

Proof. There exists $f \in X$, $f \notin U$, and since U is closed we have that

$$\beta := \inf_{\varphi\in U} ||f - \varphi|| > 0\,.$$

Now choose $g \in U$ such that

$$\beta \leq ||f - g|| \leq \frac{\beta}{\alpha}$$

and define

$$\psi := \frac{f - g}{||f - g||}\,.$$

Then $||\psi|| = 1$ and for every $\varphi \in U$ we have, since $g + ||f - g||\,\varphi \in U$, that

$$||\psi - \varphi|| = \frac{1}{||f - g||}\, ||f - (g + ||f - g||\,\varphi)|| \geq \frac{\beta}{||f - g||} \geq \alpha\,.$$

□

The Riesz lemma is the key step in the proof of a series of basic results on compact operators that will be needed in the sequel. The following is the first of these results and will be used in the following chapter on ill-posed problems.

Theorem 1.20. *Let X be a normed space. Then the identity operator $I : X \to X$ is a compact operator if and only if X has finite dimension.*

Proof. Assume that I is a compact operator and X is not finite dimensional. Choose $\varphi_1 \in X$ with $||\varphi_1|| = 1$. Then $U_1 := \mathrm{span}\{\varphi_1\}$ is a closed subspace of X and by the Riesz lemma there exists $\varphi_2 \in X$, $||\varphi_2|| = 1$, with $||\varphi_2 - \varphi_1|| \geq \frac{1}{2}$. Now let $U_2 := \mathrm{span}\{\varphi_1, \varphi_2\}$. Using the Riesz lemma again, there exists $\varphi_3 \in X$, $||\varphi_3|| = 1$, and $||\varphi_3 - \varphi_1|| \geq \frac{1}{2}$, $||\varphi_3 - \varphi_2|| \geq \frac{1}{2}$. Continuing in this manner, we obtain a sequence $\{\varphi_n\}$ in X such that $||\varphi_n|| = 1$ and $||\varphi_n - \varphi_m|| \geq \frac{1}{2}$ for $n \neq m$. Hence $\{\varphi_n\}$ does not contain a convergent subsequence, i.e. $I : X \to X$ is not compact. This is a contradiction to our assumption. Hence if I is a compact operator, then X has finite dimension. Conversely, if X has finite dimension, $I(X)$ is finite-dimensional and by the Bolzano-Weierstrass theorem $I(X)$ is relatively compact, i.e. $I : X \to X$ is a compact operator. □

The next theorem, due to Riesz [101], is one of the most celebrated theorems in all of mathematics, having its origin in Fredholm's seminal paper of 1903 [44].

Theorem 1.21 (Riesz Theorem). *Let $A : X \to X$ be a compact operator on a normed space X. Then either 1) the homogeneous equation*

$$\varphi - A\varphi = 0$$

has a nontrivial solution $\varphi \in X$ or 2) for each $f \in X$ the equation

$$\varphi - A\varphi = f$$

has a unique solution $\varphi \in X$. If $I - A$ is injective (and hence bijective), then $(I - A)^{-1} : X \to X$ is bounded.

Proof. The proof will be divided into four steps.
Step 1: Let $L := I - A$ and let $N(L) := \{\varphi \in X : L\varphi = 0\}$ be the *null space* of L. We will show that there exists a positive constant C such that

$$\inf_{\chi \in N(L)} ||\varphi - \chi|| \leq C\, ||L\varphi||$$

for all $\varphi \in X$. Suppose this is not true. Then there exists a sequence $\{\varphi_n\}$ in X such that $||L\varphi_n|| = 1$ and $d_n := \inf_{\chi \in N(L)} ||\varphi_n - \chi|| \to \infty$. Choose $\{\chi_n\} \subset N(L)$ such that $d_n \leq ||\varphi_n - \chi_n|| \leq 2d_n$ and set

$$\psi_n := \frac{\varphi_n - \chi_n}{||\varphi_n - \chi_n||}\,.$$

Then $||\psi_n|| = 1$ and $||L\psi_n|| \leq d_n^{-1} \to 0$. But since A is compact, by passing to a subsequence if necessary, we may assume that the sequence $\{A\psi_n\}$ converges to an element $\varphi_0 \in X$. Since $\psi_n = (L + A)\psi_n$, we have that $\{\psi_n\}$ converges to φ_0 and hence $\varphi_0 \in N(L)$. But

$$\begin{aligned}\inf_{\chi \in N(L)} ||\psi_n - \chi|| &= ||\varphi_n - \chi_n||^{-1} \inf_{\chi \in N(L)} ||\varphi_n - \chi_n - ||\varphi_n - \chi_n||\, \chi|| \\ &= ||\varphi_n - \chi_n||^{-1} \inf_{\chi \in N(L)} ||\varphi_n - \chi|| \geq \frac{1}{2}\end{aligned}$$

which contradicts the fact that $\psi_n \to \varphi_0 \in N(L)$.

Step 2: We next show that the range of L is a closed subspace of X. $L(X) := \{x \in X : x = L\varphi \text{ for some } \varphi \in X\}$ is clearly a subspace. Hence if $\{\varphi_n\}$ is a sequence in X such that $\{L\varphi_n\}$ converges to an element $f \in X$, we must show that $f = L\varphi$ for some $\varphi \in X$. By the above result the sequence $\{d_n\}$ where $d_n := \inf_{\chi \in N(L)} ||\varphi_n - \chi||$ is bounded. Choosing $\chi_n \in N(L)$ as above and writing $\tilde{\varphi}_n := \varphi_n - \chi_n$, we have that $\{\tilde{\varphi}_n\}$ is bounded and $L\tilde{\varphi}_n \to f$. Since A is compact, by passing to a subsequence if necessary, we may assume that $\{A\tilde{\varphi}_n\}$ converges to an element $\tilde{\varphi}_0 \in X$. Hence $\tilde{\varphi}_n$ converges to $f + \varphi_0$ and by the continuity of L we have that $L(f + \varphi_0) = f$. Hence $L(X)$ is closed.
Step 3: The next step is to show that if $N(L) = \{0\}$ then $L(X) = X$, i.e. if case 1) of the theorem does not hold then case 2) is true. To this end, we note that from our previous result the sets $L^n(X), n = 1, 2, \cdots$, form a non-increasing sequence of closed subspaces of X. Suppose that no two of these spaces coincide. Then each is a proper subspace of its predecessor. Hence, by the Riesz lemma, there exists a sequence $\{\psi_n\}$ in X such that $\psi_n \in L^n(X)$, $||\psi_n|| = 1$, and $||\psi_n - \psi|| \geq \frac{1}{2}$ for all $\psi \in L^{n+1}(X)$. Thus if $m > n$ then

$$A\psi_n - A\psi_m = \psi_n - (\psi_m + L\psi_n - L\psi_m)$$

and $\psi_m + L\psi_n - L\psi_m \in L^{n+1}(X)$ since

$$\psi_m + L\psi_n - L\psi_m = L^{n+1}(L^{m-n-1}\varphi_m + \varphi_n - L^{m-n}\varphi_m)\,.$$

Hence $||A\psi_n - A\psi_m|| \geq \frac{1}{2}$ contrary to the compactness of A. Thus we can conclude that there exists an integer n_0 such that $L^n(X) = L^{n_0}(X)$ for all $n \geq n_0$. Now let $\varphi \in X$. Then $L^{n_0}\varphi \in L^{n_0}(X) = L^{n_0+1}(X)$ and so $L^{n_0}\varphi = L^{n_0+1}\psi$ for some $\psi \in X$, i.e. $L^{n_0}(\varphi - L\psi) = 0$. But since $N(L) = \{0\}$ we have that $N(L^{n_0}) = 0$ and hence $\varphi = L\psi$. Thus $X = L(X)$.
Step 4: We now come to the final step, which is to show that if $L(X) = X$ then $N(L) = 0$, i.e. either case 1) or case 2) of the theorem is true. To show this, we first note that by the continuity of L we have that $N(L^n)$ is a closed subspace for $n = 1, 2, \cdots$. An analogous argument to that used in Step 3 shows that there exists an integer n_0 such that $N(L^n) = N(L^{n_0})$ for all $n \geq n_0$. Hence, if $L(X) = X$ then $\varphi \in N(L^{n_0})$ satisfies $\varphi = L^{n_0}\psi$ for some $\psi \in X$ and thus $L^{2n_0}\psi = 0$. Thus $\psi \in N(L^{2n_0}) = N(L^{n_0})$ and hence $\varphi = L^{n_0}\psi = 0$. Since $L\varphi = 0$ implies that $L^{n_0}\varphi = 0$, the proof of Step 4 is now complete.

The fact that $(I - A)^{-1}$ is bounded in case 2) follows from Step 1 since in this case $N(L) = \{0\}$. □

Let $A : X \to X$ be a compact operator of a normed space into itself. A complex number λ is called an *eigenvalue* of A with *eigenelement* $\varphi \in X$ if there exists $\varphi \in X$, $\varphi \neq 0$, such that $A\varphi = \lambda\varphi$. It is easily seen that eigenelements corresponding to different eigenvalues must be linearly independent. We call the dimension of the null space of $L_\lambda := \lambda I - A$ the multiplicity of λ. If $\lambda \neq 0$ is not an eigenvalue of A, it follows from the Riesz theorem that the *resolvent operator* $(\lambda I - A)^{-1}$ is a well defined bounded linear operator mapping X onto itself. On the other hand, if $\lambda = 0$ then A^{-1} cannot be bounded on $A(X)$ unless X is finite dimensional since if it were then $I = A^{-1}A$ would be compact.

Theorem 1.22. *Let $A : X \to X$ be a compact operator on a normed space X. Then A has at most a countable set of eigenvalues having no limit points except possibly $\lambda = 0$. Each non-zero eigenvalue has finite multiplicity.*

Proof. Suppose there exists a sequence $\{\lambda_n\}$ of not necessarily distinct non-zero eigenvalues with corresponding linearly independent eigenelements $\{\varphi_n\}_1^\infty$ such that $\lambda_n \to \lambda \neq 0$. Let

$$U_n := \operatorname{span}\{\varphi_1, \cdots, \varphi_n\}\,.$$

Then, by the Riesz lemma, there exists a sequence $\{\psi_n\}$ such that $\psi_n \in U_n$, $||\psi_n|| = 1$ and $||\psi_n - \psi|| \geq \frac{1}{2}$ for every $\psi \in U_{n-1}$, $n = 2, 3, \cdots$. If $n > m$, we have that

$$\begin{aligned} \lambda_n^{-1}A\psi_n - \lambda_m^{-1}A\psi_m &= \psi_n + \left(-\psi_n - \lambda_n^{-1}L_{\lambda_n}\psi_n + \lambda_m^{-1}L_{\lambda_m}\psi_m\right) \\ &= \psi_n - \psi \end{aligned}$$

where $\psi \in U_{n-1}$ since if $\psi_n = \sum_1^n \beta_j\varphi_j$ then

$$\psi_n - \lambda_n^{-1}A\psi_n = \sum_1^n \beta_j \left(1 - \lambda_n^{-1}\lambda_j\right)\varphi_j \in U_{n-1}$$

and similarly $L_{\lambda_m}\psi_m \in U_{m-1}$. Hence, we have that $\left|\left|\lambda_n^{-1}A\psi_n - \lambda_m^{-1}A\psi_m\right|\right| \geq \frac{1}{2}$ which, since $\lambda_n \to \lambda \neq 0$, contradicts the compactness of the operator A. Hence our initial assumption is false and this implies the validity of the theorem. $\square$

1.3 The Adjoint Operator

We now assume that X is a Hilbert space and first characterize the class of bounded linear functionals on X.

Theorem 1.23 (Riesz Representation Theorem). *Let X be a Hilbert space. Then for each bounded linear functional $F : X \to \mathbb{C}$ there exists a unique $f \in X$ such that*

$$F(\varphi) = (\varphi, f)$$

for every $\varphi \in X$. Furthermore, $||f|| = ||F||$.

Proof. We first show the uniqueness of the representation. This is easy since if $(\varphi, f_1) = (\varphi, f_2)$ for every $\varphi \in X$ then $(\varphi, f_1 - f_2) = 0$ for every $\varphi \in X$ and setting $\varphi = f_1 - f_2$ we have that $||f_1 - f_2||^2 = 0$. Hence, $f_1 = f_2$.

We now turn to the existence of f. If $F = 0$ we can choose $f = 0$. Hence assume $F \neq 0$ and choose $w \in X$ such that $F(w) \neq 0$. Since F is continuous, $N(F) = \{\varphi \in X : F(\varphi) = 0\}$ is a closed (and hence complete) subspace of X. Hence by Theorem 1.12 there exists a unique best approximation v to w with respect to $N(F)$, and by Theorem 1.11 we have that $w - v \perp N(F)$. Then for $g := w - v$ we have that

$$(F(g)\varphi - F(\varphi)g, g) = 0$$

for every $\varphi \in X$ since $F(g)\varphi - F(\varphi)g \in N(F)$ for every $\varphi \in X$. Hence

$$F(\varphi) = \left(\varphi, \frac{\overline{F(g)}g}{||g||^2}\right)$$

for every $\varphi \in X$, i.e.

$$f := \frac{\overline{F(g)}g}{||g||^2}$$

is the element we are seeking.

Finally, to show that $||f|| = ||F||$, we note that by the Cauchy-Schwarz inequality we have that $|F(\varphi)| \leq ||f||\,||\varphi||$ for every $\varphi \in X$ and hence $||F|| \leq ||f||$. On the other hand, $F(f) = (f, f) = ||f||^2$ and hence $||f|| \leq ||F||$. We can now conclude that $||F|| = ||f||$. □

Armed with the Riesz representation theorem we can now define the *adjoint operator* A^* of A.

Theorem 1.24. *Let X and Y be Hilbert spaces and let $A : X \to Y$ be a bounded linear operator. Then there exists a uniquely determined linear operator $A^* : Y \to X$ such that $(A\varphi, \psi) = (\varphi, A^*\psi)$ for every $\varphi \in X$ and $\psi \in Y$. A^* is called the* adjoint *of A and is a bounded linear operator satisfying $||A^*|| = ||A||$.*

Proof. For each $\psi \in Y$ the mapping $\varphi \mapsto (A\varphi, \psi)$ defines a bounded linear functional on X since

$$|(A\varphi, \psi)| \leq ||A||\,||\varphi||\,||\psi|| \ .$$

Hence by the Riesz representation theorem we can write $(A\varphi, \psi) = (\varphi, f)$ for some $f \in X$. We now define $A^* : Y \to X$ by $A^*\psi = f$. The operator A^* is unique since if $0 = (\varphi, (A_1^* - A_2^*)\psi)$ for every $\varphi \in X$ then setting $\varphi = (A_1^* - A_2^*)\psi$ we have that $||(A_1^* - A_2^*)\psi||^2 = 0$ for every $\psi \in Y$ and hence $A_1^* = A_2^*$. To show that A^* is linear, we observe that

$$\begin{aligned}
(\varphi, \beta_1 A^*\psi_1 + \beta_2 A^*\psi_2) &= \bar{\beta}_1 (\varphi, A^*\psi_1) + \bar{\beta}_2 (\varphi, A^*\psi_2) \\
&= \bar{\beta}_1 (A\varphi, \psi_1) + \bar{\beta}_2 (A\varphi, \psi_2) \\
&= (A\varphi, \beta_1\psi_1 + \beta_2\psi_2) \\
&= (\varphi, A^* (\beta_1\psi_1 + \beta_2\psi_2))
\end{aligned}$$

for every $\varphi \in X$, $\psi_1, \psi_2 \in Y$ and $\beta_1, \beta_2 \in \mathbb{C}$. Hence $\beta_1 A^*\psi_1 + \beta_2 A^*\psi_2 = A^*(\beta_1\psi_1 + \beta_2\psi_2)$, i.e. A^* is linear. To show that A^* is bounded, we note that by the Cauchy-Schwarz inequality we have that

$$||A^*\psi||^2 = (A^*\psi, A^*\psi) = (AA^*\psi, \psi) \leq ||A||\,||A^*\psi||\,||\psi||$$

for every $\psi \in Y$. Hence $||A^*|| \leq ||A||$. Conversely, since A is the adjoint of A^*, we also have that $||A|| \leq ||A^*||$ and hence $||A^*|| = ||A||$. □

Theorem 1.25. *Let X and Y be Hilbert spaces and let $A : X \to Y$ be a compact operator. Then $A^* : Y \to X$ is also a compact operator.*

Proof. Let $||\psi_n|| \leq C$ for some positive constant C. Then, since A^* is bounded, $AA^* : Y \to Y$ is a compact operator. Hence, by passing to a subsequence if necessary, we may assume that the sequence $\{AA^*\psi_n\}$ converges in Y. But

$$\begin{aligned} ||A^*(\psi_n - \psi_m)||^2 &= (AA^*(\psi_n - \psi_m), \psi_n - \psi_m) \\ &\leq 2C\,||AA^*(\psi_n - \psi_m)||\,, \end{aligned}$$

i.e. $\{A^*\psi_n\}$ is a Cauchy sequence and hence convergent. We can now conclude that A^* is a compact operator. □

The following theorem will be important to us in the next chapter of this book. We first need a lemma.

Lemma 1.26. *Let U be a closed subspace of a Hilbert space X. Then $U^{\perp\perp} = U$.*

Proof. Since U is a closed subspace, we have that $X = U \oplus U^{\perp}$ and $X = U^{\perp} \oplus U^{\perp\perp}$. Hence for $\varphi \in X$ we have that $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in U$ and $\varphi_2 \in U^{\perp}$ and $\varphi = \psi_1 + \psi_2$ where $\psi_1 \in U^{\perp\perp}$ and $\psi_2 \in U^{\perp}$. In particular, $0 = (\varphi_1 - \psi_1) + (\varphi_2 - \psi_2)$ and since it is easily verified that $U \subseteq U^{\perp\perp}$ we have that $\varphi_1 - \psi_1 = \psi_2 - \varphi_2 \in U^{\perp}$. But $\varphi_1 - \psi_1 \in U^{\perp\perp}$ and hence $\varphi_1 = \psi_1$. We can now conclude that $U^{\perp\perp} = U$. □

Theorem 1.27. *Let X and Y be Hilbert spaces. Then for a bounded linear operator $A : X \to Y$ we have that if $A(X) := \{y \in Y : y = Ax$ for some $x \in X\}$ is the* range *of A then*

$$A(X)^{\perp} = N(A^*) \,\text{and}\, N(A^*)^{\perp} = \overline{A(X)}\,.$$

Proof. We have that $g \in A(X)^{\perp}$ if and only if $(A\varphi, g) = 0$ for every $\varphi \in X$. Since $(A\varphi, g) = (\varphi, A^*g)$ we can now conclude that $A^*g = 0$, i.e. $g \in N(A^*)$. On the other hand, by Lemma 1.26, $\overline{A(X)} = \overline{A(X)}^{\perp\perp} = N(A^*)^{\perp}$ since $A(X)^{\perp} = \overline{A(X)}^{\perp} = N(A^*)$. □

The next theorem is one of the jewels of functional analysis and will play a central role in the next chapter of the book. We note that a bounded linear operator $A : X \to X$ on a Hilbert space X is said to be *self-adjoint* if $A = A^*$, i.e. $(A\varphi, \psi) = (\varphi, A\psi)$ for all $\varphi, \psi \in X$.

Theorem 1.28 (Hilbert-Schmidt Theorem). *Let $A : X \to X$ be a compact, self-adjoint operator on a Hilbert space X. Then, if $A \neq 0$, A has at least one eigenvalue different from zero, all the eigenvalues of A are real and X has an orthonormal basis consisting of eigenelements of A.*

Proof. It is a simple consequence of the self-adjointness of A that 1) eigenelements corresponding to different eigenvalues are orthogonal and 2) all eigenvalues are real. Hence the first serious problem to face is to show that $A \neq 0$ has at least one eigenvalue different from zero. To this end, let $\lambda = ||A|| > 0$ and consider the operator $T := \lambda^2 I - A^2$. We will show that $\pm\lambda$ is an eigenvalue of A. To show this, we first note that for all $\varphi \in X$ we have that

$$\begin{aligned}(T\varphi, \varphi) = ((\lambda^2 I - A^2)\varphi, \varphi) &= \lambda^2 \left|\left|\varphi\right|\right|^2 - (A^2\varphi, \varphi) \\ &= \lambda^2 \left|\left|\varphi\right|\right|^2 - \left|\left|A\varphi\right|\right|^2 \geq 0\,.\end{aligned}$$

Now choose a sequence $\{\varphi_n\} \subset X$ such that $||\varphi_n|| = 1$ and $||A\varphi_n|| \to \lambda$ as $n \to \infty$. Then, by the above identity, $(T\varphi_n, \varphi_n) \to 0$ as $n \to \infty$. To proceed further, we first define a new inner product $\langle \cdot, \cdot \rangle$ on X by

$$\langle \varphi, \psi \rangle := (T\varphi, \psi)\,.$$

The fact that $\langle \cdot, \cdot \rangle$ defines an inner product follows easily from the fact that A, and hence T, is self-adjoint and the fact that $(T\varphi, \varphi) \geq 0$ for all $\varphi \in X$. We now have from the Cauch-Schwarz inequality that

$$\begin{aligned}\left|\left|T\varphi_n\right|\right|^2 &= (T\varphi_n, T\varphi_n) = \langle \varphi_n, T\varphi_n \rangle \\ &\leq \langle \varphi_n, \varphi_n \rangle^{\frac{1}{2}} \langle T\varphi_n, T\varphi_n \rangle^{\frac{1}{2}} \\ &= (T\varphi_n, \varphi_n)^{\frac{1}{2}} (T^2\varphi_n, T\varphi)^{\frac{1}{2}} \\ &\leq (T\varphi_n, \varphi_n)^{\frac{1}{2}} \left|\left|T^2\varphi_n\right|\right|^{\frac{1}{2}} \left|\left|T\varphi_n\right|\right|^{\frac{1}{2}} \\ &\leq \left|\left|T\right|\right|^{\frac{3}{2}} (T\varphi_n, \varphi_n)^{\frac{1}{2}}\,.\end{aligned}$$

But $(T\varphi_n, \varphi_n) \to 0$ as $n \to \infty$ and hence by the above inequality $T\varphi_n \to 0$ as $n \to \infty$. Since A is compact, by passing to a subsequence if necessary, we may assume that $\{A\varphi_n\}$ converges to a limit φ which satisfies $||\varphi|| = \lim_{n\to\infty} ||A\varphi_n|| = \lambda > 0$ and $T\varphi = \lim_{n\to\infty} TA\varphi_n = \lim_{n\to\infty} AT\varphi_n = 0$, i.e. $\varphi \neq 0$ and

$$T\varphi = (\lambda I + A)(\lambda I - A)\varphi = 0\,.$$

Thus either $A\varphi = \lambda\varphi$ or $\lambda\varphi - A\varphi \neq 0$ and $A\psi = -\lambda\psi$ for $\psi = \lambda\varphi - A\varphi$. Thus either λ or $-\lambda$ is a nonzero eigenvalue of A.

We now complete the theorem by showing that X has an orthonormal basis consisting of eigenvectors of A. We first note that if Y is a subspace of X such that $A(Y) \subset Y$ then by the self-adjointness of A we have that $A(Y^\perp) \subset Y^\perp$. In particular, let Y be the closed linear span of all the eigenelements of A. The restriction of A to the nullspace of $L := \lambda I - A$ is the identity operator on the closed subspace $N(L)$. Since the restriction of A to $N(L)$ is compact from $N(L)$ onto $N(L)$, we can conclude from Theorem 1.21 that $N(L)$ has finite dimension. Now pick an orthonormal basis for each eigenspace of A and take their union. Since eigenelements corresponding to different eigenvalues are orthogonal, this union is an orthonormal basis for Y. We now note that $A : Y^\perp \to Y^\perp$ is a compact operator which has no eigenvalues since all the eigenelements of A belong to Y. But this is impossible by the first part of our proof unless either A restricted to $Y^\perp$ is the zero operator or $Y^\perp = \{0\}$. If A restricted to $Y^\perp$ is the zero operator, then $Y^\perp = \{0\}$ since otherwise nonzero elements of $Y^\perp$ would be eigenelements of A corresponding to the eigenvalue zero and hence in Y, a contradiction. Thus in either case $Y^\perp = \{0\}$, i.e. $Y = X$, and the proof is complete. □

1.4 The Sobolev Space $H^p[0, 2\pi]$

For the proper study of inverse problems it is necessary to consider function spaces that are larger than the classes of continuous and continuously differentiable functions. In particular, Sobolev spaces are the natural spaces to consider in order to apply the tools of functional analysis presented above. Hence, in this and the following section, we will present the rudiments of the theory of Sobolev spaces. Our presentation will closely follow the excellent introductory treatment of such spaces by Kress [75] which avoids the use of Fourier transforms in $L^2(\mathbb{R}^n)$ but instead relies on the elementary theory of Fourier series. This simplification is made possible by restricting attention to planar domains having C^2 boundaries and has the drawback of not being able to achieve the depth of a more sophisticated treatment such as that presented in [85]. However, the limited results we shall present will be sufficient for the purposes of this book.

We begin with the fact that the orthonormal system $\left\{\frac{1}{\sqrt{2\pi}} e^{imt}\right\}_{-\infty}^{\infty}$ is complete in $L^2[0, 2\pi]$ [3]. Hence, by Theorem 1.13, for $\varphi \in L^2[0, 2\pi]$ we have that in the sense of mean square convergence

$$\varphi(t) = \sum_{-\infty}^{\infty} a_m e^{imt}$$

where the Fourier coefficients a_m are given by

$$a_m := \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) e^{-imt} \, dt \, .$$

If we let $(\cdot, \cdot)$ denote the usual L^2-inner product with associated norm $||\cdot||$ then by Parseval's equality we have that

$$\sum_{-\infty}^{\infty} |a_m|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(t)|^2 \, dt$$
$$= \frac{1}{2\pi} \, ||\varphi||^2 \, .$$

Now let $0 \leq p < \infty$. Then we define $H^p[0, 2\pi]$ to be the space of all functions $\varphi \in L^2[0, 2\pi]$ such that

$$\sum_{-\infty}^{\infty} (1 + m^2)^p \, |a_m|^2 < \infty$$

where the a_m are the Fourier coefficients of φ. The space $H^p = H^p[0, 2\pi]$ is called a *Sobolev space.* Note that $H^0[0, 2\pi] = L^2[0, 2\pi]$.

Theorem 1.29. $H^p[0, 2\pi]$ *is a Hilbert space with inner product*

$$(\varphi, \psi)_p := \sum_{-\infty}^{\infty} (1 + m^2)^p a_m \bar{b}_m$$

where the a_m, b_m *are the Fourier coefficients of* φ, ψ *respectively. The trigonometric polynomials are dense in* $H^p[0, 2\pi]$.

Proof. If is easily verified that H^p is a vector space and $(\cdot, \cdot)_p$ is an inner product. Note that the fact that $(\cdot, \cdot)_p$ is well defined follows from the Cauchy-Schwarz inequality

$$\left| \sum_{-\infty}^{\infty} (1 + m^2)^p a_m \bar{b}_m \right|^2 \leq \sum_{-\infty}^{\infty} (1 + m^2)^p \, |a_m|^2 \sum_{-\infty}^{\infty} (1 + m^2)^p \, |b_m|^2 \, .$$

To show that H^p is complete, let $\{\varphi_n\}$ be a Cauchy sequence, i.e.

$$\sum_{-\infty}^{\infty} (1 + m^2)^p \, |a_{m,n} - a_{m,k}|^2 < \epsilon^2$$

for all $n, k \geq N = N(\epsilon)$ where $a_{m,n}$ are the Fourier coefficients of φ_n. In particular,

$$\sum_{-M_1}^{M_2} (1 + m^2) \, |a_{m,n} - a_{m,k}|^2 < \epsilon^2 \tag{1.3}$$

for all M_1, M_2 and $n, k \geq N(\epsilon)$. Since $\mathbb{C}$ is complete, there exists a sequence $\{a_m\}$ in $\mathbb{C}$ such that $a_{m,n} \to a_m$ as $n \to \infty$ for each fixed m. Letting $k \to \infty$ in (1.3) implies that

$$\sum_{-M_1}^{M_2} (1 + m^2)^p \, |a_{m,n} - a_m|^2 \leq \epsilon^2$$

for all $n \geq N(\epsilon)$ and all M_1 and M_2. Hence

$$\sum_{-\infty}^{\infty}(1+m^2)^p \left|a_{m,n} - a_m\right|^2 \leq \epsilon^2 \tag{1.4}$$

for all $n \geq N(\epsilon)$. Defining

$$f_m(t) := e^{imt}$$

and

$$\varphi := \sum_{-\infty}^{\infty} a_m f_m \,,$$

we have by (1.4) and the triangle inequality that

$$\left[\sum_{-\infty}^{\infty}(1+m^2)^p \left|a_m\right|^2\right]^{\frac{1}{2}} \leq \epsilon + \left[\sum_{-\infty}^{\infty}(1+m^2)^p \left|a_{m,n}\right|^2\right] < \infty \,,$$

i.e. $\varphi \in H^p$. From (1.4) we can conclude that $||\varphi - \varphi_n|| \to 0$ as $n \to \infty$ and hence H^p is complete.

To prove the last statement of the theorem, let $\varphi \in H^p$ with Fourier coefficients a_m. Then for

$$\varphi_n := \sum_{-n}^{n} a_m f_m$$

we have that

$$||\varphi - \varphi_n||_p^2 = \sum_{|m|=n+1}^{\infty} (1+m^2)^p \left|a_m\right|^2 \to 0$$

as $n \to \infty$ since the full series is convergent. From this we can conclude that the trigonometric polynomials are dense in H^p. □

Theorem 1.30 (Rellich's Theorem). *If $q > p$ then $H^q[0, 2\pi]$ is dense in $H^p[0, 2\pi]$ and the imbedding operator $I : H^q \to H^p$ is compact.*

Proof. Since $(1+m^2)^p \leq (1+m^2)^q$ for $0 \leq p < q < \infty$, it follows that $H^q \subset H^p$ and $||\varphi||_p \leq ||\varphi||_q$ for every $\varphi \in H^q$. The denseness of H^q in H^p follows from the denseness of trigonometric polynomials in H^p.

To show that $I : H^q \to H^p$ is a compact operator, define $I_n : H^q \to H^p$ by

$$I_n\varphi := \sum_{-n}^{n} a_m f_m$$

for $\varphi \in H^q$ having Fourier coefficients a_m. Then

$$\begin{aligned}||(I_n - I)\varphi||_p^2 &= \sum_{|m|=n+1}^{\infty} (1+m^2)^p |a_m|^2 \\ &\leq \frac{1}{(1+n^2)^{q-p}} \sum_{|m|=n+1}^{\infty} (1+m^2)^q |a_m|^2 \\ &\leq \frac{1}{(1+n^2)^{q-p}} ||\varphi||_q^2 .\end{aligned}$$

Since I_n has finite dimensional range, I_n is a compact operator and from the above inequality we have that $||I_n - I|| \leq (1+n^2)^{\frac{(p-q)}{2}} \to 0$ as $n \to \infty$. Hence I is compact by Theorem 1.17. □

Theorem 1.31 (Sobolev Imbedding Theorem). *Let $p > \frac{1}{2}$ and $\varphi \in H^p[0, 2\pi]$. Then φ coincides almost everywhere with a continuous and 2π-periodic function (i.e. the difference between φ and this function is a function η such that $||\eta||_p = 0$).*

Proof. For $\varphi \in H^p[0, 2\pi]$ we have that for $p > \frac{1}{2}$

$$\left[\sum_{-\infty}^{\infty} \left|a_m e^{imt}\right|\right]^2 \leq \sum_{-\infty}^{\infty} \frac{1}{(1+m^2)^p} \sum_{-\infty}^{\infty} (1+m^2)^p |a_m|^2$$

by the Cauchy-Schwarz inequality. Hence the Fourier series for φ is absolutely and uniformly convergent and thus coincides with a continuous 2π-periodic function. Since the Fourier series for φ agrees with φ almost everywhere (as defined in the theorem), the proof is complete. □

Definition 1.32. *For $0 \leq p < \infty$, $H^{-p} = H^{-p}[0, 2\pi]$ is defined to be the* dual space *of $H^p[0, 2\pi]$, i.e. the space of bounded linear functionals defined on $H^p[0, 2\pi]$.*

Recall that for F a bounded linear functional defined on $H^p[0, 2\pi]$, the norm of F is defined by

$$||F||_p := \sup_{\substack{\varphi \in H^p \\ ||\varphi||_p = 1}} |F\varphi|$$

The following theorem gives an explicit expression for $||F||$ and a characterization of H^{-p}.

Theorem 1.33. *For $F \in H^{-p}[0, 2\pi]$ the norm is given by*

$$||F||_p = \left[\sum_{-\infty}^{\infty} (1+m^2)^{-p} |c_m|^2\right]^{\frac{1}{2}}$$

where $c_m = F(f_m)$. Conversely, to each sequence $\{c_m\}$ in $\mathbb{C}$ satisfying

$$\sum_{-\infty}^{\infty}(1+m^2)^{-p}\left|c_m\right|^2 < \infty\,,$$

there exists a bounded linear functional $F \in H^{-p}[0, 2\pi]$ with $F(f_m) = c_m$.

Proof. Assume that $\{c_m\}$ satisfies the inequality of the theorem and define $F : H^p \to \mathbb{C}$ by

$$F(\varphi) := \sum_{-\infty}^{\infty} a_m c_m$$

for $\varphi \in H^p$ with Fourier coefficients a_m. Then F is well defined since by the Cauchy-Schwarz inequality

$$\left|F(\varphi)\right|^2 \le \sum_{-\infty}^{\infty}(1+m^2)^{-p}\left|c_m\right|^2 \sum_{-\infty}^{\infty}(1+m^2)^{p}\left|a_m\right|^2$$

and furthermore

$$\left|\left|F\right|\right|_p \le \left[\sum_{-\infty}^{\infty}(1+m^2)^{-p}\left|c_m\right|^2\right]^{\frac{1}{2}}\,.$$

On the other hand, let $F \in H^{-p}$ such that $F(f_m) = c_m$ and define φ_n by

$$\varphi_n := \sum_{-n}^{n}(1+m^2)^{-p}\bar{c}_m f_m\,.$$

Then

$$\left|\left|\varphi_n\right|\right|_p = \left[\sum_{-n}^{n}(1+m^2)^{-p}\left|c_m\right|^2\right]^{\frac{1}{2}}$$

and hence

$$\left|\left|F\right|\right|_p \ge \frac{\left|F(\varphi_n)\right|}{\left|\left|\varphi_n\right|\right|_p} = \left[\sum_{-n}^{n}(1+m^2)^{-p}\left|c_m\right|^2\right]^{\frac{1}{2}}\,.$$

By the calculation in the first part of the theorem we can now conclude that

$$\left|\left|F\right|\right|_p = \left[\sum_{-\infty}^{\infty}(1+m^2)^{-p}\left|c_m\right|^2\right]^{\frac{1}{2}}\,.$$

□

It follows from Theorem 1.33 that Rellich's theorem remains valid for $-\infty < p, q < \infty$.

Theorem 1.34. *For $g \in L^2[0, 2\pi]$, the* duality pairing

$$G(\varphi) := \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) g(t)\, dt, \quad \varphi \in H^p$$

defines a bounded linear functional on $H^p[0, 2\pi]$, i.e. $G \in H^{-p}[0, 2\pi]$. In particular, $L^2[0, 2\pi]$ may be viewed as a subspace of the dual space $H^{-p}[0, 2\pi]$, $0 \le p < \infty$, and the trigonometric polynomials are dense in $H^{-p}[0, 2\pi]$.

Proof. Let b_m be the Fourier coefficients of g. Then since $G(f_m) = b_m$, by the second part of Theorem 1.33 we have that $G \in H^{-p}$. Now let $F \in H^{-p}$ with $F(f_m) = c_m$ and define $F_n \in H^{-p}$ by

$$F_n(\varphi) := \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) g_n(t)\, dt$$

where

$$g_n := \sum_{-n}^{n} c_m \bar{f}_m \,.$$

Then

$$||F - F_n||_p^2 = \sum_{|m|=n+1}^{\infty} (1 + m^2)^{-p} \, |c_m|^2$$

tends to zero as n tends to infinity which implies that the trigonometric polynomials are dense in $H^{-p}[0, 2\pi]$. □

The above duality pairing can be extended to bounded linear functionals in H^{-p}. In particular, for $\varphi \in H^p$ and $g \in H^{-p}$ we define the integral

$$\int_0^{2\pi} \varphi(t) g(t)\, dt$$

to be $g(\varphi)$. We also note that H^{-p} becomes a Hilbert space by extending the inner product previously defined for $p \ge 0$ to $p < 0$.

More generally, if X is a norm space with dual space X^*, then for $g \in X^*$ and $\varphi \in X$ we define the duality pairing $\langle g, \varphi \rangle$ by $\langle g, \varphi \rangle := g(\varphi)$.

1.5 The Sobolev Space $H^p(\partial D)$

We now want to define Sobolev spaces on the boundary ∂D of a planar domain D, Sobolev spaces defined on D and the relationship between these two spaces. To this end let ∂D be the boundary of a simply connected bounded domain $D \subset \mathbb{R}^2$ such that ∂D is a class C^k, i.e. ∂D has a k-times continuously differentiable 2π-periodic representation $\partial D = \{x(t) : t \in [0, 2\pi), x \in C^k[0, 2\pi]\}$.

Then for $0 \leq p \leq k$ we can define the Sobolev space $H^p(\partial D)$ as the space of all functions $\varphi \in L^2(\partial D)$ such that $\varphi(x(t)) \in H^p[0, 2\pi]$. The inner product and norm on $H^p(\partial D)$ are defined via the inner product on $H^p[0, 2\pi]$ by

$$(\varphi, \psi)_{H^p(\partial D)} := (\varphi(x(t)), \psi(x(t)))_{H^p[0,2\pi]} \, .$$

It can be shown (Theorem 8.14 of [75]) that the above definitions are invariant with respect to parameterization.

The Sobolev space $H^1(D)$ for a bounded domain $D \subset \mathbb{R}^2$ with ∂D of class C^1 is defined as the completion of the space $C^1(\bar{D})$ with respect to the norm

$$||u||_{H^1(D)} := \left[\int_D (|u(x)|^2 + |\mathrm{grad}\, u(x)|^2) \, dx \right]^{\frac{1}{2}} \, .$$

It is easily seen that $H^1(D)$ is a subspace of $L^2(D)$. The main purpose of this section is to show that functions in $H^1(D)$ have a meaning when restricted to ∂D, i.e. the *trace* of functions in $H^1(D)$ to the boundary ∂D is well defined. To this end we will need the following theorem from calculus [3]:

Theorem 1.35 (Dini's Theorem). *If $\{\varphi_n\}_1^\infty$ is a sequence of real valued continuous functions converging pointwise to a continuous limit function φ on a compact set D and if $\varphi_n(x) \geq \varphi_{n+1}(x)$ for each $x \in D$ and every $n = 1, 2, \cdots$ then $\varphi_n \to \varphi$ uniformly on D.*

Making use of Dini's theorem, we can now prove the following basic result called the *trace theorem*. In the study of partial differential equations, trace theorems play an important role, and we shall encounter another of these theorems in Chapter 5 of this book.

Theorem 1.36. *Let $D \subset \mathbb{R}^2$ be a simply connected bounded domain with ∂D in class C^2. Then there exists a positive constant C such that*

$$||u||_{H^{\frac{1}{2}}(\partial D)} \leq C \, ||u||_{H^1(D)}$$

for all $u \in H^1(D)$, i.e. for $u \in H^1(D)$ the operator $u \to u|_{\partial D}$ is well defined and bounded from $H^1(D)$ into $H^{\frac{1}{2}}(\partial D)$.

Proof. We first consider continuously differentiable functions u defined in the strip $\mathbb{R} \times [0, 1]$ that are 2π-periodic with respect to the first variable. Let $Q := [0, 2\pi) \times [0, 1]$ and for $0 \leq \eta \leq 1$ define

$$a_m(\eta) := \frac{1}{2\pi} \int_0^{2\pi} u(t, \eta) e^{-imt} \, dt \, .$$

Then by Parseval's equality we have that

$$\sum_{-\infty}^{\infty} |a_m(\eta)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |u(t, \eta)|^2 \, dt, \quad 0 \leq \eta \leq 1 \, .$$

By Dini's theorem this series is uniformly convergent. Hence we can integrate term by term to obtain

$$\sum_{-\infty}^{\infty} \int_0^1 |a_m(\eta)|^2 \, d\eta = \frac{1}{2\pi} ||u||^2_{L^2(Q)} .$$

Similarly, from

$$a_m'(\eta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial \eta}(t,\eta) e^{-imt} \, dt$$

and

$$im \, a_m(\eta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial t}(t,\eta) e^{-imt} \, dt$$

we see that

$$\sum_{-\infty}^{\infty} \int_0^1 |a_m'(\eta)|^2 \, d\eta = \frac{1}{2\pi} \left|\left| \frac{\partial u}{\partial \eta} \right|\right|^2_{L^2(Q)}$$

and

$$\sum_{-\infty}^{\infty} \int_0^1 m^2 \, |a_m(\eta)|^2 \, d\eta = \frac{1}{2\pi} \left|\left| \frac{\partial u}{\partial t} \right|\right|^2_{L^2(Q)} .$$

We now assume that $u(\cdot, 1) = 0$. Then from the Cauchy-Schwarz inequality and the fact that $a_m(1) = 0$ for all m we have that

$$\begin{aligned}
||u(\cdot,0)||^2_{H^{\frac{1}{2}}[0,2\pi]} &= \sum_{-\infty}^{\infty} (1+m^2)^{\frac{1}{2}} \, |a_m(0)|^2 \\
&= 2 \sum_{-\infty}^{\infty} (1+m^2)^{\frac{1}{2}} \mathrm{Re} \int_1^0 a_m'(\eta) \overline{a_m(\eta)} \, d\eta \qquad (1.5) \\
&\leq 2 \sum_{-\infty}^{\infty} \left[\int_0^1 |a_m'(\eta)|^2 \, d\eta \right]^{\frac{1}{2}} \left[(1+m^2) \int_0^1 |a_m(\eta)|^2 \, d\eta \right]^{\frac{1}{2}} \\
&\leq 2 \left[\sum_{-\infty}^{\infty} \int_0^1 |a_m'(\eta)|^2 \, d\eta \right]^{\frac{1}{2}} \left[\sum_{-\infty}^{\infty} (1+m^2) \int_0^1 |a_m(\eta)|^2 \, d\eta \right]^{\frac{1}{2}} \\
&= \frac{1}{\pi} \left|\left| \frac{\partial u}{\partial \eta} \right|\right|_{L^2(Q)} \left[||u||^2_{L^2(Q)} + \left|\left| \frac{\partial u}{\partial t} \right|\right|^2_{L^2(Q)} \right]^{\frac{1}{2}} \\
&\leq \frac{1}{\pi} \, ||u||^2_{H^1(Q)} .
\end{aligned}$$

We now return to the domain D and choose a parallel strip $D_h := \{x + \eta h \nu(x) : x \in \partial D, \eta \in [0,1]\}$ where ν is the unit inner normal to ∂D, $h > 0$, such that each $y \in D_h$ is uniquely representable through projection onto ∂D in the form $y = x + \eta h \nu(x)$ with $x \in \partial D$, $\eta \in [0,1]$. Let ∂D_h denote the inner boundary of D_h. By parameterizing $\partial D = \{x(t) : 0 \leq t \leq 2\pi\}$ we have a parameterization of D_h in the form

$$x(t,\eta) = x(t) + \eta h\nu(x(t)), \quad 0 \leq t < 2\pi, \quad 0 \leq \eta \leq 1\,.$$

The inequality (1.5) now shows that for all $u \in C^1(D_h)$ with $u = 0$ on ∂D_h we have that

$$\begin{aligned} ||u||_{H^{\frac{1}{2}}(\partial D)} = ||u(x(t))||_{H^{\frac{1}{2}}[0,2\pi]} &\leq \frac{1}{\sqrt{\pi}}\,||u(x(t,\eta))||_{H^1(Q)} \\ &\leq C\,||u||_{H^1(D_h)} \end{aligned}$$

where C is a positive constant depending on bounds for the first derivatives of the mappings $x(t,\eta)$ and its inverse.

We next extend this estimate to arbitrary $u \in C^1(\bar{D})$. To this end, choose a function $g \in C^1(\bar{D})$ such that $g(y) = 0$ for $y \notin D_h$ and $g(y) = f(\eta)$ for $y = x + \eta h\nu(x) \in D_h$ where

$$f(\eta) := (1-\eta)^2(1+3\eta)\,.$$

Then for $f(0) = f'(0) = 1$ and $f(1) = f'(1) = 0$ which implies that

$$||u||_{H^{\frac{1}{2}}(\partial D)} = ||gu||_{H^{\frac{1}{2}}(\partial D)} \leq C\,||gu||_{H^1(D)} \leq C_1\,||u||_{H^1(D)}$$

for all $u \in C^1(\bar{D})$ where C_1 is a positive constant depending on bounds for g and its first derivatives.

We have now established the desired inequality for $u \in C^1(\bar{D})$, i.e. $A : u \mapsto u|_{\partial D}$ is a bounded operator from $C^1(\bar{D})$ into $H^{\frac{1}{2}}(\partial D)$. It can be easily shown [79] that if X is a dense subspace of a normed space $\hat{X}$ and Y is a Banach space then, if $A : X \to Y$ is a bounded linear operator, A can be extended to a bounded linear operator $\hat{A} : \hat{X} \to Y$ where $||\hat{A}|| = ||A||$. The desired inequality now follows from this result by extending the operator A from $C^1(\bar{D})$ to $H^1(D)$. □

We note that in the above proof ∂D must be in class C^2 since $\nu = \nu(x)$ must be continuously differentiable.

2

Ill-Posed Problems

For problems in mathematical physics, Hadamard postulated three properties which he deemed to be of central importance:

1. Existence of a solution.
2. Uniqueness of a solution.
3. Continuous dependence of the solution on the data.

A problem satisfying all three of these requirements is called well-posed. To be more precise, we make the following definition: Let $A : U \to V$ be an operator from a subset U of a normed space X into a subset V of a normed space Y. The equation $A\varphi = f$ is called *well-posed* if A is bijective and $A^{-1} : V \to U$ is continuous. Otherwise $A\varphi = f$ is called *ill-posed* or *improperly posed*. Contrary to Hadamard's point of view, in recent years it has become clear that many important problems of mathematical physics are in fact ill-posed! In particular, all of the inverse scattering problems considered in this book are ill-posed and for this reason we devote a short chapter to the mathematical theory of ill-posed problems. But first we present a simple example of an ill-posed problem.

Example 2.1. Consider the initial-boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in} \quad [0,\pi] \times [0,T]$$

$$u(0,t) = u(\pi,t) = 0 \quad , \quad 0 \le t \le T$$

$$u(x,0) = \varphi(x) \quad , \quad 0 \le x \le \pi$$

where $\varphi \in C[0,\pi]$ is a given function. Then, by separation of variables, we obtain the solution

$$u(x,t) = \sum_1^\infty a_n e^{-n^2 t} \sin nx$$

$$a_n = \frac{2}{\pi} \int_0^\pi \varphi(y) \sin ny \, dy \,,$$

and it is not difficult to show that this solution is unique and depends continuously on the initial data with respect to the maximum norm, i.e.

$$\max_{[0,\pi]\times[0,T]} |u(x,t)| \le C \max_{[0,\pi]} |\varphi(x)|$$

for some positive constant C [24]. Now consider the *inverse problem* of determining φ from $f := u(\cdot, T)$. In this case

$$u(x,t) = \sum_1^\infty b_n e^{n^2(T-t)} \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^\pi f(y) \sin ny \, dy$$

and hence

$$||\varphi||^2 = \frac{2}{\pi} \sum_1^\infty |b_n|^2 e^{2n^2T}$$

which is infinite unless the b_n decay extremely rapidly. Even if this is the case, small perturbations of f (and hence of the b_n) will result in the non-existence of a solution! Note that the inverse problem can be written as an integral equation of the first kind with smooth kernel:

$$\int_0^\pi K(x,y)\varphi(y)\, dy = f(x) \quad , \quad 0 \le x \le \pi$$

where

$$K(x,y) = \frac{2}{\pi} \sum_1^\infty e^{-n^2T} \sin nx \sin ny \quad , \quad 0 \le x, y \le \pi\,.$$

In particular the above integral operator is compact in any reasonable function space, for example $L^2[0,\pi]$. □

Theorem 2.2. *Let X and Y be normed spaces and let $A : X \to Y$ be a compact operator. Then $A\varphi = f$ is ill-posed if X is not of finite dimension.*

Proof. Assume A^{-1} exists and is continuous. Then $I = A^{-1}A : X \to X$ is compact and hence by Theorem 1.20 X is finite dimensional. □

We will now proceed, again following [75], to present the basic mathematical ideas for treating ill-posed problems. For a more detailed discussion we refer the reader to [46, 65, 75], and, in particular, [43].

2.1 Regularization Methods

Methods for contructing a stable approximate solution to an ill-posed problem are called *regularization* methods. In particular, for A a bounded linear operator, we want to approximate the solution φ of $A\varphi = f$ from a knowledge of a perturbed right hand side with a known error level

$$\left|\left|f - f^\delta\right|\right| \leq \delta\,.$$

When $f \in A(X)$ then if A is injective there exists a unique solution φ of $A\varphi = f$. However, in general we cannot expect that $f^\delta \in A(X)$. How do we construct a reasonable approximation φ^δ to φ that depends continuously on f^δ?

Definition 2.3. *Let X and Y be normed spaces and let $A : X \to Y$ be an injective bounded linear operator. Then a family of bounded linear operators $R_\alpha : Y \to X$, $\alpha > 0$, such that*

$$\lim_{\alpha\to 0} R_\alpha A\varphi = \varphi$$

for every $\varphi \in X$ is called a regularization scheme *for A. The parameter α is called the* regularization *parameter .*

We clearly have that $R_\alpha f \to A^{-1}f$ as $\alpha \to 0$ for every $f \in A(X)$. The following theorem shows that for compact operators this convergence cannot be uniform.

Theorem 2.4. *Let X and Y be normed spaces, $A : X \to Y$ an injective compact operator and assume X has infinite dimension. Then the operators R_α cannot be uniformly bounded with respect to α as $\alpha \to 0$ and $R_\alpha A$ cannot be norm convergent as $\alpha \to 0$.*

Proof. Assume $||R_\alpha|| \leq C$ as $\alpha \to 0$. Then since $R_\alpha f \to A^{-1}f$ as $\alpha \to 0$ for every $f \in A(X)$ we have that $\left|\left|A^{-1}f\right|\right| \leq C\,||f||$ and hence A^{-1} is bounded on $A(X)$. But this implies $I = A^{-1}A$ is compact on X which contradicts the fact that X has infinite dimension.

Now assume that $R_\alpha A$ is norm convergent as $\alpha \to 0$, i.e. $||R_\alpha A - I|| \to 0$ as $\alpha \to 0$. Then there exists $\alpha > 0$ such that $||R_\alpha A - I|| < \frac{1}{2}$ and hence for every $f \in A(X)$ we have that

$$\begin{aligned}\left|\left|A^{-1}f\right|\right| &= \left|\left|A^{-1}f - R_\alpha A A^{-1}f + R_\alpha f\right|\right| \\ &\leq \left|\left|A^{-1}f - R_\alpha A A^{-1}f\right|\right| + ||R_\alpha f|| \\ &\leq ||I - R_\alpha A||\,\left|\left|A^{-1}f\right|\right| + ||R_\alpha||\,||f|| \\ &\leq \frac{1}{2}\left|\left|A^{-1}f\right|\right| + ||R_\alpha||\,||f||\,.\end{aligned}$$

Hence $\left|\left|A^{-1}f\right|\right| \leq 2\,||R_\alpha||\,||f||$, i.e. $A^{-1} : A(X) \to X$ is bounded and we again have arrived at a contradiction. □

A regularization scheme approximates the solution φ of $A\varphi = f$ by

$$\varphi_\alpha^\delta := R_\alpha f^\delta\,.$$

Writing

$$\varphi_\alpha^\delta - \varphi = R_\alpha f^\delta - R_\alpha f + R_\alpha A\varphi - \varphi\,,$$

we have the estimate

$$\left|\left|\varphi_\alpha^\delta - \varphi\right|\right| \le \delta\,||R_\alpha|| + ||R_\alpha A\varphi - \varphi||\,.$$

By Theorem 2.4 the first term on the right hand side is large for α small whereas the second term on the right hand side is large if α is not small! So how do we choose α? A reasonable strategy is to choose $\alpha = \alpha(\delta)$ such that $\varphi_\alpha^\delta \to \varphi$ as $\delta \to 0$.

Definition 2.5. *A* strategy *for a regularization scheme R_α, $\alpha > 0$, i.e. a method for choosing the regularization parameter $\alpha = \alpha(\delta)$, is called* regular *if for every $f \in A(X)$ and all $f^\delta \in Y$ such that $\left|\left|f^\delta - f\right|\right| \le \delta$ we have that*

$$R_{\alpha(\delta)} f^\delta \to A^{-1} f$$

as $\delta \to 0$.

A natural strategy for choosing $\alpha = \alpha(\delta)$ is the *discrepancy principle* of Morozov [89], i.e. the residual $\left|\left|A\varphi_\alpha^\delta - f^\delta\right|\right|$ should not be smaller than the accuracy of the measurements of f. In particular $\alpha = \alpha(\delta)$ should be chosen such that $\left|\left|AR_\alpha f^\delta - f^\delta\right|\right| = \gamma\delta$ for some constant $\gamma \ge 1$. Given a regularization scheme, the question of course is whether or not such a strategy is regular.

2.2 Singular Value Decomposition

From now on X and Y will always be infinite dimensional Hilbert spaces and $A : X \to Y$, $A \ne 0$, will always be a compact operator. Note that $A^*A : X \to X$ is compact and self-adjoint. Hence by the Hilbert-Schmidt theorem there exist at most a countable set of eigenvalues $\{\lambda_n\}_1^\infty$, of A^*A and if $A^*A\varphi_n = \lambda_n\varphi_n$ then $(A^*A\varphi_n, \varphi_n) = \lambda_n\,||\varphi_n||^2$, i.e. $||A\varphi_n||^2 = \lambda_n\,||\varphi_n||^2$ which implies that $\lambda_n \ge 0$ for $n = 1, 2, \cdots$. The nonnegative square roots of the eigenvalues of A^*A are called the *singular values* of A.

Theorem 2.6. *Let $\{\mu_n\}_1^\infty$ be the sequence of nonzero singular values of the compact operator $A : X \to Y$ ordered such that*

$$\mu_1 \ge \mu_2 \ge \mu_3 \ge \cdots\,.$$

Then there exist orthonormal sequences $\{\varphi_n\}_1^\infty$ in X and $\{g_n\}_1^\infty$ in Y such that

$$A\varphi_n = \mu_n g_n \quad , \quad A^* g_n = \mu_n \varphi_n\,.$$

For every $\varphi \in X$ we have the singular value decomposition

$$\varphi = \sum_1^\infty (\varphi, \varphi_n)\varphi_n + P\varphi$$

where $P : X \to N(A)$ is the orthogonal projection operator of X onto $N(A)$ and

$$A\varphi = \sum_1^\infty \mu_n(\varphi, \varphi_n)g_n \, .$$

The system (μ_n, φ_n, g_n) is called a singular system *of A.*

Proof. Let $\{\varphi_n\}_1^\infty$ be the orthonormal eigenelements of A^*A corresponding to $\{\mu_n\}_1^\infty$, i.e.

$$A^*A\varphi = \mu_n^2\varphi_n$$

and define a second orthonormal sequence by

$$g_n := \frac{1}{\mu_n}A\varphi_n \, .$$

Then $A\varphi_n = \mu_n g_n$ and $A^*g_n = \mu_n\varphi_n$. The Hilbert-Schmidt theorem implies that

$$\varphi = \sum_1^\infty (\varphi, \varphi_n)\varphi_n + P\varphi$$

where $P : X \to N(A^*A)$ is the orthogonal projection operator of X onto $N(A^*A)$. But $\psi \in N(A^*A)$ implies that $(A\psi, A\psi) = (\psi, A^*A\psi) = 0$ and hence $N(A^*A) = N(A)$. Finally, applying A to the above expansion (first apply A to the partial sum and then take the limit), we have that

$$A\varphi = \sum_1^\infty \mu_n(\varphi, \varphi_n)g_n \, .$$

□

We now come to the main result we will need to study compact operator equations of the first kind, i.e. equations of the form $A\varphi = f$ where A is a compact operator.

Theorem 2.7 (Picard's Theorem). *Let $A : X \to Y$ be a compact operator with singular system (μ_n, φ_n, g_n). Then the equation $A\varphi = f$ is solvable if and only if $f \in N(A^*)^\perp$ and*

$$\sum_1^\infty \frac{1}{\mu_n^2} \left|(f, g_n)\right|^2 < \infty \, . \tag{2.1}$$

In this case a solution to $A\varphi = f$ is given by

$$\varphi = \sum_1^\infty \frac{1}{\mu_n}(f, g_n)\varphi_n \, .$$

Proof. The necessity of $f \in N(A^*)^\perp$ follows from Theorem 1.27. If φ is a solution of $A\varphi = f$ then

$$\mu_n(\varphi, \varphi_n) = (\varphi, A^* g_n) = (A\varphi, g_n) = (f, g_n)\,.$$

But from the singular value decomposition of φ we have that

$$||\varphi||^2 = \sum_1^\infty |(\varphi, \varphi_n)|^2 + ||P\varphi||^2$$

and hence

$$\sum_1^\infty \frac{1}{\mu_n^2}\,|(f, g_n)|^2 = \sum_1^\infty |(\varphi, \varphi_n)|^2 \leq ||\varphi||^2$$

which implies the necessity of condition (2.1).

Conversely, assume that $f \in N(A^*)^\perp$ and (2.1) is satisfied. Then from (2.1) we have that

$$\varphi := \sum_1^\infty \frac{1}{\mu_n}(f, g_n)\varphi_n$$

converges in the Hilbert space X. Applying A to this series we have that

$$A\varphi = \sum_1^\infty (f, g_n) g_n\,.$$

But, since $f \in N(A^*)^\perp$, this is the singular value decomposition of f corresponding to the operator A^* and hence $A\varphi = f$. □

Note that Picard's theorem illustrates the ill-posed nature of the equation $A\varphi = f$. In particular, setting $f^\delta = f + \delta g_n$ we obtain a solution of $A\varphi^\delta = f^\delta$ given by $\varphi^\delta = \varphi + \delta\varphi_n/\mu_n$. Hence, if $A(X)$ is not finite dimensional,

$$\frac{||\varphi^\delta - \varphi||}{||f^\delta - f||} = \frac{1}{\mu_n} \to \infty$$

since by Theorem 1.14 we have that $\mu_n \to 0$. We say that $A\varphi = f$ is *mildly ill-posed* if the singular values decay slowly to zero and *severely ill-posed* if they decay very rapidly (for example exponentially). All of the inverse scattering problems considered in this book are severely ill-posed.

From now on, in order to focus on ill-posed problems, we will always assume that $A(X)$ is infinite dimensional, i.e. the set of singular values is an infinite set.

Example 2.8. Consider the case of the backwards heat equation discussed in Example 2.1. The problem considered in this example is equivalent to solving the compact operator equation $A\varphi = f$ where

$$(A\varphi)(x) := \int_0^\pi K(x,y)\varphi(y)\,dy \quad , \quad 0 \le x \le \pi$$

and

$$K(x,y) := \frac{2}{\pi}\sum_1^\infty e^{-n^2T}\sin nx \sin ny\,.$$

Then A is easily seen to be self-adjoint with eigenvalues given by $\lambda_n = e^{-n^2T}$. Hence $\mu_n = \lambda_n$ and the compact operator equation $A\varphi = f$ is severely ill-posed. □

Picard's theorem suggests trying to regularize $A\varphi = f$ by damping or filtering out the influence of the higher order terms in the solution φ given by

$$\varphi = \sum_1^\infty \frac{1}{\mu_n}(f, g_n)\varphi_n\,.$$

The following theorem does exactly that. We will subsequently consider two specific regularization schemes by making specific choices of the function q that appears in the theorem.

Theorem 2.9. *Let $A : X \to Y$ be an injective compact operator with singular system (μ_n, φ_n, g_n) and let $q : (0,\infty) \times (0, ||A||] \to \mathbb{R}$ be a bounded function such that for every $\alpha > 0$ there exists a positive constant $c(\alpha)$ such that*

$$|q(\alpha,\mu)| \le c(\alpha)\mu \quad , \quad 0 < \mu \le ||A||\;,$$

and

$$\lim_{\alpha\to 0} q(\alpha,\mu) = 1 \quad , \quad 0 < \mu \le ||A||\;.$$

Then the bounded linear operators $R_\alpha : Y \to X$, $\alpha > 0$, defined by

$$R_\alpha f := \sum_1^\infty \frac{1}{\mu_n} q(\alpha,\mu_n)(f, g_n)\varphi_n$$

for $f \in Y$ describe a regularization scheme with

$$||R_\alpha|| \le c(\alpha)\,.$$

Proof. Noting that from the singular value decomposition of f with respect to the operator A^* we have that

$$||f||^2 = \sum_1^\infty |(f, g_n)|^2 + ||Pf||^2$$

where $P : X \to N(A^*)$ is the orthogonal projection of X onto $N(A^*)$, we see that for every $f \in Y$ we have that

$$\begin{aligned}||R_\alpha f||^2 &= \sum_1^\infty \frac{1}{\mu_n^2}\left|q(\alpha,\mu_n)\right|^2 \left|(f,g_n)\right|^2 \\ &\le |c(\alpha)|^2 \sum_1^\infty |(f,g_n)|^2 \\ &\le |c(\alpha)|^2\, ||f||^2\end{aligned}$$

and hence $||R_\alpha|| \le c(\alpha)$. From

$$\begin{aligned}(R_\alpha A\varphi,\varphi_n) &= \frac{1}{\mu_n} q(\alpha,\mu_n)(A\varphi, g_n) \\ &= q(\alpha,\mu_n)(\varphi,\varphi_n)\end{aligned}$$

and the singular value decomposition for $R_\alpha A\varphi - \varphi$ we obtain, using the fact that A is injective, that

$$\begin{aligned}||R_\alpha A\varphi-\varphi||^2 &= \sum_1^\infty |(R_\alpha A\varphi-\varphi,\varphi_n)|^2 \\ &= \sum_1^\infty |q(\alpha,\mu_n)-1|^2\, |(\varphi,\varphi_n)|^2 \,.\end{aligned}$$

Now let $\varphi \in X$, $\varphi \ne 0$, and let M be a bound for q. We first note that for every $\epsilon > 0$ there exists $N = N(\epsilon)$ such that

$$\sum_{N+1}^\infty |(\varphi,\varphi_n)|^2 < \frac{\epsilon}{2(M+1)^2}\,.$$

Since $\lim_{\alpha\to 0} q(\alpha,\mu) = 1$, there exists $\alpha_0 = \alpha_0(\epsilon)$ such that

$$|q(\alpha,\mu_n)-1|^2 < \frac{\epsilon}{2\,||\varphi||^2}$$

for $n = 1, 2, \cdots, N$ and all α such that $0 < \alpha \le \alpha_0$. We now have that, for $0 < \alpha \le \alpha_0$,

$$\begin{aligned}||R_\alpha A\varphi-\varphi||^2 &= \sum_1^N |q(\alpha,\mu_n)-1|^2\, |(\varphi,\varphi_n)|^2 \\ &\quad + \sum_{N+1}^\infty |q(\alpha,\mu_n)-1|^2\, |(\varphi,\varphi_n)|^2 \\ &\le \frac{\epsilon}{2\,||\varphi||^2}\sum_1^N |(\varphi,\varphi_n)|^2 + \frac{\epsilon}{2} \quad .\end{aligned}$$

But, since A is injective,

$$||\varphi||^2 = \sum_1^\infty |(\varphi, \varphi_n)|^2$$

and hence $||R_\alpha A\varphi - \varphi||^2 \leq \epsilon$ for $0 < \alpha \leq \alpha_0$. We can now conclude that $R_\alpha A\varphi \to \varphi$ as $\alpha \to 0$ for every $\varphi \in X$ and the theorem is proved. □

A particular choice of q now leads to our first regularization scheme, the *spectral cut-off* method .

Theorem 2.10. *Let $A : X \to Y$ be an injective compact operator with singular system (μ_n, φ_n, g_n). Then the* spectral cut-off

$$R_m f := \sum_{\mu_n \geq \mu_m} \frac{1}{\mu_n}(f, g_n)\varphi_n$$

describes a regularization scheme with regularization parameter $m \to \infty$ and $||R_m|| = 1/\mu_m$.

Proof. Choose q such that $q(m, \mu) = 1$ for $\mu \geq \mu_m$ and $q(m, \mu) = 0$ for $\mu < \mu_m$. Then since $\mu_m \to 0$ as $m \to \infty$ the conditions of the previous theorem are clearly satisfied with $c(m) = \frac{1}{\mu_m}$. Hence $||R_m|| \leq \frac{1}{\mu_m}$. Equality follows from the identity $R_m g_m = \varphi_m/\mu_m$. □

We conclude this section by establishing a discrepancy principle for the spectral cut-off regularization scheme.

Theorem 2.11. *Let $A : X \to Y$ be an injective compact operator with dense range in Y, let $f \in Y$ and $\delta > 0$. Then there exists a smallest integer m such that*

$$||AR_m f - f|| \leq \delta\,.$$

Proof. Since $\overline{A(X)} = Y$, A^* is injective. Hence the singular value decomposition with the singular system (μ_n, g_n, φ_n) for A^* implies that for every $f \in Y$ we have that

$$f = \sum_1^\infty (f, g_n) g_n\,. \tag{2.2}$$

Hence

$$||(AR_m - I)f||^2 = \sum_{\mu_n < \mu_m} |(f, g_n)|^2 \to 0 \tag{2.3}$$

as $m \to \infty$. In particular, there exists a smallest integer $m = m(\delta)$ such that $||AR_m f - f|| \leq \delta$. □

Note that from (2.2) and (2.3) we have that

$$||AR_m f - f||^2 = ||f||^2 - \sum_{\mu_n \geq \mu_m} |(f, g_n)|^2\,. \tag{2.4}$$

In particular, $m(\delta)$ is determined by the condition that $m(\delta)$ is the smallest value of m such that the right hand side of (2.4) is less than or equal to δ^2. For example, in the case of the backwards heat equation (Example 2.1) we have that $g_n(x) = \sqrt{2/\pi}\sin nx$ and hence m is determined by the condition that m is the smallest integer such that

$$||f||^2 - \sum_1^m |b_n|^2 \leq \delta^2$$

where the b_n are the Fourier coefficients of f.

It can be shown that the above discrepancy principle for the spectral cut-off method is regular (Theorem 15.26 of [75]).

2.3 Tikhonov Regularization

We now introduce and study the most popular regularization scheme in the field of ill-posed problems.

Theorem 2.12. *Let $A : X \to Y$ be a compact operator. Then for every $\alpha > 0$ the operator $\alpha I + A^*A : X \to X$ is bijective and has a bounded inverse. Furthermore, if A is injective then*

$$R_\alpha := (\alpha I + A^*A)^{-1}A^*$$

describes a regularization scheme with $||R_\alpha|| \leq 1/2\sqrt{\alpha}$.

Proof. From

$$\alpha\, ||\varphi||^2 \leq (\alpha\varphi + A^*A\varphi, \varphi)$$

for $\varphi \in X$ we can conclude that for $\alpha > 0$ the operator $\alpha I + A^*A$ is injective. Hence, since A^*A is a compact operator, by the Riesz theorem we have that $(\alpha I + A^*A)^{-1}$ exists and is bounded.

Now assume that A is injective and let (μ_n, φ_n, g_n) be a singular system for A. Then for $f \in Y$ the unique solution φ_α of

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$$

is given by

$$\varphi_\alpha = \sum_1^\infty \frac{\mu_n}{\alpha + \mu_n^2}(f, g_n)\varphi_n\,,$$

i.e. R_α can be written in the form

$$R_\alpha f = \sum_1^\infty \frac{1}{\mu_n} q(\alpha, \mu_n)(f, g_n)\varphi_n$$

where

$$q(\alpha, \mu) = \frac{\mu^2}{\alpha + \mu^2} .$$

Since $0 < q(\alpha, \mu) < 1$ and $\sqrt{\alpha}\mu \leq \left(\alpha + \mu^2\right)/2$, we have that $|q(\alpha, \mu)| \leq \mu/2\sqrt{\alpha}$ and the theorem follows from Theorem 2.9. $\square$

The next theorem shows that the function $\varphi_\alpha = R_\alpha f$ can be obtained as the solution of an optimization problem.

Theorem 2.13. *Let $A : X \to Y$ be a compact operator and let $\alpha > 0$. Then for every $f \in Y$ there exists a unique $\varphi_\alpha \in X$ such that*

$$||A\varphi_\alpha - f||^2 + \alpha\, ||\varphi_\alpha||^2 = \inf_{\varphi \in X} \left\{ ||A\varphi - f||^2 + \alpha\, ||\varphi||^2 \right\} .$$

The minimizer is the unique solution of $\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$.

Proof. From

$$\begin{aligned} ||A\varphi - f||^2 + \alpha\, ||\varphi||^2 = {} & ||A\varphi_\alpha - f||^2 + \alpha\, ||\varphi_\alpha||^2 \\ & + 2\mathrm{Re}(\varphi - \varphi_\alpha, \alpha\varphi_\alpha + A^*A\varphi_\alpha - A^*f) \\ & + ||A(\varphi - \varphi_\alpha)||^2 + \alpha\, ||\varphi - \varphi_\alpha||^2 \end{aligned}$$

which is valid for every $\varphi, \varphi_\alpha \in X$, we see that if φ_α satisfies $\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$ then φ_α minimizes the *Tikhonov functional*

$$||A\varphi - f||^2 + \alpha\, ||\varphi||^2 .$$

On the other hand, if φ_α is a minimizer of the Tikhonov functional, set

$$\psi := \alpha\varphi_\alpha + A^*A\varphi_\alpha - A^*f$$

and assume that $\psi \neq 0$. Then for $\varphi := \varphi_\alpha - t\psi$, t a real number, we have that

$$\begin{aligned} ||A\varphi - f||^2 + \alpha\, ||\varphi||^2 = {} & ||A\varphi_\alpha - f||^2 + \alpha\, ||\varphi_\alpha||^2 \\ & - 2t\, ||\psi||^2 + t^2(||A\psi||^2 + \alpha\, ||\psi||^2) . \end{aligned} \tag{2.5}$$

The minimum of the right hand side of (2.5) occurs when

$$t = \frac{||\psi||^2}{||A\psi||^2 + \alpha\, ||\psi||^2}$$

and for this t we have that $||A\varphi - f||^2 + \alpha\, ||\varphi||^2 < ||A\varphi_\alpha - f||^2 + \alpha\, ||\varphi_\alpha||^2$ which contradicts the definition of φ_α. Hence $\psi = 0$, i.e. $\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$. $\square$

By the interpretation of Tikhonov regularization as the minimizer of the Tikhonov functional, its solution φ_α keeps the residual $||A\varphi_\alpha - f||^2$ small and is stabilized through the penalty term $\alpha\,||\varphi_\alpha||^2$. This suggests the following two constrained optimization problems:

Minimum Norm Solution: For a given $\delta > 0$ minimize $||\varphi||$ such that $||A\varphi - f|| \leq \delta$.
Quasi-Solutions: For a given $\rho > 0$ minimize $||A\varphi - f||$ such that $||\varphi|| \leq \rho$.

We begin with the idea of a minimum norm solution and view this as a discrepancy principle for choosing φ in Tikhonov regularization.

Theorem 2.14. *Let $A : X \to Y$ be an injective compact operator with dense range in Y and let $f \in Y$ with $||f|| > \delta > 0$. Then there exists a unique α such that*

$$||AR_\alpha f - f|| = \delta\,.$$

Proof. We have to show that

$$F(\alpha) := ||AR_\alpha f - f||^2 - \delta^2$$

has a unique zero. As in Theorem 2.11 we have that

$$f = \sum_1^\infty (f, g_n) g_n$$

and for $\varphi_\alpha = R_\alpha f$ we have that

$$\varphi_\alpha = \sum_1^\infty \frac{\mu_n}{\alpha + \mu_n^2}(f, g_n)\varphi_n\,.$$

Hence

$$F(\alpha) = \sum_1^\infty \frac{\alpha^2}{(\alpha + \mu_n^2)^2}\,|(f, g_n)|^2 - \delta^2\,.$$

Since F is a continuous function of α and strictly monotonically increasing with limits $F(\alpha) \to -\delta^2$ as $\alpha \to 0$ and $F(\alpha) \to ||f||^2 - \delta^2 > 0$ as $\alpha \to \infty$, F has exactly one zero $\alpha = \alpha(\delta)$. □

In order to prove the regularity of the above discrepancy principle for Tikhonov regularization, we need to introduce the concept of *weak convergence.*

Definition 2.15. *A sequence $\{\varphi_n\}$ in X is said to be* weakly convergent *to $\varphi \in X$ if*

$$\lim_{n\to\infty} (\psi, \varphi_n) = (\psi, \varphi)$$

for every $\psi \in X$ and we write $\varphi_n \rightharpoonup \varphi$, $n \to \infty$.

Note that norm convergence $\varphi_n \to \varphi$, $n \to \infty$, always implies weak convergence but, as the following example shows, the converse is generally false.

Example 2.16. Let ℓ^2 be the space of all sequences $\{a_n\}_1^\infty$, $a_n \in \mathbb{C}$, such that

$$\sum_1^\infty |a_n|^2 < \infty . \tag{2.6}$$

It is easily shown that, with componentwise addition and scalar multiplication, ℓ^2 is a Hilbert space with inner product

$$(a, b) = \sum_1^\infty a_n \bar{b}_n$$

where $a = \{a_n\}_1^\infty$ and $b = \{b_n\}_1^\infty$. In ℓ^2 we now define the sequence $\{\varphi_n\}$ by $\varphi_n = (0, 0, 0, \cdots, 1, 0, \cdots)$ where the one appears in the n^{th} entry. Then $\{\varphi_n\}$ is not norm convergent since $||\varphi_n - \varphi_m|| = \sqrt{2}$ for $m \neq n$ and hence $\{\varphi_n\}$ is not a Cauchy sequence. On the other hand, for $\psi = \{a_n\} \in \ell^2$ we have that $(\psi, \varphi_n) = a_n \to 0$ as $n \to \infty$ due to the convergence of the series in (2.6). Hence $\{\varphi_n\}$ is weakly convergent to zero in ℓ^2.

Theorem 2.17. *Every bounded sequence in a Hilbert space contains a weakly convergent subsequence.*

Proof. Let $\{\varphi_n\}$ be a bounded sequence, $||\varphi_n|| \leq C$. Then for each integer m the sequence (φ_m, φ_n) is bounded for all n. Hence by the Bolzano-Weierstrass theorem and a diagonalization process (c.f. the proof of Theorem 1.17) we can select a subsequence $\{\varphi_{n(k)}\}$ such that $(\varphi_m, \varphi_{n(k)})$ converges as $k \to \infty$ for every integer m. Thus the linear functional F defined by

$$F(\psi) := \lim_{k \to \infty} (\psi, \varphi_{n(k)})$$

is well defined on $U := \text{span}\{\varphi_m\}$ and, by continuity, on $\bar{U}$. Now let $P : X \to \bar{U}$ be the orthogonal projection operator and for arbitrary $\psi \in X$ write $\psi = P\psi + (I - P)\psi$. For arbitrary $\psi \in X$ define $F(\psi)$ by

$$\begin{aligned} F(\psi) := \lim_{k \to \infty} (\psi, \varphi_{n(k)}) &= \lim_{k \to \infty} \left[(P\psi, \varphi_{n(k)}) + \left((I - P)\, \psi, \varphi_{n(k)} \right) \right] \\ &= \lim_{k \to \infty} \left(P\psi, \varphi_{n(k)} \right) \end{aligned}$$

where we have used the easily verifiable fact that P is self-adjoint. Thus F is defined on all of X. Furthermore, $||F|| \leq C$. Hence, by the Riesz representation theorem, there exists a unique $\varphi \in X$ such that $F(\psi) = (\psi, \varphi)$ for every $\psi \in X$. We can now conclude that $\lim_{k \to \infty} (\psi, \varphi_{n(k)}) = (\psi, \varphi)$ for every $\psi \in X$, i.e. $\varphi_{n(k)}$ is weakly convergent to φ as $k \to \infty$. $\square$

We are now in a position to show that the discrepancy principle of Theorem 2.14 is regular.

Theorem 2.18. *Let $A : X \to Y$ be an injective compact operator with dense range in Y. Let $f \in A(X)$ and $f^\delta \in Y$ satisfy $\left\|f^\delta - f\right\| \le \delta < \left\|f^\delta\right\|$ with $\delta > 0$. Then there exists a unique $\alpha = \alpha(\delta)$ such that*

$$\left\|AR_{\alpha(\delta)}f^\delta - f^\delta\right\| = \delta$$

and

$$R_{\alpha(\delta)}f^\delta \to A^{-1}f$$

as $\delta \to 0$.

Proof. In view of Theorem 2.14, we only need to establish convergence. Since $\varphi^\delta = R_{\alpha(\delta)}f^\delta$ minimizes the Tikhonov functional, we have that

$$\begin{aligned}\delta^2 + \alpha\left\|\varphi^\delta\right\|^2 &= \left\|A\varphi^\delta - f^\delta\right\|^2 + \alpha\left\|\varphi^\delta\right\|^2\\ &\le \left\|AA^{-1}f - f^\delta\right\|^2 + \alpha\left\|A^{-1}f\right\|^2\\ &\le \delta^2 + \alpha\left\|A^{-1}f\right\|^2\end{aligned}$$

and hence $\left\|\varphi^\delta\right\| \le \left\|A^{-1}f\right\|$. Now let $g \in Y$. Then

$$\begin{aligned}\left|(A\varphi^\delta - f, g)\right| &\le \left(\left\|A\varphi^\delta - f^\delta\right\| + \left\|f^\delta - f\right\|\right)\|g\|\\ &\le 2\delta\|g\| \to 0\end{aligned} \tag{2.7}$$

as $\delta \to 0$. Since A is injective, $A^*(Y)$ is dense in X and hence for every $\psi \in X$ there exists a sequence $\{g_n\}$ in Y such that $A^*g_n \to \psi$. Then

$$(\varphi^\delta - \varphi, \psi) = (\varphi^\delta - \varphi, A^*g_n) + (\varphi^\delta - \varphi, \psi - A^*g_n) \tag{2.8}$$

and, for every $\epsilon > 0$,

$$\left|(\varphi^\delta - \varphi, \psi - A^*g_n)\right| \le \left\|\varphi^\delta - \varphi\right\|\|\psi - A^*g_n\| < \frac{\epsilon}{2} \tag{2.9}$$

for all $\delta > 0$ and $N > N_0$ since $\left\|\varphi^\delta - \varphi\right\|$ is bounded. Hence, for $N > N_0$ and δ sufficiently small, we have from (2.7) - (2.9) that

$$\begin{aligned}\left|(\varphi^\delta - \varphi, \psi)\right| &\le \left|(\varphi^\delta - \varphi, A^*g_n)\right| + \left|(\varphi^\delta - \varphi, \psi - A^*g_n)\right|\\ &\le \left|(A\varphi^\delta - f, g_n)\right| + \frac{\epsilon}{2}\\ &\le \epsilon\end{aligned}$$

where we have set $f = A\varphi$. We can now conclude that $\varphi^\delta \rightharpoonup A^{-1}f$ as $\delta \to 0$. Then, again using the fact that $\left\|\varphi^\delta\right\| \le \left\|A^{-1}f\right\|$, we have that

$$\begin{aligned}\left\|\varphi^\delta - A^{-1}f\right\|^2 &= \left\|\varphi^\delta\right\|^2 - 2\mathrm{Re}\left(\varphi^\delta, A^{-1}f\right) + \left\|A^{-1}f\right\|^2\\ &\le 2\left(\left\|A^{-1}f\right\|^2 - \mathrm{Re}\left(\varphi^\delta, A^{-1}f\right)\right) \to 0\end{aligned} \tag{2.10}$$

as $\delta \to 0$ and the proof is complete. $\square$

Under additional conditions on f, which may be viewed as a regularity condition on f, we can obtain results on the order of convergence.

Theorem 2.19. *Under the assumptions of Theorem 2.18, if $f \in AA^*(Y)$ then*

$$\left|\left|\varphi^\delta - A^{-1}f\right|\right| = O\left(\delta^{1/2}\right) \quad , \quad \delta \to 0\,.$$

Proof. We have that $A^{-1}f = A^*g$ for some $g \in Y$. Then from (2.10) we have that

$$\begin{aligned}
\left|\left|\varphi^\delta - A^{-1}f\right|\right|^2 &\leq 2\left(\left|\left|A^{-1}f\right|\right|^2 - \mathrm{Re}\left(\varphi^\delta, A^{-1}f\right)\right) \\
&= 2\mathrm{Re}\left(A^{-1}f - \varphi^\delta, A^{-1}f\right) \\
&= 2\mathrm{Re}\left(f - A\varphi^\delta, g\right) \\
&\leq 2\left(\left|\left|f - f^\delta\right|\right| + \left|\left|f^\delta - A\varphi^\delta\right|\right|\right)||g|| \\
&\leq 4\delta\,||g||
\end{aligned}$$

and the theorem follows. □

Tikhonov regularization methods also apply to the case when both the operator and the right hand side are perturbed, i.e. both the operator and the right hand side are "noisy". In particular, consider the operator equation $A_h\varphi = f^\delta$, $A_h : X \to Y$, where $||A_h - A|| \leq h$ and $\left|\left|f - f^\delta\right|\right| \leq \delta$ respectively. Then the Tikhonov regularization operator is given by

$$R_\alpha := \left(\alpha I + A_h^* A_h\right)^{-1} A_h^*$$

and the regularization solution $\varphi^\alpha := R_\alpha f^\delta$ is found by minimizing the Tikhonov functional

$$\left|\left|A_h\varphi - f^\delta\right|\right| + \alpha\,||\varphi||\ .$$

The regularization parameter $\alpha = \alpha(\delta, h)$ is determined from the equation

$$\left|\left|A_h\varphi_\alpha - f^\delta\right|\right|^2 = \left(\delta + h\,||\varphi_\alpha||^2\right)\ .$$

Then all of the results obtained above in the case when A is not noisy can be generalized to the present case when both A and f are noisy. For details we refer the reader to [89].

We now turn our attention to the *method of quasi-solutions.*

Theorem 2.20. *Let $A : X \to Y$ be an injective compact operator and let $\rho > 0$. Then for every $f \in Y$ there exists a unique $\varphi_0 \in X$ with $||\varphi_0|| = \rho$ such that*

$$||A\varphi_0 - f|| \leq ||A\varphi - f||$$

for all φ satisfying $||\varphi|| \leq \rho$. The element φ_0 is called the quasi-solution *of $A\varphi = f$ with constraint ρ.*

Proof. We note that φ_0 is a quasi-solution with constraint ρ if and only if $A\varphi_0$ is a best approximation to f with respect to the set $V := \{A\varphi : ||\varphi|| \le \rho\}$. Since A is linear, V is clearly convex, i.e. $\lambda\varphi_1+(1-\lambda)\varphi_2 \in V$ for all $\varphi_1, \varphi_2 \in V$ and $0 \le \lambda \le 1$. Suppose there were two best approximations to f, i.e. there exist $v_1, v_2 \in V$ such that

$$||f - v_1|| = ||f - v_2|| = \inf_{v \in V} ||f - v|| \ .$$

Then, since V is convex, $\frac{1}{2}(v_1 + v_2) \in V$ and hence

$$\left|\left| f - \frac{v_1 + v_2}{2} \right|\right| \ge ||f - v_1|| \ .$$

By the parallelogram equality we now have that

$$\begin{aligned} ||v_1 - v_2||^2 &= 2\,||f - v_1||^2 + 2\,||f - v_2||^2 \\ &\quad - 4 \left|\left| f - \frac{v_1 + v_2}{2} \right|\right|^2 \\ &\le 0 \end{aligned}$$

and hence $v_1 = v_2$. Thus if there were two quasi-solutions φ_1 and φ_2 then $A\varphi_1 = A\varphi_2$. But since A is injective $\varphi_1 = \varphi_2$, i.e. the quasi-solution, if it exists, is unique.

To prove the existence of a quasi-solution, let $\{\varphi_n\}$ be a minimizing sequence, i.e. $||\varphi_n|| \le \rho$ and

$$\lim_{n \to \infty} ||A\varphi_n - f|| = \inf_{||\varphi|| \le \rho} ||A\varphi - f|| \ . \tag{2.11}$$

By Theorem 2.17 there exists a weakly convergent subsequence of $\{\varphi_n\}$ and without loss of generality we assume that $\varphi_n \rightharpoonup \varphi_0$ as $n \to \infty$ for some $\varphi_0 \in X$. We will show that $A\varphi_n \to A\varphi_0$ as $n \to \infty$. Since for every $\varphi \in X$ we have that

$$\lim_{n \to \infty} (A\varphi_n, \varphi) = \lim_{n \to \infty} (\varphi_n, A^*\varphi) = (\varphi_0, A^*\varphi) = (A\varphi_0, \varphi)$$

we can conclude that $A\varphi_n \rightharpoonup A\varphi_0$. Now suppose that $A\varphi_n$ does not converge to $A\varphi_0$. Then $\{A\varphi_n\}$ has a subsequence such that $\left|\left|A\varphi_{n(k)} - A\varphi_0\right|\right| \ge \delta$ for some $\delta > 0$. Since $||\varphi_n|| \le \rho$ and A is compact, $\{A\varphi_{n(k)}\}$ has a convergent subsequence which we again call $\{A\varphi_{n(k)}\}$. But since convergent sequences are also weakly convergent and have the same limit, $A\varphi_{n(k)} \to A\varphi_0$ which is a contradiction. Hence $A\varphi_n \to A\varphi_0$. From (2.11) we can now conclude that

$$||A\varphi_0 - f|| = \inf_{||\varphi|| \le \rho} ||A\varphi - f||$$

and since $||\varphi_0||^2 = \lim_{n\to\infty} (\varphi_n, \varphi_0) \le \rho\,||\varphi_0||$ we have that $||\varphi_0|| \le \rho$. This completes the proof of the theorem. □

We next show that under appropriate assumptions the method of quasi-solutions is regular.

Theorem 2.21. *Let $A : X \to Y$ be an injective compact operator with dense range and let $f \in A(X)$ and $\rho \geq \|A^{-1}f\|$. For $f^\delta \in Y$ with $\|f^\delta - f\| \leq \delta$, let φ^δ be the quasi-solution to $A\varphi = f^\delta$ with constraint ρ. Then $\varphi^\delta \rightharpoonup A^{-1}f$ as $\delta \to 0$ and if $\rho = \|A^{-1}f\|$ then $\varphi^\delta \to A^{-1}f$ as $\delta \to 0$.*

Proof. Let $g \in Y$. Then since $\|A^{-1}f\| \leq \rho$ and $\|A\varphi^\delta - f^\delta\| \leq \|A\varphi - f^\delta\|$ for $f = A\varphi$ we have that

$$\begin{aligned} \left|\left(A\varphi^\delta - f, g\right)\right| &\leq \left(\|A\varphi^\delta - f^\delta\| + \|f^\delta - f\|\right)\|g\| \\ &\leq \left(\|AA^{-1}f - f^\delta\| + \|f^\delta - f\|\right)\|g\| \\ &\leq 2\delta\,\|g\|\,. \end{aligned} \tag{2.12}$$

Hence $\left(A\varphi^\delta - f, g\right) = \left(\varphi^\delta - A^{-1}f, A^*g\right) \to 0$ as $\delta \to 0$ for every $g \in Y$. Since A is injective, $A^*(Y)$ is dense in X and we can conclude that $\varphi^\delta \rightharpoonup A^{-1}f$ as $\delta \to 0$ (c.f. the proof of Theorem 2.18).

When $\rho = \|A^{-1}f\|$ we have (using $\|\varphi^\delta\| \leq \rho = \|A^{-1}f\|$) that

$$\begin{aligned} \|\varphi^\delta - A^{-1}f\|^2 &= \|\varphi^\delta\|^2 - 2\mathrm{Re}\left(\varphi^\delta, A^{-1}f\right) + \|A^{-1}f\|^2 \\ &\leq 2\mathrm{Re}\left(A^{-1}f - \varphi^\delta, A^{-1}f\right) \to 0 \end{aligned} \tag{2.13}$$

as $\delta \to 0$. □

Note that for regularity we need to know a priori the norm of the solution to the noise-free equation.

Theorem 2.22. *Under the assumptions of Theorem 2.21, if $f \in AA^*(Y)$ and $\rho = \|A^{-1}f\|$ then*

$$\|\varphi^\delta - A^{-1}f\| = O\left(\delta^{1/2}\right) \quad , \quad \delta \to 0\,.$$

Proof. We can write $A^{-1}f = A^*g$ for some $g \in Y$. From (2.12) and (2.13) we have that $\|\varphi^\delta - A^{-1}f\|^2 \leq 2\mathrm{Re}\left(f - A\varphi^\delta, g\right) \leq 4\delta\,\|g\|$ and the theorem follows. □

3

Scattering by an Imperfect Conductor

In this chapter we consider a very simple scattering problem corresponding to the scattering of a time harmonic plane wave by an imperfect conductor. Although the problem is simple compared to most problems in scattering theory, its mathematical resolution took many years to be accomplished and was the focus of energy of some of the outstanding mathematicians of the twentieth century, in particular Kupradze, Rellich, Vekua, Müller and Weyl. Indeed the solution of the full three dimensional problem was not fully realized until 1981 (c.f. Sect. 9.5 of [33]). Here we will content ourselves with the two dimensional scalar problem and its solution by the method of integral equations. As will be seen, the main difficulty of this approach is the presence of eigenvalues of the interior Dirichlet problem for the Helmholtz equation and we will overcome this difficulty by using the ideas of Jones [64], Ursell [110] and Kleinman and Roach [73].

The plan of this chapter is as follows. We begin by considering Maxwell's equations and then derive the scalar impedance boundary value problem corresponding to the scattering of a time harmonic plane wave by an imperfectly conducting infinite cylinder. After a brief detour to discuss the relevant properties of Bessel and Hankel functions that will be needed in the sequel, we proceed to show that our scattering problem is well posed by deriving Rellich's lemma and using the method of modified single layer potentials. We will conclude this chapter by giving a brief discussion on weak solutions of the Helmholtz equation. (This theme will be revisited in greater detail in Chap. 5).

3.1 Maxwell's Equations

Consider electromagnetic wave propagation in a homogeneous, isotropic, nonconducting medium in $\mathbb{R}^3$ with electric permittivity ϵ and magnetic permeability μ. A time harmonic electromagnetic wave with frequency $\omega > 0$ is described by the electric and magnetic fields

$$\begin{aligned}\mathcal{E}(x,t) &= \epsilon^{-1/2}E(x)e^{-i\omega t}\\ \mathcal{H}(x,t) &= \mu^{-1/2}H(x)e^{-i\omega t}\end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^3$ and $\mathcal{E}$, $\mathcal{H}$ satisfy *Maxwell's equations*

$$\begin{aligned}\operatorname{curl}\mathcal{E} + \mu\frac{\partial \mathcal{H}}{\partial t} &= 0\\ \operatorname{curl}\mathcal{H} - \epsilon\frac{\partial \mathcal{E}}{\partial t} &= 0\,.\end{aligned} \tag{3.2}$$

In particular, from (3.1) and (3.2) we see that E and H must satisfy

$$\begin{aligned}\operatorname{curl}E - ikH &= 0\\ \operatorname{curl}H + ikE &= 0\end{aligned} \tag{3.3}$$

where the *wave number* k is defined by $k = \omega\sqrt{\epsilon\mu}$.

Now assume that a time harmonic electromagnetic plane wave (factoring out $e^{-i\omega t}$)

$$\begin{aligned}E^i(x) = E^i(x;d,p) &= \frac{1}{k^2}\operatorname{curl}\operatorname{curl} p e^{ikx\cdot d}\\ H^i(x) = H^i(x;d,p) &= \frac{1}{ik}\operatorname{curl} p e^{ikx\cdot d}\end{aligned} \tag{3.4}$$

where d is a constant unit vector and p is the (constant) polarization vector is an incident field that is scattered by an obstacle D that is an *imperfect conductor*, i.e. the electromagnetic field penetrates D by only a small amount. Let the total fields E and H be given by

$$\begin{aligned}E &= E^i + E^s\\ H &= H^i + H^s\end{aligned} \tag{3.5}$$

where $E^s(x) = E^s(x;d,p)$ and $H^s(x) = H^s(x;d,p)$ are the scattered fields that arise due to the presence of the obstacle D. Then E^s, H^s must be an "outgoing" wave that satisfies the *Silver-Müller radiation condition*

$$\lim_{r\to\infty}(H^s \times x - rE^s) = 0 \tag{3.6}$$

where $r = |x|$. Since D is an imperfect conductor, on the boundary ∂D the field E must satisfy the boundary condition

$$\nu \times \operatorname{curl}E - i\lambda(\nu \times E) \times \nu = 0 \tag{3.7}$$

where $\lambda = \lambda(x) > 0$ is the surface impedance defined on ∂D. Then the mathematical problem associated with the scattering of time harmonic plane waves by an imperfect conductor is to find a solution E, H of Maxwell's equations (3.3) in the exterior of D such that (3.4)–(3.7) are satisfied. In particular, (3.3)–(3.7) defines a *scattering problem* for Maxwell's equations.

Now consider the scattering due to an infinite cylinder with cross section D and axis on the x_3-coordinate axis where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Assume $E = (0, 0, E_3)$, $p = (0, 0, 1)$ and $d = (d_1, d_2, 0)$, i.e.

$$E^i(x) = e^{ikx\cdot d}\hat{e}_3$$

where $\hat{e}_3$ is the unit vector in the positive x_3 direction. Then E and H will be independent of x_3 and from Maxwell's equations we have that $H = (H_1, H_2, 0)$ where E_3, H_1 and H_2 satisfy

$$\begin{aligned} \frac{\partial E_3}{\partial x_2} &= ikH_1 \\ \frac{\partial E_3}{\partial x_1} &= -ikH_2 \\ \frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} &= -ikE_3\,. \end{aligned}$$

In particular,

$$\Delta E_3 + k^2 E_3 = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}\,. \tag{3.8}$$

In order for E_3^s to be "outgoing" we require that E_3^s satisfy the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial E_3^s}{\partial r} - ikE_3^s\right) = 0\,. \tag{3.9}$$

Finally, we need to determine the boundary condition satisfied by

$$E_3(x) = e^{ikx\cdot d} + E_3^s(x) \tag{3.10}$$

where now $x \in \mathbb{R}^2$. To this end, we compute for $E = (0, 0, E_3)$ and $\nu = (\nu_1, \nu_2, 0)$ that $\nu \times \operatorname{curl} E = (0, 0, -\partial E_3/\partial\nu)$ and $(\nu \times E) \times \nu = E$. This then implies that (3.7) becomes

$$\frac{\partial E_3}{\partial \nu} + i\lambda E_3 = 0\,. \tag{3.11}$$

Equations (3.8)–(3.11) provide the mathematical formulation of the scattering of a time harmonic electromagnetic plane wave by an imperfectly conducting infinite cylinder and it is this problem that will concern us for the rest of this chapter.

3.2 Bessel Functions

We begin our study of the scattering problem (3.8)–(3.11) by examining special solutions of the *Helmholtz equation* (3.8). In particular, if we look for solutions of (3.8) in the form

$$E_3(x) = y(kr)e^{in\theta} \quad , n = 0, \pm 1, \pm 2, \cdots$$

where (r, θ) are cylindrical coordinates, we find that $y(r)$ is a solution of *Bessel's equation*

$$y'' + \frac{1}{r}y' + \left(1 - \frac{\nu^2}{r^2}\right) y = 0 \tag{3.12}$$

for $\nu = n$. For arbitrary real ν, we see by direct calculation and the ratio test that

$$J_\nu(r) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{r}{2}\right)^{\nu+2k} \tag{3.13}$$

where Γ denotes the gamma function is a solution of Bessel's equation for $0 \le r < \infty$. J_ν is called a *Bessel function* of order ν. For $\nu = -n, n = 1, 2, \cdots$, the first n terms of (3.13) vanish and hence

$$\begin{aligned} J_{-n}(r) &= \sum_{k=n}^{\infty} \frac{(-1)^k}{k!(k-n)!} \left(\frac{r}{2}\right)^{-n+2k} \\ &= \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(n+s)!s!} \left(\frac{r}{2}\right)^{n+2s} \\ &= (-1)^n J_n(r) \end{aligned}$$

which shows that J_n and J_{-n} are linearly dependent. However if $\nu \neq n$ then it is easily seen that J_ν and $J_{-\nu}$ are linearly independent solutions of Bessel's equation.

Unfortunately, we are interested precisely in the case when $\nu = n$ and hence we must find a second linearly independent solution of Bessel's equation. This is easily done using Frobenius' method and for $n = 0, 1, 2, \cdots$ we obtain the desired second solution to be given by

$$\begin{aligned} Y_n(r) := \frac{2}{\pi} J_n(r) \log \frac{r}{2} - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{r}{2}\right)^{2k-n} \\ - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{r}{2}\right)^{n+2k}}{k!\,(n+k)!} \left[\psi(k+1) + \psi(k+n+1)\right] \end{aligned} \tag{3.14}$$

where $\psi(1) = -\gamma$, $\psi(m+1) = \gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ for $m = 1, 2, \cdots$, $\gamma = 0.57721566\cdots$ is Euler's constant and the finite sum is set equal to zero if $n = 0$. From (3.13) and (3.14) we see that

$$J_n(r) = \frac{1}{n!} \left(\frac{r}{2}\right)^n \left[1 + O(r^2)\right] \quad , \quad r \to 0 \tag{3.15}$$

and, for $n \ge 1$,

$$Y_n(r) = -\frac{(n-1)!}{\pi} \left(\frac{r}{2}\right)^{-n} \begin{cases} 1 + O(r \log r), & n = 1 \\ 1 + O(r^2), & n > 1 \end{cases} \quad , \quad r \to 0 \tag{3.16}$$

whereas for $n = 0$ we have that

$$Y_0(r) = \frac{2}{\pi} \log r + O(1) \quad , \quad r \to 0 \, . \tag{3.17}$$

Note that in (3.15) and (3.16) the constant implicit in the order term is independent of n for $n > 1$. Finally, for n a positive integer we define Y_{-n} by

$$Y_{-n}(r) = (-1)^n Y_n(r)$$

which implies that J_n and Y_n are linearly independent for all integers $n = 0, \pm 1, \pm 2, \cdots$. The function Y_n is called the *Neumann function* of order n.

Of considerable importance to us in the sequel are the *Hankel functions* $H_n^{(1)}$ and $H_n^{(2)}$ of the first and second kind of order n respectively which are defined by

$$\begin{aligned} H_n^{(1)}(r) &:= J_n(r) + iY_n(r) \\ H_n^{(2)}(r) &:= J_n(r) - iY_n(r) \end{aligned} \tag{3.18}$$

for $n = 0, \pm 1, \pm 2, \cdots$, $0 < r < \infty$. $H_n^{(1)}$ and $H_n^{(2)}$ clearly define a second pair of linearly independent solutions to Bessel's equation.

Now let y_1 and y_2 be any two solutions of Bessel's equation

$$(ry_1')' + \left(r - \frac{\nu^2}{r} \right) y_1 = 0 \tag{3.19}$$

$$(ry_2')' + \left(r - \frac{\nu^2}{r} \right) y_2 = 0 \tag{3.20}$$

and define the *Wronskian* by

$$W(y_1, y_2) := \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \, .$$

Then multiplying (3.19) by y_2 and subtracting it from (3.20) multiplied by y_1 we see that

$$\frac{d}{dr}(rW) = 0$$

and hence

$$W(y_1, y_2) = \frac{C}{r}$$

where C is a constant. The constant C can be computed by

$$C = \lim_{r \to 0} rW(y_1, y_2) \, .$$

In particular, making use of (3.15)–(3.18) we find that

$$W(J_n, H_n^{(1)}) = \frac{2i}{\pi r} \tag{3.21}$$

$$W(H_n^{(1)}, H_n^{(2)}) = -\frac{4i}{\pi r} \,. \tag{3.22}$$

We now note that for $0 \le r < \infty$, $0 < |t| < \infty$, we have that

$$e^{rt/2} e^{-r/2t} = \sum_{j=0}^{\infty} \frac{r^j t^j}{2^j j!} \sum_{k=0}^{\infty} \frac{(-1)^k r^k}{2^k t^k k!}$$

and, setting $j - k = n$, we have that

$$\begin{aligned} e^{r/2(t-1/t)} &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{\infty} \frac{(-1)^k r^{n+2k}}{2^{n+2k}(n+k)!k!} \right] t^n \\ &= \sum_{-\infty}^{\infty} J_n(r) t^n \,. \end{aligned} \tag{3.23}$$

Setting $t = ie^{i\theta}$ in (3.23) gives the *Jacobi-Anger expansion*

$$e^{ir\cos\theta} = \sum_{-\infty}^{\infty} i^n J_n(r) e^{in\theta} \,. \tag{3.24}$$

In the remaining chapters of this book we will often be interested in entire solutions of the Helmholtz equation of the form

$$v_g(x) := \int_0^{2\pi} e^{ikr\cos(\theta-\phi)} g(\phi)\, d\phi \tag{3.25}$$

where $g \in L^2[0, 2\pi]$. The function v_g is called a *Herglotz wave function* with kernel g. These functions were first introduced by Herglotz in a lecture in 1945 in Göttingen and were subsequently studied by Magnus [84], Müller [90] and Hartman and Wilcox [57]. From (3.25) and the Jacobi-Anger expansion, we see that since g has the Fourier expansion

$$g(\phi) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} a_n (-i)^n e^{in\phi}$$

where

$$\sum_{-\infty}^{\infty} |a_n|^2 < \infty \,, \tag{3.26}$$

v_g is a Herglotz wave function if and only if v_g has an expansion of the form

$$v_g(x) = \sum_{-\infty}^{\infty} a_n J_n(kr) e^{in\theta}$$

such that (3.26) is valid. Note that v_g is identically zero if and only if $g = 0$.

Finally, we note the asymptotic relations [82]

$$\begin{aligned} J_n(r) &= \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(r^{-3/2}), \quad r \to \infty \\ H_n^{(1)}(r) &= \sqrt{\frac{2}{\pi r}} \exp i\left(r - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(r^{-3/2}), \quad r \to \infty \end{aligned} \tag{3.27}$$

and the *addition formula* [82]

$$H_0^{(1)}(k\,|x-y|) = \sum_{-\infty}^{\infty} H_n^{(1)}(k\,|x|) J_n(k\,|y|) e^{in\theta} \tag{3.28}$$

which is uniformly convergent together with its first derivatives on compact subsets of $|x| > |y|$ and θ denotes the angle between x and y.

3.3 The Direct Scattering Problem

We will now show that the scattering problem for an imperfect conductor in $\mathbb{R}^2$ is well-posed. We will always assume that $D \subset \mathbb{R}^2$ is a bounded domain containing the origin with connected complement such that ∂D is in class C^2. Our aim is to show the existence of a unique solution $u \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$ of the exterior *impedance boundary value problem*

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \tag{3.29}$$

$$u(x) = e^{ikx\cdot d} + u^s(x) \tag{3.30}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{3.31}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \tag{3.32}$$

where $\lambda \in C(\partial D), \lambda(x) > 0$ for $x \in \partial D$, ν is the unit outward nomal to ∂D and the Sommerfeld radiation condition (3.31) is assumed to hold uniformly in θ where $k > 0$ is the wave number and (r, θ) are polar coordinates. We also want to show that the solution u of (3.29)–(3.32) depends continuously on the incident field u^i in an appropriate norm.

We define the (radiating) *fundamental solution* to the Helmholtz equation by

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k\,|x-y|) \tag{3.33}$$

and note that $\Phi(x, y)$ satisfies the Sommerfeld radiation condition with respect to both x and y and as $|x-y| \to 0$ we have that

$$\Phi(x, y) = \frac{1}{2\pi} \log\frac{1}{|x-y|} + O(1)\,. \tag{3.34}$$

Theorem 3.1 (Representation Theorem). *Let $u^s \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$ be a solution of the Helmholtz equation in the exterior of D satisfying the Sommerfeld radiation condition. Then for $x \in \mathbb{R}^2 \setminus \bar{D}$ we have that*

$$u^s(x) = \int_{\partial D} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y) .$$

Proof. Let $x \in \mathbb{R}^2 \setminus \bar{D}$ and circumscribe it with a disk

$$\Omega_{x,\epsilon} := \{ y : |x-y| < \epsilon \}$$

where $\Omega_{x,\epsilon} \subset \mathbb{R}^2 \setminus \bar{D}$. Let Ω_R be a disk of radius R centered at the origin and containing D and $\Omega_{x,\epsilon}$ in its interior. Then from Green's second identity we have that

$$\int_{\partial D + \partial \Omega_{x,\epsilon} + \partial \Omega_R} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y) = 0 .$$

From the definition of the Hankel function, we have that

$$\frac{d}{dr} H_0^{(1)}(r) = -H_1^{(1)}(r)$$

and hence on $\partial \Omega_{x,\epsilon}$ we have that

$$\frac{\partial}{\partial \nu(y)} \Phi(x,y) = \frac{1}{2\pi} \frac{1}{|x-y|} + O(|x-y| \log |x-y|) . \tag{3.35}$$

Using (3.34) and (3.35) and letting $\epsilon \to 0$ we see that

$$\begin{aligned} u^s(x) = & \int_{\partial D} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y) \\ & - \int_{|y|=R} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y) \end{aligned} \tag{3.36}$$

where as usual ν is the unit outward normal to the boundary of the (interior) domain. Hence to establish the theorem we must show that the second integral tends to zero as $R \to \infty$.

We first show that

$$\lim_{R \to \infty} \int_{|y|=R} |u^s|^2 \, ds = O(1) . \tag{3.37}$$

To this end, from the Sommerfeld radiation condition we have that

$$\begin{aligned} 0 &= \lim_{R \to \infty} \int_{|y|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds \\ &= \lim_{R \to \infty} \int_{|y|=R} \left(\left| \frac{\partial u^s}{\partial r} \right|^2 + k^2 |u^s|^2 + 2k \, \mathrm{Im} \left(u^s \frac{\partial \overline{u^s}}{\partial r} \right) \right) ds . \end{aligned} \tag{3.38}$$

Green's first identity applied to $D_R = \Omega_R \setminus \bar{D}$ gives

$$\int_{|y|=R} u^s \frac{\partial \overline{u^s}}{\partial r}\, ds = \int_{\partial D} u^s \frac{\partial \overline{u^s}}{\partial \nu}\, ds - k^2 \int_{D_R} |u^s|^2 \, dy + \int_{D_R} |\operatorname{grad} u^s|^2 \, dy$$

and hence from (3.38) we have that

$$\lim_{R\to\infty} \int_{|y|=R} \left(\left| \frac{\partial u^s}{\partial r} \right|^2 + k^2 |u^s|^2 \right) ds = -2k \operatorname{Im} \int_{\partial D} u^s \frac{\partial \overline{u^s}}{\partial \nu}\, ds \tag{3.39}$$

and from this we can conclude that (3.37) is true.

To complete the proof, we now note the identity

$$\begin{aligned}
&\int_{|y|=R} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y) = \\
&\quad = \int_{|y|=R} u^s(y) \left(\frac{\partial}{\partial |y|} \Phi(x,y) - ik\Phi(x,y) \right) ds(y) \\
&\quad - \int_{|y|=R} \Phi(x,y) \left(\frac{\partial u^s}{\partial |y|}(y) - iku^s(y) \right) ds(y)\,.
\end{aligned} \tag{3.40}$$

Applying the Cauchy-Schwarz inequality to each of the integrals on the right hand side of (3.40) and using (3.37), the facts that $\Phi(x,y) = O(1/\sqrt{R})$ and Φ and u^s satisfy the Sommerfeld radiation condition we have that

$$\lim_{R\to\infty} \int_{|y|=R} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y) = 0$$

and the proof is complete. □

Now let D be a bounded domain with C^2 boundary ∂D and $u \in C^2(D) \cap C^1(\bar{D})$ a solution of the Helmholtz equation in D. Then by using the techniques of the proof of the above theorem it can easily be shown that for $x \in D$ we have the *representation formula*

$$u(x) = \int_{\partial D} \left(\frac{\partial u}{\partial \nu}(y) \Phi(x,y) - u(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) \right) ds(y)\,. \tag{3.41}$$

Hence, since $\Phi(x,y)$ is a real-analytic function of x_1 and x_2 where $x = (x_1, x_2)$ and $x \neq y$, we have that u is real-analytic in D. This proves the following theorem:

Theorem 3.2. *Solutions of the Helmholtz equation are real-analytic functions of their independent variables.*

The identity theorem for real-analytic functions [63] and Theorem 3.2 imply that solutions of the Helmholtz equation satisfy the *unique continuation principle*, i.e. if u is a solution of the Helmholtz equation in a domain D and

$u(x) = 0$ for x in a neighborhood of a point $x_0 \in D$ then $u(x) = 0$ for all x in D.

We are now in a position to show that if a solution to the scattering problem (3.29–(3.32) exists, it is unique.

Theorem 3.3. *Let $u^s \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$ be a solution of the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{D}$ satisfying the Sommerfeld radiation condition and the boundary condition $\partial u^s/\partial\nu + i\lambda u^s = 0$ on ∂D. Then $u^s = 0$.*

Proof. Let Ω be a disk centered at the origin and containing D in its interior. Then from Green's second identity, the fact that R and λ are real and hence

$$\frac{\partial u^s}{\partial\nu} + i\lambda u^s = \frac{\partial\overline{u^s}}{\partial\nu} - i\lambda\overline{u^s} = 0 \quad \text{on } \partial D$$

we have that

$$\begin{aligned}\int_{\partial\Omega}\left(\overline{u^s}\frac{\partial u^s}{\partial r} - u^s\frac{\partial\overline{u^s}}{\partial r}\right) ds &= \int_{\partial D}\left(\overline{u^s}\frac{\partial u^s}{\partial\nu} - u^s\frac{\partial\overline{u^s}}{\partial\nu}\right) ds \\ &= -2i\int_{\partial D}\lambda\left|u^s\right|^2 ds\,.\end{aligned} \tag{3.42}$$

But, since by Theorem 3.2 $u^s \in C^\infty(\mathbb{R}^2 \setminus \bar{D})$ (in fact real-analytic), we have that for $x \in \mathbb{R}^2 \setminus \Omega$ u^s can be expanded in a Fourier series

$$\begin{aligned}u^s(r,\theta) &= \sum_{-\infty}^{\infty} a_n(r)e^{in\theta} \\ a_n(r) &= \frac{1}{2\pi}\int_0^{2\pi} u^s(r,\theta)e^{-in\theta}\,d\theta\end{aligned} \tag{3.43}$$

where the series and its derivatives with respect to r are absolutely and uniformly convergent on compact subsets of $\mathbb{R}^2\setminus\Omega$. In particular, it can be verified directly that $a_n(r)$ is a solution of Bessel's equation and, since u^s satisfies the Sommerfeld radiation condition,

$$a_n(r) = \alpha_n H_n^{(1)}(kr) \tag{3.44}$$

where the α_n are constants. Substituting (3.43) and (3.44) into (3.42) and integrating termwise, we see from the fact that $\overline{H_n^{(1)}(kr)} = H_n^{(2)}(kr)$ and the Wronskian formula (3.22) that

$$8i\sum_{-\infty}^{\infty}\left|\alpha_n\right|^2 = -2i\int_{\partial D}\lambda\left|u^s\right|^2 ds\,.$$

Since $\lambda > 0$, we can now conclude that $\alpha_n = 0$ for every integer n and hence $u^s(x) = 0$ for $x \in \mathbb{R}^2 \setminus \Omega$. By Theorem 3.2 and the identity theorem for real-analytic functions we can now conclude that $u^s(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$. □

Corollary 3.4. *If the solution of the scattering problem (3.29)–(3.32) exists, it is unique.*

Proof. If two solutions u_1 and u_2 exist, then their difference $u^s = u_1 - u_2$ satisfies the hypothesis of Theorem 3.3 and hence $u^s = 0$, i.e. $u_1 = u_2$. □

The next theorem is a classic result in scattering theory that was first proved by Rellich [100] and Vekua [111] in 1943. Due perhaps to wartime conditions, Vekua's paper remained unknown in the west and the result is commonly attributed only to Rellich.

Theorem 3.5 (Rellich's Lemma). *Let $u \in C^2(\mathbb{R}^2 \setminus \bar{D})$ be a solution of the Helmholtz equation satisfying*

$$\lim_{R\to\infty} \int_{|y|=R} |u|^2 \, ds = 0 \, .$$

Then $u = 0$ in $\mathbb{R}^2 \setminus \bar{D}$.

Proof. Let Ω be a disk centered at the origin and containing D in its interior. Then, as in Theorem 3.3, we have that for $x \in \mathbb{R}^2 \setminus \Omega$

$$u(r,\theta) = \sum_{-\infty}^{\infty} a_n(r) e^{in\theta}$$

$$a_n(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r,\theta) e^{-in\theta} \, d\theta$$

and $a_n(r)$ is a solution of Bessel's equation, i.e.

$$a_n(r) = \alpha_n H_n^{(1)}(kr) + \beta_n H_n^{(2)}(kr) \tag{3.45}$$

where the α_n and β_n are constants. By Parseval's equality we have that

$$\int_{|y|=R} |u|^2 \, ds = 2\pi R \sum_{-\infty}^{\infty} |a_n(R)|^2$$

and hence, from the hypothesis of the theorem,

$$\lim_{R\to\infty} R \, |a_n(R)|^2 = 0 \, . \tag{3.46}$$

From (3.45), the asymptotic expansion of $H_n^{(1)}(kr)$ given by (3.27) and the fact that $\overline{H_n^{(1)}(kr)} = H_n^{(2)}(kr)$, we see from (3.46) that $\alpha_n = \beta_n = 0$ for every n and hence $u = 0$ in $\mathbb{R}^2 \setminus \Omega$. By Theorem 3.2 and the identity theorem for real-analytic functions we can now conclude as in Theorem 3.3 that $u(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$. □

Theorem 3.6. *Let $u^s \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$ be a radiating solution of the Helmholtz equation such that*

$$\operatorname{Im} \int\limits_{\partial D} u^s \frac{\partial \overline{u}^s}{\partial \nu}\, ds \geq 0.$$

Then $u^s = 0$ in $\mathbb{R}^2 \setminus \bar{D}$.

Proof. This follows from the identity (3.39) and Rellich's lemma. □

We now want to use the method of integral equations to establish the existence of a solution to the scattering problem (3.29)–(3.32). To this end, we note that the *single layer potential*

$$u^s(x) = \int_{\partial D} \varphi(y)\Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^2 \setminus \partial D \tag{3.47}$$

with continuous density φ satisfies the Sommerfeld radiation condition, is a solution of the Helmholtz equation in $\mathbb{R}^2 \setminus \partial D$, is continuous in $\mathbb{R}^2$ and satisfies the discontinuity property [75, 85]

$$\frac{\partial u^s_\pm}{\partial \nu}(x) = \int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(x)} \Phi(x,y)\, ds(y) \mp \frac{1}{2}\varphi(x), \quad x \in \partial D$$

where

$$\frac{\partial u^s_\pm}{\partial \nu}(x) := \lim_{h \to 0} \nu(x) \cdot \nabla u\,(x \pm h\nu(x))\,.$$

(For future reference, we note that these properties of the single layer potential are also valid for $\varphi \in H^{-1/2}(\partial D)$ where the integrals are interpreted in the sense of duality pairing [75, 85].) In particular, (3.47) will solve the scattering problem (3.29)–(3.32) provided

$$\begin{aligned} \varphi(x) - 2\int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(x)} \Phi(x,y)\, ds(y) - 2i\lambda(x) \int_{\partial D} \varphi(y)\Phi(x,y)\, ds(y) \\ = 2\left[\frac{\partial u^i}{\partial \nu}(x) + i\lambda(x) u^i(x)\right], \qquad x \in \partial D \end{aligned} \tag{3.48}$$

where $u^i(x) = e^{ikx\cdot d}$. Hence to establish the existence of a solution to the scattering problem (3.29)–(3.32) it suffices to show the existence of a solution to (3.48) in the normed space $C(\partial D)$ (Example 1.3).

To this end, we first note that the integral operators in (3.48) are compact. This can easily be shown by approximating each of the kernels $K(x,y)$ in (3.48) by

$$K_n(x,y) := \begin{cases} h(n\,|x-y|)K(x,y), & x \neq y \\ 0, & x = y \end{cases}$$

where

$$h(t) := \begin{cases} 0, & 0 \le t \le \frac{1}{2} \\ 2t - 1, & \frac{1}{2} \le t \le 1 \\ 1, & 1 \le t < \infty \end{cases}$$

and using Theorem 1.17 and the fact that integral operators with continuous kernels are compact operators on $C(\partial D)$ (c.f. Theorem 2.21 of [75]). Hence, by the Riesz theorem, it suffices to show that the homogeneous equation has only the trivial solution. But this is in general not the case! In particular, let k^2 be a Dirichlet eigenvalue, i.e. their exists $u \in C^2(D) \cap C(\bar{D})$, u not identically zero, such that

$$\Delta u + k^2 u = 0 \quad \text{in } D$$

$$u = 0 \quad \text{on } \partial D\,.$$

It can be shown that $u \in C^1(\bar{D})$ [30] and $\partial u/\partial\nu$ is not identically zero since, if it were, then by the the representation formula (3.41) u would be identically zero which it is not by assumption. Hence for $\varphi := \partial u/\partial\nu$ we have from Green's second identity that

$$\int_{\partial D} \varphi(y)\Phi(x,y)\, ds(y) = 0, \quad x \in \mathbb{R}^2 \setminus \bar{D} \tag{3.49}$$

and, by continuity, for $x \in \mathbb{R}^2 \setminus D$. Hence, using the previously stated discontinuity properties for single layer potentials, we have that

$$\varphi(x) - 2\int_{\partial D} \varphi(y)\frac{\partial}{\partial\nu(x)}\Phi(x,y)\, ds(y) = 0, \quad x \in \partial D\,. \tag{3.50}$$

(3.49) and (3.50) now imply that φ is a nontrivial solution of the homogeneous equation corresponding to (3.48). Thus we cannot use the Riesz theorem to establish the existence of a solution to (3.48).

In order to obtain an integral equation that is uniquely solvable for all values of the wave number k, we need to modify the kernel of the representation (3.47). We will do this following the ideas of [64, 73] and [110]. We begin by defining the function $\chi = \chi(x,y)$ by

$$\chi(x,y) := \frac{i}{4}\sum_{-\infty}^{\infty} a_n H_n^{(1)}(kr)H_n^{(1)}(kr_y)e^{in(\theta-\theta_y)} \tag{3.51}$$

where x has polar coordinates (r,θ), y has polar coordinates (r_y,θ_y) and the coefficients a_n are chosen such that the series converges for $|x|, |y| > R$ where $\Omega_R := \{x : |x| \le R\} \subset D$. The fact that this can be done follows from (3.15), (3.16), (3.18) and the fact that

$$H_{-n}^{(1)}(kr) = (-1)^n H_n^{(1)}(kr)$$

for $n = 0, 1, 2, 3, \cdots$. In particular these equations imply that

$$\left|H_n^{(1)}(kr)\right| = O\left(\frac{2^{|n|}\,(|n|-1)!}{(kr)^{|n|}}\right)$$

for $n = \pm 1, \pm 2, \cdots$ and r on compact subsets of $(0, \infty)$. Defining

$$\Gamma(x,y) := \Phi(x,y) + \chi(x,y)\ ,$$

we now see that the *modified single layer potential*

$$u^s(x) := \int_{\partial D} \varphi(y)\Gamma(x,y)\, ds(y) \tag{3.52}$$

for continuous density φ and $x \in \mathbb{R}^2 \setminus (\partial D \cup \Omega_R)$ satisfies the Sommerfeld radiation condition, is a solution of the Helmholtz equation in $\mathbb{R}^2 \setminus (\partial D \cup \Omega_R)$ and satisfies the same discontinuity properties as the single layer potential (3.47). Hence (3.52) will solve the scattering problem (3.29)–(3.32) provided φ satisfies (3.48) with Φ replaced by Γ. By the Riesz theorem, a solution of this equation exists if the corresponding homogeneous equation only has the trivial solution.

Let φ be a solution of this homogeneous equation. Then (3.52) will be a solution of (3.29) - (3.32) with $e^{ikx\cdot d}$ set equal to zero and hence by Corollary 3.4 we have that if u^s is defined by (3.52) then $u^s(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$. By the continuity of (3.52) across ∂D, u^s is a solution of the Helmholtz equation in $D \setminus \Omega_R$, $u^s \in C^2(D \setminus \bar{\Omega}_R) \cap C(\bar{D} \setminus \Omega_R)$ and $u^s(x) = 0$ for $x \in \partial D$. From (3.51), (3.52) and the addition formula for Bessel functions, we see that there exist constants α_n such that for $R_1 \le |x| \le R_2$ where $R < R_1 < R_2$ and $\{x : |x| < R_2\} \subset D$ we can represent u^s in the form

$$u^s(x) = \sum_{-\infty}^{\infty} \alpha_n \left\{J_n(kr) + a_n H_n^{(1)}(kr)\right\} e^{in\theta}\ .$$

Since

$$u^s_+(x) := \lim_{\substack{x\to\partial D\\ x\in D}} u^s(x)$$

$$\frac{\partial u^s_+}{\partial\nu}(x) := \lim_{\substack{x\to\partial D\\ x\in D}} \frac{\partial u^s}{\partial\nu}(x)$$

exist and are continuous, we can apply Green's second identity to u^s and $\bar{u^s}$ over $D \setminus \{x : |x| \le R_1\}$ and use the Wronskian relations (3.21), (3.22) to see that

$$0 = \int_{\partial D}\left(u^s_+\frac{\partial \bar{u}^s_+}{\partial\nu} - \bar{u}^s_+\frac{\partial u^s_+}{\partial\nu}\right) ds = \int_{|x|=R_1}\left(u^s\frac{\partial \bar{u}^s}{\partial\nu} - \bar{u}^s\frac{\partial u^s}{\partial\nu}\right) ds$$

$$= 2i\sum_{-\infty}^{\infty}|\alpha_n|^2\left(1 - |1+2a_n|^2\right)\ .$$

Hence, if either $|1 + 2a_n| < 1$ or $|1 + 2a_n| > 1$ for $n = 0, \pm 1, \pm 2, \cdots$ then $\alpha_n = 0$ for $n = 0, \pm 1, \pm 2, \cdots$, i.e. $u^s(x) = 0$ for $R_1 \le |x| \le R_2$. By Theorem 3.2 and the identity theorem for real-analytic functions, we can now conclude that $u^s(x) = 0$ for $x \in D \setminus \Omega_R$. Recalling that $u^s(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$, we now see from the discontinuity property of single layer potentials that

$$0 = \frac{\partial u_-^s}{\partial \nu} - \frac{\partial u_+^s}{\partial \nu}(x) = \varphi(x) \,,$$

i.e. the homogeneous equation under consideration only has the trivial solution $\varphi = 0$. Hence, by the Riesz theorem, the corresponding inhomogeneous equation has a unique solution which depend continuously on the right hand side.

Theorem 3.7. *There exists a unique solution of the scattering problem (3.29)–(3.32) which depends continuously on $u^i(x) = e^{ikx \cdot d}$ in $C^1(\partial D)$.*

It is often important to find a solution of (3.29)–(3.32) in a larger space than $C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$. To this end, let $\Omega_R := \{x : |x| < R\}$ and define the Sobolev spaces

$$\begin{aligned} H^1_{loc}(\mathbb{R}^2 \setminus \bar{D}) := \{ & u : u \in H^1\left((\mathbb{R}^2 \setminus \bar{D}) \cap \Omega_R\right) \text{ for every } R > 0 \\ & \text{such that } (\mathbb{R}^2 \setminus D) \cap \Omega_R \neq \emptyset\} \end{aligned}$$

$$\begin{aligned} H^1_{com}(\mathbb{R}^2 \setminus \bar{D}) := \{ & u : u \in H^1(\mathbb{R}^2 \setminus \bar{D}), u \text{ is identically} \\ & \text{zero outside some ball centered at} \\ & \text{the origin}\} \,. \end{aligned}$$

We recall that $H^{-p}(\partial D)$, $0 \le p < \infty$, is the dual space of $H^p(\partial D)$ and, for $f \in H^{-p}(\partial D)$ and $v \in H^p(\partial D)$,

$$\int_{\partial D} f v \, ds := f(v)$$

is defined by duality pairing.
Then, for $f \in H^{-1/2}(\partial D)$, a *weak solution* of

$$\Delta u + k^2 u = 0 \quad \text{ in } \mathbb{R}^2 \setminus \bar{D} \tag{3.53}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0 \tag{3.54}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = f \quad \text{ on } \partial D \tag{3.55}$$

is defined to be a function $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ such that

$$- \int_{\mathbb{R}^2 \setminus \bar{D}} \left(\nabla u \cdot \nabla v - k^2 uv \right) \, dx + i \int_{\partial D} \lambda uv \, ds = \int_{\partial D} f v \, ds \tag{3.56}$$

for all $v \in H^1_{com}(\mathbb{R}^2 \setminus \bar{D})$ such that u satisfies the Sommerfeld radiation condition (3.54). Note that by the trace theorem we have that $v|_{\partial D} \in H^{1/2}(\partial D)$ is well defined and hence the integral on the right hand side of (3.56) is well defined by duality pairing. The radiation condition also makes sense in the weak case since, by regularity results for elliptic equations [85], any weak solution is automatically infinitely differentiable in $\mathbb{R}^2 \setminus \bar{D}$. It is easily verified that if $u \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$ is a solution of (3.53)–(3.55) then u is also a weak solution of (3.53)–(3.55), i.e. u satisfies (3.56). The following theorem will be proved in Chap. 8:

Theorem 3.8. *There exists a unique weak solution of the scattering problem (3.53)–(3.55) and the mapping taking the boundary data $f \in H^{-1/2}(\partial D)$ onto the solution $u \in H^1((\mathbb{R}^2 \setminus \bar{D}) \setminus \bar{\Omega}_R)$ is bounded for every R such that $(\mathbb{R}^2 \setminus \bar{D}) \cap \Omega_R \neq \emptyset$.*

In an analogous manner, we can define a weak solution of the Helmholtz equation in a bounded domain D to be any function $u \in H^1(D)$ such that

$$\int_D \left(\nabla u \cdot \nabla v - k^2 uv\right)\, dx = 0$$

for all $v \in H^1(D)$ such that $v = 0$ on ∂D in the sense of the trace theorem. The following theorems will be useful in the sequel, but we will delay their proofs until Chap. 5 where they will constitute a basic part of the analysis of that chapter.

Theorem 3.9. *Let D be a bounded domain with C^2 boundary ∂D such that k^2 is not a Dirichlet eigenvalue for D. Then for every $f \in H^{1/2}(\partial D)$ there exists a unique weak solution $u \in H^1(D)$ of the Helmholtz equation in D such that $u = f$ on ∂D in the sense of the trace theorem. Furthermore, the mapping taking f onto u is bounded.*

Theorem 3.10. *Let $u \in H^1(D)$ and $\Delta u \in L^2(D)$ in a bounded domain D with C^2 boundary ∂D having unit outward normal ν. Then there exists a positive constant C independent of u such that*

$$\left|\left|\frac{\partial u}{\partial \nu}\right|\right|_{H^{-1/2}(\partial D)} \leq C\, ||u||_{H^1(D)} \,.$$

Finally, we note that Green's identities and the representation formulas for exterior and interior domains remain valid for weak solutions of the Helmholtz equation and we refer the reader to Chap. 5 for a proof of this fact.

4

The Inverse Scattering Problem for an Imperfect Conductor

We are now in a position to introduce the inverse scattering problem for an imperfect conductor, in particular given the far field pattern of the scattered field to determine the support of the scattering object D and the surface impedance λ. Our approach to this problem is based on the *linear sampling method* in inverse scattering theory that was first introduced by Colton and Kirsch [29] and Colton, Piana and Potthast [37]. As will become clear in subsequent chapters, the advantage of this method for solving the inverse scattering problem is that in order to determine the support of the scattering object it is not necessary to have any a priori information on the physical properties of the scatterer. In particular, the relevant equation that needs to be solved is the same for the case of an imperfect conductor as it is for anisotropic media and partially coated obstacles that we will consider in the chapters which follow. Of course, for the specific inverse scattering problem we are considering in this chapter, there are alternate approaches than the one we are using and for one such alternate approach we refer the reader to [77].

The plan of our chapter is as follows. We first introduce the far field pattern corresponding to the scattering of an incident plane wave by a perfect conductor and prove the reciprocity principle. We then use this principle to show that the far field operator having the far field pattern as kernel is injective with dense range. After showing that the solution of the inverse scattering problem is unique, we then use the properties of the far field operator to establish the linear sampling method for determining the support of the scattering object and conclude by giving a method for determining the surface impedance λ. As we will see in Chap. 8, the methods used in this chapter carry over immediately to the case of partially coated perfect conductors, i.e. the case when the impedance boundary condition is imposed on only a portion of the boundary with the remaining portion being subject to a Dirichlet boundary condition.

4.1 Far Field Patterns

The inverse scattering problems we will be considering in this book all assume that the given data is the asymptotic behavior of the scattered field corresponding to an incident plane wave. Hence, our analysis of the inverse scattering problem must begin with a derivation of precisely what this asymptotic behavior is. To this end, we first recall the scattering problem under consideration, i.e. to find $u^s \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$ such that

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \tag{4.1}$$

$$u(x) = e^{ikx\cdot d} + u^s(x) \tag{4.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{4.3}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \quad \text{on } \partial D \tag{4.4}$$

where ν is the unit outward normal to ∂D and $\lambda = \lambda(x)$ is a real valued, positive and continuous function defined on ∂D. Then from the asymptotic behavior (3.27) of the Hankel function, the estimate

$$\begin{aligned} |x-y| &= \left(r^2 - 2rr_y\cos(\theta-\theta_y) + r_y^2\right)^{1/2} \\ &= r\left(1 - \frac{2r_y}{r}\cos(\theta-\theta_y) + \frac{r_y^2}{r^2}\right)^{1/2} \\ &= r - r_y\cos(\theta-\theta_y) + O\left(\frac{1}{r}\right) \end{aligned}$$

where (r_y, θ_y) are the polar coordinates of y and (r, θ) are the polar coordinates of x, we see from the Representation Theorem 3.1 that the solution u^s of (4.1) - (4.4) has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\theta, \phi) + O(r^{-3/2}) \tag{4.5}$$

where $d = (\cos\phi, \sin\phi)$, k is fixed and

$$u_\infty(\theta,\phi) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \left(u^s \frac{\partial}{\partial \nu_y} e^{-ikr_y\cos(\theta-\theta_y)} - \frac{\partial u^s}{\partial \nu_y} e^{-ikr_y\cos(\theta-\theta_y)}\right) ds(y)\,. \tag{4.6}$$

The function u_∞ is called the *far field pattern* corresponding to the scattering problem (4.1) - (4.4).

Theorem 4.1. *Suppose the far field pattern corresponding to* (4.1) - (4.4) *vanishes identically. Then $u^s(x) = 0$ for $x \in \mathbb{R}^2 \setminus D$.*

Proof. We have that

$$\int_{|y|=R} |u^s|^2 \, ds = \int_{-\pi}^{\pi} |u_\infty(\theta,\phi)|^2 \, d\theta + O\left(\frac{1}{R}\right)$$

as $R \to \infty$. If $u_\infty = 0$ then by Rellich's lemma $u^s(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$ and by continuity for $x \in \mathbb{R}^2 \setminus D$. □

We can now consider the inverse scattering problem corresponding to the direct scattering problem (4.1) - (4.4). There are in fact three different inverse scattering problems we could consider!

1. Given u_∞ and λ, determine D.
2. Given u_∞ and D, determine λ.
3. Given u_∞, determine D and λ.

From a practical point of view, the third problem is clearly the most realistic one since in general one cannot expect to know either D or λ a priori. Hence, in what follows, we shall only be concerned with the third problem and will refer to this as the inverse scattering problem. Note that the far field pattern of

$$u_n(r,\theta) = \frac{1}{n} H_n^{(1)}(kr) e^{in\theta} \quad , n > 0$$

is

$$u_{n,\infty}(\theta) = \frac{1}{n}\sqrt{\frac{2}{\pi}} e^{-i\pi/4}(-1)^n e^{in\theta} \, .$$

Hence $u_{n,\infty} \to 0$ as $n \to \infty$ in $L^2[0,2\pi]$ whereas since

$$H_n^{(1)}(kr) \sim \frac{-2^n(n-1)!}{\pi(kr)^n} \quad , n \to \infty$$

u_n will not converge in any reasonable norm. This suggests that the problem of determining u^s from u^∞ is severely ill-posed and in particular we can expect that the inverse scattering problem is also ill-posed. Further evidence in this direction is the fact that from (4.6) we see that u_∞ is an infinitely differentiable function of θ and since in general a measured far field pattern does not have this property we have that a solution does not exist to the inverse scattering problem for the case of "noisy" data.

We begin our study of the inverse scattering problem by deriving the following basic property of the far field pattern.

Theorem 4.2 (Reciprocity Relation). *Let $u_\infty(\theta,\phi)$ be the far field pattern corresponding to the scattering problem* (4.1)-(4.4). *Then $u_\infty(\theta,\phi) = u_\infty(\phi+\pi,\theta+\pi)$.*

Proof. For convenience we write $u_\infty(\hat{x},d) = u_\infty(\theta,\phi)$ where $\hat{x} = x/|x|$, e.g.

$$e^{-ikr_y\cos(\theta-\theta_y)} = e^{-iky\cdot\hat{x}} = u^i(y,-\hat{x})$$

when $u^i(x,d) = e^{ikx\cdot d}$ denotes the incident field. Then from Green's second identity we have that

$$\int_{\partial D} \left(u^i(y,d) \frac{\partial}{\partial \nu} u^i(y,-\hat{x}) - u^i(y,-\hat{x}) \frac{\partial}{\partial \nu} u^i(y,d) \right) ds(y) = 0 \tag{4.7}$$

and, using Green's second identity again, deforming ∂D to $\{x : |x| = r\}$ and letting $r \to \infty$ we have that

$$\int_{\partial D} \left(u^s(y,d) \frac{\partial}{\partial \nu} u^s(-y,\hat{x}) - u^s(y,-\hat{x}) \frac{\partial}{\partial \nu} u^s(y,d) \right) ds(y) = 0\,. \tag{4.8}$$

From (4.6) we have that

$$\begin{aligned} &\sqrt{8\pi k}\, e^{-i\pi/4}\, u_\infty(\hat{x},d) = \\ &\qquad \int_{\partial D} \left(u^s(y,d) \frac{\partial}{\partial \nu} u^i(y,-\hat{x}) - u^i(y,-\hat{x}) \frac{\partial}{\partial \nu} u^s(y,d) \right) ds(y) \end{aligned} \tag{4.9}$$

and, interchanging the roles of $\hat{x}$ and d,

$$\begin{aligned} &\sqrt{8\pi k}\, e^{-i\pi/4}\, u_\infty(-d,-\hat{x}) = \\ &\qquad \int_{\partial D} \left(u^s(y,-\hat{x}) \frac{\partial}{\partial \nu} u^i(y,d) - u^i(y,d) \frac{\partial}{\partial \nu} u^s(y,-\hat{x}) \right) ds(y)\,. \end{aligned} \tag{4.10}$$

Now subtract (4.10) from the sum of (4.7), (4.8) and (4.9) to obtain

$$\begin{aligned} &\sqrt{8\pi k}\, e^{-i\pi/4}\, (u_\infty(\hat{x},d) - u_\infty(-d,-\hat{x})) = \\ &\qquad = \int_{\partial D} \left(u(y,d) \frac{\partial}{\partial \nu} u(y,-\hat{x}) - u(y,-\hat{x}) \frac{\partial}{\partial \nu} u(y,d) \right) ds(y) \\ &\qquad = 0 \end{aligned}$$

by the boundary condition (4.4). Hence $u_\infty(\hat{x},d) = u_\infty(-d,-\hat{x})$ and this implies the theorem. $\square$

We now define the *far field operator* $F : L^2[0,2\pi] \to L^2[0,2\pi]$ by

$$(Fg)(\theta) := \int_0^{2\pi} u_\infty(\theta,\phi) g(\phi)\, d\phi\,. \tag{4.11}$$

From the representation (4.6) for u_∞ and the fact that u^s depends continuously on u^i in $C^1(\partial D)$ we see that $u_\infty(\theta,\phi)$ is continuous on $[0,2\pi] \times [0,2\pi]$.

Theorem 4.3. *The far field operator corresponding to the scattering problem* (4.1) - (4.4) *is injective with dense range.*

Proof. Using the reciprocity relation, we see that if F^* denotes the adjoint of F then

$$\begin{aligned}(F^* g)(\theta) &= \int_0^{2\pi} \overline{u_\infty(\phi,\theta)}\, g(\phi)\, d\phi \\ &= \int_0^{2\pi} \overline{u_\infty(\theta+\pi,\phi+\pi)}\, g(\phi)\, d\phi \\ &= \int_0^{2\pi} \overline{u_\infty(\theta+\pi,\phi)}\, g(\phi-\pi)\, d\phi\end{aligned}$$

where we view u_∞ and g as periodic functions of period 2π. We now see that

$$(F^* g)(\theta) = \overline{(Fh)(\theta+\pi)}$$

where $h(\phi) = \overline{g(\phi-\pi)}$. Hence F is injective if and only if F^* is injective. By Theorem 1.27 we now see that the theorem will follow if we can show that F is injective.

To this end, suppose $Fg = 0$ for $g \neq 0$. Then, by superposition, there exists a Herglotz wave function v_g with kernel g such that the far field pattern v_∞ corresponding to this Herglotz wave function as incident field is identically zero. By Rellich's lemma the scattered field $v^s(x)$ corresponding to v_∞ is identically zero for $x \in \mathbb{R}^2 \setminus \bar{D}$ and the boundary condition (4.4) now implies that

$$\frac{\partial v_g}{\partial \nu} + i\lambda v_g = 0 \quad \text{on } \partial D\,.$$

Since v_g is a solution of the Helmholtz equation in D, we have from Green's second identity applied to v_g and $\bar{v}_g$ that

$$2i \int_{\partial D} \lambda \left|v_g\right|^2 \, ds = 0\,.$$

Hence $v_g = 0$ on ∂D and by the boundary condition satisfied by v_g on ∂D we also have that $\partial v_g/\partial\nu = 0$ on ∂D. The representation formula (3.41) for solutions of the Helmholtz equation in interior domains now shows that $v_g(x) = 0$ for $x \in D$ and hence $g = 0$, a contradiction. Hence $Fg = 0$ implies that $g = 0$, i.e. F is injective, and the theorem follows. □

4.2 Uniqueness Theorems for the Inverse Problem

Our first aim in this section is to show that D is uniquely determined from $u_\infty(\theta,\phi)$ for θ and ϕ in $[0, 2\pi]$ without knowing λ a priori. Our proof is due to Kirsch and Kress [71].

Lemma 4.4. *Assume that k^2 is not a Dirichlet eigenvalue for the bounded domain B with C^2 boundary ∂B and that $\mathbb{R}^2 \setminus \bar{B}$ is connected. Let $u^i(x,d) =$*

$e^{ikx\cdot d}$. *Then the restriction of* $\{u^i(\cdot,d) : |d| = 1\}$ *to* ∂B *is complete in* $H^{1/2}(\partial B)$, *i.e.*

$$\overline{\text{span}\,\{u^i(\cdot,d)|_{\partial B} : |d| = 1\}} = H^{1/2}(\partial B)\,.$$

Proof. Let $\varphi \in H^{-1/2}(\partial B)$ satisfy

$$\int_{\partial B} \varphi(y)\, e^{-iky\cdot d}\, ds(y) = 0 \tag{4.12}$$

for all d such that $|d| = 1$. By duality pairing, to prove the lemma it suffices to show that $\varphi = 0$. To this end, we see that (4.12) implies that the single layer potential

$$u(x) := \int_{\partial B} \varphi(y)\Phi(x,y)\, ds(y) \quad , x \in \mathbb{R}^2 \setminus \partial B$$

has vanishing far field pattern $u_\infty = 0$. Hence, by Rellich's lemma, $u(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{B}$. It can easily be shown that in this case $\varphi \in C(\partial B)$ (c.f. Theorem 4.10 in the next section of this chapter for the analysis in a related case) and since in this case the single layer potential is continuous across ∂B, u satisfies the homogeneous Dirichlet problem in B. Thus, since k^2 is not a Dirichlet eigenvalue for B, $u(x) = 0$ for $x \in B$. From the discontinuity property of the normal derivative of the single layer pontential (see Sect. 3.3), we can now conclude that

$$0 = \frac{\partial u^-}{\partial \nu} - \frac{\partial u^+}{\partial \nu} = \varphi$$

and the proof is finished. □

Theorem 4.5. *Assume that* D_1 *and* D_2 *are two scattering obstacles with corresponding surface impedances* λ_1 *and* λ_2 *such that for a fixed wave number the far field patterns for both scatterers coincide for all incident directions* d. *Then* $D_1 = D_2$.

Proof. By Rellich's lemma we can conclude that the scattered fields $u^s(\cdot,d)$ corresponding to the incident fields $u^i(x,d) = e^{ikx\cdot d}$ coincide in the unbounded component G of the complement of $\bar{D}_1 \cup \bar{D}_2$. Choose $x_0 \in G$ and consider the two exterior boundary value problems

$$\Delta w_j^s + k^2 w_j^s = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}_j \tag{4.13}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w_j^s}{\partial r} - ikw_j^s\right) = 0 \tag{4.14}$$

$$\frac{\partial}{\partial \nu}\left[w_j^s + \Phi(\cdot,x_0)\right] + i\lambda_j\left[w_j^s + \Phi(\cdot,x_0)\right] = 0 \quad \text{on } \partial D_j \tag{4.15}$$

for $j = 1, 2$.

We will first show that $w_1^s(x) = w_2^s(x)$ for $x \in G$. To this end, choose a bounded domain B such that $\mathbb{R}^2 \setminus \bar{B}$ is connected, $\bar{D}_1 \cup \bar{D}_2 \subset B$, $x_0 \notin \bar{B}$ and k^2 is not a Dirichlet eigenvalue for B. Then by Lemma 4.4 there exists a sequence $\{v_n\}$ in span $\left\{u^i(\cdot, d) : |d| = 1\right\}$ such that

$$||v_n - \Phi(\cdot, x_0)||_{H^{1/2}(\partial B)} \to 0 \quad , n \to \infty .$$

From Theorem 3.9 one can conclude that $v_n \to \Phi(\cdot, x_0)$ and $\operatorname{grad} v_n \to \operatorname{grad} \Phi(\cdot, x_0)$ as $n \to \infty$ uniformly on $\bar{D}_1 \cup \bar{D}_2$. Since the v_n are linear combinations of plane waves, the corresponding scattered fields $v_{n,1}^s$ and $v_{n,2}^s$ for D_1 and D_2 respectively coincide on G. But from Theorem 3.7 we have that $v_{n,j}^s \to w_j^s$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}^2 \setminus \bar{D}_j$ for $j = 1, 2$ and hence $w_1^s(x) = w_2^s(x)$ for $x \in G$.

Now assume that $D_1 \neq D_2$. Then, without loss of generality, there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \bar{D}_2$ (see Figure 4.1). We can choose $h > 0$ such that

$$x_n := x^* + \frac{h}{n} \nu(x^*) \quad , n = 1, 2, \cdots$$

is contained in G and consider the solutions $w_{n,j}^s$ to the scattering problem (4.13)–(4.15) with x_0 replaced by x_n. Then $w_{n,1}^s(x) = w_{n,2}^s(x)$ for $x \in G$. But,

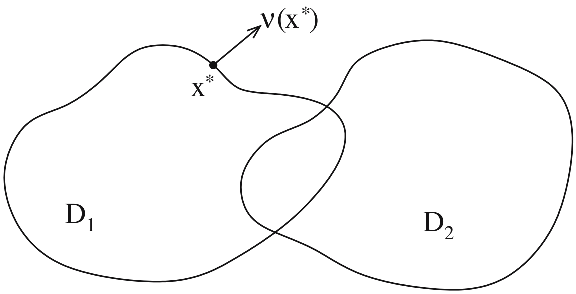

Fig. 4.1.

considering $w_n^s = w_{n,2}^s$ as the scattered field corresponding to D_2, we see that

$$\frac{\partial w_n^s}{\partial \nu}(x^*) + i\lambda_1(x^*) w_n^s(x^*) \tag{4.16}$$

remains bounded as $n \to \infty$. On the other hand, considering $w_n^s = w_{n,1}^s$ as the scattered field corresponding to D_1, we have that

$$\frac{\partial w_n^s}{\partial \nu}(x^*) + i\lambda_1(x^*) w_n^s(x^*) = -\left(\frac{\partial \Phi}{\partial \nu}(x^*, x_n) + i\lambda_1(x^*) \Phi(x^*, x_0) \right)$$

and hence (4.16) becomes unbounded as $n \to \infty$. This is a contradiction and hence $D_1 = D_2$. □

We now want to show that the far field pattern u_∞ not only uniquely determines D but the surface impedance $\lambda = \lambda(x)$ as well [77]. To this end, we first need the following lemma [65].

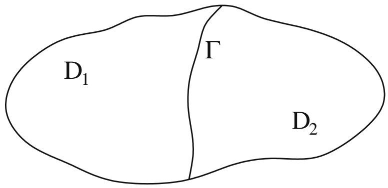

Fig. 4.2.

Lemma 4.6. *Let $D \subset R^2$ be a domain that is decomposed into two disjoint subdomains D_1 and D_2 with common boundary $\Gamma := \partial D_1 \cap \partial D_2$ (see Figure 4.2). Assume that ∂D is in class C^2. Suppose $u_j \in C^2(D_j) \cap C^1(\bar{D}_j)$ satisfies*

$$\Delta u_j + k^2 u_j = 0 \quad \text{in } D_j$$

and $u_1 = u_2$ on Γ and $\partial u_1/\partial\nu = \partial u_2/\partial\nu$ on Γ where ν is the unit outward normal to Γ considered as part of ∂D_1. Then the function

$$u(x) := \begin{cases} u_1(x), & x \in \bar{D}_1 \\ u_2(x), & x \in \bar{D}_2 \end{cases}$$

is a solution to the Helmholtz equation in $D = D_1 \cup D_2 \cup \Gamma$.

Proof. Fix $x_0 \in \Gamma \cap D$ and let $\Omega := \{x : |x - x_0| < \epsilon\} \subset D$. Let $\Omega_j := \Omega \cap D_j$ and let $x \in \Omega_1$. Then by the representation formula (3.41) we have that

$$u_1(x) = \int_{\partial\Omega_1} \left[\frac{\partial u_1}{\partial\nu}(y)\Phi(x,y) - u_1(y)\frac{\partial}{\partial\nu(y)}\Phi(x,y) \right] ds(y)$$

for $x \in \Omega_1$. On the other hand,

$$0 = \int_{\partial\Omega_2} \left[\frac{\partial u_2}{\partial\nu}(y)\Phi(x,y) - u_2(y)\frac{\partial}{\partial\nu(y)}\Phi(x,y) \right] ds(y)$$

for $x \in \Omega_1$. Now add these two equations together, noting that the contributions on $\Gamma \cap \Omega$ cancel, to arrive at

$$u_1(x) = \int_{\partial\Omega} \left[\frac{\partial u}{\partial\nu}(y)\Phi(x,y) - u(y)\frac{\partial}{\partial\nu(y)}\Phi(x,y) \right] ds(y) \tag{4.17}$$

for $x \in \Omega_1$. Similarly,

$$u_2(x) = \int_{\partial\Omega} \left[\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial}{\partial \nu(y)}\Phi(x,y) \right] ds(y) \tag{4.18}$$

for $x \in \Omega_2$. Now note that the right hand sides of (4.17) and (4.18) coincide and define a solution of the Helmholtz equation in Ω and the lemma follows. □

Theorem 4.7. *Assume that D_1 and D_2 are two scattering obstacles with corresponding surface impedances λ_1 and λ_2 such that for a fixed wave number the far field patterns coincide for all incident directions d. Then $D_1 = D_2$ and $\lambda_1 = \lambda_2$.*

Proof. By Theorem 4.5 we have that $D_1 = D_2$. Hence it only remains to show that $\lambda_1(x) = \lambda_2(x)$ for $x \in \partial D$ where $D = D_1 = D_2$. Let u_1 and u_2 be the solutions of (4.1) – (4.4) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively. Then, by Rellich's lemma, $u_1(x) = u_2(x)$ for $x \in \mathbb{R}^2 \setminus \bar{D}$ and hence $u_1 = u_2$ and $\partial u_1/\partial\nu = \partial u_2/\partial\nu$ on ∂D. From the boundary conditions

$$\frac{\partial u_j}{\partial \nu} + i\lambda_j u_j = 0 \quad \text{on } \partial D \tag{4.19}$$

for $j = 1, 2$ we have that

$$(\lambda_1 - \lambda_2)u_1 = 0 \quad \text{on } \partial D\,. \tag{4.20}$$

Now suppose that $u_1 = 0$ on an arc $\Gamma \subset \partial D$. Then from (4.19) we have that $\partial u_1/\partial\nu = 0$ on Γ and by Lemma 4.6 we have that

$$u(x) = \begin{cases} u_1(x) & , x \in \mathbb{R}^2 \setminus D \\ 0 & , x \in D \end{cases}$$

defines a solution of the Helmholtz equation in $(\mathbb{R}^2 \setminus D) \cup \Gamma \cup D$. By the fact that solutions of the Helmholtz equation are real analytic, we can now conclude that $u_1(x) = 0$ for $x \in \mathbb{R}^2 \setminus D$. But

$$u_1(x) = e^{ikx\cdot d} + u_1^s(x)$$

and u_1^s satisfies the Sommerfeld radiation condition but $e^{ikx\cdot d}$ does not. This is a contradiction and hence u_1 cannot vanish on any arc $\Gamma \subset \partial D$. Thus if $x \in \partial D$ there exists a sequence $\{x_n\} \subset \partial D$ such that $x_n \to x$ as $n \to \infty$ and $u_1(x_n) \neq 0$ for every n. From (4.20) we have that $\lambda_1(x_n) = \lambda_2(x_n)$ for every n and, since λ_1 and λ_2 are continuous functions we have that $\lambda_1(x) = \lambda_2(x)$. Since $x \in \partial D$ was an arbitrary point, the theorem is proved. □

4.3 The Linear Sampling Method

We shall now give an algorithm for determining the scattering obstacle D from a knowledge of the far field pattern corresponding to the scattering problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \tag{4.21}$$

$$u(x) = e^{ikx\cdot d} + u^s(x) \tag{4.22}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{4.23}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \quad \text{on } \partial D \tag{4.24}$$

where $\lambda \in C(\partial D)$, $\lambda(x) > 0$ for $x \in \partial D$ and it is not assumed that λ is known a priori. The algorithm we have in mind is the *linear sampling method* and was first introduced by Colton and Kirsch [29] and Colton, Piana and Potthast [37]. For survey papers discussing this method we refer the reader to [25] and [28].

We begin our discussion of the linear sampling method by considering the general scattering problem

$$\Delta w + k^2 w = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \tag{4.25}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial r} - ikw\right) = 0 \tag{4.26}$$

$$\frac{\partial w}{\partial \nu} + i\lambda w = f \quad \text{on } \partial D \tag{4.27}$$

where $f \in H^{-1/2}(\partial D)$, i.e. we are considering weak solutions of (4.25) – (4.27). The *boundary operator* $B : H^{-1/2}(\partial D) \to L^2[0, 2\pi]$ is now defined to be the linear operator mapping f onto the far field pattern w_∞ corresponding to (4.25) – (4.27).

Theorem 4.8. *The boundary operator B is compact, injective and has dense range in $L^2[0, 2\pi]$.*

Proof. By representing w in the form of a modified single layer pontential

$$w(x) = \int_{\partial D} \varphi(y)\Gamma(x, y)\, ds(y) \tag{4.28}$$

as discussed in Sect. 3.3 and generalizing the analysis given there for $\varphi \in C(\partial D)$ to the present case $\varphi \in H^{-1/2}(\partial D)$ it can be shown by the Riesz theorem that there exists a unique density $\varphi \in H^{-1/2}(\partial D)$ such that w, as defined by (4.28), satisfies (4.25) – (4.27) and the mapping $f \to \varphi$ is bounded in $H^{-1/2}(\partial D)$. From (4.28) we have that the far field pattern w_∞ is given by

$$w_\infty(\hat{x}) = \int_{\partial D} \varphi(y)\Gamma_\infty(\hat{x}, y)\, ds(y) \tag{4.29}$$

where $\hat{x} = x/|x|$ and Γ_∞ is the far field pattern of Γ. Viewing $\Gamma_\infty(\hat{x}, \cdot)$ as a function in $H^1(\partial D)$, we see that for $\varphi \in H^{-1}(\partial D)$ we have

$$\left| \int_{\partial D} \varphi(y) \left[\Gamma_\infty(\hat{x}_1, y) - \Gamma_\infty(\hat{x}_2, y) \right] ds(y) \right| \\ \leq ||\varphi||_{H^{-1}(\partial D)} \, ||\Gamma_\infty(\hat{x}_1, \cdot) - \Gamma_\infty(\hat{x}_2, \cdot)||_{H^1(\partial D)}$$

and hence (4.29) defines a bounded operator from $H^{-1}(\partial D)$ to $C[0, 2\pi]$. Parameterizing ∂D and using Rellich's theorem, we see that the imbedding operator from $H^{-1}(\partial D)$ to $H^{-1/2}(\partial D)$ is compact and (4.29) defines a compact operator from $H^{-1/2}(\partial D)$ to $C[0, 2\pi]$. This implies that (4.29) is also compact from $H^{-1/2}(\partial D)$ to $L^2[0, 2\pi]$. Since $f \to \varphi$ is bounded in $H^{-1/2}(\partial D)$, we can now conclude that $B : H^{-1/2}(\partial D) \to L^2[0, 2\pi]$ is compact.

Now suppose that the far field pattern w_∞ corresponding to (4.25) – (4.27) vanishes. Then by Rellich's lemma we have that $w(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$ and from the weak formulation (3.56) we see that

$$\int_{\partial D} f v \, ds = 0$$

for all $v \in H^1_{com}(\mathbb{R}^2 \setminus \bar{D})$, i.e., from the trace theorem, for every $v \in H^{1/2}(\partial D)$. Hence, by duality pairing, $f = 0$ and this implies that B is injective.

To show that B has dense range, let

$$u_{n,\infty}(\theta) = \sum_{-n}^{n} a_l e^{il\theta} \, .$$

Then $u_{n,\infty}$ is the far field pattern of

$$u_n(r, \theta) = \sum_{-n}^{n} a_l \gamma_l^{-1} H_l^{(1)}(kr) e^{il\theta}$$

where

$$\gamma_l = \sqrt{\frac{2}{\pi}} \exp\left[-i \left(\frac{l\pi}{2} + \frac{\pi}{4} \right) \right]$$

and u_n satisfies (4.25) – (4.27) for

$$f = \left(\frac{\partial u_n}{\partial \nu} + i\lambda u_n \right) \bigg|_{\partial D} \, .$$

Since f is continuous and hence in $H^{-1/2}(\partial D)$, we can conclude by the completeness of the trigonometric polynomials in $L^2[0, 2\pi]$ that B has dense range. □

The following theorem will provide the key ingredient of the linear sampling method for determining D from the far field pattern u_∞.

Theorem 4.9. *If $\Phi_\infty(\hat{x}, z)$ is the far field pattern of the fundamental solution $\Phi(x, z)$, then $\Phi_\infty(\hat{x}, z)$ is in the range of B if and only if $z \in D$.*

Proof. If $z \in D$ then $\Phi(\cdot, z)$ is the solution of (4.25) – (4.27) with

$$f = \left(\frac{\partial \Phi}{\partial \nu} + i\lambda\Phi \right)\Big|_{\partial D} \tag{4.30}$$

and $Bf = \Phi_\infty$. If $z \in \mathbb{R}^2 \setminus D$ and Φ_∞ is in the range of B then by Rellich's lemma $\Phi(\cdot, z)$ is a weak solution of (4.25) – (4.27) with f again given by (4.30). But Φ is not in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \bar{D})$ and hence this is not possible. Thus if $z \in \mathbb{R}^2 \setminus D$ then Φ_∞ is not in the range of B. $\square$

Now let v_g be a Herglotz wave function with kernel $g \in L^2[0, 2\pi]$ and define the operator $H : L^2[0, 2\pi] \to H^{-1/2}(\partial D)$ by

$$Hg := \left(\frac{\partial v_g}{\partial \nu} + i\lambda v_g \right)\Big|_{\partial D}.$$

The importance of the operator H follows from the fact that the far field operator F is easily seen to have the factorization

$$F = -BH\,.$$

The following theorem was first proved in [39] (see also [34]).

Theorem 4.10. *The operator H is bounded, injective and has dense range in $H^{-1/2}(\partial D)$.*

Proof. From the definition of H and v_g, H is clearly bounded and injectivity follows from the uniqueness of the solution to the interior impedance problem (see the end of Sect. 4.1). In order to show that the range is dense, it suffices to show that if

$$u_n(x) := J_n(kr)e^{in\theta}$$

then the set

$$\left\{ \left(\frac{\partial u_n}{\partial \nu} + i\lambda u_n \right)\Big|_{\partial D} : n = 0, \pm 1, \pm 2, \cdots \right\}$$

is complete in $H^{-1/2}(\partial D)$. By duality pairing, this requires us to show that if $g \in H^{1/2}(\partial D)$ and

$$\int_{\partial D} g(y) \left(\frac{\partial}{\partial \nu} + i\lambda \right) u_n(y)\, ds(y) = 0 \tag{4.31}$$

for $n = 0, \pm 1, \pm 2, \cdots$ then $g = 0$.

Suppose that (4.31) is valid for some $g \in H^{1/2}(\partial D)$ and let Ω_R be a disk centered at the origin of radius R and containing D in its interior. Then from (4.31) and the addition formula for Bessel functions, we can conclude that

$$u(x) := \int_{\partial D} g(y) \left(\frac{\partial}{\partial \nu(y)} + i\lambda \right) \Phi(x, y)\, ds(y) \tag{4.32}$$

is identically zero for $x \in \mathbb{R}^2 \setminus \bar{\Omega}_R$. By Theorem 3.2 we can conclude that $u(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$. We now make use of the fact that the double layer potential

$$v(x) := \int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(y)} \Phi(x, y)\, ds(y) \quad , x \in \mathbb{R}^2 \setminus \partial D$$

with continuous density φ satisfies the discontinuity property

$$v_\pm(x) = \int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(y)} \Phi(x, y)\, ds(y) \pm \frac{1}{2}\varphi(x) \quad , x \in \partial D$$

where $\pm$ denotes the limits as $x \longrightarrow \partial D$ from outside and inside D respectively and that

$$\frac{\partial v_+}{\partial \nu}(x) = \frac{\partial v_-}{\partial \nu}(x) \quad , x \in \partial D\,.$$

Furthermore, these properties remain valid for $\varphi \in H^{1/2}(\partial D)$ where the integrals are interpreted in the sense of duality pairing [75, 85]. Hence, since $u(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$, we have that

$$0 = g(x) + 2 \int_{\partial D} g(y) \left(\frac{\partial}{\partial \nu(y)} + i\lambda \right) \Phi(x, y)\, ds(y) \quad , x \in \partial D\,.$$

Since $\partial/\partial\nu(y)\Phi(x, y)$ is continuous and $\Phi(x, y) = O(\log |x - y|)$, we can now easily verify that g is continuous.

We now return to (4.32) and use the discontinuity properties of double and single layer potentials with continuous densities to conclude that

$$u_+ - u_- = g$$

$$\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} = -i\lambda g$$

and, since $u_+ = \partial u_+/\partial \nu = 0$, we have that

$$\frac{\partial u_-}{\partial \nu} + i\lambda u_- = 0 \quad \text{on } \partial D\,.$$

We can now conclude as we did at the end of Sect. 4.1 that $u(x) = 0$ for $x \in D$ and since $u(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$ we now have that $0 = u_+ - u_- = g$ and the theorem follows. □

In order to derive an algorithm for determining D, we now introduce the *far field equation*

$$\int_0^{2\pi} u_\infty(\theta, \phi) g(\phi)\, d\phi = \gamma \exp\left(-ikr_z \cos(\theta - \theta_z)\right) \tag{4.33}$$

where (r_z, θ_z) are the polar coordinates of a point $z \in \mathbb{R}^2$ and

$$\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$$

or, in simpler notation,

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z)$$

where $\hat{x} = (\cos\theta, \sin\theta)$ and

$$\Phi_\infty(\hat{x}, z) = \gamma e^{-ik\hat{x}\cdot z}$$

is the far field pattern of the fundamental solution $\Phi(x, z)$. The following theorem provides the mathematical basis of the linear sampling method [8].

Theorem 4.11. *Let u_∞ be the far field pattern corresponding to the scattering problem* (4.21) – (4.24) *with associated far field operator F.*

1. *If $z \in D$ then for every $\epsilon > 0$ there exists $g_z^\epsilon := g_z \in L^2[0, 2\pi]$ satisfying the inequality*
$$||Fg_z - \Phi_\infty(\cdot, z)||_{L^2[0,2\pi]} < \epsilon$$
such that
$$\lim_{z\to\partial D} ||g_z||_{L^2[0,2\pi]} = \infty$$
and
$$\lim_{z\to\partial D} ||v_{g_z}||_{H^1(D)} = \infty$$
where v_{g_z} is the Herglotz wave function with kernel g_z.
2. *If $z \notin D$ then for every $\epsilon > 0$ and $\delta > 0$ there exists $g_z^{\epsilon,\delta} := g_z \in L^2[0, 2\pi]$ satisfying the inequality*
$$||Fg_z - \Phi_\infty(\cdot, z)||_{L^2[0,2\pi]} < \epsilon + \delta$$
such that
$$\lim_{\delta\to 0} ||g_z||_{L^2[0,2\pi]} = \infty$$
and
$$\lim_{\delta\to 0} ||v_{g_z}||_{H^1(D)} = \infty$$
where v_{g_z} is the Herglotz wave function with kernel g_z.

Proof. Assume $z \in D$. Then by Theorem 4.9 there exists $f_z \in H^{-1/2}(\partial D)$ such that $Bf_z = -\Phi_\infty(\cdot, z)$. By Theorem 4.10 we see that for every $\epsilon > 0$ there exists a Herglotz wave function with kernel $g_z \in L^2[0, 2\pi]$ such that

$$||Hg_z - f_z||_{H^{-1/2}(\partial D)} < \epsilon \tag{4.34}$$

and, from the continuity of the operator B, there exists a positive constant C independent of ϵ such that

$$||BHg_z - Bf_z||_{L^2[0,2\pi]} < C\epsilon\,.$$

Hence, since $F = -BH$, we have that

$$||Fg_z - \Phi_\infty||_{L^2[0,2\pi]} < C\epsilon\,.$$

Since $f_z = -\left(\frac{\partial}{\partial\nu} + i\lambda\right)\Phi(\cdot, z)|_{\partial D}$, when $z \to \partial D$ $||f_z||_{H^{-1/2}(\partial D)}$ tends to infinity since otherwise $\Phi(\cdot, z)$, $z \in \partial D$, would be a solution of (4.25) – (4.27) for $f = f_z$, i.e. $\Phi(\cdot, z) \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$, a contradiction. It now follows from (4.34) that $||Hg_z||_{H^{-1/2}(\partial D)} \to \infty$ and hence from Theorem 3.10 and Theorem 1.36 we have that $||v_{g_z}||_{H^1(D)} \to \infty$. We can now conclude from this fact that $||g_z||_{L^2[0,2\pi]}$ also tends to infinity.

Now assume that $z \notin D$. Then $-\Phi_\infty(\cdot, z)$ is not in the range of B. However the range of B is dense in $L^2[0, 2\pi]$ and hence for every $\delta > 0$ there exists a regularized "solution" f_z^α of $Bf = -\Phi_\infty(\cdot, z)$ given by

$$f_z^\alpha = -\sum_1^\infty \frac{\mu_n}{\alpha + \mu_n^2}(\Phi_\infty, g_n)\varphi_n$$

where (μ_n, φ_n, g_n) is a singular system for B such that

$$||Bf_z^\alpha + \Phi_\infty(\cdot, z)||_{L^2[0,2\pi]} < \delta\,. \tag{4.35}$$

This follows from the fact that f_z^α minimizes the Tikhonov functional. From Picard's theorem we see further that $||f_z^\alpha||_{H^{-1/2}(\partial D)} \to \infty$ as $\alpha \to 0$ since $-\Phi_\infty(\cdot, z)$ is not in the range of B. We can again find $g_z^\alpha \in L^2[0, 2\pi]$ such that

$$||Bf_z^\alpha - BHg_z^\alpha||_{L^2[0,2\pi]} < \epsilon \tag{4.36}$$

for ϵ arbitrarily small. Hence, from (4.35) and (4.36) we have that

$$\begin{aligned}
&||Fg_z^\alpha - \Phi_\infty(\cdot, z)||_{L^2[0,2\pi]} = ||BHg_z^\alpha + \Phi_\infty(\cdot, z)||_{L^2[0,2\pi]} \\
&\leq ||BHg_z^\alpha - Bf_z^\alpha||_{L^2[0,2\pi]} + ||Bf_z^\alpha + \Phi_\infty(\cdot, z)||_{L^2[0,2\pi]} \\
&< \epsilon + \delta\,.
\end{aligned}$$

The facts that $||f_z^\alpha||_{H^{-1/2}(\partial D)} \to \infty$ as $\alpha \to 0$ and f_z^α is approximated by Hg_z^α in $H^{-1/2}(\partial D)$ now imply (using Theorem 3.10 and Theorem 1.36 again) that

$$\lim_{\alpha\to 0} ||v_{g_z^\alpha}||_{H^1(D)} = \infty$$

and hence

$$\lim_{\alpha\to 0} ||g_z^\alpha||_{L^2[0,2\pi]} = \infty\,.$$

The theorem now follows by noting that $\lim_{\delta\to 0}\alpha(\delta) = 0$. □

The *linear sampling method* is based on numerically determining the function g_z in the above theorem and hence the scattering object D. However, at this point, the numerical scheme that is used is rather ad hoc since in general the far field equation has no solution even in the case of "noise-free" data u_∞. Nevertheless, the procedure that has been used to determine g_z has been proven to be numerically quite successful and is as follows:

1. Select a grid of "sampling points" in a region known to contain D.
2. Use Tikhonov regularization and the Morozov discrepancy principle to compute an approximate solution g_z to the far field equation for each z in the above grid. In the case when $\lambda = 0$, a justification for using such a procedure to construct g_z has been given in [5] but the general case remains open. It is of course possible to use other regularization schemes to reconstruct g_z and investigations in this direction have reported on in [109].
3. Choose a cut-off value C and assert that $z \in D$ if and only if $||g_z|| \leq C$. The choice of C is heuristic but becomes empirically easier to choose when the frequency becomes higher [23].

For numerical examples using the above numerical strategy, we refer the reader to Chap. 8 as well as to the papers in the references.

4.4 Determination of the Surface Impedance

Having determined the scattering object D (without needing to know λ a priori!) we now want to determine λ. We shall do this following the ideas of [11] and note that the method we will present is also valid when the impedance boundary condition is only imposed on part of the boundary and on the other part the total field u is required to satisfy a Dirichlet boundary condition corresponding to that portion of the boundary being a perfect conductor (see Chap. 8).

We begin by noting that there exists a unique weak solution $u_z \in H^1(D)$ of the interior impedance problem (see Chap. 8)

$$\Delta u_z + k^2 u_z = 0 \quad \text{in } D \tag{4.37}$$

$$\frac{\partial u_z}{\partial \nu} + i\lambda u_z = -\left(\frac{\partial}{\partial \nu}\Phi(\cdot, z) + i\lambda\Phi(\cdot, z)\right) \quad \text{on } \partial D \tag{4.38}$$

where $z \in D$ and as before $\lambda = \lambda(x) \in C(\partial D)$, $\lambda(x) > 0$ for $x \in \partial D$. From Sect. 4.3 we have that the following theorem is valid:

Theorem 4.12. *Let $\epsilon > 0$, $z \in D$ and u_z be the solution of (4.37), (4.38). Then there exists a Herglotz wave function v_{g_z} with kernel $g_z \in L^2[0, 2\pi]$ such that*

$$||u_z - v_{g_z}||_{H^1(D)} \leq \epsilon$$

and there exists a constant $C > 0$, independent of ϵ, such that

$$||Fg_z - \Phi(\cdot, z)||_{L^2[0,2\pi]} \leq C\epsilon\,.$$

Now define

$$w_z := u_z + \Phi(\cdot, z)\,.$$

Lemma 4.13. *For every $z_1, z_2 \in D$ we have that*

$$2 \int_{\partial D} w_{z_1} \lambda \bar{w}_{z_2} \, ds = -4\pi k \, |\gamma|^2 \, J_0(k\,|z_1 - z_2|) - i \left(\overline{u_{z_2}(z_1)} - u_{z_1}(z_2) \right)$$

where $\gamma = e^{i\pi/4}/\sqrt{8\pi k}$ and J_0 is a Bessel function of order zero.

Proof. We have previously noted that Green's second identity remains valid for weak solutions of the Helmholtz equation. In particular,

$$\begin{aligned} 2i \int_{\partial D} w_{z_1} \lambda \bar{w}_{z_2} \, ds &= \int_{\partial D} \left(w_{z_1} \frac{\partial \bar{w}_{z_2}}{\partial \nu} - \bar{w}_{z_2} \frac{\partial w_{z_1}}{\partial \nu} \right) ds \\ &= \int_{\partial D} \left(\Phi(\cdot, z_1) \frac{\partial}{\partial \nu} \overline{\Phi(\cdot, z_2)} - \overline{\Phi(\cdot, z_2)} \frac{\partial}{\partial \nu} \Phi(\cdot, z_1) \right) ds \\ &+ \int_{\partial D} \left(u_{z_1} \frac{\partial}{\partial \nu} \overline{\Phi(\cdot, z_2)} - \overline{\Phi(\cdot, z_2)} \frac{\partial u_{z_1}}{\partial \nu} \right) ds \\ &+ \int_{\partial D} \left(\Phi(\cdot, z_1) \frac{\partial \bar{u}_{z_2}}{\partial \nu} - \bar{u}_{z_2} \frac{\partial}{\partial \nu} \Phi(\cdot, z_1) \right) ds \, . \end{aligned}$$

But

$$\int_{\partial D} \left(\Phi(\cdot, z_1) \frac{\partial}{\partial \nu} \overline{\Phi(\cdot, z_2)} - \overline{\Phi(\cdot, z_2)} \frac{\partial}{\partial \nu} \Phi(\cdot, z_1) \right) ds =$$

$$\begin{aligned} &= -2ik \int_{|\hat{x}|=1} \Phi_\infty(\hat{x}, z_1) \overline{\Phi_\infty(\hat{x}, z_2)} \, ds(\hat{x}) \\ &= -2ik \, |\gamma|^2 \int_{|\hat{x}|=1} e^{-ik\hat{x}\cdot z_1} e^{ik\hat{x}\cdot z_2} \, ds(\hat{x}) \\ &= -4ik\pi \, |\gamma|^2 \, J_0(k\,|z_1 - z_2|) \end{aligned}$$

from the Jacobi–Anger expansion (3.24). From the representation formula (3.41) we now obtain

$$2i \int_{\partial D} w_{z_1} \lambda \bar{w}_{z_2} \, ds = -4ik\pi \, |\gamma|^2 \, J_0(k\,|z_1 - z_2|) + \overline{u_{z_2}(z_1)} - u_{z_1}(z_2)$$

and the lemma follows by dividing both sides by i. □

Assuming D is connected, let $\Omega_r \subset D$ be a disk of radius r contained in D and define the set W by

$$W := \left\{ f \in L^2(\partial D) : f = w_z|_{\partial D} \; , z \in \Omega_r \right\} .$$

The following theorem will give us a constructive method for determining the maximum of λ.

Remark 4.14. If D is not connected, the theorem remains true if we replace Ω_r by a union of discs where each component contains one disc from the union.

Theorem 4.15. *Let $\lambda = \lambda(x)$ be the surface impedance of the scattering problem (4.21)–(4.24). Then*

$$\max_{x\in\partial D} \lambda(x) = \sup_{\substack{z_i\in\Omega_r\\ \alpha_i\in\mathbb{C}}} \frac{\sum_{i,j} \alpha_i\bar{\alpha}_j \left[-4\pi\,|\gamma|^2\, J_0\left(k\,|z_i - z_j|\right) - i\left(\overline{u_{z_j}(z_i)} - u_{z_i}(z_j)\right)\right]}{2\left|\left|\sum_i \alpha_i\left(u_{z_i} + \Phi(\cdot, z_i)\right)\right|\right|^2_{L^2(\partial D)}}$$

where the sums are arbitrary finite sums.

Proof. It is easy to see that

$$\max_{x\in\partial D} \lambda(x) = \sup_{f\in L^2(\partial D)} \frac{1}{||f||^2_{L^2(\partial D)}} \int_{\partial D} \lambda\,|f|^2\, ds\,.$$

Hence the theorem will follow from Lemma 4.13 if we can show that W is complete in $L^2(\partial D)$ (first fix z_1 and then z_2 and consider linear combinations of w_z for $z \in \Omega_r$).

To show that W is complete in $L^2(\partial D)$, let $\varphi \in L^2(\partial D)$ be such that for every $z \in \Omega_r$ we have

$$\int_{\partial D} w_z\varphi\, ds = 0\,.$$

We want to show that $\varphi = 0$. To this end, let v be the (weak) solution of the interior impedance problem

$$\Delta v + k^2 v = 0 \quad \text{ in } D$$
$$\frac{\partial v}{\partial\nu} + i\lambda v = \varphi \quad \text{ on } \partial D\,.$$

Then for every $z \in \Omega_r$ we have that

$$\begin{aligned}
0 = \int_{\partial D} w_z\varphi\, ds &= \int_{\partial D} w_z\left(\frac{\partial v}{\partial\nu} + i\lambda v\right) ds \\
&= \int_{\partial D}\left(u_z\frac{\partial v}{\partial\nu} + i\lambda u_z v + \Phi(\cdot, z)\frac{\partial v}{\partial\nu} + i\lambda\Phi(\cdot, z)v\right) ds \\
&= \int_{\partial D}\left(u_z\frac{\partial v}{\partial\nu} + v\left(-\frac{\partial u_z}{\partial\nu} - \frac{\partial}{\partial\nu}\Phi(\cdot, z) - i\lambda\Phi(\cdot, z)\right)\right) ds \\
&\quad + \int_{\partial D}\left(\Phi(\cdot, z)\frac{\partial v}{\partial\nu} + i\lambda v\Phi(\cdot, z)\right) ds \\
&= \int_{\partial D}\left(\Phi(\cdot, z)\frac{\partial v}{\partial\nu} - v\frac{\partial}{\partial\nu}\Phi(\cdot, z)\right) ds \\
&= v(z)\,.
\end{aligned}$$

Since v is a solution of the Helmholtz equation in D, and hence real-analytic by Theorem 3.2, we now have that $v(z) = 0$ for every $z \in D$ and hence by the trace theorem and Theorem 3.10 we have that $\varphi = 0$. $\square$

Given that D is known (e.g. by the linear sampling method) we can now approximate u_z by the Herglotz wave function v_{g_z} with kernel g_z being the approximate solution of the far field equation given by the first part of Theorem 4.11. By Theorem 4.15 this in turn provides us with an approximation to $\max_{x\in\partial D} \lambda(x)$. In the special case when $\lambda(x) = \lambda$ is constant, we can set $z_1 = z_2 = z_0 \in \Omega_r$ in Lemma 4.13 to arrive at

$$\lambda = \frac{-2\pi k\,|\gamma|^2 - \mathrm{Im}(u_{z_0}(z_0))}{||u_{z_0} + \Phi(\cdot, z)||^2_{L^2(\partial D)}} .$$

Numerical examples using this formula will be provided in Chap. 8 when we consider mixed boundary value problems in scattering theory for which the same formula is valid.

4.5 Limited Aperture Data

In many cases of practical interest, the far field data $u_\infty(\theta, \phi) = u_\infty(\hat{x}, d)$ where $\hat{x} = (\cos\theta, \sin\theta)$ and $d = (\cos\phi, \sin\phi)$ is only known for $\hat{x}$ and d on subsets of the unit circle, i.e. we are concerned with limited aperture scattering data. In order to handle this case, we note that from the proof of Theorem 4.11 the function $g_z \in L^2[0, 2\pi]$ of this theorem is the kernel of a Herglotz wave function that approximates a solution to the Helmholtz equation in D with respect to the $H^1(D)$ norm (c.f. Theorem 4.12). Therefore to treat the case of limited aperture far field data it suffices to show that if Ω_R is a disk of radius R centered at the origin then a Herglotz wave function can be approximated in $H^1(\Omega_R)$ by a Herglotz wave function with kernel supported in a subset Γ_0 of $L^2[0, 2\pi]$. This new Herglotz wave function and its kernel can now be used in place of g_z and v_{g_z} in Theorem 4.11 where $||Fg_z - \Phi_\infty(\cdot, x)||_{L^2[0,2\pi]}$ is replaced by $||Fg_z - \Phi_\infty(\cdot, z)||_{\Gamma_1}$ where Γ_1 is a subset of $L^2[0, 2\pi]$ and $||g_z||_{L^2[0,2\pi]}$ is replaced by $||g_z||_{\Gamma_0}$. In particular, the far field equation (4.33) now becomes

$$\int_{\Gamma_0} u_\infty(\theta, \phi) g(\phi)\, d\phi = \gamma \exp\left(-ikr_z \cos(\theta - \theta_z)\right) \ , \ \theta \in \Gamma_1 \ .$$

We now proceed to prove the above approximation property [10]. Assuming that k^2 is not a Dirichlet eigenvalue for the disk Ω_R (this is not a restriction since we can always find a disk containing D that has this property), by the trace theorem it suffices to show that the set of functions

$$v_g(x) := \int_{|d|=1} g(d) e^{ikx\cdot d}\, ds(d)$$

where g is a square integrable function on the unit circle with support in some subinterval of the unit circle is complete in $H^{1/2}(\partial\Omega_R)$. With a slight abuse of notation we call this subinterval Γ_0. Hence, using duality pairing, we must show that if $\varphi \in H^{-1/2}(\partial\Omega_R)$ satisfies

$$\int_{\partial\Omega_R} \varphi(x) \left[\int_{\Gamma_0} g(d) e^{ikx\cdot d}\, ds(d) \right] ds(x) = 0$$

for every $g \in L^2(\Gamma_0)$ then $\varphi = 0$. To this end, we interchange the order of integration (which is valid for $\varphi \in H^{-1/2}(\partial\Omega_R)$ and $g \in L^2(\Gamma_0)$ since φ is a bounded linear functional on $H^{1/2}(\partial\Omega_R)$) to arrive at

$$\int_{\Gamma_0} g(d) \left[\int_{\partial\Omega_R} \varphi(x) e^{ikx\cdot d}\, ds(x) \right] ds(d) = 0$$

for every $g \in L^2(\Gamma_0)$. This in turn implies (taking conjugates) that the far field pattern $(S\bar{\varphi})_\infty$ of the single layer potential

$$(S\bar{\varphi})(y) := \int_{\partial\Omega_R} \overline{\varphi(x)} \Phi(x,y)\, ds(x) \ , \ y \in \mathbb{R}^2 \setminus \bar{\Omega}_R$$

satisfies

$$(S\bar{\varphi})_\infty(d) := \gamma \int_{\partial\Omega_R} \overline{\varphi(x)} e^{-ikx\cdot d}\, ds(x) = 0$$

for $d \in \Gamma_0$ where

$$\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \ .$$

By analyticity we can conclude that $(S\bar{\varphi})_\infty = 0$ for all vectors d on the unit circle. Arguing now as in the proof of Lemma 4.4, we can conclude that $\bar{\varphi} = 0$ and hence $\varphi = 0$. □

In conclusion, we mention that it is also possible to consider inverse scattering problems for D in a piecewise homogeneous background medium instead of only a homogeneous background [25, 28, 42]. To do this requires a knowledge of the Green's function for the piecewise homogeneous background medium. In some circumstances however, the need to know the Green's function can be avoided and for partial progress in this direction we refer the reader to [18] and [26].

5

Scattering by an Orthotropic Medium

Until now the reader has been introduced only to the scattering of time-harmonic electromagnetic waves by an imperfect conductor. We will now consider the scattering of electromagnetic waves by a penetrable orthotropic inhomogeneity imbedded in a homogeneous background. As in the previous chapter, we will confine ourselves to the scalar case that corresponds to the scattering of electromagnetic waves by an orthotropic infinite cylinder. The direct scattering problem is now modeled by a transmission problem for the Helmholtz equation outside the scatterer and an equation with non constant coefficients inside the scatterer. This chapter is devoted to the analysis of the solution to the direct problem.

After a brief discussion of the derivation of the equations that govern the scattering of electromagnetic waves by an orthotropic infinite cylinder, we proceed with the solution of the corresponding transmission problem. The integral equation method used by Piana [94] and Potthast [95] to solve the forward problem in this case is only valid under restrictive assumptions. Hence, following [55], we propose here a variational method and find a solution to the problem in a larger space than the space of twice continuously differentiable functions. In order to build the analytical frame work for this variational method, we first extend the discussion of Sobolev spaces and weak solutions initiated in Sect. 1.5 and 3.3. This is followed by a proof of the celebrated Lax-Milgram lemma and the investigation of the Dirichlet to Neumann map. Included are several simple examples of the use of variational methods for solving boundary value problems. We conclude our chapter with a solvability result for the direct problem.

5.1 Maxwell Equations for an Orthotropic Medium

We begin by considering electromagnetic waves propagating in an inhomogeneous anisotropic medium in $\mathbb{R}^3$ with electric permittivity $\epsilon = \epsilon(x)$, magnetic permeability $\mu = \mu(x)$ and electric conductivity $\sigma = \sigma(x)$. As the reader knows from Chap. 3, the electromagnetic wave is described by the electric field $\mathcal{E}$ and the magnetic field H satisfying the *Maxwell equations*

$$\operatorname{curl}\mathcal{E} + \mu\frac{\partial\mathcal{H}}{\partial t} = 0, \qquad \operatorname{curl}\mathcal{H} - \epsilon\frac{\partial\mathcal{E}}{\partial t} = \sigma\mathcal{E}.$$

For time harmonic electromagnetic waves of the form

$$\mathcal{E}(x,t) = \tilde{E}(x)e^{-i\omega t}, \qquad \mathcal{H}(x,t) = \tilde{H}(x)e^{-i\omega t}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent parts $\tilde{E}$ and $\tilde{H}$ satisfy

$$\begin{aligned}&\operatorname{curl}\tilde{E} - i\omega\mu(x)\tilde{H} = 0\\ &\operatorname{curl}\tilde{H} + (i\omega\epsilon(x) - \sigma(x))\tilde{E} = 0.\end{aligned}$$

Now let us suppose that the inhomogeneity occupies an infinitely long conducting cylinder. Let D be the cross section of this cylinder having a C^2 boundary ∂D with ν being the unit outward normal to ∂D. We assume that the axis of the cylinder coincides with the z-axis. We further assume that the conductor is imbedded in a non-conducting homogeneous background, i.e. the electric permittivity $\epsilon_0 > 0$ and the magnetic permeability $\mu_0 > 0$ of the background medium are positive constants while the conductivity $\sigma_0 = 0$. Next we define

$$\tilde{E}^{int,ext} = \frac{1}{\sqrt{\epsilon_0}}E^{int,ext}, \qquad \tilde{H}^{int,ext} = \frac{1}{\sqrt{\mu_0}}H^{int,ext}, \qquad k^2 = \epsilon_0\mu_0\omega^2,$$

$$\mathcal{A}(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right), \qquad \mathcal{N}(x) = \frac{1}{\mu_0}\mu(x)$$

where $\tilde{E}^{ext}, \tilde{H}^{ext}$ and $\tilde{E}^{int}, \tilde{H}^{int}$ denote the electric and magnetic fields in the exterior medium and inside the conductor, respectively. For an orthotropic medium we have that the matrices $\mathcal{A}$ and $\mathcal{N}$ are independent of the z-coordinate and are of the form

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & a\end{pmatrix} \qquad \mathcal{N} = \begin{pmatrix} n_{11} & n_{12} & 0\\ n_{21} & n_{22} & 0\\ 0 & 0 & n\end{pmatrix}.$$

In particular, the field E^{int}, H^{int} inside the conductor satisfies

$$\operatorname{curl}E^{int} - ik\mathcal{N}H^{int} = 0, \qquad \operatorname{curl}H^{int} + ik\mathcal{A}E^{int} = 0 \tag{5.1}$$

and the field E^{ext}, H^{ext} outside the conductor satisfies

$$\operatorname{curl}E^{ext} - ikH^{ext} = 0, \qquad \operatorname{curl}H^{ext} + ikE^{ext} = 0. \tag{5.2}$$

Across the boundary of the conductor we have the continuity of the tangential component of both the electric and magnetic fields. Assuming that $\mathcal{A}$

is invertible, and using $ikE^{int} = \mathcal{A}^{-1}\mathrm{curl}\, H^{int}$ and $ikE^{ext} = \mathrm{curl}\, H^{ext}$, the Maxwell equations become

$$\mathrm{curl}\, \mathcal{A}^{-1}\mathrm{curl}\, H^{int} - k^2 \mathcal{N} H^{int} = 0 \tag{5.3}$$

for the magnetic field inside the conductor and

$$\mathrm{curl}\,\mathrm{curl}\, H^{ext} - k^2 H^{ext} = 0 \tag{5.4}$$

for the magnetic field outside the conductor. If the scattering is due to a given time harmonic incident field E^i, H^i we have that

$$E^{ext} = E^s + E^i, \qquad H^{ext} = H^s + H^i$$

where E^s, H^s denotes the scattered field. In general the incident field E^i, H^i is an entire solution to (5.2). In particular, in the case of incident plane waves, E^i, H^i is given by (3.4). The scattered field E^s, H^s satisfies the Silver-Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0$$

uniformly in $\hat{x} = x/|x|$ and $r = |x|$.
Now let us assume that the incident wave propagates perpendicular to the axis of the cylinder and is polarized perpendicular to the axis of the cylinder such that

$$H^i(x) = (0,\, 0,\, u^i), \qquad H^s(x) = (0,\, 0,\, u^s), \qquad H^{int}(x) = (0,\, 0,\, v).$$

By elementary vector analysis it can be seen that (5.3) is equivalent to

$$\nabla \cdot A\nabla v + k^2 n v = 0 \qquad \text{in } D \tag{5.5}$$

where

$$A := \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

Analogously, (5.4) is equivalent to the Helmholtz equation

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in } \mathbb{R}^2 \setminus \overline{D}. \tag{5.6}$$

The transmission conditions $\nu \times (H^s + H^i) = \nu \times H^{int}$ and $\nu \times \mathrm{curl}\, (H^s + H^i) = \nu \times \mathcal{A}^{-1}\mathrm{curl}\, H^{int}$ on the boundary of the conductor become

$$v - u^s = u^i \quad \text{and} \quad \nu \cdot A\nabla v - \nu \cdot \nabla u^s = \nu \cdot \nabla u^i \qquad \text{on} \quad \partial D. \tag{5.7}$$

Finally, the $\mathbb{R}^2$ analogue of the Silver-Müller radiation condition is the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

which holds uniformly in $\hat{x} = x/|x|$.

Summarizing the above discussion we have that the scattering of incident time harmonic electromagnetic waves by an orthotropic cylindrical conductor is modeled by the following transmission problem in $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a nonempty, open, and bounded set having C^2 boundary ∂D such that the exterior domain $\mathbb{R}^2 \setminus \bar{D}$ is connected. The unit normal vector to ∂D, which is directed into the exterior of D, is denoted by ν. On $\bar{D}$ we have a matrix-valued function $A : \bar{D} \to \mathbb{C}^{2\times 2}$, $A = (a_{jk})_{j,k=1,2}$ with continuously differentiable functions $a_{jk} \in C^1(\bar{D})$. By $\mathrm{Re}(A)$ we mean the matrix-valued function having as entries the real parts $\mathrm{Re}(a_{jk})$, and similarly we define $\mathrm{Im}(A)$. We suppose that $\mathrm{Re}(A(x))$ and $\mathrm{Im}(A(x))$, $x \in \bar{D}$, are symmetric matrices which satisfy $\bar{\xi} \cdot \mathrm{Im}(A)\,\xi \ \le 0$ and $\bar{\xi} \cdot \mathrm{Re}(A)\,\xi \ge \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^3$ and $x \in \overline{D}$ where γ is a positive constant. Note that due to the symmetry of A, $\mathrm{Im}\left(\bar{\xi} \cdot A\,\xi\right) = \bar{\xi} \cdot \mathrm{Im}(A)\,\xi$ and $\mathrm{Re}\left(\bar{\xi} \cdot A\,\xi\right) = \bar{\xi} \cdot \mathrm{Re}(A)\,\xi$. We further assume that $n \in C(\bar{D})$ with $\mathrm{Im}(n) \ge 0$.

For functions $u \in C^1(\mathbb{R}^2 \setminus D)$ and $v \in C^1(\bar{D})$ we define the normal and conormal derivative by

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h\to +0} \nu(x) \cdot \nabla u(x + h\nu(x)), \qquad x \in \partial D$$

and

$$\frac{\partial v}{\partial \nu_A}(x) = \lim_{h\to +0} \nu(x) \cdot A(x)\nabla v(x - h\nu(x)), \qquad x \in \partial D$$

respectively. Then the scattering of a time harmonic incident field u^i by an orthotropic inhomogeneity in $\mathbb{R}^2$ can be mathematically formulated as the problem of finding v, u such that

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad \text{in} \quad D \tag{5.8}$$

$$\Delta u^s + k^2\, u^s = 0 \qquad \text{in} \quad \mathbb{R}^2 \setminus \bar{D} \tag{5.9}$$

$$v - u^s = u^i \qquad \text{on} \quad \partial D \tag{5.10}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \qquad \text{on} \quad \partial D \tag{5.11}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \tag{5.12}$$

The aim of this chapter is to establish the existence of a unique solution to the scattering problem (5.8)–(5.12). In most applications the material properties of the inhomogeneity do not change continuously to those of the background medium and hence the integral equation methods used in [94] and [95] are not applicable. Hence we will introduce a variational method to solve our problem. Since variational methods are well suited to Hilbert spaces, in the next section we reformulate our scattering problem in appropriate Sobolev spaces. To this end we need to extend the discussion on Sobolev spaces given in Sect. 1.5.

5.2 Mathematical Formulation of the Direct Scattering Problem

In the context of variational methods, one naturally seeks a solution to a linear second order elliptic boundary value problem in the space of functions that are square integrable and have square integrable first partial derivatives. Let D be an open nonempty bounded subset of $\mathbb{R}^2$ with smooth boundary ∂D. In Sect. 1.5 we have introduced the Sobolev spaces $H^1(D)$, $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$. The reader has already encountered the connection between $H^{\frac{1}{2}}(\partial D)$ and $H^1(D)$, namely $H^{\frac{1}{2}}(\partial D)$ is the trace space of $H^1(D)$. More specifically, for functions defined in $\bar{D}$ the values on the boundary are defined and the restriction of the function to the boundary ∂D is called the *trace*. The operator mapping a function onto its trace is called the *trace operator*. Theorem 1.36 states that the trace operator can be extended as a continuous mapping $\gamma_0 : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ and this extension has a continuous right inverse (see also Theorem 3.37 in [85]). The latter means that for any $f \in H^{\frac{1}{2}}(\partial D)$ there exists a $u \in H^1(D)$ such that $\gamma_0 u = f$ and $\|u\|_{H^1(D)} \leq C\|f\|_{H^{\frac{1}{2}}(\partial D)}$ where C is a positive constant independent of f.
For any integer $r \geq 0$ we let

$$C^r(D) := \{u : \partial^\alpha u \text{ exists and is continuous on } D \text{ for } |\alpha| \leq r\},$$

$$C^r(\bar{D}) := \{u|_{\bar{D}} : u \in C^r(\mathbb{R}^2)\}$$

and put

$$C^\infty(D) = \bigcap_{r\geq 0} C^r(D) \qquad C^\infty(\bar{D}) = \bigcap_{r\geq 0} C^r(\bar{D}).$$

In Sect. 1.5 $H^1(D)$ is naturally defined as the completion of $C^1(\bar{D})$ with respect to the norm

$$\|u\|^2_{H^1(D)} := \|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)}.$$

Note that $H^1(D)$ is a Hilbert space with the inner product

$$(u, v)_{H^1(D)} := (u, v)_{L^2(D)} + (\nabla u, \nabla v)_{L^2(D)}.$$

It can be shown that $C^\infty(\bar{D})$ is dense in $H^1(D)$. The proof of this result can be found in [85].
Since $H^1(D)$ is a subspace of $L^2(D)$ we can consider the *imbedding* map $\mathcal{I} : H^1(D) \to L^2(D)$ defined by $\mathcal{I}(u) = u \in L^2(D)$ for $u \in H^1(D)$. Obviously $\mathcal{I}$ is a bounded linear operator. The following two lemmas are particular cases of the well known *Rellich compactness theorem*.

Lemma 5.1. *The imbedding $\mathcal{I} : H^1(D) \to L^2(D)$ is compact.*

In the sequel we also need to consider the Sobolev space $H^2(D)$ which is the space of functions $u \in H^1(D)$ such that u_x and u_y are also in $H^1(D)$. Similarly, $H^2(D)$ can be defined as the completion of $C^2(\bar{D})$ (or $C^\infty(\bar{D})$) with respect to the norm

$$\|u\|^2_{H^2(D)} = \|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)} + \|u_{xx}\|^2_{L^2(D)} + \|u_{xy}\|^2_{L^2(D)} + \|u_{yy}\|^2_{L^2(D)}.$$

Lemma 5.2. *The imbedding $\mathcal{I} : H^2(D) \to H^1(D)$ is a compact operator.*

The proof of the Rellich compactness theorem can be found for instance in [47] or [85]. For the special case of $H^p[0, 2\pi]$ this result is proved in Theorem 1.30.

We now define

$$C_0^\infty(D) := \{u : \, u \in C_K^\infty(D) \text{ for some compact subset } K \text{ of } D\}$$

where

$$C_K^\infty(D) := \{u \in C^\infty(D) : \, \operatorname{supp} u \subseteq K\}$$

and the support of u, denoted by supp u, is the closure in D of the set $\{x \in D \, : \, u(x) \neq 0\}$. The completion of $C_0^\infty(D)$ in $H^1(D)$ is denoted by $H_0^1(D)$ and can be characterized by

$$H_0^1(D) := \{u \in H^1(D) : \, u|_{\partial D} = 0\}.$$

This space equipped with the inner product of $H^1(D)$ is also a Hilbert space. The following inequality, known as *Poincaré's inequality*, holds for functions in $H_0^1(D)$

$$\|u\|_{L^2(D)} \leq C\|\nabla u\|_{L^2(D)}$$

where the constant $C > 0$ depends only on D [63].

Remark 5.3. Our presentation of Sobolev spaces is by no means complete. A systematic treatment of Sobolev spaces requires the use of the Fourier transform and distribution theory and we refer the reader to Chap. 3 in [85] for this material.

For later use we recall the following classical result from real analysis.

Lemma 5.4. *Let G be a closed subset of $\mathbb{R}^2$. For each $\epsilon > 0$, there exists a $\chi_\epsilon \in C^\infty(\mathbb{R}^2)$ satisfying*

$$\begin{aligned} \chi_\epsilon(x) &= 1 && \text{if} \quad x \in G, \\ 0 \leq \chi_\epsilon(x) &\leq 1 && \text{if} \quad 0 < dist(x, G) < \epsilon, \\ \chi_\epsilon(x) &= 0 && \text{if} \quad dist(x, G) > \epsilon, \end{aligned}$$

where $dist(x, G)$ denotes the distance of x from G.

The function $\chi_\epsilon(x)$ defined in the above lemma is called a *cut-off function* for G. It is used to smooth out the characteristic function of a set.

Having in mind the solution of the scattering problem in Sect. 5.1, we now extend the definition of the conormal derivative $\partial u/\partial \nu_A$ to functions $u \in H^1(D, \Delta_A)$ where

$$H^1(D, \Delta_A) := \{u \in H^1(D) : \ \nabla \cdot A\nabla u \in L^2(D)\}$$

equipped with the graph norm

$$\|u\|^2_{H^1(D,\Delta_A)} := \|u\|^2_{H^1(D)} + \|\nabla \cdot A\nabla u\|^2_{L^2(D)}.$$

In particular, we have the following *trace theorem*:

Theorem 5.5. *The mapping $\gamma_1 : u \to \partial u/\partial \nu_A := \nu \cdot A\nabla u$ defined in $C^\infty(\bar{D})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_1, from $H^1(D, \Delta_A)$ to $H^{-\frac{1}{2}}(\partial D)$.*

Proof. . Let $\phi \in C^\infty(\bar{D})$ and $u \in C^\infty(\bar{D})$. The divergence theorem then becomes

$$\int_{\partial D} \phi\, \nu \cdot A\nabla u\, ds = \int_D \nabla\phi \cdot A\nabla u\, dx + \int_D \phi\, \nabla \cdot A\nabla u\, dx.$$

As $C^\infty(\bar{D})$ is dense in $H^1(D)$, this equality is still valid for $\phi \in H^1(D)$ and $u \in C^\infty(\bar{D})$. Therefore

$$\left| \int_{\partial D} \phi\, \nu \cdot A\nabla u\, ds \right| \le C\|u\|_{H^1(D,\Delta_A)}\|\phi\|_{H^1(D)} \quad \forall \phi \in H^1(D), \quad \forall u \in C^\infty(\bar{D})$$

where C is a positive constant independent of ϕ and u but depending on A and D. Now let f be an element of $H^{\frac{1}{2}}(\partial D)$. There exists a $\phi \in H^1(D)$ such that $\gamma_0\phi = f$ where γ_0 is the trace operator on ∂D. Then the above inequality implies that

$$\left| \int_{\partial D} f\, \nu \cdot A\nabla u\, ds \right| \le C\|u\|_{H^1(D,\Delta_A)}\|f\|_{H^{\frac{1}{2}}(\partial D)} \quad \forall f \in H^{\frac{1}{2}}(\partial D), \quad \forall u \in C^\infty(\bar{D}).$$

Therefore the mapping

$$f \to \int_{\partial D} f\, \nu \cdot A\nabla u\, ds \qquad f \in H^{\frac{1}{2}}(D)$$

defines a continuous linear functional and

$$\|\nu \cdot A\nabla u\|_{H^{-\frac{1}{2}}(\partial D)} \le C\|u\|_{H^1(D,\Delta_A)}.$$

Therefore the linear mapping $\gamma_1 : u \to \nu \cdot A\nabla u$ defined on $C^\infty(\bar{D})$ is continuous with respect to the norm of $H^1(D,\Delta_A)$. Since $C^\infty(\bar{D})$ is dense in $H^1(D,\Delta_A)$, γ_1 can be extended by continuity to a bounded linear mapping (still called γ_1) from $H^1(D,\Delta_A)$ to $H^{-\frac{1}{2}}(\partial D)$. □

As a consequence of the above theorem we can now extend the divergence theorem to a wider space of functions.

Corollary 5.6. *Let $u \in H^1(D)$ such that $\nabla \cdot A\nabla u \in L^2(D)$ and $v \in H^1(D)$. Then*

$$\int\limits_D \nabla v \cdot A\nabla u \, dx + \int\limits_D v \, \nabla \cdot A\nabla u \, dx = \int\limits_{\partial D} v \, \nu \cdot A\nabla u \, ds.$$

Remark 5.7. With the help of a cutoff function for a neighborhood of ∂D we can in a similar way as in Theorem 5.5 define $\partial u/\partial\nu_A$ for $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ such that $\nabla \cdot A\nabla v \in L^2_{loc}(\mathbb{R}^2 \setminus \bar{D})$ (see Sect. 3.3 for the definition of H^1_{loc}-spaces).

Remark 5.8. Setting $A = I$ in Theorem 5.5 and Corollary 5.6 we have that $\partial u/\partial\nu$ is well defined in $H^{-\frac{1}{2}}(\partial D)$ for functions $u \in H^1(D,\Delta) := \{u \in H^1(D) : \ \Delta u \in L^2(D)\}$. Furthermore the following Green's identity holds:

$$\int\limits_D \nabla v \cdot \nabla u \, dx + \int\limits_D v \, \Delta u \, dx = \int\limits_{\partial D} v \, \frac{\partial u}{\partial \nu} \, ds \qquad u \in H^1(D,\Delta), \, v \in H^1(D).$$

In particular, Theorem 3.1 and equation (3.41) are valid for H^1-solutions to the Helmholtz equation.

We are now ready to formulate the direct scattering problem for an orthotropic medium in $\mathbb{R}^2$ in suitable Sobolev spaces. Assume that A, n and D satisfy the assumptions of Sect. 5.1. Given $f \in H^{\frac{1}{2}}(\partial D)$ and $h \in H^{-\frac{1}{2}}(\partial D)$, find $u \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ and $v \in H^1(D)$ such that

$$\nabla \cdot A\nabla v + k^2 n \, v = 0 \qquad \text{in} \quad D \tag{5.13}$$

$$\Delta u + k^2 \, u = 0 \qquad \text{in} \quad \mathbb{R}^2 \setminus \bar{D} \tag{5.14}$$

$$v - u = f \qquad \text{on} \quad \partial D \tag{5.15}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{5.16}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0. \tag{5.17}$$

The scattering problem (5.8)–(5.12) is a particular case of (5.13)–(5.17). In particular the scattered field u^s and the interior field v satisfy (5.13)–(5.17) with $u = u^s$, $f = u^i|_{\partial D}$ and $h := \left.\dfrac{\partial u^i}{\partial \nu}\right|_{\partial D}$ where the incident wave u^i is such that

$$\Delta u^i + k^2 u^i = 0 \qquad \text{in } \mathbb{R}^2.$$

Note that the boundary conditions (5.15) and (5.16) are assumed in the sense of the trace operator as discussed above and u and v satisfy (5.13) and (5.14), respectively, in the weak sense. The reader has already met in Sect. 3.3 the concept of a weak solution in the context of the impedance boundary value problem for the Helmholtz equation. In the next section we provide a more systematic discussion of weak solutions and variational methods for finding weak solutions of boundary value problems.

5.3 Variational Methods

We will start this section with an important result from functional analysis namely the *Lax-Milgram lemma.* Let X be a Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\, \cdot)$.

Definition 5.9. *A mapping $a(\cdot,\, \cdot) : X \times X \to \mathbb{C}$ is called a* sesquilinear form *if*

$$\begin{aligned} a(\lambda_1 u_1 + \lambda_2 u_2, v) &= \lambda_1 a(u_1, v) + \lambda_2 a(u_2, v) \\ &\qquad \textit{for all}\ \lambda_1, \lambda_2 \in \mathbb{C},\ u_1, u_2, v \in X \\ a(u, \mu_1 v_1 + \mu_2 v_2) &= \bar{\mu}_1 a(u, v_1) + \bar{\mu}_2 a(u, v_2) \\ &\qquad \textit{for all}\ \mu_1, \mu_2 \in \mathbb{C},\ u, v_1, v_2 \in X \end{aligned}$$

with the bar denoting the complex conjugation.

Definition 5.10. *A mapping $F : X \to \mathbb{C}$ is called a* conjugate linear functional *if*

$$F(\mu_1 v_1 + \mu_2 v_2) = \bar{\mu}_1 F(v_1) + \bar{\mu}_2 F(v_2)\ \textit{for all}\ \mu_1, \mu_2 \in \mathbb{C},\ v_1, v_2 \in X.$$

As will be seen later, we will be interested in solving the following problem: *Given a conjugate linear functional $F : X \to \mathbb{C}$ and a sesquilinear form $a(\cdot,\, \cdot)$ on $X \times X$, find $u \in X$ such that*

$$a(u, v) = F(v) \qquad \textit{for all} \quad v \in X. \tag{5.18}$$

The solution to this problem is provided by:

Theorem 5.11 (Lax-Milgram Lemma). *Assume that $a : X \times X \to \mathbb{C}$ is a sesquilinear form (not necessarily symmetric) for which there exist constants $\alpha, \beta > 0$ such that*

$$|a(u, v)| \leq \alpha \|u\| \, \|v\| \qquad \textit{for all} \quad u \in X, \ v \in X \tag{5.19}$$

and

$$a(u, u) \geq \beta \|u\|^2 \qquad \textit{for all} \quad u \in X. \tag{5.20}$$

Then for every bounded conjugate linear functional $F : X \to \mathbb{C}$ there exists a unique element $u \in X$ such that

$$a(u, v) = F(v) \qquad \textit{for all} \quad v \in X. \tag{5.21}$$

Furthermore $\|u\| \leq C\|F\|$ where $C > 0$ is a constant independent of F.

Proof. For each fixed element $u \in X$, the mapping $v \to a(u, v)$ is a bounded conjugate linear functional on X and hence the Riesz representation theorem asserts the existence of a unique element $w \in X$ satisfying

$$a(u, v) = (w, v) \qquad \text{for all} \quad v \in X.$$

Thus we can define an operator $A : X \to X$ mapping u to w such that

$$a(u, v) = (Au, v) \qquad \text{for all} \quad u, v \in X.$$

1. We first claim that $A : X \to X$ is a bounded linear operator. Indeed, if $\lambda_1, \lambda_2 \in \mathbb{C}$ and $u_1, u_2 \in X$ we see by using the properties of the inner product in a Hilbert space that for each $v \in X$ we have

$$\begin{aligned}
(A(\lambda_1 u_1 + \lambda_2 u_2), v) &= a((\lambda_1 u_1 + \lambda_2 u_2), v) \\
&= \lambda_1 a(u_1, v) + \lambda_2 a(u_2, v) \\
&= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) \\
&= (\lambda_1 Au_1 + \lambda_2 Au_2, v)
\end{aligned}$$

   Since this holds for arbitrary $u_1, u_2, v \in X$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ we have established linearity. Furthermore

$$\|Au\|^2 = (Au, Au) = a(u, Au) \leq \alpha \|u\| \, \|Au\|.$$

   Consequently $\|Au\| \leq \alpha \|u\|$ for all $u \in X$ and so A is bounded.
2. Next we show that A is one to one and the range of A is equal to X. To prove this, we compute

$$\beta \|u\|^2 \leq a(u, v) = (Au, u) \leq \|Au\| \, \|u\|.$$

   Hence $\beta \|u\| \leq \|Au\|$. This inequality implies that A is one to one and the range of A is closed in X. Now let $w \in A(X)^\perp$ and observe that $\beta \|w\|^2 \leq a(w, w) = (Aw, w) = 0$ which implies that $w = 0$. Since $A(X)$ is closed we can now conclude that $A(X) = X$.

3. Next, once more from the Riesz representation theorem, there exists a unique $\tilde{w} \in X$ such that
$$F(v) = (\tilde{w}, v) \qquad \text{for all} \quad v \in X$$
and $\|\tilde{w}\| = \|F\|$. We then use part 2 of this proof to find a $u \in X$ satisfying $Au = \tilde{w}$. Then
$$a(u, v) = (Au, v) = (\tilde{w}, v) = F(v) \qquad \text{for all} \quad v \in X$$
which proves the solvability of (5.21). Furthermore, we have that
$$\|u\| \leq \frac{1}{\beta}\|Au\| = \frac{1}{\beta}\|\tilde{w}\| = \frac{1}{\beta}\|F\|.$$
4. Finally we show that there is at most one element $u \in X$ satisfying (5.21). If there exist $u \in X$ and $\tilde{u} \in X$ such that
$$a(u, v) = F(v) \quad \text{and} \quad a(\tilde{u}, v) = F(v) \qquad \text{for all} \quad v \in X$$
then
$$a(u - \tilde{u}, v) = 0 \qquad \text{for all} \quad v \in X.$$
Hence, setting $v = u - \tilde{u}$ we obtain
$$\beta\|u - \tilde{u}\|^2 \leq a(u - \tilde{u}, u - \tilde{u}) = 0$$
whence $u = \tilde{u}$.

□

Remark 5.12. If a sesquilinear form $a(\cdot,\cdot)$ satisfies (5.19) it is said that $a(\cdot,\cdot)$ is *continuous*. A sesquilinear form $a(\cdot,\cdot)$ satisfying (5.21) is called *strictly coercive*.

Example 5.13. As an example of an application of the Lax-Milgram lemma we consider the existence of a unique weak solution to the Dirichlet problem for the Poisson equation: Given $f \in H^{\frac{1}{2}}(\partial D)$ and $\rho \in L^2(D)$ find $u \in H^1(D)$ such that
$$\begin{cases} \Delta u = \rho & \text{in } D \\ u = f & \text{on } \partial D. \end{cases} \tag{5.22}$$
In order to motivate the definition of a $H^1(D)$ weak solution to the above Dirichlet problem, let us consider first $u \in C^2(D) \cap C^1(\bar{D})$ satisfying $\Delta u = \rho$. Multiplying $\Delta u = \rho$ by $\bar{v} \in C_0^\infty(D)$ and using Green's first identity we obtain
$$\int_D \nabla u \cdot \nabla \bar{v}\, dx = \int_D \rho \bar{v}\, dx \tag{5.23}$$
which makes sense for $u \in H^1(D)$ and $v \in H_0^1(D)$ as well. Note that the boundary terms disappear when we apply Green's identity due to the fact

that $v = 0$ on ∂D. Now we will use (5.23) to define a weak solution. To this end we set $X = H_0(D)$ and define

$$a(w, v) = (\nabla u, \nabla v)_{L^2(D)}, \qquad w, v \in X.$$

In particular, it is clear that

$$|a(w, v)| \leq \|\nabla w\|_{L^2(D)} \|\nabla v\|_{L^2(D)} \leq \|w\|_{H^1(D)} \|v\|_{H^1(D)}.$$

Furthermore from Poincaré's inequality there exists a constant $C > 0$ depending only on D such that

$$a(w, w) = \|\nabla w\|^2_{L^2(D)} \geq C \|w\|^2_{H^1(D)}$$

whence $a(\cdot, \cdot)$ satisfies the assumptions of the Lax-Milgram lemma.
Let now $u_0 \in H^1(D)$ be such that $u_0 = f$ on ∂D and $\|u_0\|_{H^1(D)} \leq C\|f\|_{H^{\frac{1}{2}}(\partial D)}$. If $u = f$ on ∂D then $u - u_0 \in H_0^1(D)$. Next we examine the following problem:
Find $u \in H^1(D)$ *such that*

$$\begin{cases} u - u_0 \in H_0^1(D) \\ a(u - u_0, v) = -a(u_0, v) + (\rho, v)_{L^2(D)} \qquad \text{for all } v \in H_0^1(D) \end{cases} \tag{5.24}$$

A solution of (5.24) is called a *weak solution* of the Dirichlet problem (5.22) and (5.24) is called the *variational form* of (5.22).
Since $a(\cdot, \cdot)$ is continuous, the mapping $F : v \to -a(u_0, v) + (\rho, v)_{L^2(D)}$ is a bounded conjugate linear functional on $H_0^1(D)$. Therefore, from the Lax-Milgram lemma, (5.24) has a unique solution $u \in H^1(D)$ which satisfies

$$\|u\|_{H^1(D)} \leq C(\|u_0\|_{H^1(D)} + \|\rho\|_{L^2(D)}) \leq \tilde{C}(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|\rho\|_{L^2(D)})$$

where the constant $\tilde{C} > 0$ is independent of f and ρ.
Obviously any $C^2(D) \cap C^1(\bar{D})$ solution of the Dirichlet problem is a weak solution. Conversely, if the weak solution u is smooth enough (which depends on the smoothness of ∂D, f and ρ - see [85]) then the weak solution satisfies (5.22) pointwise. Indeed, taking a function $v \in C_0^\infty(D)$ in (5.24) we see that

$$\int_D (\Delta u - \rho)\, v\, dx = 0 \qquad \text{for all } v \in C_0^\infty(D)$$

and hence $\Delta u = \rho$ almost everywhere in D. Furthermore $u - u_0 \in H_0^1(D)$ if and only if $u = u_0$ on ∂D, whence $u = f$ on ∂D.

Now we return to the abstract variational problem (5.18) and consider it in the following form: Find $u \in X$ such that

$$a(u, v) + b(u, v) = F(v) \qquad \text{for all} \quad v \in X \tag{5.25}$$

where X is a Hilbert space, $a, b : X \times X \to \mathbb{C}$ are two continuous sesquilinear forms and F is a bounded conjugate linear functional on X. In addition

1. Assume that the continuous sesqilinear form $a(\cdot,\,\cdot)$ is strictly coercive, i.e. $a_1(u,u) \geq \alpha\|u\|^2$ for some positive constant α. From the Lax-Milgram lemma we then have that there exists a bijective bounded linear operator $A : X \to X$ with bounded inverse satisfying
$$a(u,v) = (Au,v) \qquad \text{for all} \quad v \in X.$$
2. Let us denote by B the bounded linear operator from X to X defined by
$$b(u,v) = (Bu,v) \qquad \text{for all} \quad v \in X.$$
The existence and the continuity of B is guaranteed by the Riesz representation theorem (see also the first part of the proof of Lax-Milgram lemma). We further assume that the operator B is compact.
3. Finally, let $w \in X$ be such that
$$F(v) = (w,v) \qquad \text{for all} \quad v \in X$$
which is uniquely provided by the Riesz representation theorem.

Under the assumptions 1-3 (5.25) equivalently reads

$$\text{Find } u \in X \text{ such that } Au + Bu = w. \tag{5.26}$$

Theorem 5.14. *Let X and Y be two Hilbert spaces and let $A : X \to Y$ be a bijective bounded linear operator with bounded inverse $A^{-1} : Y \to X$, and $B : X \to Y$ a compact linear operator. Then $A + B$ is injective if and only if it is surjective. If $A + B$ is injective (and hence bijective) then the inverse $(A + B)^{-1} : Y \to X$ is bounded.*

Proof. Since A^{-1} exists, we have that $A+B = A(I-(-A^{-1})B)$. Furthermore, since A is a bijection, $(I - (-A^{-1})B)$ is injective and surjective if and only if $A+B$ is injective and surjective. Next we observe that $(-A^{-1})B$ is a compact operator since it is the product of a compact operator and a bounded operator. The result of the theorem now follows from Theorem 1.21 and the fact that $(A + B)^{-1} = (I - (-A^{-1})B)^{-1}A^{-1}$. □

Example 5.15. Consider now the Dirichlet problem for the Helmholtz equation in a bounded domain D: Given $f \in H^{\frac{1}{2}}(\partial D)$ find $u \in H^1(D)$ such that

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases} \tag{5.27}$$

where k is real. Following Example 5.13, we can write this problem in the following variational form: *Find $u \in H^1(D)$ such that*

$$\begin{cases} u - u_0 \in H^1_0(D) & \\ a(u - u_0, v) = -a(u_0, v) & \text{for all } v \in H^1_0(D) \end{cases} \tag{5.28}$$

where u_0 is a function in $H^1(D)$ such that $u_0 = f$ on ∂D and $\|u_0\|_{H^1(D)} \leq C\|f\|_{H^{\frac{1}{2}}(\partial D)}$, and the sesquilinear form $a(\cdot,\,\cdot)$ is defined by

$$a(w,v) := \int_D \left(\nabla w \cdot \nabla \bar{v} - k^2 w\bar{v}\right) dx, \qquad w, v \in H_0^1(D).$$

Obviously $a(\cdot,\,\cdot)$ is continuous but not strictly coercive. Defining

$$a_1(w,v) := \int_D \nabla w \cdot \nabla \bar{v}\, dx, \qquad w, v \in H_0^1(D)$$

and

$$a_2(w,v) := -k^2 \int_D w\bar{v}\, dx, \qquad w, v \in H_0^1(D)$$

we have that

$$a(w,v) = a_1(w,v) + a_2(w,v)$$

where now $a_1(\cdot,\,\cdot)$ is strictly coercive in $H_0^1(D) \times H_0^1(D)$ (see Example 5.13). Let $A : H_0^1(D) \to H_0^1(D)$ and $B : H_0^1(D) \to H_0^1(D)$ be bounded linear operators defined by $(Au, v) = a_1(u,\, v)$ and

$$(Bu, v) = \int_D u\bar{v}\, dx \qquad \text{for all} \quad v \in H_0^1(D),$$

respectively. In particular A is bounded and has a bounded inverse. We claim that $B : H_0^1(D) \to H_0^1(D)$ is compact. To prove this claim we take two bounded sequences $\{\psi_j\}, \{\phi_j\} \in H_0^1(D)$. Then by Theorem 2.17 we can extract subsequences, still denoted by $\{\psi_j\}$ and $\{\phi_j\}$, which converge weakly to ψ and ϕ in $H_0^1(D)$ respectively. Since from Lemma 5.1 the imbedding from $H_0^1(D)$ to $L^2(D)$ is compact there again exist subsequences, still denoted by $\{\psi_j\}$ and $\{\phi_j\}$, converging strongly to ψ and ϕ in $L^2(D)$ respectively, i.e. $\|\psi_j\|_{L^2(D)} \to \|\psi\|_{L^2(D)}$ and $\|\phi_j\|_{L^2(D)} \to \|\phi\|_{L^2(D)}$. Hence, from the definition of B, $B\psi_j$ is weakly convergent in $H_0^1(D)$ and $(B\psi_j,\, \phi_j) \to (B\psi,\, \phi)$. Consequently, setting $\phi_j = B\psi_j$ we obtain that $\|B\psi_j\|_{H_0^1(D)} \to \|B\psi\|_{H_0^1(D)}$. Hence we have shown that for each bounded sequence $\{\psi_j\}$ in $H_0^1(D)$, $\{B\psi_j\}$ contains a convergent subsequence which proves that B is compact.
We can now apply Theorem 5.14 to (5.28). In particular the injectivity of $A - k^2B$ implies the existence of a unique solution to (5.28). The injectivity of $A - k^2B$ is equivalent to the fact that the only function $u \in H_0^1(D)$ that satisfies

$$a(u,v) = 0 \qquad \text{for all} \quad v \in H_0^1(D)$$

is $u \equiv 0$. This is the uniqueness question for a weak solution to the Dirichlet boundary value problem for the Helmholtz equation. The values of k^2 for which there exists a nonzero function $u \in H_0^1(D)$ satisfying

$$\Delta u + k^2 u = 0 \qquad \text{in } D$$

(in the weak sense) are called the *Dirichlet eigenvalues* of $-\Delta$ and the corresponding nonzero solutions are called the *eigensolutions* for $-\Delta$. Note that the zero boundary condition is incorporated in the space $H_0^1(D)$.
Summarizing the above analysis, we have shown that if k^2 is not a Dirichlet eigenvalue for $-\Delta$ then (5.27) has a unique solution in $H^1(D)$.

Theorem 5.16. *There exists an orthonormal basis u_j for $H_0^1(D)$ consisting of eigensolutions for $-\Delta$. The corresponding eigenvalues k^2 are all positive and accumulate only at $+\infty$*

Proof. In Example 5.15 we have shown that $u \in H_0^1(D)$ satisfies

$$\Delta u + k^2 u = 0 \qquad \text{in } D$$

if and only if u is a solution to the operator equation $Au - k^2 Bu = 0$ where $A : H_0^1(D) \to H_0^1(D)$ and $B : H_0^1(D) \to H_0^1(D)$ are the bijective operator and compact operator, respectively, constructed in Example 5.15. The equation $Au - k^2 Bu = 0$ can be written as

$$\left(\frac{1}{k^2} I - A^{-1}B\right) u = 0 \qquad u \in H_0^1(D).$$

It is easily verifiable that A (and hence A^{-1}) is self-adjoint and that A^{-1} and B commute. Since B is self-adjoint we can conclude that $A^{-1}B$ is self-adjoint. Now noting that $A^{-1}B : H_0^1(D) \to H_0^1(D)$ is compact since it is a product of a compact operator and a bounded operator, the result follows from the Hilbert-Schmidt theorem. □

Remark 5.17. The results of Example 5.13 and Example 5.15 are valid as well if D is not simply connected, i.e. $\mathbb{R}^2 \setminus \bar{D}$ is not connected.

The boundary value problems arising in scattering theory are formulated in unbounded domains. In order to solve such problems by using variational techniques developed in this section, we need to write it as an equivalent problem in a bounded domain. In particular, introducing a large open disc Ω_R centered at the origin that contains $\bar{D}$, where D is the support of the scatterer, we first solve the problem in $\Omega_R \setminus \bar{D}$ (or in Ω_R in the case of transmission problems) by using variational methods. Having solved this problem, we then want to extend the solution outside Ω_R to a solution of the original problem. The main question here is what boundary condition should we impose on the artificial boundary $\partial\Omega_R$ to enable such an extension? To find the appropriate boundary conditions on $\partial\Omega_R$ we introduce the *Dirichlet to Neumann map.* We first formalize the definition of a *radiating solution* to the Helmholtz equation.

Definition 5.18. *A solution u to the Helmholtz equation whose domain of definition contains the exterior of some disk is called radiating if it satisfies the Sommerfeld radiation condition*

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0$$

where $r = |x|$ and the limit is assumed to hold uniformly in all directions $x/|x|$.

Definition 5.19. *The Dirichlet to Neumann map T is defined by*

$$T : w \to \frac{\partial w}{\partial \nu} \qquad on \quad \partial\Omega_R$$

where w is a radiating solution to the Helmholtz equation $\Delta w + k^2 w = 0$, $\partial\Omega_R$ is the boundary of some disk of radius R and ν is the outward unit normal to $\partial\Omega_R$.

Taking advantage of the fact that Ω_R is a disk, by separating variables as in Sect. 3.2 we can find a solution to the exterior Dirichlet problem outside Ω_R in the form of a series expansion involving Hankel functions. Making use of this expansion we can establish the following important properties of the Dirichlet to Neumann map.

Theorem 5.20. *The Dirichlet to Neumann map T is a bounded linear operator from $H^{\frac{1}{2}}(\partial\Omega_R)$ to $H^{-\frac{1}{2}}(\partial\Omega_R)$. Furthermore there exists a bounded operator $T_0 : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{-\frac{1}{2}}(\partial\Omega_R)$ satisfying*

$$-\int_{\partial\Omega_R} T_0 w\, \overline{w}\, ds \geq C\|w\|^2_{H^{\frac{1}{2}}(\partial\Omega_R)} \tag{5.29}$$

for some constant $C > 0$, such that $T - T_0 : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{-\frac{1}{2}}(\partial\Omega_R)$ is compact.

Proof. Let w be a radiating solution to the Helmholtz equation outside Ω_R and let (r,θ) denote polar coordinates in $\mathbb{R}^2$. Then from Sect. 3.2 we have that

$$w(r,\theta) = \sum_{-\infty}^{\infty} \alpha_n H_n^{(1)}(kr)e^{in\theta}, \quad r \geq R \text{ and } 0 \leq \theta \leq 2\pi$$

where $H_n^{(1)}(kr)$ are the Hankel functions of the first kind of order n. Hence T maps the Dirichlet data of $w|_{\partial\Omega_R}$ given by

$$w|_{\partial\Omega_R} = \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

with coefficients $a_n := \alpha_n H_n^{(1)}(kR)$ onto the corresponding Neumann data given by

$$Tw = \sum_{-\infty}^{\infty} a_n \gamma_n e^{in\theta}$$

where

$$\gamma_n := \frac{k H_n^{(1)'}(kR)}{H_n^{(1)}(kR)}, \qquad n = 0, 1, \dots .$$

The Hankel functions and their derivatives do not have real zeros since otherwise the Wronskian (3.22) would vanish. From this we observe that T is bijective. In view of the asymptotic formulas for the Hankel functions developed in Sect. 3.2 we see that

$$c_1 n \le |\gamma_n| \le c_2 n, \qquad n = \pm 1, \pm 2, \dots$$

and some constants $0 < c_1 < c_2$. From this the boundness of $T : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{-\frac{1}{2}}(\partial\Omega_R)$ is obvious since from Theorem 1.31 for $p \in \mathbb{R}$ the norm on $H^p(\partial\Omega_R)$ can be described in terms of the Fourier coefficients

$$\|w\|^2_{H^p(\partial\Omega_R)} = \sum_{-\infty}^{\infty} (1+n^2)^p |\alpha_n|^2 .$$

For the limiting operator $T_0 : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{-\frac{1}{2}}(\partial\Omega_R)$ given by

$$T_0 w = -\sum_{-\infty}^{\infty} \frac{n}{R} a_n e^{in\theta}$$

we clearly have

$$-\int_{\Omega_R} T_0 w \, \overline{w} \, ds = \sum_{-\infty}^{\infty} \frac{n}{R} |a_n|^2$$

with the integral to be understood as the duality pairing between $H^{\frac{1}{2}}(\partial\Omega_R)$ and $H^{-\frac{1}{2}}(\partial\Omega_R)$. Hence

$$-\int_{\partial\Omega_R} T_0 w \, \overline{w} \, ds \ge C \|w\|^2_{H^{\frac{1}{2}}(\partial\Omega_R)}$$

for some constant $C > 0$. Finally, from the series expansions for the Bessel and Neumann functions (see Sect. 3.2) for fixed k we derive

$$\gamma_n = -\frac{n}{R}\left\{1 + O\left(\frac{1}{n}\right)\right\}, \qquad n \to \infty .$$

This implies that $T - T_0$ is compact from $H^{\frac{1}{2}}(\partial\Omega_R)$ into $H^{-\frac{1}{2}}(\partial\Omega_R)$ since it is bounded from $H^{\frac{1}{2}}(\partial\Omega_R)$ into $H^{\frac{1}{2}}(\partial\Omega_R)$ and the imbedding from $H^{\frac{1}{2}}(\partial\Omega_R)$ into $H^{-\frac{1}{2}}(\partial\Omega_R)$ is compact by Rellich's theorem 1.30. This proves the theorem. $\square$

Example 5.21. We consider the problem of finding a weak solution to the exterior Dirichlet problem for the Helmholtz equation: Given $f \in H^{\frac{1}{2}}(\partial D)$ find $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ such that

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^2 \setminus \bar{D} \\ u = f & \text{on } \partial D \\ \lim\limits_{r\to\infty} \sqrt{r}\left(\dfrac{\partial u}{\partial r} - iku\right) = 0 & \end{cases} \tag{5.30}$$

Instead of (5.30) we solve an equivalent problem in the bounded domain $\Omega_R \setminus \bar{D}$, namely find $u \in H^1(\Omega_R \setminus \bar{D})$ such that

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_R \setminus \bar{D} \\ u = f & \text{on } \partial D \\ \dfrac{\partial u}{\partial \nu} = Tu & \text{on } \partial \Omega_R \end{cases} \tag{5.31}$$

where $f \in H^{\frac{1}{2}}(\partial D)$ is the given boundary data, T is the Dirichlet to Neumann map and Ω_R is a large disk containing $\bar{D}$.

Lemma 5.22. *The problems (5.30) and (5.31) are equivalent.*

Proof. First let $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ be a solution of (5.30). Then the restriction of u to $\Omega_R \setminus \bar{D}$ is in $H^1(\Omega_R \setminus \bar{D})$ and is a solution of (5.31). Conversely let $u \in H^1(\Omega_R \setminus \bar{D})$ be a solution to (5.31). In order to define u in all of $\mathbb{R}^2 \setminus \bar{D}$ we construct the radiating solution $\tilde{u}$ of the Helmholtz equation outside Ω_R such that $\tilde{u} = u$ on $\partial \Omega_R$. This solution can be constructed in the form of a series expansion in terms of Hankel functions in the same way as in the proof of Theorem 5.20. Hence we have that $Tu = \dfrac{\partial \tilde{u}}{\partial \nu}$. By using Green's second identity for the radiating solution $\tilde{u}$ and the fundamental solution $\Phi(x,y)$ (which is also a radiating solution) we obtain that

$$\int\limits_{\partial \Omega_R} \left[(Tu)(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu} \right] ds_y = 0, \qquad x \in \Omega_R.$$

Consequently the representation formula (3.41) (see Remark 5.8) and the fact that $\dfrac{\partial u}{\partial \nu} = Tu$ imply

$$\begin{aligned} u(x) &= \int\limits_{\partial D} \left[u(y)\frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial u}{\partial \nu}\Phi(x,y) \right] ds_y \\ &\quad - \int\limits_{\partial \Omega_R} \left[u(y)\frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial u}{\partial \nu}\Phi(x,y) \right] ds_y \\ &= \int\limits_{\partial D} \left[u(y)\frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial u}{\partial \nu}\Phi(x,y) \right] ds_y. \end{aligned}$$

Therefore u coincides with the radiating solution to the Helmhltz equation in the exterior of $\bar{D}$. Hence a solution of (5.30) can be derived from a solution of (5.31). □

Next we formulate (5.31) as a variational problem. To this end we define the Hilbert space

$$X := \{u \in H^1(\Omega_R \setminus \bar{D}) : \; u = 0 \quad \text{on} \;\; \partial D\}$$

and the sesquilinar from $a(\cdot,\, \cdot)$ by

$$a(u,v) = \int\limits_{\Omega_R \setminus \bar{D}} \left(\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v}\right)\, dx - \int\limits_{\partial \Omega_R} T u\, \overline{v}\, ds$$

which is obtained by multiplying the Helmohltz equation in (5.31) by a test function $v \in X$, integrating by parts and using the boundary condition $\partial u / \partial \nu = Tu$ on $\partial \Omega_R$ and the zero boundary condition on ∂D. Now let $u_0 \in H^1(\Omega_R \setminus \bar{D})$ be such that $u_0 = f$ on ∂D. Then the variational formulation of (5.31) reads: *Find* $u \in H^1(\Omega_R \setminus \bar{D})$ *such that*

$$\begin{cases} \qquad\qquad u - u_0 \in X \\ \\ a(u - u_0, v) = -a(u_0, v) \qquad \text{for all } v \in X. \end{cases} \tag{5.32}$$

To analyze (5.32) we define

$$a_1(w,v) = \int\limits_{\Omega_R \setminus \bar{D}} \left(\nabla w \cdot \nabla \overline{v} + w \overline{v}\right)\, dx - \int\limits_{\partial \Omega_R} T_0 w\, \overline{v}\, ds$$

and

$$a_2(w,v) = -(k^2 + 1) \int\limits_{\Omega_R \setminus \bar{D}} w \overline{v}\, dx - \int\limits_{\partial \Omega_R} (T - T_0) w\, \overline{v}\, ds$$

where T_0 is the operator defined in Theorem 5.20 and write the equation in (5.32) as

$$a_1(u - u_0, v) + a_2(u - u_0, v) = F(v), \qquad \text{for all } v \in X$$

with $F(v) := a(u_0, v)$. Since T is a bounded operator from $H^{\frac{1}{2}}(\partial \Omega_R)$ to $H^{-\frac{1}{2}}(\partial \Omega_R)$, F is a bounded conjugate linear functional on X and both $a_1(\cdot,\, \cdot)$ and $a_2(\cdot,\, \cdot)$ are continuous on $X \times X$. In addition, using (5.29), we see that

$$a_1(w,\, w) \geq C \|w\|^2_{H^1(\Omega_R \setminus \bar{D})}.$$

Note that including a L^2-inner product term in $a_1(\cdot,\, \cdot)$ is important since the Poincaré inequality does not hold in X any longer. Furthermore, due to the compact embedding of $H^1(\Omega_R \setminus \bar{D})$ into $L^2(\Omega_R \setminus \bar{D})$ and the fact that $T - T_0 : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{-\frac{1}{2}}(\partial\Omega_R)$ is compact, $a_2(\cdot, \cdot)$ gives rise to a compact operator $B : X \to X$ (see Example 5.15). Hence from Theorem 5.14 we conclude that the uniqueness of a solution to (5.31) implies the existence of a solution to (5.31) and consequently from Lemma 5.22 the existence of a weak solution to (5.30). To prove the uniqueness of a solution to (5.31) we first observe that according to Lemma 5.22 a solution to the homogeneous problem (5.31) ($f = 0$) can be extended to a solution of the homogeneous problem (5.30). Now let u be a solution to the homogeneous problem (5.30). Then Green's first identity and the boundary condition imply

$$\int_{\partial\Omega_R} \frac{\partial u}{\partial \nu}\overline{u}\, ds = \int_{\partial D} \frac{\partial u}{\partial \nu}\overline{u}\, ds + \int_{\Omega_R\setminus\bar{D}} \left(|\nabla u|^2 - k^2|u|^2\right)\, dx \tag{5.33}$$

$$= \int_{\Omega_R\setminus\bar{D}} \left(|\nabla u|^2 - k^2|u|^2\right)\, dx \tag{5.34}$$

whence

$$\mathrm{Im}\left(\int_{\partial\Omega_R} \frac{\partial u}{\partial \nu}\overline{u}\, ds\right) = 0.$$

From Theorem 3.6 we conclude that $u = 0$ in $\mathbb{R}^2 \setminus \bar{D}$ which proves the uniqueness and therefore the existence of a unique weak solution to the exterior Dirichlet problem for the Helmholtz equation. Note that in the above proof of uniqueness we have used the fact that off the boundary a $H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ solution of the Helmholtz equation is real-analytic. This can be seen from the Green's representation formula as in Theorem 3.2 which is also valid for radiating solutions to the Helmholtz equation in $H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ (see Remark 5.8).

In this section we have developed variational techniques for finding weak solutions to boundary value problems for partial differential equations. As the reader has already seen, in scattering problems the boundary conditions are typically the traces of real-analytic solutions e.g. plane waves. Hence, provided that the boundary of the scattering object is smooth, one would expect that the scattered field is in fact smooth. It can be shown that if the boundary, the boundary conditions and the coefficients of the equations are smooth enough, a weak solution is in fact C^2 inside the domain and C^1 up to the boundary. This general statement falls in the class of so called regularity results for the solutions of boundary value problems for elliptic partial differential equations. Precise formulation of such results can be found in any classical book of partial differential equations (c.f. [47] and [85]).

5.4 Solution of the Direct Scattering Problem

We now turn our attention to the main goal of this chapter, the solution of the scattering problem (5.13)–(5.17). Following Hähner [55], we shall use the variational techniques developed in Sect. 5.3 to find a solution to this problem. In order to arrive at a variational formulation of (5.13)–(5.17), we introduce a large open disk Ω_R centered at the origin containing $\bar{D}$ and consider the following problem: Given $f \in H^{\frac{1}{2}}(\partial D)$ and $h \in H^{-\frac{1}{2}}(\partial D)$, find $u \in H^1(\Omega_R \setminus \bar{D})$ and $v \in H^1(D)$ such that

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad \text{in} \quad D \tag{5.35}$$

$$\Delta u + k^2\, u = 0 \qquad \text{in} \quad \Omega_R \setminus \bar{D} \tag{5.36}$$

$$v - u = f \qquad \text{on} \quad \partial D \tag{5.37}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{5.38}$$

$$\frac{\partial u}{\partial \nu} = Tu \qquad \text{on} \quad \partial \Omega_R \tag{5.39}$$

where T is the Dirichlet to Neumann operator defined in Definition 5.19.
We note that exactly in the same way as in the proof of Lemma 5.22 one can show that a solution u, v to (5.35)–(5.39) can be extended to a solution to the scattering problem (5.13)–(5.17) and conversely a solution u, v to the scattering problem (5.13)–(5.17) is such that v and u restricted to $\Omega_R \setminus \bar{D}$ solves (5.35)–(5.39).
Next let $u_f \in H^1(\Omega_R \setminus \bar{D})$ be the unique solution to the following Dirichlet boundary value problem:

$$\Delta u_f + k^2 u_f = 0 \quad \text{in } \Omega_R \setminus \bar{D}, \qquad u_f = f \quad \text{on } \partial D, \qquad u_f = 0 \quad \text{on } \partial \Omega_R.$$

The existence of a unique solution to this problem is shown in Example 5.15 (see also Remark 5.17). Note that we can always choose Ω_R such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in $\Omega_R \setminus \bar{D}$. An equivalent variational formulation of (5.35)–(5.39) is: Find $w \in H^1(\Omega_R)$ such that

$$\int\limits_D \left(\nabla\overline{\phi} \cdot A\nabla w - k^2 n \overline{\phi} w\right) dx + \int\limits_{\Omega_R \setminus \bar{D}} \left(\nabla\overline{\phi} \cdot \nabla w - k^2 \overline{\phi} w\right) dx \tag{5.40}$$

$$- \int\limits_{\partial \Omega_R} \overline{\phi}\, Tw\, ds = \int\limits_{\partial D} \overline{\phi} h\, ds - \int\limits_{\partial \Omega_R} \overline{\phi}\, Tu_f\, ds + \int\limits_{\Omega_R \setminus \bar{D}} \left(\nabla\overline{\phi} \cdot \nabla u_f - k^2 \overline{\phi} u_f\right) dx$$

for all $\phi \in H^1(\Omega_R)$. With the help of Green's first identity (see Corollary 5.6 and Remark 5.8) it is easy to see that $v := w|_D$ and $u := w|_{\Omega_R \setminus \bar{D}} - u_f$ satisfy (5.35)–(5.39). Conversely, multiplying the equations in (5.35)–(5.39) by a test function and using the transmission conditions one can show that $w = u$ in D and $w = u + u_f$ in $\Omega_R \setminus \bar{D}$ is such that $w \in H^1(\Omega_R)$ and satisfies (5.40),

where v, u is a solution of (5.35)–(5.39).
Next we define the following continuous sesquilinear forms on $H^1(\Omega_R) \times H^1(\Omega_R)$:

$$a_1(\psi,\phi) := \int\limits_D \left(\nabla\overline{\phi}\cdot A\nabla\psi + \overline{\phi}\,\psi\right)\,dx + \int\limits_{\Omega_R\setminus\bar{D}} \left(\nabla\overline{\phi}\cdot\nabla\psi + \overline{\phi}\,\psi\right)dx$$
$$- \int\limits_{\partial\Omega_R} \overline{\phi}\,T_0\psi\,ds \qquad\qquad \phi,\psi\in H^1(\Omega_R)$$

and

$$a_2(\psi,\phi) := -\int\limits_D (nk^2+1)\overline{\phi}\psi\,dx - \int\limits_{\Omega_R\setminus\bar{D}} (k^2+1)\overline{\phi}\,\psi\,dx$$
$$- \int\limits_{\partial\Omega_R} \overline{\phi}\,(T-T_0)\psi\,ds \qquad\qquad \phi,\psi\in H^1(\Omega_R)$$

where the operator T_0 is the operator defined in Theorem 5.20. Furthermore, we define the bounded conjugate linear functional F on $H^1(\Omega_R)$ by

$$F(\phi) := \int\limits_{\partial D} \overline{\phi}h\,ds - \int\limits_{\partial\Omega_R} \overline{\phi}\,Tu_f\,ds + \int\limits_{\Omega_R\setminus\bar{D}} \left(\nabla\overline{\phi}\cdot\nabla u_f - k^2\overline{\phi}u_f\right)dx.$$

Then (5.40) can be written as the problem of finding $w\in H^1(\Omega_R)$ such that

$$a_1(w,\phi)+a_2(w,\phi)=f(\phi) \qquad \text{for all } \phi\in H^1(\Omega_R).$$

From the assumption $\bar{\xi}\cdot\mathrm{Re}(A)\,\xi \ge \gamma|\xi|^2$, for all $\xi\in\mathbb{C}^3$ and $x\in\overline{D}$ and (5.29), we can conclude that the sesquilinear form $a_1(\cdot,\cdot)$ is strictly coercive. Hence as a consequence of the Lax-Milgram lemma the operator $A : H^1(\Omega_R)\to H^1(\Omega_R)$ defined by $a_1(w,\phi) = (Aw,\phi)_{H^1(\Omega_R)}$ is invertible with bounded inverse. Furthermore, due to the compact imbedding of $H^1(\Omega_R)$ into $L^2(\Omega_R)$ and the fact that $T-T_0 : H^{\frac{1}{2}}(\partial\Omega_R)\to H^{-\frac{1}{2}}(\partial\Omega_R)$ is compact (see Theorem 5.20), we can show exactly in the same way as in Example 5.15 that the operator $B : H^1(\Omega_R)\to H^1(\Omega_R)$ defined by $a_2(w,\phi) = (Bw,\phi)_{H^1(\Omega_R)}$ is compact. Finally, by Theorem 5.14, the uniqueness of a solution to (5.35)–(5.39) implies that a solution exists.

Lemma 5.23. *The problems (5.35)–(5.39) and (5.13)–(5.17) have at most one solution.*

Proof. According to our previous remarks, a solution to the homogeneous problem (5.35)–(5.39) ($f=h=0$) can be extended to a solution $v\in H^1(D)$ and $u\in H^1_{loc}(\mathbb{R}^2\setminus\bar{D})$ to the homogeneous problem (5.13)–(5.17). Therefore it suffices to prove uniqueness for (5.13)–(5.17). Green's first identity and the transmission conditions imply that

$$\begin{aligned}\int\limits_{\partial\Omega_R} \overline{u}\frac{\partial u}{\partial\nu}\,ds &= \int\limits_{\partial D} \overline{u}\frac{\partial u}{\partial\nu}\,ds + \int\limits_{\Omega_R\setminus\overline{D}} \left(|\nabla u|^2 - k^2|u|^2\right)^2\,dx\\ &= \int\limits_{D} \left(\nabla\overline{v}\cdot A\nabla v - k^2 n|v|^2\right)^2\,dx + \int\limits_{\Omega_R\setminus\overline{D}} \left(|\nabla u|^2 - k^2|u|^2\right)^2\,dx.\end{aligned}$$

Now since $\bar{\xi}\cdot\mathrm{Im}(A)\,\xi \leq 0$ for all $\xi\in\mathbb{C}^2$ and $\mathrm{Im}(n)>0$ for $x\in\overline{D}$ we conclude that

$$\mathrm{Im}\left(\int\limits_{\partial\Omega_R} \overline{u}\frac{\partial u}{\partial\nu}\,ds\right) \leq 0,$$

which from Theorem 3.6 implies that $u=0$ in $\mathbb{R}^2\setminus\bar{D}$. From the transmission conditions we can now conclude that $v=0$ and $\partial v/\partial\nu_A=0$ on ∂D.
In order to conclude that $v=0$ in D we employ a unique continuation principle. To this end we extend $\mathrm{Re}(A)$ to a real, symmetric, positive definite, and continuously differentiable matrix-valued function in $\overline{\Omega}_R$ and $\mathrm{Im}(A)$ to a real, symmetric, continuously differentiable matrix-valued function which is compactly supported in Ω_R. We also choose a continuously differentiable extension of n into $\overline{\Omega}_R$ and define $v=0$ in $\Omega_R\setminus\bar{D}$. Since $v=0$ and $\partial v/\partial\nu_A=0$ on ∂D then $v\in H^1(\Omega_R)$ and satisfies $\nabla\cdot A\nabla v+k^2 nv=0$ in Ω_R. Then by the regularity result in the interior of Ω_R (see Theorem 5.25), v is smooth enough to apply the unique continuation principle (see Theorem 17.2.6 in [59]). In particular since $v=0$ in $\Omega_R\setminus\bar{D}$ then $v=0$ in Ω_R. This proves the uniqueness. □

Summarizing the above analysis, we have proved the following theorem on the existence, uniqueness and continuous dependence on the data of a solution to the direct scattering problem for an orthotropic medium in $\mathbb{R}^2$.

Theorem 5.24. *Assume that D, A and n satisfy the assumptions in Sect. 5.1 and let $f\in H^{\frac{1}{2}}(\partial D)$ and $h\in H^{-\frac{1}{2}}(\partial D)$ be given. Then the transmission problem (5.13)–(5.17) has a unique solution $v\in H^1(D)$ and $u\in H^1_{loc}(\mathbb{R}^2\setminus\bar{D})$ which satisfy*

$$\|v\|_{H^1(D)} + \|u\|_{H^1(\Omega_R\setminus\bar{D})} \leq C\left(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\right) \tag{5.41}$$

with $C>0$ a positive constant independent of f and h.

Note that the a priori estimate (5.41) is obtained using the fact that by a duality argument $\|F\|$ is bounded by $\|h\|_{H^{-\frac{1}{2}}(\partial D)}$ and $\|u_f\|_{H^1(\Omega_R\setminus\bar{D})}$ which in turn is bounded by $\|f\|_{H^{\frac{1}{2}}(\partial D)}$ (see Example 5.15).

We end this section by stating two regularity results from the general theory of partial differential equations formulated for our transmission problem. The proofs of these results are rather technical and beyond the scope of this book.

Let D_1 and D_2 be bounded, open subsets of $\mathbb{R}^2$ such that $\bar{D}_1 \subset D_2$ and assume that A is a matrix-valued function with continuously differentiable entrees $a_{jk} \in C^1(\bar{D}_2)$ and $n \in C^1(\bar{D}_2)$. Furthermore, suppose that A is symmetric and satisfies $\bar{\xi} \cdot \mathrm{Re}(A)\, \xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^3$ and $x \in \overline{D}_2$ for some constant $\gamma > 0$.

Theorem 5.25. *If $u \in H^1(D_2)$ and $q \in L^2(D_2)$ satisfy*

$$\nabla \cdot A\nabla u + k^2 n u = q$$

then $u \in H^2(D_1)$ and

$$\|u\|_{H^2(D_1)} \leq C \left(\|u\|_{H^1(D_2)} + \|q\|_{L^2(D_2)}\right)$$

where $C > 0$ depends only on γ, D_1 and D_2.

For a proof of this theorem in a more general formulation see Theorem 4.16 in [85] or Theorem 15.1 in [45]. Note also that a more general interior regularity theorem shows that if the entries of A and n are smoother then C^1 and q is smoother then L^2 then one can improve the regularity of u and this eventually leads to a C^2 solution in the interior of D_2.

For later use, in the next theorem we state a local boundary regularity result for the solution of the transmission problem (5.13)–(5.17). By $\Omega_\epsilon(z)$ we denote an open ball centered at $z \in \mathbb{R}^2$ of radius ϵ.

Theorem 5.26. *Assume $z \in \partial D$, and let $u^i \in H^1(D)$ such that $\Delta u^i \in L^2(D)$. Define $f := u^i$ and $h := \partial u^i/\partial \nu$ on ∂D .*

1. *If for some $\epsilon > 0$ the incident wave u^i is also defined in $\Omega_{2\epsilon}(z)$ and the restriction of u^i to $\Omega_{2\epsilon}(z)$ is in $H^2(\Omega_{2\epsilon}(z))$ then the solution u to (5.13)–(5.17) satisfies $u \in H^2((\mathbb{R}^2 \setminus \overline{D}) \cap \Omega_\epsilon(z))$ and there is a positive constant C such that*

$$\|u\|_{H^2((\mathbb{R}^2\setminus\overline{D})\cap\Omega_\epsilon(z))} \leq C \left(\|u^i\|_{H^2(\Omega_{2\epsilon}(z))} + \|u^i\|_{H^1(D)}\right).$$

2. *If for some $\epsilon > 0$ the incident wave u^i is also defined in $\Omega_R \setminus \Omega_\epsilon(z)$ and the restriction of u^i to $\Omega_R \setminus \Omega_\epsilon(z)$ is in $H^2(\Omega_R \setminus \Omega_\epsilon(z))$ then the solution u to (5.13)–(5.17) satisfies $u \in H^2(\mathbb{R}^2 \setminus (\overline{D} \cup \Omega_{2\epsilon}(z)))$ and there is a positive constant C such that*

$$\|u\|_{H^2(\mathbb{R}^2\setminus(\overline{D}\cup\Omega_{2\epsilon}(z)))} \leq C \left(\|u^i\|_{H^2(\Omega_R\setminus\Omega_\epsilon(z))} + \|u^i\|_{H^1(D)}\right).$$

This result is proved in Theorem 2 in [55]. The proof employs the interior regularity result stated in Theorem 5.25 and techniques from Theorem 8.8 in [47].

6

The Inverse Scattering Problem for an Orthotropic Medium

In this chapter we extend the results of Chap. 4 to the case of the inverse scattering problem for an inhomogeneous orthotropic medium. The inverse problem we shall consider in this chapter is to determine the *support* of the orthotropic inhomogeneity given the far field pattern of the scattered field for many incident directions.

The investigation of the inverse problem is based on the analysis of a non-standard boundary value problem called the *interior transmission problem.* This problem plays the same role for the inhomogeneous medium problem as the interior impedance problem plays in the solution of the inverse problem for an imperfect conductor studied in Chap. 4. Having discussed the well-posedness of the interior transmission problem and the countability of the transmission eigenvalues following [13], we proceed with a uniqueness result for the inverse problem. We will present here a proof due to Hähner [55] which is based on the use of a regularity result for the solution of the interior transmission problem. Finally, in the last section of this chapter, we derive the linear sampling method for finding an approximation to the support of the inhomogeneity. Although the analysis of the justification of the linear sampling method refers to the scattering problem for an orthotropic medium, the implementation of the method does not rely on any a priori knowledge of the physical properties of the scattering object. In particular, we show that the far field equation we used in Chap. 4 to determine the shape of an imperfect conductor can also be used in the present case where the corresponding far field pattern is used for the kernel of this equation.

6.1 Formulation of the Inverse Problem

Let D be the support and A and n the constitutive parameters of a bounded orthotropic inhomogeneous medium in $\mathbb{R}^2$ where D, A and n satisfy the assumptions given in Sect. 5.1. The scattering of a time harmonic incident plane wave $u^i := e^{ikx \cdot d}$ by the inhomogeneity D is described by the transmission problem (5.13)–(5.17) with $f := e^{ikx\cdot d}$ and $h := \partial e^{ikx\cdot d}/\partial\nu$ which we recall here for reader's convenience:

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad \text{in} \quad D \tag{6.1}$$

$$\Delta u^s + k^2\, u^s = 0 \qquad \text{in} \quad \mathbb{R}^2 \setminus \bar{D} \tag{6.2}$$

$$v - u^s = e^{ikx\cdot d} \qquad \text{on} \quad \partial D \tag{6.3}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial e^{ikx\cdot d}}{\partial \nu} \qquad \text{on} \quad \partial D \tag{6.4}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{6.5}$$

where $k > 0$ is the (fixed) wave number, $d := (\cos\phi, \sin\phi)$ is the incident direction, $x = (x_1, x_2) \in \mathbb{R}^2$ and $r = |x|$. In particular, the interior field $v(\cdot) := v(\cdot, \phi)$ and scattered field $u^s(\cdot) := u^s(\cdot, \phi)$ depend on the incident angle ϕ. The radiating scattered field u^s again has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\theta, \phi) + O(r^{-3/2}), \qquad r \to \infty$$

where the function $u_\infty(\cdot, \phi)$ defined on $[0, 2\pi]$ is the *far field pattern* corresponding to the scattering problem (6.1)–(6.5) and the unit vector $\hat{x} := (\cos\theta, \sin\theta)$ is the observation direction. In the same way as in Theorem 4.2 it can be shown that the far field pattern $u_\infty(\theta, \phi)$ corresponding to (6.1)–(6.5) satisfies the reciprocity relation $u_\infty(\theta, \phi) = u_\infty(\phi+\pi, \theta+\pi)$ and is given by

$$u_\infty(\theta, \phi) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int\limits_{\partial B} \left(u^s(y) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu} - e^{-ik\hat{x}\cdot y} \frac{\partial u^s(y)}{\partial \nu} \right) ds(y) \tag{6.6}$$

where ∂B is the boundary of a bounded domain containing D (it can also be ∂D).

The following result can be obtained as a consequence of Rellich's lemma (see Theorem 4.1):

Theorem 6.1. *Suppose that the far field pattern u_∞ corresponding to (6.1)–(6.5) satisfies $u_\infty = 0$ for a fixed angle ϕ and all θ in $[0, 2\pi]$. Then $u^s = 0$ in $\mathbb{R}^2 \setminus \bar{D}$.*

Note that by the analyticity of the far field pattern Theorem 6.1 holds if $u_\infty = 0$ only for a subinterval of $[0, 2\pi]$.

The *inverse scattering problem* we are concerned with is to determine D from a knowledge of the far field pattern $u_\infty(\theta, \phi)$ for all incident angles $\phi \in [0, 2\pi]$ and all observation angles $\theta \in [0, 2\pi]$. We remark that for an orthotropic medium standard examples [51, 94] show that A and n are not in fact uniquely determined from the far field pattern $u_\infty(\theta, \phi)$ for all $\phi \in [0, 2\pi]$

and $\theta \in [0,\, 2\pi]$, but rather what is possible to determine is the support of the inhomogeneity D.

We now consider the *far field operator* $F : L^2[0,\, 2\pi] \to L^2[0,\, 2\pi]$ corresponding to (6.1)–(6.5) defined by

$$(Fg)(\theta) := \int_0^{2\pi} u_\infty(\theta, \phi) g(\phi) d\phi \,. \tag{6.7}$$

As the reader has already seen in Chap. 4, the far field operator will play a central role in the solution of the inverse problem. The first problem to resolve is that of injectivity and the denseness of the range of the far field operator. We recall a Herglotz function with kernel $g \in L^2[0,\, 2\pi]$ is given by

$$v_g(x) := \int_0^{2\pi} e^{ikx\cdot d} g(\phi)\, d\phi \tag{6.8}$$

where $d = (\cos\phi, \sin\phi)$. Note that by superposition, Fg is the far field pattern of the solution to (6.1)–(6.5) with $e^{ikx\cdot d}$ replaced by v_g. For future reference we note that

$$\tilde{v}_g(x) := \int_0^{2\pi} e^{-ikx\cdot d} g(\phi)\, d\phi \tag{6.9}$$

is also a Herglotz wave function with kernel $g(\phi - \pi)$.

Theorem 6.2. *The far field operator F corresponding to the scattering problem (6.1)–(6.5) is injective with dense range if and only if there does not exist a Herglotz wave function v_g such that the pair v, v_g is a solution to*

$$\nabla \cdot A \nabla v + k^2 n\, v = 0 \quad \textit{and} \quad \Delta v_g + k^2\, v_g = 0 \qquad \textit{in} \quad D \tag{6.10}$$

$$v = v_g \qquad \textit{and} \quad \frac{\partial v}{\partial \nu_A} = \frac{\partial v_g}{\partial \nu} \qquad \textit{on} \quad \partial D \tag{6.11}$$

Proof. Exactly in the same way as in Theorem 4.3, one can show that the far field operator F is injective if and only if its adjoint operator F^* is injective. Since $\mathrm{N}(F^*)^\perp = \overline{F(L^2[0,\, 2\pi])}$, to prove the theorem we must only show that F is injective. But $Fg = 0$ with $g \neq 0$ is equivalent to the existence of a nonzero Herglotz wave function v_g with kernel g for which the far field pattern u_∞ corresponding to (6.1)–(6.5) with $e^{ikx\cdot d}$ replaced by v_g vanishes. By Rellich's lemma we have that $u^s = 0$ in $\mathbb{R}^2 \setminus \bar{D}$ and hence the transmission conditions imply that

$$v = v_g \qquad \text{and} \qquad \frac{\partial v}{\partial \nu_A} = \frac{\partial v_g}{\partial \nu} \qquad \text{on} \quad \partial D.$$

Since v_g is a solution of the Helmholtz equation we have that v and v_g satisfy (6.10) as well. This proves the theorem. $\square$

Motivated by Theorem 6.2 we now define the *interior transmission problem* associated with the transmission problem (5.13)–(5.17).

Interior Transmission Problem. *Given* $f \in H^{\frac{1}{2}}(\partial D)$ *and* $h \in H^{-\frac{1}{2}}(\partial D)$, *find two functions* $v \in H^1(D)$ *and* $w \in H^1(D)$ *satisfying*

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad in \quad D \tag{6.12}$$

$$\Delta w + k^2\, w = 0 \qquad in \quad D \tag{6.13}$$

$$v - w = f \qquad on \quad \partial D \tag{6.14}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial w}{\partial \nu} = h \qquad on \quad \partial D \tag{6.15}$$

The boundary value problem (6.12)–(6.13) with $f = 0$ and $h = 0$ is called the *homogeneous interior transmission problem.*

Definition 6.3. *Values of* k^2 *for which the homogeneous interior transmission problem has a nontrivial solution are called transmission eigenvalues.*

In particular Theorem 6.2 states that if k^2 is not a transmission eigenvalue then the range of the far field operator is dense.

6.2 The Interior Transmission Problem

As seen above, the interior transmission problem appears naturally in scattering problems for an inhomogeneous medium. Our approach to studying the interior transmission problem is based on [13] and [19]. Of particular concern to us in this section is the countability of the transmission eigenvalues. For more information on the interior transmission problem and other solution approaches we refer the reader to Chap. 8 in [33] and [103] which deal with (6.12)–(6.15) when $A = I$ (which is excluded in our analysis).

We begin by establishing the uniqueness of a solution to the interior transmission problem.

Theorem 6.4. *If either* $Im(n) > 0$ *or* $Im\left(\bar{\xi} \cdot A\,\xi\right) < 0$ *at a point* $x_0 \in D$, *then the interior transmission problem (6.12)–(6.15) has at most one solution.*

Proof. Let v and w be a solution of the homogeneous interior transmission problem (i.e. $f = h = 0$). Applying the divergence theorem to $\overline{v}$ and $A\nabla v$ (see Corollary 5.6), using the boundary condition and applying Green's first identity to $\overline{w}$ and w (see Remark 5.8) we obtain

$$\int_D \nabla\overline{v} \cdot A\nabla v \, dy - \int_D k^2 n|v|^2 \, dy = \int_{\partial D} \overline{v} \cdot \frac{\partial v}{\partial \nu_A} \, dy = \int_D |\nabla w|^2 \, dy - \int_D k^2 |w|^2 \, dy.$$

Hence

$$\operatorname{Im}\left(\int_D \nabla\bar{v}\cdot A\nabla v\,dy\right)=0 \quad\text{and}\quad \operatorname{Im}\left(\int_D n|v|^2\,dy\right)=0\,. \tag{6.16}$$

If $\operatorname{Im}(n)>0$ at a point $x_0\in D$, and hence by continuity in a small disk $\Omega_\epsilon(x_0)$, then the second equality of (6.16) and the unique continuation principle (Theorem 17.2.6 in [59]) imply that $v\equiv 0$ in D. In the case when $\operatorname{Im}\left(\bar{\xi}\cdot A\,\xi\right)<0$ at a point $x_0\in D$ for all $\xi\in\mathbb{C}^2$, and hence by continuity in a small ball $\Omega_\epsilon(x_0)$, from the first equality of (6.16) we obtain that $\nabla v\equiv 0$ in $\Omega_\epsilon(x_0)$ and from (6.12) $v\equiv 0$ in $\Omega_\epsilon(x_0)$, whence again from the unique continuation principle $v\equiv 0$ in D. From the boundary conditions (6.13) and (6.14), and the integral representation formula, w also vanishes in D. □

We now proceed to the solvability of the interior transmission problem. In the following analysis we assume without loss of generality that D is simply connected. We first study an intermediate problem called the *modified interior transmission problem*, which turns out to be a compact perturbation of our original transmission problem.

The modified interior transmission problem is given $f\in H^{\frac{1}{2}}(\partial D)$, $h\in H^{-\frac{1}{2}}(\partial D)$, a real valued function $m\in C(\bar{D})$, and two functions $\rho_1\in L^2(D)$ and $\rho_2\in L^2(D)$ find $v\in H^1(D)$ and $w\in H^1(D)$ satisfying

$$\nabla\cdot A\nabla v-m\,v=\rho_1 \qquad \text{in}\quad D \tag{6.17}$$

$$\Delta w-w=\rho_2 \qquad \text{in}\quad D \tag{6.18}$$

$$v-w=f \qquad \text{on}\quad \partial D \tag{6.19}$$

$$\frac{\partial v}{\partial\nu_A}-\frac{\partial w}{\partial\nu}=h \qquad \text{on}\quad \partial D. \tag{6.20}$$

We now reformulate (6.17)–(6.20) as an equivalent variational problem of the form (5.18). To this end, we define the Hilbert space

$$W(D):=\left\{\mathbf{w}\in\left(L^2(D)\right)^2:\nabla\cdot\mathbf{w}\in L^2(D)\quad\text{and}\quad\nabla\times\mathbf{w}=0\right\}$$

equipped with the inner product

$$(\mathbf{w}_1,\,\mathbf{w}_2)_W=(\mathbf{w}_1,\,\mathbf{w}_2)_{L^2(D)}+(\nabla\cdot\mathbf{w}_1,\,\nabla\cdot\mathbf{w}_2)_{L^2(D)}$$

and the norm

$$\|\mathbf{w}\|_W^2=\|\mathbf{w}\|_{L^2(D)}^2+\|\nabla\cdot\mathbf{w}\|_{L^2(D)}^2.$$

We denote by $\langle\cdot,\,\cdot\rangle$ the duality pairing between $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$. The duality pairing

$$\langle\varphi,\,\boldsymbol{\psi}\cdot\nu\rangle=\int_D\varphi\,\nabla\cdot\boldsymbol{\psi}\,dx+\int_D\nabla\varphi\cdot\boldsymbol{\psi}\,dx, \tag{6.21}$$

for $(\varphi, \boldsymbol{\psi}) \in H^1(D) \times W(D)$ will be of particular interest in the sequel.
We next introduce the sesquilinear form $\mathcal{A}$ defined on $\{H^1(D) \times W(D)\}^2$ by

$$\mathcal{A}(U,V) = \int_D A\nabla v \cdot \nabla \bar{\varphi}\, dx + \int_D m\, v\, \bar{\varphi}\, dx + \int_D \nabla \cdot \mathbf{w}\, \nabla \cdot \bar{\boldsymbol{\psi}}\, dx + \int_D \mathbf{w} \cdot \bar{\boldsymbol{\psi}}\, dx \\ - \left\langle v,\, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle - \langle \bar{\varphi},\, \mathbf{w} \cdot \nu \rangle \tag{6.22}$$

where $U := (v, \mathbf{w})$ and $V := (\varphi, \boldsymbol{\psi})$ are in $H^1(D) \times W(D)$. We denote by $L : H^1(D) \times W(D) \to \mathbb{C}$ the bounded conjugate linear functional given by

$$L(V) = \int_D (\rho_1\, \bar{\varphi} + \rho_2\, \nabla \cdot \bar{\boldsymbol{\psi}})\, dx + \langle \bar{\varphi},\, h \rangle - \left\langle f,\, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle\ . \tag{6.23}$$

Then the variational formulation of the problem (6.17)–(6.20) is to find $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$ such that

$$\mathcal{A}(U,V) = L(V), \qquad \text{for all} \quad V \in H^1(D) \times W(D)\ . \tag{6.24}$$

The following theorem proves the equivalence between problems (6.17)–(6.20) and (6.24).

Theorem 6.5. *The problem (6.17)–(6.20) has a unique solution $(v, w) \in H^1(D) \times H^1(D)$ if and only if the problem (6.24) has a unique solution $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$. Moreover if (v, w) is the unique solution to (6.17)–(6.20) then $U = (v, \nabla w)$ is the unique solution to (6.24). Conversely, if $U = (v, \mathbf{w})$ is the unique solution to (6.24) then the unique solution (v, w) to (6.17)–(6.20) is such that $\mathbf{w} = \nabla w$.*

Proof. We first prove the equivalence between the existence of a solution (v, w) to (6.17)–(6.20) and the existence of a solution $U = (v, \mathbf{w})$ to (6.24).

1. Assume that (v, w) is a solution to (6.17)–(6.20) and set $\mathbf{w} = \nabla w$. From (6.18) we see that since $\nabla \mathbf{w} = w + \rho_2 \in L^2(D)$ then $\mathbf{w} \in W(D)$. Taking the L^2 scalar product of (6.18) with $\nabla \cdot \boldsymbol{\psi}$ for some $\boldsymbol{\psi} \in W(D)$ and using (6.21) we see that

$$\int_D \nabla \cdot \mathbf{w}\, \nabla \cdot \bar{\boldsymbol{\psi}}\, dx + \int_D \mathbf{w} \cdot \bar{\boldsymbol{\psi}}\, dx - \left\langle w,\, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle = \int_D \rho_2\, \nabla \cdot \bar{\boldsymbol{\psi}}\, dx.$$

Hence, by (6.19)

$$\int_D \nabla \cdot \mathbf{w}\, \nabla \cdot \bar{\boldsymbol{\psi}}\, dx + \int_D \mathbf{w} \cdot \bar{\boldsymbol{\psi}}\, dx - \left\langle v,\, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle \\ = - \left\langle f,\, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle + \int_D \rho_2\, \nabla \cdot \bar{\boldsymbol{\psi}}\, dx\ . \tag{6.25}$$

We now take the L^2 scalar product of (6.17) with φ in $H^1(D)$ and integrate by parts. Using the boundary condition (6.20) we see that

$$\int_D A\nabla v\cdot\nabla\bar\varphi\,dx+\int_D m\,v\,\bar\varphi\,dx-\langle\bar\varphi,\,\mathbf{w}\cdot\nu\rangle=\langle\bar\varphi,\,h\rangle+\int_D \rho_1\,\bar\varphi\,dx\,. \tag{6.26}$$

Finally, adding (6.25) and (6.26) we have that $U=(v,\nabla w)$ is a solution to (6.24).

2. Now assume that $U=(v,\mathbf{w})\in H^1(D)\times W(D)$ is a solution to (6.24). Since $\nabla\times\mathbf{w}=0$ and D is simply connected we deduce the existence of a function $w\in H^1(D)$ such that $\mathbf{w}=\nabla w$ where w is determined up to an additive constant. As we shall see later, this constant can be adjusted so that (v,w) is a solution to (6.17)–(6.20). Obviously, if U satisfies (6.24) then $(v,\mathbf{w})$ satisfies (6.25) and (6.26) for all $(\varphi,\boldsymbol{\psi})\in H^1(D)\times W(D)$. One can easily see from (6.26) that the pair (v,w) satisfies

$$\nabla\cdot A\nabla v-m\,v=\rho_1\qquad\text{in}\quad D \tag{6.27}$$

$$\frac{\partial v}{\partial\nu_A}-\frac{\partial w}{\partial\nu}=h\qquad\text{on}\quad\partial D. \tag{6.28}$$

On the other hand, substituting for $\mathbf{w}$ in (6.25) and using the duality identity (6.21) in the second integral we have that

$$\int_D(\Delta w-w)\,\nabla\cdot\bar{\boldsymbol{\psi}}\,dx+\big\langle w-v,\,\bar{\boldsymbol{\psi}}\cdot\nu\big\rangle \tag{6.29}$$

$$=-\big\langle f,\,\bar{\boldsymbol{\psi}}\cdot\nu\big\rangle+\int_D\rho_2\,\nabla\cdot\bar{\boldsymbol{\psi}}\,dx$$

for all $\boldsymbol{\psi}$ in $W(D)$.

Now consider a function $\phi\in L^2_0(D)=\left\{\phi\in L^2(D)\;:\;\int_D\phi\,dx=0\right\}$ and let $\chi\in H^1(D)$ be a solution to

$$\begin{cases}\Delta\chi=\bar\phi\;\;\text{in } D\\ \dfrac{\partial\chi}{\partial\nu}=0\;\;\text{on }\partial D.\end{cases} \tag{6.30}$$

The existence of a solution of the above Neumann boundary value problem can be established by the variational methods developed in Chap. 5 (see Example 5.13). We leave it to the reader as an exercise [85]. Taking $\boldsymbol{\psi}=\nabla\chi$ in (6.29) (note that from (6.30) $\nabla\cdot\bar{\boldsymbol{\psi}}=\phi$ in D and $\bar{\boldsymbol{\psi}}\cdot\nu=0$ on ∂D) we have that

$$\int_D(\Delta w\;-\;w-\rho_2)\,\phi\,dx=0\qquad\text{for all}\quad\phi\in L^2_0(D)$$

which implies the existence of a constant c_1 such that

$$\Delta w - w - \rho_2 = c_1 \quad \text{in } D\,. \tag{6.31}$$

We now take $\phi \in L^2_0(\partial D)$ and let $\sigma \in H^1(D)$ be a solution to

$$\begin{cases} \Delta\sigma = 0 \ \text{ in } D \\ \dfrac{\partial\sigma}{\partial\nu} = \bar{\phi} \ \text{ on } \partial D. \end{cases} \tag{6.32}$$

Taking $\boldsymbol{\psi} = \nabla\sigma$ in (6.25) (note that (6.32) implies that $\nabla \cdot \bar{\boldsymbol{\psi}} = 0$ in D and $\bar{\boldsymbol{\psi}} \cdot \nu = \phi$ on ∂D) we have that

$$\int_{\partial D} (w - v + f)\,\phi\, ds = 0 \qquad \text{for all} \quad \phi \in L^2_0(\partial D)$$

which implies the existence of a constant c_2 such that

$$w - v + f = c_2 \quad \text{on } \partial D\,. \tag{6.33}$$

Substituting (6.31) and (6.33) into (6.29) and using (6.21) we see that

$$(c_1 - c_2) \int_D \nabla \cdot \bar{\boldsymbol{\psi}}\, dx = 0 \quad \forall\, \boldsymbol{\psi} \in W(D)$$

which implies $c_1 = c_2 = c$ (take for instance $\boldsymbol{\psi} = \nabla\varrho$ where $\varrho \in H^1_0(D)$ and $\Delta\varrho = 1$ in D). Equations (6.27), (6.31), and (6.33) show that $(v, w-c)$ is a solution to (6.17)–(6.20).

We next consider the uniqueness equivalence between (6.17)–(6.20) and (6.24).

3. Assume that (6.17)–(6.20) has at most one solution. Let $U_1 = (v_1, \mathbf{w}_1)$ and $U_2 = (v_2, \mathbf{w}_2)$ be two solutions to (6.24). From step 2 above we deduce the existence of w_1 and w_2 in $H^1(D)$ such that $\mathbf{w}_1 = \nabla w_1$ and $\mathbf{w}_2 = \nabla w_2$ and (v_1, w_1) and (v_2, w_2) are solutions to (6.17)–(6.20), whence $(v_1, w_1) = (v_2, w_2)$ and $(v_1, \mathbf{w}_1) = (v_2, \mathbf{w}_2)$.
4. Finally assume that (6.24) has at most one solution and consider two solutions (v_1, w_1) and (v_2, w_2) to (6.17)–(6.20). We can deduce from step 1 above that $(v_1, \nabla w_1)$ and $(v_2, \nabla w_2)$ are two solutions to (6.24). Hence $v_1 = v_2$ and $w = w_1 - w_2$ is a function in $H^1(D)$ that satisfies

$$\begin{cases} \Delta w - w = 0 \quad \text{in } D \\ w = \dfrac{\partial w}{\partial\nu} = 0 \quad \text{on } \partial D \end{cases}$$

which implies $w = 0$.

$\square$

We now investigate the modified interior transmission problem in the variational formulation (6.24).

Theorem 6.6. *Assume that there exists a constant $\gamma > 1$ such that, for $x \in D$,*

$$Re\left(\bar{\xi} \cdot A(x)\,\xi\right) \geq \gamma|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{C}^2 \quad \text{and} \quad m(x) \geq \gamma\,. \tag{6.34}$$

Then problem (6.24) has a unique solution $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$. This solution satisfies the a priori estimate

$$\begin{aligned} \|v\|_{H^1(D)} + \|\mathbf{w}\|_W \leq 2C\frac{\gamma+1}{\gamma-1}\Big(&\|\rho_1\|_{L^2(D)} + \|\rho_2\|_{L^2(D)} \\ &+ \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\Big) \end{aligned} \tag{6.35}$$

where the constant $C > 0$ is independent of ρ_1, ρ_2, f, h and γ.

Proof. The trace theorems (see Sect. 5.2) and Schwarz's inequality ensure the continuity of the conjugate linear functional L on $H^1(D) \times W(D)$ and the existence of a constant c independent of ρ_1, ρ_2, f and h such that

$$\|L\| \leq C\left(\|\rho_1\|_{L^2} + \|\rho_2\|_{L^2} + \|f\|_{H^{\frac{1}{2}}} + \|h\|_{H^{-\frac{1}{2}}}\right)\,. \tag{6.36}$$

On the other hand, if $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$ then, by assumption (6.34),

$$|\mathcal{A}(U,U)| \geq \gamma\,\|v\|^2_{H^1} + \|\mathbf{w}\|^2_W - 2\,\mathcal{R}e\left(\langle\bar{v},\,\mathbf{w}\rangle\right)\,. \tag{6.37}$$

According to the duality identity (6.21), one has by Schwarz's inequality that

$$|\langle\bar{v},\,\mathbf{w}\rangle| \leq \|v\|_{H^1}\,\|\mathbf{w}\|_W$$

and therefore

$$|\mathcal{A}(U,U)| \geq \gamma\,\|v\|^2_{H^1} + \|\mathbf{w}\|^2_W - 2\,\|v\|_{H^1}\,\|\mathbf{w}\|_W\,.$$

Using the identity $\gamma x^2 + y^2 - 2xy = \frac{\gamma+1}{2}\left(x - \frac{2}{\gamma+1}\,y\right)^2 + \frac{\gamma-1}{2}x^2 + \frac{\gamma-1}{\gamma+1}y^2$, we conclude that

$$|\mathcal{A}(U,U)| \geq \frac{\gamma-1}{\gamma+1}\left(\|\mathbf{w}\|^2_W + \|v\|^2_{H^1}\right),$$

whence $\mathcal{A}$ is coercive. The continuity of $\mathcal{A}$ follows easily from Schwarz's inequality, the trace theorem and Theorem 5.5. Theorem 6.6 is now a direct consequence of the Lax-Milgram lemma applied to (6.24). □

Theorem 6.7. *Assume that there exists a constant $\gamma > 1$ such that, for $x \in D$,*

$$Re\left(\bar{\xi} \cdot A(x)\,\xi\right) \geq \gamma|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{C}^2 \quad \text{and} \quad m(x) \geq \gamma\,. \tag{6.38}$$

Then the modified interior transmission problem (6.17)–(6.20) has a unique solution (v, w) *that satisfies*

$$\|v\|_{H^1(D)} + \|w\|_{H^1(D)} \le C\frac{\gamma+1}{\gamma-1}\Big(\|\rho_1\|_{L^2(D)} + \|\rho_2\|_{L^2(D)} + \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\Big) \tag{6.39}$$

where the constant $C > 0$ *in independent of* ρ_1, ρ_2, f, h *and* γ.

Proof. The existence and uniqueness of a solution follows from Theorem 6.5 and Theorem 6.6. The a priori estimate (6.39) can be obtained directly from (6.17)–(6.20) but can also be deduced from (6.35) as follows. Theorem 6.5 tells us that $(v, \nabla w)$ is the unique solution to (6.24). Hence, according to (6.35),

$$\|v\|_{H^1} + \|\nabla w\|_{L^2} \le C_1\frac{\gamma+1}{\gamma-1}\Big(\|\rho_1\|_{L^2} + \|\rho_2\|_{L^2} + \|f\|_{H^{\frac{1}{2}}} + \|h\|_{H^{-\frac{1}{2}}}\Big)\,.$$

From Poincaré's inequality in Sect. 5.2 we can write

$$\|w\|_{H^1(D)} \le C_2\Big(\|\nabla w\|_{L^2(D)} + \|w\|_{L^2(\partial D)}\Big)\,.$$

Now by using the boundary condition (6.19) and the trace theorem we obtain that

$$\|w\|_{H^1(D)} \le C_2\Big(\|\nabla w\|_{L^2(D)} + \|v\|_{H^1(D)} + \|f\|_{L^2(\partial D)}\Big)$$

for some positive constant C_2. The constants C_1 and C_2 can then be adjusted so that (6.39) holds. □

Now we are ready to show the existence of a solution to the interior transmission problem (6.12)–(6.15).

Theorem 6.8. *Assume that either* $Im(n) > 0$ *or* $Im\big(\bar{\xi}\cdot A\,\xi\big) < 0$ *at a point* $x_0 \in D$ *and that there exists a constant* $\gamma > 1$ *such that, for* $x \in D$,

$$Re\big(\bar{\xi}\cdot A(x)\,\xi\big) \ge \gamma|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{C}^2\,. \tag{6.40}$$

Then (6.12)–(6.15) has a unique solution $(v, w) \in H^1(D) \times H^1(D)$. *This solution satisfies the a priori estimate*

$$\|v\|_{H^1(D)} + \|w\|_{H^1(D)} \le C\Big(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\Big) \tag{6.41}$$

where the constant $C > 0$ *is independent of* f *and* h.

Proof. Set

$$\mathcal{X}(D) = \big\{(v, w) \in H^1(D) \times H^1(D)\ :\ \nabla\cdot A\nabla v \in L^2(D) \text{ and } \Delta w \in L^2(D)\big\}$$

and consider the operator $\mathcal{G}$ from $\mathcal{X}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$\mathcal{G}(v,w) = \left(\nabla \cdot A\nabla v - mv, \Delta w - w, (v-w)_{|\partial D}, \left(\frac{\partial v}{\partial \nu} - \frac{\partial w}{\partial \nu} \right)_{|\partial D} \right) \tag{6.42}$$

where $m \in C(\bar{D})$ and $m > 1$. Obviously $\mathcal{G}$ is continuous and from Theorem 6.7 we know that the inverse of $\mathcal{G}$ exists and is continuous. Now consider the operator $\mathcal{T}$ from $\mathcal{X}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$\mathcal{T}(v,w) = \big((k^2\, n + m)v,\ (k^2\ + 1)w,\ 0,\ 0\big)$$

From the compact embedding of $H^1(D)$ into $L^2(D)$ (see Sect. 5.2), the operator $\mathcal{T}$ is compact. Theorem 6.4 implies that $\mathcal{G} + \mathcal{T}$ is injective and therefore from Theorem 5.14 we can deduce the existence and the continuity of $(\mathcal{G}+\mathcal{T})^{-1}$, which means in particular the existence of a unique solution to the interior transmission problem (6.12)–(6.15) that satisfies the a priori estimate (6.41). $\square$

In general we cannot conclude the solvability of the interior transmission problem if A and n do not satisfy the assumptions of the previous theorem. In particular, if $\mathrm{Im}(A) = 0$ and $\mathrm{Im}(n) = 0$ in D, k^2 may be a transmission eigenvalue (see Definition 6.3). Do transmission eigenvalue exist and if so do they form a discrete set? We can only provide a partial answer to the above question in the following theorem.

Theorem 6.9. *Assume that Im$(n) = 0$ and Im$(A) = 0$ in D and that there exists a constant $\gamma > 1$ such that, for $x \in D$,*

$$\bar{\xi} \cdot A(x)\,\xi \geq \gamma|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^2 \quad \textit{and} \quad n(x) \geq \gamma.$$

Then the set of transmission eigenvalues is either empty or forms a discrete set.

Proof. Consider the operator $\mathcal{G}$ defined by (6.42) with $m = n$ and the compact operator $\mathcal{T}$ from $\mathcal{X}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$\mathcal{T}(w,v) = (nw, v,\ 0,\ 0)\,.$$

We want to prove that the operator $\mathcal{G} + (k^2+1)\mathcal{T}$ is invertible for all $k \in \mathbb{C}\setminus S$ where S is an empty or discrete subset of $\mathbb{C}$. Since $\mathcal{G}$ is bijective (Theorem 6.7), this is equivalent to showing that $(I + (k^2+1)\mathcal{G}^{-1}\,\mathcal{T})^{-1}$ exists, where I is the identity operator from $\mathcal{X}(D)$ into $\mathcal{X}(D)$. The fact that this operator exists and is bounded except for at most a discrete set of k values follows immediately from Theorem 1.22. Note that $(k^2+1)G^{-1}\mathcal{T}$ is a compact operator. $\square$

In general it is not known if transmission eigenvalues exist. The only known result on the existence of transmission eigenvalues is for the case when $A = I$ and $n(x) = n(r)$ (see Theorem 8.13 in [33])

The above analysis of the interior transmission problem requires that the matrix A satisfies

$$\mathrm{Re}\left(\bar{\xi}\cdot A(x)\,\xi\right) \geq \gamma|\xi|^2 \ \text{ for all } \ \xi \in \mathbb{C}^2, \ \ x \in D \ \text{ and some constant } \ \gamma > 1$$

that is $\|\mathrm{Re}(A)\| > 1$. The case of $\mathrm{Re}(A)$ positive definite such that $\|\mathrm{Re}(A)\| < 1$ is considered in [19]. By modifying the variational approach of Theorem 6.5 and Theorem 6.6 one can prove the following results.

Theorem 6.10. *Assume that there exists a constant $\gamma > 1$ such that, for $x \in D$,*

$$Re\left(\bar{\xi}\cdot (A(x))^{-1}\,\xi\right) \geq \gamma|\xi|^2 \ \ \text{for all} \ \ \xi \in \mathbb{C}^2 \quad \text{and} \quad \gamma^{-1} \leq m < 1.$$

Then (6.17)–(6.20) has a unique solution (v, w) that satisfies

$$\|v\|_{H^1(D)} + \|w\|_{H^1(D)} \leq C\left(\|\rho_1\|_{L^2(D)} + \|\rho_2\|_{L^2(D)} + \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\right)$$

where the constant $C > 0$ in independent of ρ_1, ρ_2, f, h.

Theorem 6.11. *Assume that either $Im(n) > 0$ or $Im\left(\bar{\xi}\cdot A\,\xi\right) < 0$ at a point $x_0 \in D$ and that there exists a constant $\gamma > 1$ such that, for $x \in D$,*

$$Re\left(\bar{\xi}\cdot (A(x))^{-1}\,\xi\right) \geq \gamma|\xi|^2 \ \ \text{for all} \ \ \xi \in \mathbb{C}^2.$$

Then (6.12)–(6.15) has a unique solution $(v, w) \in H^1(D) \times H^1(D)$. This solution satisfies the a priori estimate

$$\|v\|_{H^1(D)} + \|w\|_{H^1(D)} \leq C\left(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\right),$$

where the constant $C > 0$ is independent of f and h.

Finally in the same way as in Theorem 6.9 one can show that in this case, under additional assumptions on n, the set of the transmission eigenvalues is at most discrete. In particular the following theorem holds.

Theorem 6.12. *Assume that $Im(n) = 0$ and $Im(A) = 0$ in D and that there exists a constant $\gamma > 1$ such that, for $x \in D$,*

$$\bar{\xi}\cdot (A(x))^{-1}\,\xi \geq \gamma|\xi|^2 \ \ \text{for all} \ \ \xi \in \mathbb{C}^2$$

and

$$\gamma^{-1} \leq n(x) < 1.$$

Then the set of the transmission eigenvalues is either empty or forms a discrete set.

We end this section by remarking that in the case of $A = I$, where I is the identity matrix, the above analysis is no longer applicable. This case can be treated either by rewriting the interior transmission problem as a boundary value problem for a fourth order partial differential equation for the difference $v - w \in H^2(D)$ [103] or by using analytic projection operators (Sect. 8.6 of [33]). The case when $\|\mathrm{Re}(A(x))\| > 1$ for $x \in D_0 \subset D$ and $\|\mathrm{Re}(A(x))\| < 1$ for $x \in D \setminus \bar{D}_0$ remains an open question.

6.3 Uniqueness

The proof of uniqueness for the inverse medium scattering problem is more complicated than for the case of scattering by an imperfect conductor considered in Chap. 4. The idea of the uniqueness proof for the inverse medium scattering problem originates from [61, 62] in which it is shown that the shape of a penetrable, inhomogeneous, isotropic medium is uniquely determined by its far field pattern for all incident plane waves. The case of an orthotropic medium is due to Hähner [55] (see also [35]), the proof of which is based on the existence of a solution to the modified interior transmission problem. We begin with a simple lemma.

Lemma 6.13. *Assume that either $\bar{\xi} \cdot Re(A)\,\xi \geq \gamma|\xi|^2$ or $\bar{\xi} \cdot Re(A^{-1})\,\xi \geq \gamma|\xi|^2$ for some $\gamma > 1$. Let $\{v_n, w_n\} \in H^1(D) \times H^1(D)$, $n \in \mathbb{N}$, be a sequence of solutions to the interior transmission problem (6.12)–(6.15) with boundary data $f_n \in H^{\frac{1}{2}}(\partial D)$, $h_n \in H^{-\frac{1}{2}}(\partial D)$. If the sequences $\{f_n\}$ and $\{h_n\}$ converge in $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$ respectively, and if the sequences $\{v_n\}$ and $\{w_n\}$ are bounded in $H^1(D)$, then there exists a subsequence $\{w_{n_k}\}$ which converges in $H^1(D)$.*

Proof. Assume first that $\bar{\xi} \cdot \mathrm{Re}(A)\,\xi \geq \gamma|\xi|^2$, $\gamma > 1$, and let $\{v_n, w_n\}$ be as in the statement of the lemma. Due to the compact imbedding of $H^1(D)$ into $L^2(D)$ we can select L^2-convergent subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$. Hence, $\{v_{n_k}\}$ and $\{w_{n_k}\}$ satisfy

$$\begin{aligned}
\nabla \cdot A\nabla v_{n_k} - \gamma v_{n_k} &= -(\gamma + k^2 n) v_{n_k} && \text{in} \quad D \\
\Delta w_{n_k} - w_{n_k} &= -(1 + k^2) w_{n_k} && \text{in} \quad D \\
v_{n_k} - w_{n_k} &= f_{n_k} && \text{on} \quad \partial D \\
\frac{\partial v_{n_k}}{\partial \nu_A} - \frac{\partial w_{n_k}}{\partial \nu} &= h_{n_k} && \text{on} \quad \partial D.
\end{aligned}$$

Then the result of the lemma follows from the a priori estimate of Theorem 6.7. In the case when $\bar{\xi} \cdot \mathrm{Re}(A^{-1})\,\xi \geq \gamma|\xi|^2$, $\gamma > 1$, we use Theorem 6.10 and $1/\gamma$ instead of γ in the above equation for v_{n_k} to obtain the same result. □

Note that in the proof of Lemma 6.13 we use the a priori estimate for the modified interior transmission problem instead of the a priori estimate for the interior transmission problem. This allows us to obtain the result without assuming that k^2 is not a transmission eigenvalue.

We are now ready to prove the uniqueness theorem.

Theorem 6.14. *Let the domains D_1 and D_2, the matrix-valued functions A_1 and A_2, and the functions n_1 and n_2 satisfy the assumptions in Sect. 5.2. Moreover assume that either $\bar{\xi} \cdot Re(A_1)\,\xi \geq \gamma|\xi|^2$ or $\bar{\xi} \cdot Re(A_1^{-1})\,\xi \geq \gamma|\xi|^2$, and either $\bar{\xi} \cdot Re(A_2)\,\xi \geq \gamma|\xi|^2$ or $\bar{\xi} \cdot Re(A_2^{-1})\,\xi \geq \gamma|\xi|^2$ for some $\gamma > 1$.*

If the far field patterns $u^1_\infty(\theta,\phi)$ and $u^2_\infty(\theta,\phi)$ corresponding to D_1, A_1, n_1 and D_2, A_2, n_2, respectively, coincide for all $\theta \in [0,\, 2\pi]$ and $\phi \in [0,\, 2\pi]$, then $D_1 = D_2$.

Proof. Denote by G the unbounded connected component of $\mathbb{R}^2 \setminus (\bar{D}_1 \cup \bar{D}_2)$ and define $D_1^e := \mathbb{R}^2 \setminus \bar{D}_1$, $D_2^e := \mathbb{R}^2 \setminus \bar{D}_2$. By Rellich's lemma we conclude that the scattered fields u_1 and u_2 which are the radiating part of the solution to (5.13)–(5.17) with D_1, A_1, n_1 and D_2, A_2, n_2, respectively, and boundary data with $f := e^{ikx\cdot d}$ and $h := \partial e^{ikx\cdot d}/\partial\nu$, $d = (\cos\phi, \sin\phi)$, coincide in G. Let $\Phi(x,z)$ denote the fundamental solution to the Helmholtz equation given by (3.33).

We now show that the scattered solutions $u_1(\cdot,z)$ and $u_2(\cdot,z)$ also coincide for the incident waves $\Phi(\cdot,z)$ with $z \in G$, i.e. for $f := \Phi(\cdot,z)$ and $h := \partial\Phi(\cdot,z)/\partial\nu$. To this end, choose a large disk Ω_R such that $\bar{D}_1 \cup \bar{D}_2 \subset \Omega_R$ and k^2 is not a Dirichlet eigenvalue for Ω_R. Then, for $z \notin \bar{\Omega}_R$, by Lemma 4.4, there exists a sequence $\{u_n^i\}$ in $\text{span}\{e^{ikx\cdot d} : |d| = 1\}$ such that

$$\|u_n^i - \Phi(\cdot,\, z)\|_{H^{\frac{1}{2}}(\partial\Omega_R)} \to 0, \quad \text{as } n \to \infty.$$

The well-possedness of the Dirichlet problem for the Helmholtz equation in Ω_R (see Example 5.13) implies that u_n^i approximates $\Phi(\cdot,\, z)$ in $H^1(\Omega_R)$. Then the continuous dependence on the data of the scattered field (5.41) together with the fact that the scattered fields corresponding to u_n^i coincide as linear combinations of scattered fields due to plane waves imply that $u_1(\cdot,z)$ and $u_2(\cdot,z)$ also coincide for a fixed $z \notin \bar{\Omega}_R$. Since $\Phi(\cdot,\, z)$ and its derivatives are real-analytic in z, we can again conclude from the well-posedness of the transmission problem (5.13)–(5.17) that $u_1(\cdot,z)$ and $u_2(\cdot,z)$ are real-analytic in z, and therefore must coincide for all $z \in G$.

Let us now assume that $\bar{D}_1$ is not included in $\bar{D}_2$. Since D_2^e is connected, we can find a point $z \in \partial D_1$ and $\epsilon > 0$ with the following properties, where $\Omega_\delta(z)$ denotes the ball of radius δ centered at z:

1. $\Omega_{8\epsilon}(z) \cap \bar{D}_2 = \emptyset$,
2. The intersection $\bar{D}_1 \cap \Omega_{8\epsilon}(z)$ is contained in the connected component of $\bar{D}_1$ to which z belongs,
3. There are points from this connected component of $\bar{D}_1$ to which z belongs which are not contained in $\bar{D}_1 \cap \bar{\Omega}_{8\epsilon}(z)$,
4. The points $z_n := z + \dfrac{\epsilon}{n}\nu(z)$ lie in G for all $n \in \mathbb{N}$, where $\nu(z)$ is the unit normal to ∂D_1 at z.

Due to the singular behavior of $\Phi(\cdot,\, z_n)$ at the point z_n, it is easy to show that $\|\Phi(\cdot,\, z_n)\|_{H^1(D_1)} \to \infty$ as $n \to \infty$. We now define

$$w^n(x) := \frac{1}{\|\Phi(\cdot,\, z_n)\|_{H^1(D_1)}}\Phi(x,\, z_n), \qquad x \in \bar{D}_1 \cup \bar{D}_2$$

and let v_1^n, u_1^n and v_2^n, u_2^n be the solutions of the scattering problem (5.13)–(5.17) with boundary data $f := w^n$ and $h := \partial w^n/\partial\nu$ corresponding to D_1 and D_2, respectively. Note that for each n, w^n is a solution of the Helmholtz equation in D_1 and D_2. Our aim is to prove that if $\bar{D}_1 \not\subset \bar{D}_2$ then the equality $u_1(\cdot, z) = u_2(\cdot, z)$ for $z \in G$ allows the selection of a subsequence $\{w^{n_k}\}$ from $\{w^n\}$ that converges to zero with respect to $H^1(D_1)$. This certainly contradicts the definition of $\{w^n\}$ as a sequence of functions with $H^1(D_1)$-norm equal to one. Note that $u_1(\cdot, z) = u_2(\cdot, z)$ obviously implies that $u_1^n = u_2^n$ in G.

We begin by noting that, since the functions $\Phi(\cdot,\, z_n)$ together with their derivatives are uniformly bounded in every compact subset of $\mathbb{R}^2 \setminus \Omega_{2\epsilon}(z)$ and $\|\Phi(\cdot,\, z_n)\|_{H^1(D_1)} \to \infty$ as $n \to \infty$, then $\|w^n\|_{H^1(D_2)} \to 0$ as $n \to \infty$. Hence, if Ω_R is a large ball containing $\bar{D}_1 \cup \bar{D}_2$, then $\|u_2^n\|_{H^1(\Omega_R \cap G)} \to 0$ as $n \to \infty$ from the a priori estimate (5.41). Since $u_1^n = u_2^n$ in G then $\|u_1^n\|_{H^1(\Omega_R \cap G)} \to 0$ as $n \to \infty$ as well. Now, with the help of a cutoff function $\chi \in C_0^\infty(\Omega_{8\epsilon}(z))$ satisfying $\chi(x) = 1$ in $\Omega_{7\epsilon}(z)$ (see Theorem 5.4), we see that $\|u_1^n\|_{H^1(\Omega_R \cap G)} \to 0$ implies that

$$(\chi u_1^n) \to 0, \qquad \frac{\partial(\chi u_1^n)}{\partial\nu} \to 0, \qquad \text{as } n \to \infty \tag{6.43}$$

with respect to the $H^{\frac{1}{2}}(\partial D)$-norm and $H^{-\frac{1}{2}}(\partial D)$-norm, respectively. Indeed, for the first convergence we simply apply the trace theorem while for the convergence of $\partial(\chi u_1^n)/\partial\nu$, we first deduce the convergence of $\Delta(\chi u_1^n)$ in $L^2(\Omega_R \cap D_1^e)$, which follows from $\Delta(\chi u_1^n) = \chi\Delta u_1^n + 2\nabla\chi \cdot \nabla u_1^n + u_1^n\Delta\chi$, and then apply Theorem 5.5. Note here that we need conditions 2 and 4 on z to ensure $\Omega_{8\epsilon}(z) \cap D_1^e = \Omega_{8\epsilon}(z) \cap G$.

We next note that in the exterior of $\Omega_{2\epsilon}(z)$ the $H^2(\Omega_R \setminus \Omega_{2\epsilon}(z))$-norms of w^n remain uniformly bounded. Then the assertion about boundary regularity of the solution to (5.13)–(5.17) stated in the second part of Theorem 5.26 implies that u_1^n is uniformly bounded with respect to the $H^2((\Omega_R \cap D_1^e) \setminus \Omega_{4\epsilon}(z))$-norm. Therefore, using the compact imbedding of $H^2(\Omega_R \cap D_1^e)$ into $H^1(\Omega_R \cap D_1^e)$, we can select a $H^1(\Omega_R \cap D_1^e)$ convergent subsequence $\{(1 - \chi)u_1^{n_k}\}$ from $\{(1 - \chi)u_1^n\}$. Hence, $\{(1 - \chi)u_1^{n_k}\}$ is a convergent sequence in $H^{\frac{1}{2}}(\partial D)$, and similarly to the above reasoning we also have that $\{\partial((1 - \chi)u_1^{n_k})/\partial\nu\}$ converges in $H^{-\frac{1}{2}}(\partial D)$. This, together with (6.43), implies that the sequences

$$\{u_1^{n_k}\} \qquad \text{and} \qquad \left\{\frac{\partial u_1^{n_k}}{\partial\nu}\right\}$$

converge in $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$, respectively.

Finally, since the functions $v_1^{n_k}$ and w^{n_k} are solutions to the interior transmission problem (6.12)–(6.15) for the domain D_1 with boundary data $f = u_1^{n_k}$ and $h = \partial u_1^{n_k}/\partial\nu$, and since the $H^1(D_1)$-norms of $v_1^{n_k}$ and w^{n_k} remain uniformly bounded, according to Lemma 6.13 we can select a subsequence of $\{w^{n_k}\}$, denoted again by $\{w^{n_k}\}$, which converges in $H^1(D_1)$ to a function $w \in H^1(D_1)$. As a limit of weak solutions to the Helmholtz equation,

$w \in H^1(D_1)$ is a weak solution to the Helmholtz equation. We also have that $w|_{D_1 \setminus \Omega_{2\epsilon}(z)} = 0$ because the functions w^{n_k} converge uniformly to zero in the exterior of $\Omega_{2\epsilon}(z)$. Hence, w must be zero in all of D_1 (here we make use of condition 3, namely the fact that the connected component of D_1 containing z has points which do not lie in the exterior of $\bar{\Omega}_{2\epsilon}(z)$). This contradicts the fact that $\|w^{n_k}\|_{H^1(D_1)} = 1$. Hence the assumption $\bar{D}_1 \not\subset \bar{D}_2$ is false.
Since we can derive the analogous contradiction for the assumption $\bar{D}_2 \not\subset \bar{D}_1$, we have proved that $D_1 = D_2$. □

6.4 The Linear Sampling Method

Having shown that the support of the inhomogeneity can be uniquely determined from the far field pattern, we now want to find an approximation to the support. To this end we will use the linear sampling method previously introduced in Chap. 4 for the inverse scattering problem for an imperfect conductor. In particular, we shall show that, providing k^2 is not a transmission eigenvalue, the boundary ∂D of the inhomogeneity D can be characterized by the solution of the far field equation (4.33) where the kernel of the far field operator is the far field pattern corresponding to (6.1)–(6.5).

Given $(f, h) \in H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$, let $(v, u) \in H^1(D) \times H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ be the unique solution to the corresponding transmission problem (5.13)–(5.17). We recall that the radiating part u has the asymptotic behavior

$$u(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}) + O(r^{-3/2}), \qquad r \to \infty, \quad \hat{x} = x/|x|$$

where u_∞ is the far field pattern corresponding to (v, u).

Definition 6.15. *The bounded linear operator $B : H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to L^2[0, 2\pi]$ maps $(f, h) \in H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ onto the far field pattern $u_\infty \in L^2[0, 2\pi]$ where (v, u) is the solution of (5.13)–(5.17) with the boundary data (f, h).*

Note that the fact that B is bounded follows directly from the well-posedness of (5.13)–(5.17).
As in the case of the scattering problem for an imperfect conductor, the operator B will play an important role in the solution of the inverse problem. In order to determine the range of the operator B, it is more convenient to consider its transpose instead of its adjoint. This is because operating with the duality relation between $H^{\frac{1}{2}}(\partial D)$, $H^{-\frac{1}{2}}(\partial D)$ is much simpler then using the corresponding inner products. In the following we will define the transpose operator and derive some useful properties of this operator.
Let X and Y be two Hilbert spaces and let X^* and Y^* be their dual spaces. For any linear mapping $A : X \to Y$, the *transpose* $A^\top : Y^* \to X^*$ is the linear mapping defined by

$$\left\langle A^\top v, u\right\rangle_{X,X^*} = \langle v, Au\rangle_{Y,Y^*}\,, \qquad \text{for all } u \in X \text{ and } v \in Y^*$$

where $\langle\cdot,\,\cdot\rangle$ denotes the duality pairing between the denoted spaces.
It can be shown (see Lemma 2.9 in [85]) that the transpose $A^\top$ is bounded if and only if A is bounded. To describe the relation between the range and the kernel of A and $A^\top$ we use the following terminology. For any subset $W \subseteq X$, the *annihilator* W^a is the closed subspace of X^* defined by

$$W^a = \{g \in X^* :\ \langle g, u\rangle = 0 \text{ for all } u \in W\}.$$

Similarly, for $V \subseteq X^*$, the annihilator ${}^aV$ is the closed subspace of X defined by

$${}^aV = \{u \in X :\ \langle g, u\rangle = 0 \text{ for all } g \in V\}.$$

Lemma 6.16. *The null space and range of A and $A^\top$ satisfy*

$$\mathrm{N}(A^\top) = A(X)^a \quad \textit{and} \quad \mathrm{N}(A) = {}^aA^\top(Y^*).$$

Proof. Applying the various definitions we obtain

$$\begin{aligned} A(X)^a &= \{g \in Y^* :\ \langle g, v\rangle = 0 \;\text{ for all }\; v \in \operatorname{range} A\} \\ &= \{g \in Y^* :\ \langle g, Au\rangle = 0 \;\text{ for all }\; u \in X\} \\ &= \{g \in Y^* :\ \left\langle A^\top g, u\right\rangle = 0 \;\text{ for all }\; u \in X\} \\ &= \{g \in Y^* :\ A^\top g = 0\} = \mathrm{N}(A^\top) \end{aligned}$$

A similar argument shows that $\mathrm{N}(A) = {}^aA^\top(Y^*)$. □

It is an easy exercise to show that a subset $W \subseteq X$ is dense if and only if $W^a = \{0\}$. In particular from Lemma 6.16 we have that

Corollary 6.17. *The operator A has dense range if and only if the transpose $A^\top$ is injective.*

With the help of the above lemma and corollary we can now prove the following result for the operator B.

Theorem 6.18. *The range of $B : H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to L^2[0,\, 2\pi]$ is dense in $L^2[0,\, 2\pi]$.*

Proof. We consider the dual operator $B^\top : L^2[0,\, 2\pi] \longrightarrow H^{-\frac{1}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ which maps a function g into $(\tilde f, \tilde h)$ such that

$$\langle B(f,h), g\rangle_{L^2\times L^2} = \left\langle f, \tilde f\right\rangle_{H^{\frac{1}{2}}\times H^{-\frac{1}{2}}} + \left\langle h, \tilde h\right\rangle_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}}$$

where $\langle\cdot,\,\cdot\rangle$ denotes the duality pairing between the denoted spaces. Now let $(\tilde v, \tilde u)$ be the unique solution of (5.13)–(5.17) with $(f, h) := (\tilde v_g|_{\partial D}, \partial\tilde v_g/\partial\nu|_{\partial D})$ where $\tilde v_g$ is the Herglotz wave function defined by (6.9). Then from (6.6) we have

$$\langle B(f,h), g\rangle = \int_0^{2\pi} u_\infty(\theta) g(\theta)\, d\theta = \int_{\partial D} \left(u(y) \frac{\partial \tilde{v}_g(y)}{\partial \nu} - \tilde{v}_g(y) \frac{\partial u(y)}{\partial \nu} \right) ds(y).$$

Since u and $\tilde{u}$ are solutions of the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{D}$ satisfying the Sommerfeld radiation condition, an application of Green's second identity implies that

$$\int_{\partial D} \left[u(y) \frac{\partial \tilde{u}(y)}{\partial \nu} - \tilde{u}(y) \frac{\partial u(y)}{\partial \nu} \right] ds(y) = 0.$$

Using the transmission conditions on the boundary for $\tilde{u}$ and $\tilde{v}$ we obtain

$$\begin{aligned}
&\langle B(f,h), g\rangle_{L^2 \times L^2} = \\
&= \int_{\partial D} \left[u(y) \left(\frac{\partial \tilde{v}_g(y)}{\partial \nu} + \frac{\partial \tilde{u}(y)}{\partial \nu} \right) - (\tilde{v}_g(y) + \tilde{u}(y)) \frac{\partial u(y)}{\partial \nu} \right] ds(y) \\
&= \int_{\partial D} \left(u(y) \frac{\partial \tilde{v}(y)}{\partial \nu_A} - \tilde{v}(y) \frac{\partial u(y)}{\partial \nu} \right) ds(y) \\
&= \int_{\partial D} \left[(v(y) - f(y)) \frac{\partial \tilde{v}(y)}{\partial \nu_A} - \tilde{v}(y) \left(\frac{\partial v(y)}{\partial \nu_A} - h(y) \right) \right] ds(y).
\end{aligned}$$

Finally, applying Green's (generalized) second identity to v and $\tilde{v}$ we have that

$$\langle B(f,h), g\rangle_{L^2 \times L^2} = \int_{\partial D} \left[f(y) \left(-\frac{\partial \tilde{v}(y)}{\partial \nu_A} \right) + \tilde{v}(y) h(y) \right] ds(y).$$

Hence the dual operator $B^\top$ can be characterized as

$$B^\top g = \left(-\left. \frac{\partial \tilde{v}}{\partial \nu_A} \right|_{\partial D}, \tilde{v}|_{\partial D} \right).$$

In the following we want to show that the operator $B^\top$ is injective. To this end let $B^\top g \equiv 0$, $g \in L^2[0, 2\pi]$. This implies that $\tilde{v} = 0$ and $\partial \tilde{v}/\partial \nu_A = 0$ on the boundary ∂D. Therefore $\tilde{u}$ satisfies the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{D}$, the Sommerfeld radiation condition and, from the transmission conditions,

$$\tilde{u} = -\tilde{v}_g \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial \nu} = -\frac{\partial \tilde{v}_g}{\partial \nu} \qquad \text{on } \partial D.$$

Thus, setting $\tilde{u} \equiv -\tilde{v}_g$ in D we have that $\tilde{u}$ can be extended to an entire solution of the Helmholtz equation satisfying the radiation condition. This is only possible if $\tilde{u}$ vanishes which implies that $\tilde{v}_g$ vanishes also and thus $g \equiv 0$, whence $B^\top$ is injective. Finally, from Corollary 6.17, we have that the range of B is dense in $L^2[0, 2\pi]$. $\square$

From Lemma 6.16 we also have that

$$\mathrm{N}(B) = B^{\top}(L^2[0,\,2\pi])^a := \left\{ (f_0, h_0) : \int\limits_{\partial D} \left(-f_0 \frac{\partial \tilde{v}}{\partial \nu_A} + h_0 \tilde{v} \right) ds = 0 \right\},$$

where $\tilde{v}$ is as in the proof of Theorem 6.18. Hence, using the divergence theorem, we see that the pairs $(v|_{\partial D}, \partial v/\partial \nu_A|_{\partial D})$, where $v \in H^1(D)$ is a solution of $\nabla \cdot A \nabla v + k^2 n\, v = 0$ in D, are in the kernel of B. So B is not injective. We will restrict the operator B in such a way that the restriction is injective and still has dense range.

To this end let us denote by $\overline{H}$ the closure in $H^1(D)$ of all Herglotz wave functions with kernel $g \in L^2[0,\,2\pi]$. Note that the space $\overline{H}$ coincides with the space of H^1 weak solutions to the Helmholtz equation. In other words, $\overline{H} = \overline{W(D)}$, where $\overline{W(D)}$ is the closure in $H^1(D)$ of $W(D)$ defined by

$$W(D) := \{u \in C^2(D) \cap C^1(\overline{D}) : \Delta u + k^2\, u = 0\}.$$

Indeed, if $u \in \overline{W(D)}$ then by seeing u as a weak solution of the interior impedance boundary value problem for the Helmholtz equation in D with $\lambda = 1$ we have from Theorem 8.4 in Chap. 8 (set $\Gamma_D = \emptyset$) that there exists a positive constant C such that

$$\|u\|_{H^1(D)} \le C \left\| \frac{\partial u}{\partial \nu} + iu \right\|_{H^{-\frac{1}{2}}(\partial D)}.$$

Then the proof of Theorem 4.10 implies that for any $\epsilon > 0$ there exists a Herglotz wave function v_g such that $\|u - v_g\|_{H^1(D)} < \epsilon$, whence $\overline{H} = \overline{W(D)}$. For later use we state this result in the following lemma.

Lemma 6.19. *Any solution to the Helmhotz equation in a bounded domain $D \subset \mathbb{R}^2$ can be approximated in the $H^1(D)$ norm by a Herglotz wave function.*

Next, we define

$$H(\partial D) := \left\{ \left(u|_{\partial D},\, \frac{\partial u}{\partial \nu}\bigg|_{\partial D} \right) : u \in \overline{H} \right\}.$$

Lemma 6.20. *$H(\partial D)$ is a closed subset of $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$.*

Proof. Consider $(f, h) \in \overline{H(\partial D)}$. There exists a sequence $\{u_n, \partial u_n/\partial \nu\}$ converging to (f, h) in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ where $u_n \in \overline{H}$. Since the sequence $\{u_n, \partial u_n/\partial \nu\}$ is bounded in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$, by considering u_n to be the solution of an impedance boundary value problem in D we can deduce that $\{u_n\}$ is bounded in $H^1(D)$. From this it follows that a subsequence (still denoted by $\{u_n\}$) converges weakly in $H^1(D)$ to a function u which is clearly in $\overline{H}$. From the continuity of the trace operators (see Theorem 1.36 and Theorem 5.5) we deduce that $\{u_n, \partial u_n/\partial \nu\}$ converges weakly in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ to $(u, \partial u/\partial \nu)$ and by the uniqueness of the limit $(f, h) = (u, \partial u/\partial \nu)$. Hence $(f, h) \in H(\partial D)$ which completes the proof. $\square$

From the above lemma, $H(\partial D)$ equipped with the induced norm from $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ is a Banach space.

Now, let B_0 denote the restriction of B to $H(\partial D)$.

Theorem 6.21. *Assume that k^2 is not a transmission eigenvalue. Then the bounded linear operator $B_0 : H(\partial D) \longrightarrow L^2[0,\, 2\pi]$ is injective and has dense range.*

Proof. Let $B_0(f,h) = 0$ for $(f,h) \in H(\partial D)$ and let (v,u) be the solution to (5.13)–(5.17) corresponding to this boundary data. Then the radiating solution to the Helmholtz equation in the exterior of D has zero far field pattern, whence $u = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$. This implies that v satisfies

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \quad \text{in} \quad D, \qquad v = f \quad \text{and} \quad \frac{\partial v}{\partial \nu} = h \quad \text{on} \quad \partial D.$$

From the definition of $H(\partial D)$, f, h are the traces on ∂D of a $H^1(D)$ solution w to the Helmholtz equation and its normal derivative, respectively. Therefore (v,w) solves the homogeneous interior transmission problem (6.12)–(6.13) and since k^2 is not a transmission eigenvalue, we have that $w \equiv 0$ and $v \equiv 0$ in D, whence $f = h = 0$.
It remains to show that the set $B_0(H(\partial D))$ is dense in $L^2[0,\, 2\pi]$. To this end, it is sufficient to show that the range of B is contained in the range of B_0 since from Theorem 6.18 the range of B is dense in $L^2[0,\, 2\pi]$. Let u_∞ be in the range of B, that is u_∞ is the far field pattern of the radiating part u of a solution (v,u) to (5.13)–(5.17). Let (v,w) be the unique solution to (6.12)–(6.13) with the boundary data $(u|_{\partial D}, \partial u/\partial\nu|_{\partial D})$. Hence (v,u) is the solution of (5.13)–(5.17) with boundary data $(w|_{\partial D}, \partial w/\partial\nu|_{\partial D}) \in H(\partial D)$ and has far field pattern coinciding with u_∞. This means that $B_0\,(w|_{\partial D}, \partial w/\partial\nu|_{\partial D}) = u_\infty$. □

Theorem 6.22. *The operator $B_0 : H(\partial D) \longrightarrow L^2[0,\, 2\pi]$ is compact.*

Proof. Given $w \in \overline{H}$ consider the solution (v,u) of (5.13)–(5.17) with boundary data $f := w|_{\partial D}$ and $h := \partial w/\partial\nu|_{\partial D}$. Let $\partial\Omega_R$ be the boundary of a disc Ω_R centered at the origin containing $\bar{D}$. The continuous dependence estimate (5.41) implies that the operator $\mathcal{G} : H(\partial D) \to H^{\frac{1}{2}}(\partial\Omega_R) \times H^{-\frac{1}{2}}(\partial\Omega_R)$ which maps

$$\left(w|_{\partial D},\, \left.\frac{\partial w}{\partial \nu}\right|_{\partial D} \right) \to \left(u|_{\partial\Omega_R},\, \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega_R} \right)$$

is bounded. Next we denote by $K : H^{\frac{1}{2}}(\partial\Omega_R) \times H^{-\frac{1}{2}}(\partial\Omega_R) \to L^2[0,\, 2\pi]$ the operator which takes $(u|_{\partial\Omega_R},\, \partial u/\partial\nu|_{\partial\Omega_R})$ to u_∞ given by

$$u_\infty(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int\limits_{\partial B} \left(u(y) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu} - e^{-ik\hat{x}\cdot y} \frac{\partial u(y)}{\partial \nu} \right) ds(y)$$

where $\hat{x} = x/|x|$. A similar argument as in the proof of Theorem 4.8 shows that K is compact. Therefore $B_0 = KG$ is compact since it is a composition of a bounded operator with a compact operator. □

For a Herglotz wave function v_g given by (6.8) with kernel $g \in L^2[0, 2\pi]$ we define $H : L^2[0, 2\pi] \to H(\partial D)$ by

$$Hg := \left(v_g\big|_{\partial D}, \frac{\partial v_g}{\partial \nu}\bigg|_{\partial D} \right)$$

Corollary 6.23. *Assume that $u_\infty \in L^2[0, 2\pi]$ is in the range of B_0. Then for every $\epsilon > 0$ there exists a $g_\epsilon \in L^2[0, 2\pi]$ such that Hg_ϵ satisfies*

$$\|B_0(Hg_\epsilon) - u_\infty\|_{L^2[0,\, 2\pi]} \le \epsilon.$$

Proof. The proof is a straight forward application of the definition of the space $H(\partial D)$, the continuity of the trace operator and the operator B_0 together with Lemma 6.19. □

Turning to our main goal of finding an approximation to the scattering obstacle D we consider the *far field equation* corresponding to the scattering by an orthotropic medium given by

$$\int_0^{2\pi} u_\infty(\theta, \phi) g(\phi) d\phi = \gamma e^{-ik\hat{x}\cdot z}, \qquad z \in \mathbb{R}^2 \tag{6.44}$$

where $u_\infty(\theta, \phi)$ is the far field pattern of the radiating part of the solution to the forward problem (6.1)–(6.5) corresponding to the incident plane wave with incident direction $d = (\cos\phi, \sin\phi)$ and observation direction $\hat{x} = (\cos\theta, \sin\theta)$. As in Chap. 4 the far field equation can be written in the form

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z), \qquad z \in \mathbb{R}^2$$

where Fg is the far field operator corresponding to the transmission problem (6.1)–(6.5), and $\Phi_\infty(\hat{x}, z)$ is the far field pattern of the fundamental solution $\Phi(x, z)$ to the Helmholtz equation in $\mathbb{R}^2$. We observe that the far field operator Fg can be factored as

$$Fg = B_0(Hg).$$

Hence the far field equation takes the form

$$\left(B_0(Hg)\right)(\hat{x}) = \Phi_\infty(\hat{x}, z), \qquad z \in \mathbb{R}^2 \,. \tag{6.45}$$

As the reader has already encountered in the case of the scattering by an imperfect conductor, the *linear sampling method* is based on the characterization of the domain D by the behavior of a solution to the far field equation (6.45). By definition $B_0(Hg)$ is the far field pattern of the solution (v, u) to the transmission problem (5.13)–(5.17) with boundary data $(f, h) := Hg$. Therefore, for $z \in D$, from Rellich's lemma the far field equation implies that this u coincides with $\Phi(\cdot, z)$ in $\mathbb{R}^2 \setminus \bar{D}$. In other words, for $z \in D$, $g \in L^2[0, 2\pi]$ is a solution to the far field equation if and only if v and $w := v_g$ solve the interior transmission problem

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad \text{in} \quad D \tag{6.46}$$

$$\Delta w + k^2\, w = 0 \qquad \text{in} \quad D \tag{6.47}$$

$$v - w = \Phi(\cdot, z) \qquad \text{on} \quad \partial D \tag{6.48}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial w}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \qquad \text{on} \quad \partial D \tag{6.49}$$

where v_g is the Herglotz wave function with kernel g. In general this is not true. However, in the following we will show that one can construct an approximate solution to the far field equation that behaves in a certain manner.

We first assume that $z \in D$ and that k^2 is not a transmission eigenvalue. Then the interior transmission problem (6.46)–(6.49) has a unique solution (v, w). In this case $(v, \Phi(\cdot, z))$ solves the transmission problem (5.13)–(5.17) with transmission conditions $f := w|_{\partial D}, h := \partial w/\partial \nu|_{\partial D}$. Since the above solution has the far field pattern $\Phi_\infty(\cdot, z)$ we can conclude that $\Phi_\infty(\cdot, z)$ is in the range of B_0. From Corollary 6.23 we can find a g_z^ϵ such that

$$\|B_0(Hg_z^\epsilon) - \Phi_\infty(\cdot, z)\|_{L^2[0,\, 2\pi]} \leq \epsilon \tag{6.50}$$

for an arbitrary small ϵ. Note that the corresponding Herglotz wave function $v_{g_z^\epsilon}$ approximates w in the $H^1(D)$ norm. We now want to show that if z approaches the boundary from the interior of D then the kernel g_z^ϵ and the corresponding Herglotz wave function blow up in the appropriate norms. To this end we choose a sequence of points $\{z_j\}$, $z_j \in D$, such that

$$z_j = z^* - \frac{R}{j}\nu(z^*), \qquad j = 1, 2, \ldots,$$

with sufficiently small R, where $z^* \in \partial D$ and $\nu(z^*)$ is the unit outward normal at z^*. We denote by (v_j, w_j) the solution of (6.46)–(6.49) corresponding to $z = z_j$. As $j \to \infty$ the points z_j approach the boundary point z^* and therefore $\|\Phi(\cdot, z_j)\|_{H^{\frac{1}{2}}(\partial D)} \to \infty$. From the trace theorem and by using the boundary conditions we can write

$$\|v_j\|_{H^1(D)} + \|w_j\|_{H^1(D)} \geq \|v_j - w_j\|_{H^{\frac{1}{2}}(\partial D)} = \|\Phi(\cdot, z_j)\|_{H^{\frac{1}{2}}(\partial D)} . \tag{6.51}$$

In particular we show that the relation (6.51) implies that

$$\lim_{j \to \infty} \|w_j\|_{H^1(D)} = \infty.$$

To this end, we assume on the contrary that

$$\|w_j\|_{H^1(D)} \leq \bar{C}, \qquad j = 1, 2, \ldots,$$

for some positive constant $\bar{C}$. From the trace theorem we have

$$\|w_j\|_{H^{\frac{1}{2}}(\partial D)} \leq \bar{C} \qquad \text{and} \qquad \|\frac{\partial w_j}{\partial \nu}\|_{H^{\frac{1}{2}}(\partial D)} \leq \bar{C}, \qquad j = 1, 2, \ldots.$$

We recall that for every j the pair $(v_j, \Phi(\cdot, z_j))$ is the solution of (5.13)–(5.17) with $(f, g) := \left(w_j|_{\partial D}, \partial w_j/\partial\nu|_{\partial D}\right)$. The a priori estimate (5.41) implies that

$$\begin{aligned} &\|v_j\|_{H^1(D)} + \|\Phi(\cdot, z_j)\|_{H^1(\Omega_R\setminus\bar{D})} \\ &\qquad \leq C\left(\|w_j\|_{H^{\frac{1}{2}}(\partial D)} + \|\frac{\partial w_j}{\partial \nu}\|_{H^{-\frac{1}{2}}(\partial D)}\right) \leq 2C\bar{C}, \end{aligned}$$

which contradicts the fact that $\|\Phi(\cdot, z_j)\|_{H^1(\Omega_R\setminus\bar{D})}$ does not remain bounded as $z_j \to z^* \in \partial D$. So we have that

$$\lim_{j\to\infty} \|w_j\|_{H^1(D)} = \infty.$$

Since for every $j = 1, 2, \ldots$ the corresponding Herglotz wave functions $v_{g^\epsilon_{z_j}}$ satisfying (6.50) approximates the solution w_j in the $H^1(D)$ norm, we conclude that

$$\lim_{j\to\infty} \|v_{g^\epsilon_{z_j}}\|_{H^1(D)} = \infty,$$

and hence

$$\lim_{j\to\infty} \|g^\epsilon_{z_j}\|_{L^2[0,\,2\pi]} = \infty.$$

Next we consider $z \in \mathbb{R}^2\setminus\bar{D}$ and again we assume that k^2 is not a transmission eigenvalue. For these points $\Phi_\infty(\cdot, z)$ does not belong to the range of the operator B_0 because $\Phi(\cdot, z)$ is not a weak solution to the Helmholtz equation in the exterior of D. But from Theorem 6.21 and Theorem 6.22 we can use Tikhonov regularization to construct a regularized solution of the equation

$$B_0(f, h) = \Phi_\infty(\cdot, z)\ . \tag{6.52}$$

In particular, if $(f^\alpha_z, h^\alpha_z) = (w^\alpha(\cdot, z)|_{\partial D}, \partial w^\alpha(\cdot, z)/\partial\nu|_{\partial D}) \in H(\partial D)$ with $w^\alpha(\cdot, z) \in \overline{H}$ is a regularized solution of (6.52) corresponding to the regularization parameter α chosen by a regular regularization strategy, we have

$$\|B_0(f^\alpha_z, h^\alpha_z) - \Phi_\infty(\cdot, z)\|_{L^2[0,\,2\pi]} \leq \delta, \tag{6.53}$$

for an arbitrary small but fixed $\delta > 0$, and

$$\lim_{\alpha\to 0}\left(\|f^\alpha_z\|_{H^{\frac{1}{2}}(\partial D)} + \|h^\alpha_z\|_{H^{-\frac{1}{2}}(\partial D)}\right) = \infty\ . \tag{6.54}$$

Note that $\alpha \to 0$ as $\delta \to 0$. Using Corollary 6.23, for every α and $\epsilon > 0$ we can find a Herglotz wave function $v_{g^{\alpha,\epsilon}_z}$ with kernel $g^{\alpha,\epsilon}_z \in L^2[0,\,2\pi]$ such that

$$\|B_0(H_{g_z^{\alpha,\epsilon}}) - B_0(f_z^\alpha, h_z^\beta)\|_{L^2[0,\,2\pi]} \le \epsilon, \tag{6.55}$$

and thus

$$\|B_0(H_{g_z^{\alpha,\epsilon}}) - \Phi_\infty(\cdot, z)\|_{L^2[0,\,2\pi]} \le \delta + \epsilon\ . \tag{6.56}$$

Moreover, we know that the Herglotz wave function $v_{g_z^{\alpha,\epsilon}}$ approximates $w^\alpha(\cdot, z)$ in $H^1(D)$. Hence the continuity of the trace operator yields

$$\|H_{g_z^{\alpha,\epsilon}} - (f_z^\alpha, h_z^\beta)\|_{H(\partial D)} \le C\|v_{g_z^{\alpha,\epsilon}} - w^\alpha(\cdot, z)\|_{H^1(D)} < \epsilon\ . \tag{6.57}$$

Finally, (6.54) and (6.57) imply that

$$\lim_{\alpha\to 0} \|H_{g_z^{\alpha,\epsilon}}\|_{H(\partial D)} = \infty \quad \text{and} \quad \lim_{\alpha\to 0} \|v_{g_z^{\alpha,\epsilon}}\|_{H^1(D)} = \infty$$

and hence

$$\lim_{\alpha\to 0} \|g_z^{\alpha,\epsilon}\|_{L^2[0,\,2\pi]} = \infty.$$

We summarize the above analysis in the following theorem.

Theorem 6.24. *Let the symmetric matrix-valued function $A = (a_{j,k})_{j,k=1,2}$, $a_{j,k} \in C^1(\bar{D})$, satisfy $\bar{\xi} \cdot Im(A)\,\xi \le 0$ and either $\bar{\xi} \cdot Re(A)\,\xi \ge \gamma|\xi|^2$ or $\bar{\xi} \cdot Re(A^{-1})\,\xi \ge \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and $x \in \overline{D}$ with a constant $\gamma > 1$. Furthermore, let $n \in C(\bar{D})$ be such that $\mathcal{I}m\,(n) \ge 0$ and D be a bounded domain having a C^2-boundary ∂D such that $\mathbb{R}^2 \setminus \bar{D}$ is connected. Assume that k^2 is not a transmission eigenvalue. Then if F is the far field operator (6.7) corresponding to the transmission problem (6.1)–(6.5), we have that*

1. *If $z \in D$ then for every $\epsilon > 0$ there exists a solution $g_z^\epsilon := g_z \in L^2[0,\,2\pi]$ satisfying the inequality*

   $$\|Fg_z - \Phi_\infty(\cdot, z)\|_{L^2[0,\,2\pi]} < \epsilon.$$

   Moreover this solution satisfies

   $$\lim_{z\to\partial D} \|g_z\|_{L^2[0,\,2\pi]} = \infty \quad \textit{and} \quad \lim_{z\to\partial D} \|v_{g_z}\|_{H^1(D)} = \infty,$$

   where v_{g_z} is the Herglotz wave function with kernel g_z.

2. *If $z \in \mathbb{R}^2 \setminus \bar{D}$ then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g_z^{\epsilon,\delta} := g_z \in L^2[0,\,2\pi]$ of the inequality*

   $$\|Fg_z - \Phi_\infty(\cdot, z)\|_{L^2[0,\,2\pi]} < \epsilon + \delta$$

   such that

   $$\lim_{\delta\to 0} \|g_z\|_{L^2[0,\,2\pi]} = \infty \quad \textit{and} \quad \lim_{\delta\to 0} \|v_{g_z}\|_{H^1(D)} = \infty,$$

   where v_{g_z} is the Herglotz wave function with kernel g_z.

The importance of Theorem 6.24 in solving the inverse scattering problem of determining the support D of an orthotropic inhomogeneity from the far field pattern is clear from our discussion in Chap. 4. In particular, by using regularization methods to solve the far field equation $Fg = \Phi_\infty(\cdot, z)$ for z on an appropriate grid containing D, an approximation to g_z can be obtained and hence ∂D can be determined by those points where $\|g_z\|_{L^2[0,\,2\pi]}$ becomes unbounded. More discussion on the numerical implementation is presented in Chap. 8.

7

The Factorization Method

The linear sampling method introduced in Chaps. 4 and 6 is based on the far field equation $Fg = \Phi_\infty(\cdot, z)$, where F is the far field operator corresponding to the scattering problem. In particular, it is shown in Theorem 4.12 and Theorem 6.24 that, in the case of noise free data, for every n there exists an approximate solution $g_n^z \in L^2[0, 2\pi]$ of the far field equation with discrepancy $1/n$ such that the sequence of Herglotz wave functions $v_{g_n^z}$ with kernel g_n^z converges (in an appropriate norm) if and only if $z \in D$ where D is the support of the scattering object. Unfortunately, since the convergence of $v_{g_n^z}$ is described in terms of a norm depending on D, $v_{g_n^z}$ cannot be used to characterize D. Instead, the linear sampling method characterizes the obstacle by the behavior of g_n^z and it is not possible to obtain any convergence result for the regularized solution g of the far field equation if the noise in the data goes to zero. It would be desirable to modify the far field equation in order to avoid this difficulty and this desire motivated Kirsch to introduce in [66] and [67] the *factorization method* for solving both the inverse obstacle scattering problem and the inverse scattering problem for a non-absorbing inhomogeneous medium. In particular, the factorization method replaces the far field operator in the far field equation by the operator $(FF^*)^{1/4}$. One can then show that $(FF^*)^{1/4}g = \Phi_\infty(\cdot, z)$ has a solution if and only if $z \in D$. Despite considerable efforts [68, 70, 49, 50], the factorization method is still limited to a restricted class of scattering problems. In particular, to date the method has not been established for the case of limited aperture data, partially coated obstacles and many of the basic scattering problems for Maxwell's equations (see Chap. 9). On the other hand, when applicable, the factorization method provides a mathematical justification for using the regularized solution of an appropriate far field equation to determine D, a feature which is in general lacking in the linear sampling method.

The plan of this chapter is as follows. We first present some preliminary mathematical result on boundary integral operators and Riesz bases in Hilbert spaces. After establishing some properties of the far field operator, we then proceed to derive the factorization method for the case of a perfect conductor.

7.1 Preliminary Results

We begin with some results on single and double layer potentials. In Sect. 3.3 and Sect. 4.3 we have introduced single and double layer potentials with continuous densities and discussed their continuity properties. In particular, if $D \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary ∂D and ν is the unit outward normal to ∂D, the single layer potential is defined by

$$(\mathcal{S}\psi)(x) := \int_{\partial D} \psi(y)\Phi(x,y)ds_y, \qquad x \in \mathbb{R}^2 \setminus \partial D \tag{7.1}$$

and the double layer potential is defined by

$$(\mathcal{D}\psi)(x) := \int_{\partial D} \psi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y, \qquad x \in \mathbb{R}^2 \setminus \partial D \tag{7.2}$$

where $\Phi(x,y) := i/4H_0^{(1)}(k|x-y|)$ is the fundamental solution to the Helmholtz equation with $H_0^{(1)}$ being a Hankel function of the first kind of order zero. For $x \in \mathbb{R}^2 \setminus \partial D$, both the single and double layer potentials are solutions to the Helmholtz equation and satisfy the Sommerfeld radiation condition. It can be shown [75, 85] that, for $-1 \leq s \leq 1$, the mapping $\mathcal{S} : H^{s-\frac{1}{2}}(\partial D) \to H^{s+1}_{loc}(\mathbb{R}^2)$ is continuous and the mappings $\mathcal{D} : H^{s+\frac{1}{2}}(\partial D) \to H^{s+1}_{loc}(\mathbb{R}^2 \setminus \bar{D})$ and $\mathcal{D} : H^{s+\frac{1}{2}}(\partial D) \to H^{s+1}(D)$ are continuous.
For smooth densities we define the restriction of $\mathcal{S}$ and $\mathcal{D}$ to the boundary ∂D by

$$(S\psi)(x) : = \int_{\partial D} \psi(y)\Phi(x,y)ds_y \qquad x \in \partial D \tag{7.3}$$

$$(K\psi)(x) : = \int_{\partial D} \psi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y \qquad x \in \partial D \tag{7.4}$$

and the restriction of the normal derivative of $\mathcal{S}$ and $\mathcal{D}$ to the boundary ∂D by

$$(K'\psi)(x) : = \frac{\partial}{\partial \nu_x}\int_{\partial D} \psi(y)\Phi(x,y)ds_y \qquad x \in \partial D \tag{7.5}$$

$$(T\psi)(x) : = \frac{\partial}{\partial \nu_x}\int_{\partial D} \psi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y. \qquad x \in \partial D. \tag{7.6}$$

It can be shown [30, 75] that for smooth densities the single layer potential and the normal derivative of the double layer potential are continuous across ∂D, i.e.

$$(\mathcal{S}\psi)_+ = (\mathcal{S}\psi)_- = S\psi \qquad \text{on } \partial D \tag{7.7}$$

$$\frac{\partial(\mathcal{D}\psi)_+}{\partial \nu} = \frac{\partial(\mathcal{D}\psi)_-}{\partial \nu} = T\psi \qquad \text{on } \partial D, \tag{7.8}$$

while the normal derivative of the single layer potential and the double layer potential are discontinuous across ∂D and satisfy the following jump relations

$$\frac{\partial(\mathcal{S}\psi)_\pm}{\partial\nu} = K'\psi \mp \frac{1}{2}\psi \qquad \text{on } \partial D \tag{7.9}$$

$$(\mathcal{D}\psi)_\pm = K\psi \pm \frac{1}{2}\psi \qquad \text{on } \partial D, \tag{7.10}$$

where the subindex $+$ and $-$ indicates that x approaches ∂D from outside and from inside D, respectively. It can be shown that for $-1 \le s \le 1$ (7.7) and (7.9) remain valid for $\psi \in H^{-\frac{1}{2}+s}(\partial D)$, while (7.8) and (7.10) are valid for $\psi \in H^{\frac{1}{2}+s}(\partial D)$, where $u_\pm$ and $\partial u_\pm(x)/\partial\nu$ are interpreted in the sense of trace theorems for $u \in H^{1+s}(\mathbb{R}^2 \setminus \bar{D})$ and $u \in H^{1+s}(D)$, respectively, (see Theorem 1.36 and Theorem 5.5 for the case of $s = 0$). Furthermore, the following operators are continuous (see [60, 85])

$$S : H^{-\frac{1}{2}+s}(\partial D) \longrightarrow H^{\frac{1}{2}+s}(\partial D) \tag{7.11}$$

$$K : H^{\frac{1}{2}+s}(\partial D) \longrightarrow H^{\frac{1}{2}+s}(\partial D) \tag{7.12}$$

$$K' : H^{-\frac{1}{2}+s}(\partial D) \longrightarrow H^{-\frac{1}{2}+s}(\partial D) \tag{7.13}$$

$$T\ :\ H^{\frac{1}{2}+s}(\partial D) \longrightarrow H^{-\frac{1}{2}+s}(\partial D) \tag{7.14}$$

for $-1 \le s \le 1$.

Definition 7.1. *Let X be a Hilbert space equipped with the operation of conjugation and let X^* be its dual. If $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^* we define*

$$(f, u) = \langle \overline{f}, u\rangle \qquad f \in X^*,\ u \in X,$$

where $\langle \overline{f}, u\rangle := \overline{\langle f, \overline{u}\rangle}$.

Definition 7.2. *Let X and Y be Hilbert spaces and $A : X \to Y$ be a linear operator. We define the adjoint operator $A^* : Y^* \to X^*$ by*

$$(A^*v, u) = (v, Au)\,, \qquad v \in Y^*, u \in X$$

where X^* and Y^* are the duals of X and Y, respectively, and $(\cdot,\cdot)$ is defined by Definition 7.1.

Note that this definition of the adjoint is consistent with that given in Chap. 1. Furthermore, up to conjugation, A^* is the same as the transpose operator $A^\top$ defined in Sect. 6.4.

Theorem 7.3. *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D.*

1. *Let S_i be the boundary operator defined by (7.3) with k replaced by i in the fundamental solution. Then S_i satisfies*

$$(S_i\psi, \psi) \ge C\|\psi\|^2_{H^{-\frac{1}{2}}(\partial D)}, \ \psi \in H^{-\frac{1}{2}}(\partial D)$$

   where $(\cdot,\,\cdot)$ is defined by Definition 7.1.

2. $S - S_i$ *is compact from* $H^{-\frac{1}{2}}(\partial D)$ *to* $H^{\frac{1}{2}}(\partial D)$.
3. S *is an isomorphism from* $H^{-\frac{1}{2}}(\partial D)$ *onto* $H^{\frac{1}{2}}(\partial D)$.
4. $Im(S\psi, \psi) = 0$ *for some* $\psi \in H^{-\frac{1}{2}}(\partial D)$ *implies* $\psi = 0$.

Proof. Let $v \in H^1_{loc}(\mathbb{R}^2 \setminus \partial D)$ be the single layer potential given by

$$v(x) := \int_{\partial D} \psi(y)\Phi(x,y)\, ds(y), \qquad \psi \in H^{-\frac{1}{2}}(\partial D), \qquad x \in \mathbb{R}^2 \setminus \partial D.$$

In particular, v satisfies the Helmholtz equation in D and $\mathbb{R}^2 \setminus \bar{D}$ and the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial v}{\partial r} - ikv \right) = 0.$$

1. Set $k = i$ in the definition of v. Applying Green's first identity to v and $\overline{v}$ in D and $\Omega_R \setminus \bar{D}$ where Ω_R is a disk of radius R centered at the origin containing D, and using (7.7) and (7.9), we have that

$$(S_i\psi, \psi) = \left\langle \overline{v}, \left(\frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu} \right) \right\rangle = \int_D \left(|\nabla v|^2 + |v|^2 \right) dx$$
$$+ \int_{\Omega_R \setminus \bar{D}} \left(|\nabla v|^2 + |v|^2 \right) dx - \int_{|x|=R} \overline{v} \frac{\partial v}{\partial r} ds.$$

From the Sommerfeld radiation condition we obtain

$$(S_i\psi, \psi) = \int_D \left(|\nabla v|^2 + |v|^2 \right) dx + \int_{\Omega_R \setminus \bar{D}} \left(|\nabla v|^2 + |v|^2 \right) dx$$
$$+ \int_{|x|=R} |v|^2 ds + o(1)$$

and letting $R \to \infty$, noting that v decays exponentially, we have that

$$(S_i\psi, \psi) = \int_{\mathbb{R}^2} \left(|\nabla v|^2 + |v|^2 \right) dx \, . \tag{7.15}$$

Furthermore, from the jump properties of v across the boundary and the trace Theorem 5.5, we can write

$$\|\psi\|_{H^{-\frac{1}{2}}(\partial D)} = \left\| \frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial D)} \le \tilde{C} \|v\|_{H^1(\mathbb{R}^2)} \tag{7.16}$$

where $\tilde{C} > 0$ and hence combining (7.15) and (7.16) we have that

$$(S_i\psi, \psi) \ge C \|\psi\|_{H^{-\frac{1}{2}}(\partial D)}, \qquad C > 0.$$

2. The kernel of $S - S_i$ is a C^∞ function in a neighborhood of $\partial D \times \partial D$ and hence by applying the same argument as in the first part of Theorem 4.8 we conclude that $S - S_i$ is compact from $H^{-\frac{1}{2}}(\partial D)$ to $H^{\frac{1}{2}}(\partial D)$.
3. Applying the Lax-Milgram lemma to the bounded and coercive sesquilinear form
$$a(\psi, \phi) := (S_i \psi, \phi)\,, \qquad \phi, \psi \in H^{-\frac{1}{2}}(\partial D)$$
we conclude that $S_i^{-1} : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ exists and is bounded. From Theorem 5.14 and using part 2, S is an isomorphism if and only if S is injective. To show that S is injective, we consider $\psi \in H^{-\frac{1}{2}}(\partial D)$ such that $S\psi = 0$. Since the single layer potential v is a radiating solution to the homogeneous Dirichlet boundary value problem in $\mathbb{R}^2 \setminus \bar{D}$, $v = 0$ in $\mathbb{R}^2 \setminus \bar{D}$. Similarly, v satisfies the homogeneous Dirichlet boundary value problem in D and from the assumption that k^2 is not a Dirichlet eigenvalue we conclude that $v = 0$ in D as well. Finally
$$\psi = \frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu} = 0$$
which proves that S is injective.
4. Let $\operatorname{Im}(S\psi, \psi) = 0$ for some $\psi \in H^{-\frac{1}{2}}(\partial D)$. The same argument as in part 1 yields
$$\begin{aligned}(S\psi, \psi) &= \left\langle \overline{v}, \left(\frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu}\right)\right\rangle = \int\limits_D \left(|\nabla v|^2 - k^2 |v|^2\right) dx \\ &+ \int\limits_{\Omega_R \setminus \bar{D}} \left(|\nabla v|^2 - k^2 |v|^2\right) dx - \int\limits_{|x|=R} \overline{v} \frac{\partial v}{\partial r} ds \\ &= \int\limits_{\Omega_R} \left(|\nabla v|^2 - k^2 |v|^2\right) dx - ik \int\limits_{|x|=R} |v|^2 \, ds + o(1), \quad R \to \infty\end{aligned}$$
Taking the imaginary part we see that
$$0 = \operatorname{Im}\,(S\psi, \psi) = -k \lim_{R\to\infty} \int\limits_{|x|=R} |v|^2 \, ds.$$
Rellich's lemma implies that v vanishes in $\mathbb{R}^2 \setminus \bar{D}$ and thus $S\psi = 0$ on ∂D by the trace theorem (Theorem 1.36). Finally, since S is an isomorphism, we can conclude that $\psi = 0$.

$\square$

Remark 7.4. Property 1 in Theorem 7.3 implies that there exists a square root $S_i^{\frac{1}{2}}$ of S_i and $S_i^{\frac{1}{2}}$ is an isomorphism from $H^{-\frac{1}{2}}(\partial D)$ onto $L^2(\partial D)$ and from $L^2(\partial D)$ onto $H^{\frac{1}{2}}(\partial D)$ (see Sect. 9.4 in [60]). Furthermore $S_i^{\frac{1}{2}}$ is positive definite using the duality defined by Definition 7.1 and self-adjoint, i.e. $S_i^{\frac{1}{2}} = S_i^{\frac{1}{2}*}$, where the adjoint operator is defined by Definition 7.2

In a similar way as in Theorem 7.3 one can show the following properties for the operator T.

Theorem 7.5. *Assume that k^2 is not a Neumann eigenvalue of $-\Delta$ in D.*

1. *Let T_i be the boundary operator defined by (7.6) with k replaced by i in the fundamental solution. Then T_i satisfies*
$$-(T_i\psi, \psi) \geq C\|\psi\|^2_{H^{\frac{1}{2}}(\partial D)} \qquad \text{for all } \psi \in H^{\frac{1}{2}}(\partial D)$$
where $(\cdot, \cdot)$ is defined by Definition 7.1.
2. *$T - T_i$ is compact from $H^{\frac{1}{2}}(\partial D)$ to $H^{-\frac{1}{2}}(\partial D)$.*
3. *T is an isomorphism from $H^{\frac{1}{2}}(\partial D)$ onto $H^{-\frac{1}{2}}(\partial D)$.*
4. *$Im(T\psi, \psi) = 0$ for some $\psi \in H^{\frac{1}{2}}(\partial D)$ implies $\psi = 0$.*

We now turn our attention to the concept of a Riesz basis in a Hilbert space. Let X be a Hilbert space. A sequence $\{\phi_n\}_1^\infty$ in said to be a *Schauder basis* for X if for each vector $u \in X$ there exists a unique sequence of complex numbers $c_1, c_2, \dots$ such that $u = \sum_1^\infty c_n\phi_n$ where the converges is understood as
$$\lim_{k\to\infty} \left\| u - \sum_1^k \phi_n \right\|_X = 0.$$
In particular, a complete orthonormal system is a Schauder basis for X. The simplest way of constructing a new basis from an old is one through an isomorphism. In particular, let $\{\phi_n\}_1^\infty$ be a basis in X and $T : X \to X$ be a bounded linear operator with bounded inverse. Then $\{\psi_n\}_1^\infty$ such that $\psi_n = T\phi_n$, $n = 1, 2, \cdots$ is also a basis for X.

Definition 7.6. *Two bases $\{\phi_n\}_1^\infty$ and $\{\psi_n\}_1^\infty$ are said to be equivalent if $\sum_1^\infty c_n\phi_n$ converges if and only if $\sum_1^\infty c_n\psi_n$ converges.*

It can be shown [115] that

Theorem 7.7. *Two bases $\{\phi_n\}_1^\infty$ and $\{\psi_n\}_1^\infty$ are equivalent if and only if there exists a bounded linear operator $T : X \to X$ with bounded inverse such that $\psi_n = T\phi_n$ for every n.*

In Hilbert spaces the most important bases are orthonormal bases thanks to their nice properties (see Theorem 1.13). Second in importance are those bases that are equivalent to some orthonormal basis. They will be called *Riesz bases.*

Definition 7.8. *A basis for a Hilbert space is a Riesz basis if it is equivalent to an orthonormal basis, that is, if it is obtained from an orthonormal basis by means of bounded invertible linear operator.*

Definition 7.9. *Two inner products $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ in a Hilbert space X are said to be equivalent if $c\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1$ for some positive constants c, C where $\|\cdot\|_j$, $j = 1, 2$, is the norm generated by $(\cdot, \cdot)_j$.*

The next theorem provides some important properties of Riesz bases.

Theorem 7.10. *Let X be a Hilbert space. Then the following statements are equivalent.*

1. *The sequence $\{\phi_n\}_1^\infty$ forms a Riesz basis for X.*
2. *There exists an equivalent inner product on X with respect to which the sequence $\{\phi_n\}_1^\infty$ becomes an orthonormal basis for X.*
3. *The sequence $\{\phi_n\}_1^\infty$ is complete in X and there exists positive constants c and C such that for an arbitrary positive integer k and arbitrary complex numbers $c_1, \ldots, c_k$ one has*

$$c \sum_1^k |c_n|^2 \le \left\| \sum_1^k c_n \phi_n \right\|^2 \le C \sum_1^k |c_n|^2.$$

Proof. $1 \Longrightarrow 2$: Since $\{\phi_n\}_1^\infty$ is a Riesz basis for X, there exists a bounded linear operator T with bounded inverse that transforms $\{\phi_n\}_1^\infty$ into some orthonormal basis $\{e_n\}_1^\infty$, i.e. $T\phi_n = e_n$, $n = 1, 2, \cdots$. Define a new inner product $(\cdot,\, \cdot)_1$ on X by setting

$$(\phi,\, \psi)_1 = (T\phi,\, T\psi), \qquad \phi, \psi \in X$$

and let $\|\cdot\|_1$ be the norm generated by this inner product. Then

$$\frac{\|\phi\|}{\|T^{-1}\|} \le \|\phi\|_1 \le \|T\| \|\phi\|$$

for every $\phi \in X$. Hence the new inner product is equivalent to the original one. Clearly,

$$(\phi_n,\, \phi_m)_1 = (T\phi_n,\, T\phi_m) = (e_n,\, e_m) = \delta_{nm}$$

for every n and m, where $\delta_{nm} = 0$ for $n \neq m$ and $\delta_{nm} = 1$ for $n = m$.
$2 \Longrightarrow 3$: Suppose that $(\cdot,\, \cdot)_1$ is an equivalent inner product on X and $\{\phi_n\}_1^\infty$ is an orthonormal basis with respect to $(\cdot,\, \cdot)_1$. From the relation

$$c\|\phi\|_1 \le \|\phi\|_2 \le C\|\phi\|_1$$

where c, C are positive constants, it follows that for arbitrary complex numbers $c_1, \ldots, c_k$ one has

$$\frac{1}{C^2} \sum_1^k |c_n|^2 \le \left\| \sum_1^k c_n \phi_n \right\|^2 \le \frac{1}{c^2} \sum_1^k |c_n|^2.$$

Clearly, from Theorem 1.13, $\{\phi_n\}_1^\infty$ is complete in X.
$3 \Longrightarrow 1$: Let $\{e_n\}_1^\infty$ be an arbitrary orthonormal basis for X. We define operators T and S on the subset of linear combinations of $\{e_n\}_1^\infty$ and $\{\phi_n\}_1^\infty$ by

$$T\sum_1^k c_n e_n = \sum_1^k c_n \phi_n, \qquad S\sum_1^k c_n \phi_n = \sum_1^k c_n e_n.$$

It follows by assumption that T and S are bounded on their domain of definition. Since both $\{e_n\}_1^\infty$ and $\{\phi_n\}_1^\infty$ are complete in X (Theorem 1.13), each of the operators T and S can be extended by continuity to bounded linear operators defined on the entire space X. It is easily seen that $ST = TS = I$, whence $T^{-1} = S$. Hence $\{\phi_n\}_1^\infty$ is a Riesz basis for X. □

For a more comprehensive study of the Riesz basis we refer the reader to [115].

We end this section with a result on Riesz basis due to Kirsch [66] which will later play an important role in the factorization method.

Theorem 7.11. *Let X be a Hilbert space. Assume that $K : X \to X$ is a compact linear operator with* $\mathrm{Im}(K\phi,\, \phi) \neq 0$ *for all $\phi \in X$, $\phi \neq 0$. Let $\{\phi_n\}_1^\infty$ be a linearly independent and complete sequence in X which is orthogonal in the sense that*

$$((I + K)\phi_n,\, \phi_m) = c_n \delta_{nm} \tag{7.17}$$

where $(\cdot,\, \cdot)$ is the inner product on X and the constants c_n are such that $\mathrm{Im}(c_n) \to 0$ *as $n \to \infty$ and there exists a positive constant $r > 0$ independent of n such that $|c_n| = r$ for all $n = 1, 2, \ldots$. Then $\{\phi_n\}_1^\infty$ is a Riesz basis.*

Proof. The proof consists of several steps.

1. We first show that the sequence $\{\phi_n\}_1^\infty$ is bounded. Assume on the contrary that there exists a subsequence, still denoted by $\{\phi_n\}_1^\infty$, such that $\|\phi_n\| \to \infty$. Set $\hat\phi_n = \phi_n/\|\phi_n\|$ and note that

$$1 + (K\hat\phi_n, \hat\phi_n) = \Big((I + K)\hat\phi_n, \hat\phi_n\Big) \to 0 \qquad \text{as } n \to \infty\,. \tag{7.18}$$

   Since $\{\hat\phi_n\}_1^\infty$ is bounded, there exists a subsequence, still denoted by $\{\hat\phi_n\}_1^\infty$, that converges weakly to a $\hat\phi \in X$. Since K is compact we have that $\|K\hat\phi_n - K\hat\phi\| \to 0$ for a further subsequence, still denoted by $\{\hat\phi_n\}_1^\infty$. Hence $(K\hat\phi_n,\, \hat\phi_n) = (K\hat\phi_n - K\hat\phi, \hat\phi_n) + (K\hat\phi, \hat\phi_n) \to (K\hat\phi, \hat\phi)$ as $n \to \infty$. Then (7.18) implies that $1 + (K\hat\phi, \hat\phi) = 0$. Taking the imaginary part we see that $\mathrm{Im}(K\hat\phi, \hat\phi) = 0$ and thus $\hat\phi = 0$ which contradicts the fact that $1 + (K\hat\phi, \hat\phi) = 0$.
2. We next show that r is the only accumulation point of $\{c_n\}_1^\infty$. To this end we notice that the conditions on c_n implies that $\pm r$ are the only possible accumulation points of the sequence $\{c_n\}_1^\infty$. Assume now that there exists a subsequence, still denoted by $\{c_n\}_1^\infty$, such that $\{c_n\}_1^\infty \to -r$ as $n \to \infty$. Since from the previous step $\{\phi_n\}_1^\infty$ is bounded there exists a subsequence, still denoted by $\{\phi_n\}_1^\infty$, such that $\phi_n \rightharpoonup \phi$ weakly. As in step 1 we conclude that $(K\phi_n,\, \phi_n) \to (K\phi, \phi)$ and thus from (7.17)

$$\mathrm{Im}(c_n) = \mathrm{Im}(K\phi_n, \phi_n) \to \mathrm{Im}(K\phi, \phi).$$

On the other hand since $\mathrm{Im}(c_n) \to 0$ we obtain that $\mathrm{Im}(K\phi, \phi) = 0$ and hence $\phi = 0$. Another application of (7.17) implies that $\|\phi_n\|^2 \to -r$ which is impossible since $r > 0$. Thus we have shown that $c_n \to r$. In particular, there exists an integer n_0 such that $\mathrm{Re}(c_n) \geq r/2$ for all $n \geq n_0$.

3. We define the closed subspace $U \subset X$ by

$$U := \{\phi \in X : \, ((I+K)\phi, \, \phi_m) = 0 \qquad \text{for } m = 1, \ldots, n_0 - 1\} .$$

We will show that the set $\{\phi_n : n \geq n_0\}$ is complete in U. To this end, we first note that from (7.17) $\phi_n \in U$ for $n \geq n_0$. For a given $\phi \in U$, since $\{\phi_n\}_1^\infty$ is complete in X, there exists $\alpha_n^{(k)} \in \mathbb{C}$, $n = 1, \ldots, k$, and $k \in \mathbb{N}$ such that

$$\sum_{n=1}^{n_0-1} \alpha_n^{(k)} \phi_n + \sum_{n=n_0}^{k} \alpha_n^{(k)} \phi_n \to \phi \quad \text{as } k \to \infty.$$

Applying $I + K$ and taking the inner product of the result with ϕ_m, $m = 1, \ldots, n_0 - 1$, from the continuity of K and of the inner product we obtain

$$\sum_{n=1}^{n_0-1} \alpha_n^{(k)} \underbrace{((I+K)\phi_n, \, \phi_m)}_{=c_n\delta_{nm}} + \sum_{n=n_0}^{k} \alpha_n^{(k)} \underbrace{((I+K)\phi_n, \, \phi_m)}_{=0} \to \underbrace{((I+K)\phi, \, \phi_m)}_{=0}$$

and thus $\alpha_n^{(k)} \to 0$ as $k \to 0$ for every $n = 1, \ldots, n_0 - 1$. This implies that

$$\sum_{n=1}^{n_0-1} \alpha_n^{(k)} \phi_n \to 0 \quad \text{as } k \to \infty,$$

whence

$$\sum_{n=n_0}^{k} \alpha_n^{(k)} \phi_n \to \phi \quad \text{as } k \to \infty$$

and hence $\mathrm{span}\{\phi_n : n \geq n_0\}$ is dense in U.

4. In the next step we show that there exists a $C > 0$ such that

$$\mathrm{Re}\left((I+K)\phi, \, \phi\right) \geq C\|\phi\|^2 \qquad \text{for all } \phi \in U . \tag{7.19}$$

To this end, we first claim that

$$\mathrm{Re}\left((I+K)\phi, \, \phi\right) > 0 \qquad \text{for all } \phi \in U.$$

Indeed, from step 2 we know that

$$\mathrm{Re}\left((I+K)\phi_n, \, \phi_n\right) = \mathrm{Re}(c_n) > 0 \qquad \text{for } n \geq n_0.$$

The orthogonality relation (7.17) yields

$$\mathrm{Re}\left((I+K)\sum_1^k \alpha_n\phi_n,\, \sum_1^k \alpha_n\phi_n\right) = \sum_1^k \mathrm{Re}(c_n)|\alpha_n|^2 > 0$$

and the completeness of $\{\phi_n : n \geq n_0\}$ in U proves the claim. Having proved that $\mathrm{Re}\,((I+K)\phi,\, \phi) > 0$, we now suppose on the contrary that (7.19) is not true. Then there exists a sequence $\{\phi^{(j)}\}$, $\phi^{(j)} \in U$, with $\|\phi^{(j)}\| = 1$ satisfying

$$\mathrm{Re}\left((I+K)\phi^{(j)},\, \phi^{(j)}\right) \to 0 \quad \text{as} \quad j \to \infty.$$

By the completeness of $\{\phi_n : n \geq n_0\}$ in U we can assume without loss of generality that $\phi^{(k)}$ is of the form

$$\phi^{(j)} = \sum_{n=n_0}^{k_j} \alpha_n^{(j)}\phi_n, \qquad \alpha_n^{(j)} \in \mathbb{C}.$$

From the orthogonality relation (7.17) we have that

$$\begin{aligned}\left((I+K)\phi^{(j)},\, \phi^{(j)}\right) &= \left((I+K)\sum_{n=n_0}^{k_j} \alpha_n^{(j)}\phi_n,\, \sum_{n=n_0}^{k_j} \alpha_n^{(j)}\phi_n\right) \\ &= \sum_{n,m=n_0}^{k_j} \alpha_n^{(j)}\overline{\alpha}_m^{(j)}\left((I+K)\phi_n,\, \phi_m\right) = \sum_{n=n_0}^{k_j} c_n|\alpha_n^{(j)}|^2.\end{aligned}$$

Taking the real part we now have that

$$\sum_{n=n_0}^{k_j} \mathrm{Re}(c_n)|\alpha_n^{(j)}|^2 \to 0 \qquad \text{as} \quad j \to \infty.$$

Since from step 2 we have that $r/2 \leq \mathrm{Re}(c_n) \leq r$, this implies that

$$\sum_{n=n_0}^{k_j} |\alpha_n^{(j)}|^2 \to 0 \qquad \text{as} \quad j \to \infty,$$

whence

$$\left((I+K)\phi^{(j)},\, \phi^{(j)}\right) \to 0 \qquad \text{as} \quad j \to \infty\,. \tag{7.20}$$

Now we proceed as in step 1, where we replace $\hat{\phi}_n$ by $\phi^{(j)}$, to conclude that a subsequence of $\phi^{(j)}$, still denoted by $\phi^{(j)}$, converges weakly to an element ϕ and consequently $(K\phi^{(j)},\, \phi^{(j)}) \to (K\phi,\, \phi)$. From (7.20) we conclude that $\mathrm{Im}(K\phi,\, \phi) = 0$ which implies that $\phi = 0$. From (7.20) again we have that $\|\phi^{(j)}\| \to 0$ which contradicts the fact that $\|\phi^{(j)}\| = 1$.

5. We now define the self-adjoint operator

$$T := I + \frac{1}{2}(K + K^*)$$

and observe that T is strictly coercive in U since

$$\begin{aligned}(T\phi, \phi) &= \frac{1}{2}\left((I+K)\phi, \phi\right) + \frac{1}{2}\left((I+K^*)\phi, \phi\right) \\ &= \frac{1}{2}\left((I+K)\phi, \phi\right) + \frac{1}{2}\left(\phi, (I+K)\phi\right) \\ &= \mathrm{Re}\left((I+K)\phi, \phi\right) \geq C\|\phi\| \qquad \text{for all } \ \phi \in U.\end{aligned}$$

Hence from the Lax-Milgram lemma T is an isomorphism on U and the bilinear form

$$(\phi, \psi)_1 := (T\phi, \psi)$$

defines an inner product on U and $(\cdot, \cdot)_1$ is equivalent to the original inner product. Furthermore, the set $\{\phi_n : \ n \geq n_0\}$ is orthogonal with respect to $(\cdot, \cdot)_1$ since

$$\begin{aligned}(\phi_n, \phi_m)_1 = (T\phi_n, \phi_m) &= \frac{1}{2}\left((I+K)\phi_n, \phi_m\right) + \frac{1}{2}\overline{((I+K)\phi_n, \phi_m)} \\ &= \mathrm{Re}(c_n)\delta_{nm} \qquad \text{for } n, m > n_0.\end{aligned}$$

Hence, $\{\phi_n/\sqrt{\mathrm{Re}(c_n)} : \ n \geq n_0\}$ is a complete orthonormal system in U. Obviously, from Theorem 1.13, for every $\phi \in U$

$$\phi = \sum_{n_0}^{\infty} \frac{(\phi, \phi_n)_1}{\mathrm{Re}(c_n)}\phi_n = \sum_{n_0}^{\infty} \frac{(T\phi, \phi_n)}{\mathrm{Re}(c_n)}\phi_n$$

and Parseval's equality gives

$$\|\phi\|^2 = \sum_{n_0}^{\infty} \frac{|(T\phi, \phi_n)|^2}{\mathrm{Re}(c_n)}.$$

In particular, from Theorem 7.10, the set $\{\phi_n : \ n \geq n_0\}$ forms a Riesz basis for U.

6. Finally, we show that every element $\phi \in X$ can be expanded in a series of the ϕ_n. Let $\phi \in X$, define

$$\phi^{\{1\}} := \sum_{1}^{n_0 - 1} \frac{((I+K)\phi, \phi_n)}{c_n}\phi_n$$

and set $\phi^{\{2\}} := \phi - \phi^{\{1\}}$. One can easily see that $\phi^{\{2\}} \in U$ since for $m = 1, \ldots n_0 - 1$

$$\left((I+K)\phi^{\{2\}},\, \phi_m\right) = ((I+K)\phi,\, \phi_m) - \sum_{1}^{n_0-1} \frac{((I+K)\phi, \phi_n)}{c_n} \underbrace{((I+K)\phi_n,\, \phi_m)}_{c_n\delta_{nm}} = 0.$$

Hence by step 5

$$\phi = \underbrace{\sum_{n_0}^{\infty} \alpha_n \phi_n}_{=\phi^{\{2\}}} + \underbrace{\sum_{1}^{n_0-1} \alpha_n \phi_n}_{=\phi^{\{1\}}} .$$

Thus $X = U \oplus V$ where V is the finite dimensional space of linear combinations of ϕ_n for $n = 1, \ldots, n_0 - 1$. From step 5, the fact that V is finite dimensional and the fact that the sum $X = U \oplus V$ is direct (i.e. every $\phi \in X$ can be uniquely written as $\phi = \phi^{\{1\}} + \phi^{\{2\}}$, where $\phi^{\{1\}} \in V$ and $\phi^{\{2\}} \in U$), it is easily seen that $\{\phi_n\}$ forms a Riesz basis for X. The proof is now finished.

□

7.2 Properties of the Far Field Operator

We shall now prove some important properties of the far field operator in the case when the scattering obstacle is a perfect conductor. In particular, consider the direct scattering problem of finding the total field u such that

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \tag{7.21}$$

$$u(x) = u^s(x) + u^i(x) \tag{7.22}$$

$$u = 0 \quad \text{on } \partial D \tag{7.23}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{7.24}$$

where $u^s := u^s(\cdot, \phi)$ is the scattered field due to the incident plane wave $u^i(x) = e^{ikx\cdot d}$ propagating in the incident direction $d = (\cos\phi, \sin\phi)$. This scattering problem is a particular case of the following exterior Dirichlet problem: given $f \in H^{\frac{1}{2}}(\partial D)$ find $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ such that

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \tag{7.25}$$

$$u = f \quad \text{on } \partial D \tag{7.26}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{7.27}$$

which is shown in Example 5.21 to be well-posed. In particular, the scattered field u^s satisfies (7.25)–(7.27) with $f = -e^{ikx\cdot d}|_{\partial D}$.

The reader has already seen that the Sommerfeld radiation condition implies that a radiating solution u to the Helmholtz equation has the asymptotic behavior

$$u(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\theta) + O(r^{-3/2}) \qquad r = |x| \to \infty \tag{7.28}$$

uniformly in all directions $\hat{x} = (\cos\theta, \sin\theta)$, where $u_\infty(\theta)$ is the far field pattern given by

$$u_\infty(\theta) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \left(u(y) \frac{\partial e^{iky\cdot\hat{x}}}{\partial \nu} - \frac{\partial u(y)}{\partial \nu} e^{iky\cdot\hat{x}} \right) ds(y)\,. \tag{7.29}$$

Now, let $F : L^2[0,\, 2\pi] \to L^2[0,\, 2\pi]$ be the far field operator corresponding to the scattering problem (7.21)–(7.24) given by

$$(Fg)(\theta) := \int_0^{2\pi} u_\infty(\theta, \phi) g(\phi)\, d\phi$$

where $u_\infty(\theta, \phi)$ is the far field pattern of $u^s(x, \phi)$.
In the same way as in Theorem 4.2 one can establish the following theorem:

Theorem 7.12. *The far field pattern $u_\infty(\theta, \phi)$ corresponding to the scattering problem (7.21)–(7.24) satisfies the reciprocity relation*

$$u_\infty(\theta, \phi) = u_\infty(\phi + \pi, \theta + \pi).$$

Using the reciprocity relation, one can now show exactly in the same way as in Theorem 4.3 that the following result is true:

Theorem 7.13. *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then the far field operator corresponding to scattering problem (7.21)–(7.24) is injective with dense range.*

We now want to establish the fact that the far field operator F corresponding to the scattering problem (7.21)–(7.24) is *normal* , i.e. $F^*F = FF^*$ where F^* is the L^2-adjoint of F. To this end, we need the following basic identity [25, 31, 32].

Theorem 7.14. *Let $F : L^2[0,\, 2\pi] \to L^2[0,\, 2\pi]$ be the far field operator corresponding to the scattering problem (7.21)–(7.24). Then for every $g, h \in L^2[0,\, 2\pi]$ we have*

$$\sqrt{2\pi k}\, e^{-i\pi/4}\, (Fg, h) = \sqrt{2\pi k}\, e^{+i\pi/4}\, (g, Fh) + ik\, (Fg, Fh)\,.$$

Proof. We first note that if u and w are two radiating solutions of the Helmholtz equation with far field patterns u_∞ and w_∞, then from Green's second identity and the uniformity of the asymptotic relation (7.28) we have that

$$\int_{\partial D} \left(u \frac{\partial \overline{w}}{\partial \nu} - \overline{w} \frac{\partial u}{\partial \nu} \right) ds = -2ik \int_0^{2\pi} u_\infty \overline{w}_\infty d\theta \, . \tag{7.30}$$

If v_g is a Herglotz wave function with kernel g given by

$$v_g(x) = \int_0^{2\pi} g(\phi) e^{ikx\cdot d} d\phi, \qquad d := (\cos\phi, \sin\phi)$$

we have

$$\begin{aligned}\int_{\partial D} \left(u \frac{\partial \overline{v_g}}{\partial \nu} - \overline{v_g} \frac{\partial u}{\partial \nu} \right) ds &= \int_0^{2\pi} \overline{g(\phi)} \int_{\partial D} \left(u \frac{\partial e^{-ikx\cdot d}}{\partial \nu} - \frac{\partial u}{\partial \nu} e^{-ikx\cdot d} \, ds \right) d\phi \\ &= \sqrt{8\pi k}\, e^{-i\pi/4} \int_0^{2\pi} \overline{g(\phi)} u_\infty(\phi) d\phi. \end{aligned} \tag{7.31}$$

Now let v_g and v_h be Herglotz functions with kernels $g, h \in L^2[0,\, 2\pi]$, respectively. Let u_g^s and u_h^s be the corresponding scattered fields, i.e. u_g^s and u_h^s satisfy (7.21)–(7.24) with u^i replaced by v_g and v_h, respectively, and denote by $u_{g,\infty}$ and $u_{h,\infty}$ the corresponding far field patterns. Then from (7.30) and (7.31) we have

$$\begin{aligned} 0 &= \int_{\partial D} \left((u_g^s + v_g) \frac{\partial \overline{(u_h^s + v_h)}}{\partial \nu} - \overline{(u_h^s + v_h)} \frac{\partial (u_g^s + v_g)}{\partial \nu} \right) ds \\ &= \int_{\partial D} \left(u_g^s \frac{\partial \overline{u_h^s}}{\partial \nu} - \overline{u_h^s} \frac{\partial u_g^s}{\partial \nu} \right) ds + \int_{\partial D} \left(u_g^s \frac{\partial \overline{v_h}}{\partial \nu} - \overline{v_h} \frac{\partial u_g^s}{\partial \nu} \right) ds \\ &+ \int_{\partial D} \left(v_g \frac{\partial \overline{u_h^s}}{\partial \nu} - \overline{u_h^s} \frac{\partial v_g}{\partial \nu} \right) ds \\ &= -2ik \int_0^{2\pi} u_{g,\infty} \overline{u_{h,\infty}} d\phi + \sqrt{8\pi k} e^{-i\pi/4} \int_0^{2\pi} \overline{h} u_{g,\infty} d\phi - \sqrt{8\pi k} e^{i\pi/4} \int_0^{2\pi} g \overline{u_{h,\infty}} d\phi \\ &= -2ik\,(Fg, Fh) + \sqrt{8\pi k}\, e^{-i\pi/4}\,(Fg, h) - \sqrt{8\pi k}\, e^{i\pi/4}\,(g, Fh)\,. \end{aligned}$$

and the proof is complete. □

Theorem 7.15. *The far field operator corresponding to the scattering problem (7.21)–(7.24) is normal, i.e.* $FF^* = F^*F$.

Proof. From Theorem 7.14 we have that

$$(g,\, ikF^*Fh) = \sqrt{2\pi k} \left(e^{+i\pi/4}\,(g, Fh) - e^{-i\pi/4}\,(g, F^*h) \right)$$

for all h and g in $L^2[0, 2\pi]$, and hence

$$ikF^*F = \sqrt{2\pi k}\left(e^{-i\pi/4}F - e^{+i\pi/4}F^*\right) . \tag{7.32}$$

Using the reciprocity relation, as in the proof of the first part of Theorem 4.3 we see that

$$(F^*g)(\theta) = \overline{RFR\overline{g}}$$

where $R : L^2[0, 2\pi] \to L^2[0, 2\pi]$ defines the reflection property $(Rg)(\phi) = g(\phi + \pi)$. From this, observing that $(Rg, Rh) = (g, h) = (\overline{h}, \overline{g})$, we find that

$$(F^*g, F^*h) = (RFR\overline{h}, RFR\overline{g}) = (FR\overline{h}, FR\overline{g}),$$

and hence, using Theorem 7.14 again,

$$\begin{aligned} ik\,(F^*g, F^*h) &= \sqrt{2\pi k}\left\{\left(e^{-i\pi/4}FR\overline{h}, R\overline{g}\right) - e^{+i\pi/4}\left(R\overline{h}, FR\overline{g}\right)\right\} \\ &= \sqrt{2\pi k}\left\{e^{-i\pi/4}\,(g, F^*h) - e^{+i\pi/4}\,(F^*g, h)\right\}. \end{aligned}$$

If we now proceed as in the derivation of (7.32) we find that

$$ikFF^* = \sqrt{2\pi k}\left(e^{-i\pi/4}F - e^{+i\pi/4}F^*\right) \tag{7.33}$$

and the proof is finished. □

Assuming that k^2 is not a Dirichlet eigenvalue for $-\Delta$, it can be shown that, since F is normal and injective, there exists a countable number of eigenvalues $\lambda_j \in \mathbb{C}$ of F with $\lambda_j \neq 0$ and the corresponding eigenvectors ψ_j form a complete orthonormal system for $L^2[0, 2\pi]$ [104]. From Theorem 7.14 we see that the eigenvalues of the far field operator F lie on the circle of radius $\sqrt{2\pi/k}$ with center at $e^{3\pi i/4}\sqrt{2\pi/k}$.

Of importance in studying the far field operator is the operator $B : H^{\frac{1}{2}}(\partial D) \to L^2[0, 2\pi]$ defined by $Bf = u_\infty$ where u_∞ is the far field pattern of the radiating solution u to (7.25)–(7.27) with boundary data $f \in H^{\frac{1}{2}}(\partial D)$. We leave to the reader as an exercise to prove, in the same way as Theorem 4.8, the following properties of the operator B.

Theorem 7.16. *Assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D. Then, the operator $B : H^{\frac{1}{2}}(\partial D) \to L^2[0, 2\pi]$ is compact, injective and has dense range in $L^2[0, 2\pi]$.*

We end this section with a factorization formula for the far field operator F in terms of the operator B and the boundary integral operator S defined by (7.3).

Lemma 7.17. *The far field operator F can be factored as*

$$F = -\overline{\gamma}^{-1}BS^*B^*$$

with $B^* : L^2[0, 2\pi] \to H^{-\frac{1}{2}}(\partial D)$ and $S^* : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ the adjoints of B and S respectively (defined by Definition 7.2) and $\gamma = e^{i\pi/4}/\sqrt{8\pi k}$.

Proof. Consider the operator $H : L^2[0, 2\pi] \to H^{\frac{1}{2}}(\partial D)$ defined by $Hg = v_g|_{\partial D}$ where v_g is the Helglotz wave function with kernel g given by

$$v_g(x) := \int_0^{2\pi} g(\theta) e^{ikx\cdot\hat{y}}\, ds \qquad \hat{y} = (\cos\theta,\, \sin\theta)\,.$$

By changing the order of integration it is easy to show that the adjoint (see Definition 7.2) $H^* : H^{-\frac{1}{2}}(\partial D) \to L^2[0, 2\pi]$ such that

$$(Hg, \varphi) = (g, H^*\varphi)$$

is given by

$$H^*\varphi(\phi) = \int_{\partial D} \varphi(y) e^{-ik\hat{x}\cdot y} ds(y), \quad \hat{x} = (\cos\phi,\, \sin\phi)\ . \tag{7.34}$$

By a superposition argument we have that

$$Fg = -BHg\,. \tag{7.35}$$

On the other hand, from the asymptotic behavior of the fundamental solution (see Sect. 4.1) we observe that $\gamma H^*\varphi$ is the far field pattern of the single layer potential $\mathcal{S}\varphi$ given by (7.1). Since $\mathcal{S}\varphi|_{\partial D} = S\varphi$ where S is given by (7.3) we can write

$$\gamma H^*\varphi = BS\varphi,$$

whence

$$H = \overline{\gamma}^{-1} S^* B^*\,. \tag{7.36}$$

Substituting H from (7.36) into (7.35) the lemma is proved. □

7.3 The Factorization Method

In this section we consider the inverse problem of determining the shape of a perfectly conducting object D from a knowledge of the far field pattern $u_\infty(\theta, \phi)$ of the scattered field $u^s(x, \phi)$ corresponding to (7.21)–(7.24). Exactly in the same way as in Theorem 4.5 one can prove the following uniqueness result.

Theorem 7.18. *Assume that D_1 and D_2 are two obstacles such that the far field patterns corresponding to the scattering problem (7.21)–(7.24) for D_1 and D_2 coincide for all incident angles $\phi \in [0, 2\pi]$. Then $D_1 = D_2$.*

We shall now use the factorization method introduced by Kirsch in [66] to reconstruct the shape of a perfect conductor from a knowledge of the far field operator.

We assume that k^2 is not a Dirichlet eigenvalue for D. From the previous section we know that there exists eigenvalues $\lambda_j \neq 0$ of F and that the corresponding eigenvectors form a complete orthonormal system in $L^2[0,\, 2\pi]$. It is easy to see that $\{|\lambda_j|, \psi_j, \mathrm{sign}(\lambda_j)\psi_j\}_1^\infty$ is a singular system for F (see Sect. 2.2), where for $z \in \mathbb{C}$ we define $\mathrm{sign}(z) = z/|z|$. From Lemma 7.17 we can write

$$-\overline{\gamma}^{-1} B S^* B^* \psi_j = \lambda_j \psi_j \qquad j = 1, 2, \cdots .$$

If we define functions $\varphi_j \in L^2[0,\, 2\pi]$ by

$$B^* \psi_j = \sqrt{\lambda_j}\varphi_j, \qquad j = 1, 2, \cdots \tag{7.37}$$

where the branch of $\sqrt{\lambda_j}$ is chosen such that $\mathrm{Im}\sqrt{\lambda_j} > 0$ (note that $\mathrm{Im}(\lambda_j) > 0$ since $\lambda_j \neq 0$ lie on the circle of radius $\sqrt{2\pi/k}$ and centered at $e^{3\pi i/4}\sqrt{2\pi/k}$), we see that

$$B S^* \varphi_j = -\overline{\gamma}\sqrt{\lambda_j}\psi_j \ . \tag{7.38}$$

Since

$$\begin{aligned}(S\varphi_j, \varphi_l) = (\varphi_j, S^*\varphi_l) &= \frac{1}{\sqrt{\lambda_j}\overline{\sqrt{\lambda_l}}} \left(B^*\psi_j, S^* B^* \psi_l\right) \\ &= \frac{1}{\sqrt{\lambda_j}\overline{\sqrt{\lambda_l}}} \left(\psi_j, B S^* B^* \psi_l\right) = -\frac{\overline{\gamma}\overline{\lambda_l}}{\sqrt{\lambda_j}\overline{\sqrt{\lambda_l}}} \left(\psi_j, \psi_l\right) \ ,\end{aligned}$$

we have that

$$(S\varphi_j, \varphi_l) = c_j \delta_{jl} \qquad \text{where} \quad c_j := -\overline{\gamma}\frac{\overline{\lambda}_j}{|\lambda_j|}, \quad j, l = 1, 2, \cdots . \tag{7.39}$$

From Sect. 7.2 we know that λ_j lies on a circle of radius $\sqrt{2\pi/k}$ and center $e^{3\pi i/4}\sqrt{2\pi/k}$ which passes through the origin. We further know that $\lambda_j \to 0$ as $j \to \infty$. Therefore, we conclude that $|c_j| = 1/\sqrt{8\pi k}$, and $\mathrm{Im}(c_j) \to 0$ as $j \to \infty$.

Let S_i again be the boundary integral operator given by (7.3) corresponding to the wave number $k = i$. Since from Remark 7.4 we have that $S_i^{\frac{1}{2}}$ is well defined and invertible, we can decompose S in the form

$$S = S_i^{\frac{1}{2}}[I + S_i^{-\frac{1}{2}}(S - S_i)S_i^{-\frac{1}{2}}]S_i^{\frac{1}{2}} = S_i^{\frac{1}{2}}[I + K]S_i^{\frac{1}{2}} \tag{7.40}$$

where

$$K := S_i^{-\frac{1}{2}}(S - S_i)S_i^{-\frac{1}{2}} \ . \tag{7.41}$$

Recall that from part 2 of Theorem 7.3 that $S - S_i : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is compact. Hence $K : L^2(\partial D) \to L^2(\partial D)$ is compact since it is the composition of bounded operators with a compact operator. Letting

$$\tilde{\varphi}_j := S_i^{\frac{1}{2}}\varphi_j \qquad j = 1, 2, \cdots , \tag{7.42}$$

the orthogonality relation (7.39) takes the form

$$((I+K)\tilde{\varphi}_j, \tilde{\varphi}_l) = c_j \delta_{jl} \qquad \text{where} \quad c_j := -\overline{\gamma} \frac{\overline{\lambda_j}}{|\lambda_j|}, \quad j, l = 1, 2, \cdots . \quad (7.43)$$

The main step toward the final result is the following theorem.

Theorem 7.19. *The set* $\{\varphi_j\}_1^\infty$ *defined by (7.37) is a Riesz basis for* $H^{-\frac{1}{2}}(\partial D)$.

Proof. We apply Theorem 7.11 to $X := L^2(\partial D)$, $K = S_i^{-\frac{1}{2}}(S - S_i)S_i^{-\frac{1}{2}}$ and the set $\{\tilde{\varphi}_j\}_1^\infty$ defined by (7.42) which is certainly linearly independent and complete in $L^2(\partial D)$ since B and B^* are injective and S and $S^{\frac{1}{2}}$ are isomorphisms. We need to verify that K satisfies $\mathrm{Im}(K\varphi, \varphi) \neq 0$ for $\varphi \neq 0$. To this end, let $\varphi \in L^2(\partial D)$ and set $\psi = S_i^{-\frac{1}{2}}\varphi$. Then $\psi \in H^{-\frac{1}{2}}(\partial D)$ and

$$(K\varphi, \varphi) = ((S - S_i)\psi, \psi) .$$

Since $(S_i\psi, \psi)$ is real valued (note that the kernel of S_i is a real valued function), then the result follows from part 5 of Theorem 7.3. Hence Theorem 7.11 implies that $\{\tilde{\varphi}_j\}_1^\infty$ is a Riesz basis for $L^2(\partial D)$. Finally, since $S_i^{\frac{1}{2}}$ is an isomorphism from $H^{-\frac{1}{2}}(\partial D)$ onto $L^2(\partial D)$, we obtain that $\{\varphi_j\}_1^\infty$ forms a Riesz basis for $H^{-\frac{1}{2}}(\partial D)$. □

Remark 7.20. Let $A : X \to X$ be a compact, self-adjoint, positive definite operator in a Hilbert space. It is easy to show that for each $r > 0$ there exists a uniquely defined compact, positive operator $A^r : X \to X$. In particular, this operator is defined in terms of the spectral decomposition

$$A^r \varphi = \sum_1^\infty \lambda_j^r \, (\varphi, \varphi_j) \, \varphi_j$$

where $\lambda_j > 0$ and φ_j, $j = 1, 2, \cdots$, are the eigenvalues and eigenvectors of A, respectively. The inverse of A^r is defined by

$$A^{-r} \varphi = \sum_1^\infty \lambda_j^{-r} \, (\varphi, \varphi_j) \, \varphi_j .$$

We are now able to prove the first main result of this section.

Theorem 7.21. *Assume that* k^2 *is not a Dirichlet eigenvalue for* $-\Delta$ *in* D. *Then the range of* $B : H^{\frac{1}{2}}(\partial D) \to L^2[0,\, 2\pi]$ *is given by*

$$B(H^{\frac{1}{2}}(\partial D)) = \left\{ \sum_1^\infty \rho_j \psi_j : \; \sum_1^\infty \frac{|\rho_j|^2}{|\lambda_j|} < \infty \right\} = (F^*F)^{\frac{1}{4}}(L^2[0,\, 2\pi]) \quad (7.44)$$

where $\{|\lambda_j|, \psi_j, \mathrm{sign}(\lambda_j)\psi_j\}_1^\infty$ *is the singular system of the far field operator* F.

Proof. First, we note that $S^* : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is an isomorphism since $S^*\varphi = \overline{S\overline{\varphi}}$. Suppose that $B\varphi = \psi$ for some $\varphi \in H^{\frac{1}{2}}(\partial D)$. Then $(S^*)^{-1}\varphi \in H^{-\frac{1}{2}}(\partial D)$ and thus $(S^*)^{-1}\varphi = \sum_1^\infty \alpha_j\varphi_j$ with $\sum_1^\infty |\alpha_j|^2 < \infty$, since $\{\varphi_j\}$ forms a Riesz basis for $H^{-\frac{1}{2}}(\partial D)$ (see Theorem 7.10). Hence, by (7.38) we have

$$\psi = B\varphi = BS^*(S^*)^{-1}\varphi = -\overline{\gamma}\sum_1^\infty \alpha_j\sqrt{\lambda_j}\psi_j = \sum_1^\infty \rho_j\psi_j$$

with $\rho_j = -\overline{\gamma}\alpha_j\sqrt{\lambda_j}$ and thus

$$\sum_1^\infty \frac{|\rho_j|^2}{|\lambda_j|} = \overline{\gamma}^2\sum_1^\infty |\alpha_j|^2 < \infty\,. \tag{7.45}$$

On the other hand, let $\psi = \sum_1^\infty \rho_j\psi_j$ with the ρ_j satisfying $\sum_1^\infty \left(|\rho_j|^2/|\lambda_j|\right) < \infty$ and define $\varphi := \sum_1^\infty \alpha_j\varphi_j$ with $\alpha_j = \overline{\gamma}^{-1}\rho_j/\sqrt{\lambda_j}$. Then $\sum_1^\infty |\alpha_j|^2 < \infty$ and hence $\varphi \in H^{-\frac{1}{2}}(\partial D)$. But $S^*\varphi \in H^{\frac{1}{2}}(\partial D)$, whence

$$B(S^*\varphi) = -\overline{\gamma}\sum_1^\infty \alpha_j\sqrt{\lambda_j}\psi_j = \sum_1^\infty \rho_j\psi_j = \psi.$$

We now observe that $\sqrt{|\lambda_j|}$ and ψ_j are the eigenvalues and eigenfunctions, respectively, of the self-adjoint operator $(F^*F)^{\frac{1}{4}}$ (see Remark 7.20). Hence Theorem 2.7 yields

$$(F^*F)^{\frac{1}{4}}(L^2[0,\,2\pi]) = \left\{\sum_1^\infty \rho_j\psi_j :\ \sum_1^\infty \frac{|\rho_j|^2}{|\lambda_j|} < \infty\right\} = B(H^{\frac{1}{2}}(\partial D))\,.$$

□

We recall that from Remark 7.20 $(F^*F)^{-\frac{1}{4}}$ is well defined.

Lemma 7.22. *The operator* $(F^*F)^{-\frac{1}{4}}B$ *is an isomorphism from* $H^{\frac{1}{2}}(\partial D)$ *onto* $L^2[0,\,2\pi]$.

Proof. Let $\{\varphi_j\}_1^\infty$ be defined by (7.37). Then from Theorem 7.10, since $S : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is an isomorphism, we have that $\{S\varphi_j\}_1^\infty$ is a Riesz basis for $H^{\frac{1}{2}}(\partial D)$. In order to show that $(F^*F)^{-\frac{1}{4}}B$ is an isomorphism, from Theorem 7.10 it suffices to show that $\left\{(F^*F)^{-\frac{1}{4}}BS\varphi_j\right\}_1^\infty$ forms a Riesz basis for $L^2[0,\,2\pi]$. To this end, using (7.37) and Lemma 7.17, we obtain

$$\begin{aligned}(F^*F)^{-\frac{1}{4}}BS\varphi_j &= \frac{1}{\sqrt{\lambda_j}}(F^*F)^{-\frac{1}{4}}BSB^*\psi_j \\ &= -\overline{\gamma}\lambda_j(F^*F)^{-\frac{1}{4}}\psi_j = -\overline{\gamma}\sqrt{\frac{\lambda_j}{|\lambda_j|}}\psi_j\,.\end{aligned} \tag{7.46}$$

The result now follows from the fact that the set $\{\psi_j\}_1^\infty$ is a complete orthonormal system in $L^2[0,\,2\pi]$. □

The following theorem gives examples of functions in the range of B. Recall that $\Phi_\infty(\hat{x}, z)$ denotes the far field pattern of the fundamental solution $\Phi(x, z)$ of the Helmholtz equation.

Theorem 7.23. *$\Phi_\infty(\cdot, z)$ is in the range of B if and only if $z \in D$.*

Proof. First take $z \in D$ and define $f := \Phi(\cdot, z)|_{\partial D}$. Then, since $\Phi(\cdot, z)$ is a solution to the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{D}$, by definition we have that $Bf = \Phi_\infty(\cdot, z)$.
Next, let $z \in \mathbb{R}^2 \setminus \bar{D}$ and assume that there exists a $f \in H^{\frac{1}{2}}(\partial D)$ such that $Bf = \Phi_\infty(\cdot, z)$. Let u be the solution of the exterior boundary value problem (7.25)–(7.27) with boundary data f. By Rellich's lemma, $u(x) = \Phi(x, z)$ for all x outside of any sphere containing D and z. If $z \notin \bar{D}$ this contradicts the fact that u is analytic in $\mathbb{R}^2 \setminus \bar{D}$ while $\Phi(x, z)$ is singular at $x = z$. If $z \in \partial D$ we have that $\Phi(x, z) = f(x)$ for $x \in \partial D$, i.e. $\Phi(\cdot, z) \in H^{\frac{1}{2}}(\partial D)$. This is a contradiction since $\nabla\Phi(\cdot, z)$ is neither in $L^2(D)$ nor in $L^2_{loc}(\mathbb{R}^2 \setminus \bar{D})$. □

Combining Theorem 7.19 and Theorem 7.23 we obtain the main result of this section.

Theorem 7.24. *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D and let F be the far field operator corresponding to (7.21)–(7.24). Then*

$$D = \left\{ z \in \mathbb{R}^2 : \sum_1^\infty \frac{|\rho_j^{(z)}|^2}{\sigma_j} < \infty \right\}$$
$$= \left\{ z \in \mathbb{R}^2 : \Phi_\infty(\cdot, z) \in (F^*F)^{\frac{1}{4}}(L^2[0, 2\pi]) \right\}$$

where $\left\{\sigma_j, \psi_j, \tilde{\psi}_j\right\}_1^\infty$ is the singular system of F, and $\rho_j^{(z)} = (\Phi_\infty(\cdot, z), \psi_j)_{L^2}$, $j = 1, 2, \cdots$, are the expansion coefficients of $\Phi_\infty(\hat{x}, z)$ with respect to $\{\psi_j\}_1^\infty$. Moreover, there exists $C > 1$ such that

$$\frac{1}{C^2}\|\Phi(\cdot, z)\|^2_{H^{\frac{1}{2}}(\partial D)} \le \sum_1^\infty \frac{|\rho_j^{(z)}|^2}{\sigma_j} \le C^2\|\Phi(\cdot, z)\|^2_{H^{\frac{1}{2}}(\partial D)}, \quad z \in D \,. \tag{7.47}$$

Proof. It only remains to prove the last estimate. From the proof of Theorem 7.19 we have that for $z \in D$

$$g := \sum_1^\infty \frac{\rho_j^{(z)}}{\sqrt{\sigma_j}}\psi_j$$

is the solution of $(F^*F)^{\frac{1}{4}}g = \Phi_\infty(\cdot, z)$. On the other hand, $\Phi_\infty(\cdot, z)$ is the far field pattern of the fundamental solution $\Phi(\cdot, z)$, i.e. if we define $f := \Phi(\cdot, z)|_{\partial D}$ then $Bf = \Phi_\infty(\cdot, z)$ and hence $g = (F^*F)^{-\frac{1}{4}}Bf$. The estimate (7.47) follows from the fact that $\|g\|^2_{L^2} = \sum_1^\infty |\rho_j^{(z)}|^2/\sigma_j$ and using Lemma 7.22 and Theorem 7.10. □

Remark 7.25. The estimate (7.47) describes how the value of the series blows up when z approaches the boundary ∂D. In particular, it is easily shown that $\|\Phi(\cdot, z)\|^2_{H^{\frac{1}{2}}(\partial D)}$ behaves as $|\ln(d(z, \partial D))|$ where $d(z, \partial D)$ denotes the distance of $z \in D$ from the boundary.

The factorization method looks for a solution to the linear equation

$$(F^*F)^{\frac{1}{4}} g = \Phi_\infty(\cdot, z) \tag{7.48}$$

which is ill posed since $(F^*F)^{\frac{1}{4}} : L^2[0, 2\pi] \to L^2[0, 2\pi]$ is compact. Therefore a regularization scheme is needed to compute the solution of (7.48). In particular, using Tikhonov regularization, a regularized solution g^α is defined as the solution of the well posed equation

$$\alpha g^\alpha + (F^*F)^{\frac{1}{2}} g^\alpha = (F^*F)^{\frac{1}{4}} \Phi_\infty(\cdot, z)$$

where $\alpha > 0$ is the regularization parameter which can be chosen according to the Morozov discrepancy principle (see Sect. 2.3) such that

$$\|(F^*F)^{\frac{1}{4}} g^\alpha - \Phi_\infty(\cdot, z)\| = \delta \|g^\alpha\|$$

with $\delta > 0$ being the error in the measured far field data. Unlike the far field equation $Fg = \Phi_\infty(\cdot, z)$ on which the linear sampling method is based, (7.48) is solvable if and only if $z \in D$. Therefore, it is possible to obtain a convergence result for the regularized solution of (7.48) when $\delta \to 0$. This is provided by the following theorem from the theory of ill-posed problems, which we recall for the reader's convenience [65].

Theorem 7.26. *Let $K_\delta : X \to Y$, $\delta \geq 0$, be a family of injective and compact operators with dense range between Hilbert spaces X and Y such that $\|K_0 - K_\delta\| \leq \delta$ for all $\delta > 0$. Furthermore, let $f \in Y$ and $(\alpha_\delta, g_\delta) \in \mathbb{R}^+ \times X$ be the regularized Tikhonov-Morozov solution of the equation $K_\delta g = f$, i.e. the solution of the system*

$$(\alpha_\delta I + K_\delta^* K_\delta) g_\delta = K_\delta^* f \qquad \|K_\delta g - f\| = \delta \|g_\delta\| .$$

Then

1. *If the noise free equation $K_0 g = f$ has a unique solution $g \in X$ then $g_\delta \to g$ as $\delta \to 0$.*
2. *If the noise free equation $K_0 g = f$ has no solution then $\|g_\delta\| \to \infty$ as $\delta \to 0$.*

7.4 Closing Remarks

The factorization method described in the previous section relies in an essential manner on the fact that the far field operator corresponding to the scattering problem is normal. Unfortunately this is not always the case. In particular, the far field operator is not normal in the case of the scattering problem for an imperfect conductor considered in Chap. 3 and the scattering problem for an absorbing inhomogeneous medium. A version of the factorization method that does not need the far field operator to be normal was introduced by Kirsch in [68, 70].

A drawback of both the linear sampling method and the factorization method is the large amount of data needed for the inversion procedure. In particular, the factorization method has not been established for limited aperture data. Although the linear sampling method is valid for limited aperture far field data (see Sect. 4.5), one still needs a multistatic set of data i.e. the far field measured at all observation directions on a subset of the unit circle with incident directions on a (possibly different) subset of the unit circle.
What happens if the far field pattern is only known for a finite number of incident waves? In certain cases it has been shown ([22, 38, 102], [106]) that only a finite number of incident plane waves is sufficient to uniquely determine the support of the scattering object. Progress has recently been made in the use of qualitative methods which use only a finite number of incident plane waves. In particular, it was shown in [80, 81] and [99] that a single or few incident waves can determine the *convex scattering support* which provides a lower bound for the convex hull of the scatterer.

8

Mixed Boundary Value Problems

This chapter is devoted to the study of mixed boundary value problems in electromagnetic scattering theory. Mixed boundary value problems typically model the scattering by objects that are coated by a thin layer of material on part of the boundary. We shall consider here two main problems: 1) the scattering by a perfect conductor that is partially coated by a thin dielectric layer and 2) the scattering by an orthotropic dielectric that is partially coated by a thin layer of highly conducting material. The first problem leads to an exterior mixed boundary value problem for the Helmholtz equation where on the coated part of the boundary the total field satisfies an impedance boundary condition and on the remaining part of the boundary the total field vanishes, while the second problem leads to a transmission problem with mixed transmission-conducting boundary conditions. In this chapter we shall present the mathematical analysis of these two mixed boundary value problems.

In the study of inverse problems for partially coated obstacles, it is important to mentioned that, in general, it is not known a priori whether or not the scattering object is coated and if so what is the extent of the coating. Hence the linear sampling method becomes the method of choice for solving inverse problems for mixed boundary value problems since it does not make use of the physical properties of the scattering object. In addition to the reconstruction of the shape of the scatterer, a main question in this chapter will be to find out whether the obstacle is coated and if so what are the electrical properties of the coating. In particular we will show that the solution of the far field equation that was used to determine the shape of the scatterer by means of the linear sampling method can also be used in conjunction with a variational method to determine the maximum value of the surface impedance of the coated portion in the case of partially coated perfect conductors and of the surface conductivity in the case of partially coated dielectrics.

Finally, we will extend the linear sampling method to the scattering problem by very thin objects, referred to as cracks, which are modeled by open arcs in $\mathbb{R}^2$.

8.1 Scattering by a Partially Coated Perfect Conductor

We consider the scattering of an electromagnetic time harmonic plane wave by a perfectly conducting infinite cylinder in $\mathbb{R}^3$ that is partially coated by a thin dielectric material. In particular, the total electromagnetic field on the uncoated part of the boundary satisfies the perfect conducting boundary condition, namely the tangential component of the electric field is zero, whereas the boundary condition on the coated part is described by an impedance boundary condition [53].

More precisely, let D denote the cross section of the infinitely long cylinder and assume that $D \subset \mathbb{R}^2$ is an open bounded region with C^2 boundary ∂D such that $\mathbb{R}^2 \setminus \bar{D}$ is connected. The boundary ∂D has the dissection $\partial D = \overline{\partial D}_D \cup \overline{\partial D}_I$, where ∂D_D and ∂D_I are disjoint, relatively open subsets (possibly disconnected) of ∂D. Let ν denote the unit outward normal to ∂D and assume that the surface impedance $\lambda \in C(\overline{\partial D_I})$ satisfies $\lambda(x) \geq \lambda_0 > 0$ for $x \in \partial D_I$. Then the total field $u = u^s + u^i$, given as the sum of the unknown scattered field u^s and the known incident field u^i, satisfies

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \bar{D}\,, \tag{8.1}$$

$$u = 0 \qquad \text{on} \qquad \partial D_D\,, \tag{8.2}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \qquad \text{on} \qquad \partial D_I\,, \tag{8.3}$$

where $k > 0$ is the wave number and u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{8.4}$$

uniformly in $\hat{x} = x/|x|$ with $r = |x|$. Note that here again the incident field u^i is usually an entire solution of the Helmholtz equation. In particular, in the case of incident plane waves, we have $u^i(x) = e^{ikx\cdot d}$ where $d := (\cos\,\phi,\, \sin\,\phi)$ is the incident direction and $x = (x_1,\, x_2) \in \mathbb{R}^2$.

Due to the boundary condition the above exterior mixed boundary value problem may not have a solution in $C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R} \setminus D)$ even for incident plane waves and analytic boundary. In particular, the solution fails to be differentiable at the boundary points of $\overline{\partial D}_D \cap \overline{\partial D}_I$. Therefore looking for a weak solution in the case of mixed boundary value problems is very natural.

In order to define a weak solution to the mixed boundary value problem in the energy space $H^1(D)$, we need to understand the respective trace spaces on parts of the boundary. To this end we now present a brief discussion of Sobolev spaces on open arcs. The classical reference for such spaces is [83]. For a systematic treatment of these spaces, we refer the reader to [85].

Let $\partial D_0 \subseteq \partial D$ be an open subset of the boundary. We define

$$H^{\frac{1}{2}}(\partial D_0) := \{u|_{\partial D_0} : u \in H^{\frac{1}{2}}(\partial D)\}$$

i.e. the space of restrictions to ∂D_0 of functions in $H^{\frac{1}{2}}(\partial D)$, and define

$$\tilde{H}^{\frac{1}{2}}(\partial D_0) := \{u \in H^{\frac{1}{2}}(\partial D) : \operatorname{supp} u \subseteq \overline{\partial D_0}\}$$

where $\operatorname{supp} u$ is the essential support of u, i.e. the largest relatively closed subset of ∂D such that $u = 0$ almost everywhere on $\partial D \setminus \operatorname{supp} u$. We can identify $\tilde{H}^{\frac{1}{2}}(\partial D_0)$ with the trace space of $H_0^1(D, \partial D \setminus \overline{\partial D_0})$ where

$$H_0^1(D, \partial D \setminus \overline{\partial D_0}) = \left\{ u \in H^1(D) : \ u|_{\partial D \setminus \overline{\partial D_0}} = 0 \text{ in the trace sense} \right\} .$$

A very important property of $\tilde{H}^{\frac{1}{2}}(\partial D_0)$ is that the extension by zero of $u \in \tilde{H}^{\frac{1}{2}}(\partial D_0)$ to the whole ∂D is in $H^{\frac{1}{2}}(\partial D)$ and the zero extension operator is bounded from $\tilde{H}^{\frac{1}{2}}(\partial D_0)$ to $H^{\frac{1}{2}}(\partial D)$. It can also be shown (c.f. Theorem A4 in [85]) that there exists a bounded extension operator $\tau : H^{\frac{1}{2}}(\partial D_0) \to H^{\frac{1}{2}}(\partial D)$. In other words for any $u \in H^{\frac{1}{2}}(\partial D_0)$ there exists an extension $\tau u \in H^{\frac{1}{2}}(\partial D)$ such that

$$\|\tau u\|_{H^{\frac{1}{2}}(\partial D)} \le C \|u\|_{H^{\frac{1}{2}}(\partial D_0)} \tag{8.5}$$

with C independent of u.

Example 8.1. Consider the step function

$$u(t) = \begin{cases} 1 & t \in [0,\, \pi] \\ 0 & t \in (\pi,\, 2\pi] \end{cases}$$

Using the definition of Sobolev spaces in terms of the Fourier coefficients (see Sect. 1.4) it is easy to show that the step function is not in $H^{\frac{1}{2}}[0,\, 2\pi]$. In particular, the Fourier coefficients of u are $a_{2k} = 0$ and $a_{2k+1} = 1/(i(2k+1)\pi)$ whence

$$\sum_{-\infty}^{\infty} \left(1 + m^2\right)^{\frac{1}{2}} |a_m|^2 = \sum_{-\infty}^{\infty} \left(1 + (2k+1)^2\right)^{\frac{1}{2}} \frac{1}{\pi^2 (2k+1)^2} = +\infty .$$

Now consider the unit circle $\partial\Omega = \{x \in \mathbb{R}^2 : \ x = (\sin t,\ \cos t),\ t \in [0,\, 2\pi]\}$ and denote by $\partial\Omega_0 = \{x \in \mathbb{R}^2 : \ x = (\sin t,\ \cos t),\ t \in [0,\, \pi]\}$ the upper half circle. Let $v : \partial\Omega_0 \to \mathbb{R}$ be the constant function $v = 1$. By definition $v \in H^{\frac{1}{2}}(\partial\Omega_0)$ since it is the restriction to $\partial\Omega_0$ of the constant function 1 defined on the whole circle $\partial\Omega$ which is in $H^{\frac{1}{2}}(\partial\Omega)$. But $v \notin \tilde{H}^{\frac{1}{2}}(\partial\Omega_0)$ since its extension by zero to the whole circle is not in $H^{\frac{1}{2}}(\partial\Omega)$ (note that the extension $\tilde{v}(\sin t,\, \cos t)$ is a step function and from the above is not in $H^{\frac{1}{2}}[0,\, 2\pi]$).

The above example shows that if $u \in \tilde{H}^{\frac{1}{2}}(\partial D_0)$ then it has a certain behavior at the boundary of ∂D_0 in ∂D. A better insight to this behavior is given in [83]. In particular, the space $\tilde{H}^{\frac{1}{2}}(\partial D_0)$ coincides with the space

$$H_{00}^{\frac{1}{2}}(\partial D_0) := \{u \in H^{\frac{1}{2}}(\partial D_0) : \ r^{-\frac{1}{2}} u \in L^2(\partial D_0)\}$$

where r is the polar radius.

Both $H^{\frac{1}{2}}(\partial D_0)$ and $\tilde{H}^{\frac{1}{2}}(\partial D_0)$ are Hilbert spaces when equipped with the restriction of the inner product of $H^{\frac{1}{2}}(\partial D)$. Hence, we can define the corresponding dual spaces

$$H^{-\frac{1}{2}}(\partial D_0) := \left(\tilde{H}^{\frac{1}{2}}(\partial D_0)\right)' = \text{the dual space of} \quad \tilde{H}^{\frac{1}{2}}(\partial D_0)$$

and

$$\tilde{H}^{-\frac{1}{2}}(\partial D_0) := \left(H^{\frac{1}{2}}(\partial D_0)\right)' = \text{the dual space of} \quad H^{\frac{1}{2}}(\partial D_0)$$

with respect to the duality pairing explained in the following.

A bounded linear functional $F \in H^{-\frac{1}{2}}(\partial D_0)$ can in fact be seen as the restriction to ∂D_0 of some $\tilde{F} \in H^{-\frac{1}{2}}(\partial D)$ in the following sense: if $\tilde{u} \in H^{\frac{1}{2}}(\partial D)$ denotes the extension by zero of $u \in \tilde{H}^{\frac{1}{2}}(\partial D_0)$, then the restriction $F := \tilde{F}|_{\partial D_0}$ is defined by

$$F(u) = \tilde{F}(\tilde{u}) \ .$$

With the above understanding, in order to unify the notations, we identify

$$H^{-\frac{1}{2}}(\partial D_0) := \{v|_{\partial D_0} : v \in H^{-\frac{1}{2}}(\partial D)\}$$

and

$$\langle v, u\rangle_{H^{-\frac{1}{2}}(\partial D_0), \tilde{H}^{\frac{1}{2}}(\partial D_0)} = \langle v, \tilde{u}\rangle_{H^{-\frac{1}{2}}(\partial D), H^{\frac{1}{2}}(\partial D)}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the denoted spaces and $\tilde{u} \in H^{\frac{1}{2}}(\partial D)$ is the extension by zero of $u \in \tilde{H}^{\frac{1}{2}}(\partial D_0)$.

For a bounded linear functional $F \in H^{-\frac{1}{2}}(\partial D)$, we define $\operatorname{supp} F$ to be the largest relatively closed subset of ∂D such that the restriction of F to $\partial D \setminus \operatorname{supp} F$ is zero. Similarly for $\tilde{H}^{\frac{1}{2}}(\partial D_0)$ we can now write

$$\tilde{H}^{-\frac{1}{2}}(\partial D_0) := \{v \in H^{-\frac{1}{2}}(\partial D) : \operatorname{supp} v \subseteq \overline{\partial D_0}\} \ .$$

Therefore, the extension by zero $\tilde{v} \in H^{-\frac{1}{2}}(\partial D)$ of $v \in \tilde{H}^{-\frac{1}{2}}(\partial D_0)$ is well defined and

$$\langle \tilde{v}, u\rangle_{H^{-\frac{1}{2}}(\partial D), H^{\frac{1}{2}}(\partial D)} = \langle v, u\rangle_{\tilde{H}^{-\frac{1}{2}}(\partial D_0), H^{\frac{1}{2}}(\partial D_0)}$$

where $u \in H^{\frac{1}{2}}(\partial D)$.

We can now formulate the following mixed boundary value problems:

Exterior mixed boundary value problem: Let $f \in H^{\frac{1}{2}}(\partial D_D)$ and $h \in H^{-\frac{1}{2}}(\partial D_I)$. Find a function $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ such that

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \bar{D} \tag{8.6}$$

$$u = f \qquad \text{on} \qquad \partial D_D \tag{8.7}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = h \qquad \text{on} \qquad \partial D_I \ , \tag{8.8}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0 \ . \tag{8.9}$$

Note that the scattering problem for a partially coated perfect conductor (8.1)–(8.4) is a particular case of (8.6)–(8.9). In particular the scattered field u^s satisfies (8.6)–(8.9) with $f := -u^i|_{\partial D_D}$ and $h := -\partial u^i/\partial \nu - i\lambda u^i|_{\partial D_I}$.

For later use we also consider the corresponding interior mixed boundary value problem.

Interior mixed boundary value problem: Let $f \in H^{\frac{1}{2}}(\partial D_D)$ and $h \in H^{-\frac{1}{2}}(\partial D_I)$. Find a function $u \in H^1(D)$ such that

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad D \tag{8.10}$$

$$u = f \qquad \text{on} \qquad \partial D_D \tag{8.11}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = h \qquad \text{on} \qquad \partial D_I \tag{8.12}$$

Theorem 8.2. *Assume that $\partial D_I \neq \emptyset$ and $\lambda \neq 0$. Then the interior mixed boundary value problem (8.10)–(8.12) has at most one solution in $H^1(D)$.*

Proof. Let u be a solution of (8.10)–(8.12) with $f \equiv 0$ and $h \equiv 0$. Then an application of Green's first identity in D yields

$$-k^2 \int_D |u|^2\, dx + \int_D |\nabla u|^2\, dx = \int_{\partial D} \frac{\partial u}{\partial \nu} \bar{u}\, ds \ , \tag{8.13}$$

and making use of homogeneous boundary condition we obtain

$$-k^2 \int_D |u|^2\, dx + \int_D |\nabla u|^2\, dx = -i \int_{\partial D_I} \lambda |u|^2\, ds \, . \tag{8.14}$$

Since λ is a real-valued function and $\lambda(x) \geq \lambda_0 > 0$, by taking the imaginary part of (8.14) we conclude that $u|_{\partial D_I} \equiv 0$ as a function in $H^{\frac{1}{2}}(\partial D_I)$ and consequently $\partial u/\partial \nu|_{\partial D_I} \equiv 0$ as a function in $H^{-\frac{1}{2}}(\partial D_I)$.
Now let Ω_ρ be a ball of radius ρ with center on ∂D_I such that $\bar{\Omega}_\rho \cap \partial D_D = \emptyset$ and define $v = u$ in $D \cap \Omega_\rho$, $v = 0$ in $(\mathbb{R}^2 \setminus \bar{D}) \cap \Omega_\rho$. Then applying Green's second identity in each of these domains to v and a test function $\varphi \in C_0^\infty(\Omega_\rho)$ we see that v is a weak solution of Helmholtz equation in Ω_ρ. Thus v is a real-analytic solution in Ω_ρ. We can now conclude that $u \equiv 0$ in Ω_ρ and thus $u \equiv 0$ in D. □

Theorem 8.3. *The exterior mixed boundary value problem (8.6)–(8.9) has at most one solution in $H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$.*

Proof. The proof of the theorem is essentially the same as the proof of Theorem 3.3. □

Theorem 8.4. *Assume that $\partial D_I \neq \emptyset$ and $\lambda \neq 0$. Then the interior mixed boundary value problem (8.10)–(8.12) has a solution which satisfies the estimate*

$$\|u\|_{H^1(D)} \leq C \left(\|f\|_{H^{\frac{1}{2}}(\partial D_D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D_I)} \right) \tag{8.15}$$

with C a positive constant independent of f and h.

Proof. To prove the theorem we use the variational approach developed in Sect. 5.3. (For a solution procedure based on integral equations of the first kind we refer the reader to [14]). Let $\tilde{f} \in H^{\frac{1}{2}}(\partial D)$ be the extension of the Dirichlet data $f \in H^{\frac{1}{2}}(\partial D_D)$ that satisfies $\|\tilde{f}\|_{H^{\frac{1}{2}}(\partial D)} \leq C\|f\|_{H^{\frac{1}{2}}(\partial D_D)}$ given by (8.5), and let $u_0 \in H^1(D)$ be such that $u_0 = \tilde{f}$ on ∂D and $\|u_0\|_{H^1(D)} \leq C\|\tilde{f}\|_{H^{\frac{1}{2}}(\partial D)}$. In particular we may choose u_0 to be a solution of $\Delta u_0 = 0$ (see Example 5.13). Defining the Sobolev space $H_0^1(D, \partial D_D)$ by

$$H_0^1(D, \partial D_D) := \left\{ u \in H^1(D) : \; u = 0 \text{ on } \partial D_D \right\}$$

equipped with the norm induced from $H^1(D)$, we observe that $w = u - u_0 \in H_0^1(D, \partial D_D)$ where $u \in H^1(D)$ is a solution to (8.10)–(8.12). Furthermore, w satisfies

$$\Delta w + k^2 w = -k^2 u_0 \;\; \text{in} \;\; D \tag{8.16}$$

and

$$\frac{\partial w}{\partial \nu} + i\lambda w = \tilde{h} \qquad \text{on} \qquad \partial D_I \tag{8.17}$$

where $\tilde{h} \in H^{-\frac{1}{2}}(\partial D_I)$ is given by

$$\tilde{h} := -\frac{\partial u_0}{\partial \nu} - i\lambda u_0 + h \; .$$

Multiplying (8.16) by a test function $\varphi \in H_0^1(D, \partial D_D)$ and using Green's first identity together with the boundary condition (8.17) we can write (8.10)–(8.12) in the following equivalent variational form: *Find $u \in H^1(D)$ such that $w = u - u_0 \in H_0^1(D, \partial D_D)$ and*

$$a(w, \varphi) = L(\varphi), \qquad \textit{for all}\, \varphi \in H_0^1(D, \partial D_D) \tag{8.18}$$

where the sesquilinear form $a(\cdot, \cdot) : H_0^1(D, \partial D_D) \times H_0^1(D, \partial D_D) \to \mathbb{C}$ is defined by

$$a(w, \varphi) := \int_D \left(\nabla w \cdot \nabla \bar{\varphi} - k^2 w \bar{\varphi} \right) \, dx + i \int_{\partial D_I} \lambda \, w \, \bar{\varphi} \, ds$$

and the conjugate linear functional $L : H_0^1(D, \partial D_D) \to \mathbb{C}$ is defined by

$$L(\bar{\varphi}) = k^2 \int_D u_0 \bar{\varphi}\, dx + \int_{\partial D_I} \tilde{h} \cdot \bar{\varphi}\, dx$$

where the integral over ∂D_I is interpreted as the duality pairing between $\tilde{h} \in H^{-\frac{1}{2}}(\partial D_I)$ and $\varphi \in \tilde{H}^{\frac{1}{2}}(\partial D_I)$ (note that $\varphi \in \tilde{H}^{\frac{1}{2}}(\partial D_I)$ since $\tilde{H}^{\frac{1}{2}}(\partial D_I)$ is the trace space of $H_0^1(D, \partial D_D)$).
Next we write $a(\cdot, \cdot)$ as the sum of two terms $a(\cdot, \cdot) = a_1(\cdot, \cdot) + a_2(\cdot, \cdot)$ where

$$a_1(w, \varphi) := \int_D (\nabla w \cdot \nabla \bar{\varphi} + w\, \bar{\varphi})\; dx + i \int_{\partial D_I} \lambda\, w\, \bar{\varphi}\, ds$$

and

$$a_2(w, \varphi) := -(k^2 + 1) \int_D w\, \bar{\varphi}\, dx\ .$$

From the Cauchy-Schwarz inequality and the trace Theorem 1.36, since λ is a bounded function on ∂D_I, we have that

$$\begin{aligned} |a_1(w, \varphi)| &\leq C_1 \|w\|_{H^1(D)} \|\varphi\|_{H^1(D)} + C_2 \|w\|_{L^2(\partial D_I)} \|\varphi\|_{L^2(\partial D_I)} \\ &\leq \tilde{C} \left(\|w\|_{H^1(D)} \|\varphi\|_{H^1(D)} + \|w\|_{H^{\frac{1}{2}}(\partial D)} \|\varphi\|_{H^{\frac{1}{2}}(\partial D)} \right) \\ &\leq C \|w\|_{H^1(D)} \|\varphi\|_{H^1(D)} \end{aligned}$$

and

$$|a_2(w, \varphi)| \leq \tilde{C} \|w\|_{L^2(D)} \|\varphi\|_{L^2(D)} \leq C \|w\|_{H^1(D)} \|\varphi\|_{H^1(D)}\ .$$

Hence $a_1(\cdot, \cdot)$ and $a_2(\cdot, \cdot)$ are bounded sesquilinear forms.

Furthermore, noting that $\varphi = 0$ on ∂D_D, we have that

$$\int_{\partial D_I} \frac{\partial u_0}{\partial \nu}\, \bar{\varphi}\, ds = \int_{\partial D} \frac{\partial u_0}{\partial \nu}\, \bar{\varphi}\, ds = \int_D \nabla u_0 \cdot \nabla \bar{\varphi}\, dx\ .$$

Therefore from the previous estimates and the trace Theorems 1.36 and 5.5 we have that

$$\begin{aligned} |L(\varphi)| &\leq C_1 \|u_0\|_{H^1(D)} \|\varphi\|_{H^1(D)} + C_2 \|u_0\|_{H^{\frac{1}{2}}(\partial D)} \|\varphi\|_{H^{\frac{1}{2}}(\partial D)} \\ &+ C_3 \|h\|_{H^{-\frac{1}{2}}(\partial D_I)} \|\varphi\|_{\tilde{H}^{\frac{1}{2}}(\partial D_I)} \\ &\leq \tilde{C} \left(\|\tilde{f}\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D_I)} \right) \|\varphi\|_{H^1(D)} \\ &\leq C \left(\|f\|_{H^{\frac{1}{2}}(\partial D_D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D_I)} \right) \|\varphi\|_{H^1(D)} \qquad \text{for all } \ \varphi \in V \end{aligned}$$

which shows that L is a bounded conjugate linear functional and

$$\|L\| \le C \left(\|f\|_{H^{\frac{1}{2}}(\partial D_D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D_I)} \right) \tag{8.19}$$

with the constant $C > 0$ independent of f and h.
Next, since $\lambda(x) \ge \lambda_0 > 0$, we can write

$$|a_1(w, w)| \ge \|w\|^2_{H^1(D)} + \lambda_0 \|w\|^2_{L^2(\partial D_I)} \ge \|w\|^2_{H^1(D)}$$

whence $a_1(\cdot, \cdot)$ is strictly coercive. Hence from the Lax-Milgram lemma there exists a bijective bounded linear operator $A : H^1_0(D, \partial D_D) \to H^1_0(D, \partial D_D)$ with bounded inverse such that $(Aw, \varphi) = a_1(w, \varphi)$ for all w and φ in $H^1_0(D, \partial D_D)$. Finally, due to the compact imbedding of $H^1(D)$ into $L^2(D)$, there exists a compact bounded linear operator $B : H^1_0(D, \partial D_D) \to H^1_0(D, \partial D_D)$ such that $(Bw, \varphi) = a_2(w, \varphi)$ for all w and φ in $H^1_0(D, \partial D_D)$ (see Example 5.15). Therefore, from Theorem 5.14 and Theorem 8.2 we obtain the existence of a unique solution to (8.18) and consequently to the interior mixed boundary value problem (8.10)–(8.12). The a priori estimate (8.15) follows from (8.19). □

Now let us consider an open disk Ω_R of radius R centered at the origin and containing $\bar{D}$.

Theorem 8.5. *The exterior mixed boundary value problem (8.6)–(8.9) has a solution which satisfies the estimate*

$$\|u\|_{H^1(\Omega_R \setminus \bar{D})} \le C \left(\|f\|_{H^{\frac{1}{2}}(\partial D_D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D_I)} \right) \tag{8.20}$$

with C a positive constant independent of f and h but depending on R.

Proof. First, exactly in the same way as in Example 5.21, we can show that the exterior mixed boundary value problem (8.6)–(8.9) is equivalent to the following problem

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad \Omega_R \setminus \bar{D}\,, \tag{8.21}$$

$$u = f \qquad \text{on} \qquad \partial D_D, \tag{8.22}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = h \qquad \text{on} \qquad \partial D_I, \tag{8.23}$$

$$\frac{\partial u}{\partial \nu} = Tu \qquad \text{on} \qquad \partial \Omega_R \tag{8.24}$$

where T is the Dirichlet to Neumann map. If $\tilde{f} \in H^{\frac{1}{2}}(\partial D)$ is the extension of $f \in H^{\frac{1}{2}}(\partial D_D)$ that satisfies (8.5) with ∂D_0 replaced by ∂D_D, we construct $u_0 \in H^1(\Omega_R \setminus \bar{D})$ such that $u_0 = \tilde{f}$ on ∂D, $u = 0$ on $\partial \Omega_R$ and $\Delta u_0 = 0$ in $\Omega_R \setminus \bar{D}$ (see Example 5.13). Then, for every solution u to (8.21)–(8.24), $w = u - u_0$ is in the Sobolev space $H^1_0(\Omega_R \setminus \bar{D}, \partial D_D)$ defined by

$$H^1_0(\Omega_R \setminus \bar{D}, \partial D_D) := \left\{ u \in H^1(\Omega_R \setminus \bar{D}) : \ u = 0 \text{ on } \partial D_D \right\}$$

and satisfies the variational equation

$$\begin{aligned}
&\int\limits_{\Omega_R\setminus\bar{D}} \left(\nabla w \cdot \nabla \bar{\varphi} - k^2 w\bar{\varphi}\right)\, ds - i \int\limits_{\partial D_I} \lambda\, w\, \bar{\varphi}\, ds - \int\limits_{\partial \Omega_R} T w\, \bar{\varphi}\, ds \\
&= k^2 \int\limits_{\Omega_R\setminus\bar{D}} u_0 \bar{\varphi}\, dx - \int\limits_{\partial D_I} \left(\frac{\partial u_0}{\partial \nu} - i\lambda u_0 + h\right) \bar{\varphi}\, ds \\
&+ \int\limits_{\partial \Omega_R} \left(T u_0 - \frac{\partial u_0}{\partial \nu}\right) \bar{\varphi}\, ds \qquad \text{for all } \varphi \in H_0^1(\Omega_R \setminus \bar{D}, \partial D_D)
\end{aligned}$$

Making use of Theorem 5.20, the assertion of the theorem can now be proven in the same way as Theorem 8.4. □

Remark 8.6. In the case when either $\partial D_I = \emptyset$ (this case corresponds to the Dirichlet boundary value problem) or $\lambda = 0$ the corresponding interior problem may not be uniquely solvable. If non uniqueness occurs, then k^2 is said to be an eigenvalue of the corresponding boundary value problem. In these cases, Theorem 8.4 holds true under the assumption that k^2 is not an eigenvalue of the corresponding boundary value problem.

Remark 8.7. Due to the change of the boundary conditions, the solution to the mixed boundary value problems (8.6)–(8.9) and (8.10)–(8.12) has a singular behavior near the boundary points in $\overline{\partial D_D} \cup \overline{\partial D_N}$. In particular, even for C^∞ boundary ∂D and analytic incident waves u^i the solution in general is not in $H^2_{loc}(\mathbb{R}^2 \setminus \bar{D})$. More precisely, the most singular term of the solution behaves like $O(r^{\frac{1}{2}})$ where (r, ϕ) denotes the local polar coordinates centered at the boundary points in $\overline{\partial D_D} \cup \overline{\partial D_N}$ [41]. This is important to take into consideration when finite element approximations are used to compute the solution.

8.2 The Inverse Scattering Problem for a Partially Coated Perfect Conductor

We now consider time harmonic incident fields given by $u^i(x) = e^{ikx\cdot d}$ with incident direction $d := (\cos\phi, \sin\phi)$ and $x = (x_1, x_2) \in \mathbb{R}^2$. The corresponding scattered field $u^s = u^s(\cdot, \phi)$ which satisfies (8.1)–(8.4) depends also on the incident angle ϕ and has the asymptotic behavior (4.5). The far field pattern $u_\infty(\theta, \phi)$, $\theta \in [0, 2\pi]$ of the scattered field defines the far field operator $F : L^2[0, 2\pi] \to L^2[0, 2\pi]$ corresponding to the scattering problem (8.1)–(8.4) by

$$(Fg)(\theta) := \int\limits_0^{2\pi} u_\infty(\theta, \phi) g(\phi) d\phi \qquad g \in L^2[0, 2\pi]\,. \tag{8.25}$$

The *inverse scattering problem* for a partially coated perfect conductor is given the far field pattern $u_\infty(\theta, \phi)$ for $\theta \in [0, 2\pi]$ and $\phi \in [0, 2\pi]$ determine *both* D and $\lambda = \lambda(x)$ for $x \in \partial D_I$.

In the same way as in the proof of Theorem 4.3, using Theorem 8.2 we can show the following result.

Theorem 8.8. *Assume that $\partial D_I \neq \emptyset$ and $\lambda \neq 0$. Then the far field operator corresponding to the scattering problem (8.1)–(8.4) is injective with dense range.*

Remark 8.9. If $\partial D_I = \emptyset$ or $\lambda = 0$, all the following results about the far field operator and the determination of D remain valid assuming the uniqueness for the corresponding interior boundary value problem. Note that the case of $\partial D_I = \emptyset$ corresponds to the scattering problem for a perfect conductor.

Concerning the unique determination of D, the following theorem can be proved in the same way as Theorem 4.5. The only change needed in the proof is that we can always choose the point x^* such that either $\Omega_\epsilon(x^*) \cap \partial D_1 \subset \partial D_{1D}$ or $\Omega_\epsilon(x^*) \cap \partial D_1 \subset \partial D_{1I}$ for some small disk $\Omega_\epsilon(x^*)$ centered at x^* of radius ϵ and satisfying $\Omega_\epsilon(x^*) \cap \bar{D}_2 = \emptyset$, whence one uses either the Dirichlet condition or impedance condition at x^* to arrive at a contraction.

Theorem 8.10. *Assume that D_1 and D_2 are two partially coated scattering obstacles with corresponding surface impedances λ_1 and λ_2 such that for a fixed wave number the far field patterns for both scatterers coincide for all incident angles ϕ. Then $D_1 = D_2$.*

Theorem 8.11. *Assume that D_1 and D_2 are two partially coated scattering obstacles with corresponding surface impedances λ_1 and λ_2 such that for a fixed wave number the far field patterns coincide for all incident angles ϕ. Then $D_1 = D_2$ and $\lambda_1 = \lambda_2$.*

Proof. By Theorem 8.10 we first have that $D_1 = D_2 = D$. Then, following the proof of Theorem 4.7 we can prove that the total fields u_1 and u_2 corresponding to λ_1 and λ_2 coincide in $\mathbb{R}^2 \setminus \bar{D}$, whence $u_1 = u_2$ and $\partial u_1/\partial\nu = \partial u_2/\partial\nu$ on ∂D. From the boundary condition we have

$$u_j = 0 \quad \text{on } \partial D_{D_j}, \qquad \frac{\partial u_j}{\partial \nu} + i\lambda_j u_j = 0 \quad \text{on } \partial D_{I_j}$$

for $j = 1, 2$. First we observe that $\partial D_{D_1} \cap \partial D_{D_2} = \emptyset$, because otherwise $u_1 = \partial u_1/\partial\nu = 0$ on an open arc $\Gamma \subset \partial D$ and a contradiction can be obtain as in the proof of Theorem 4.7. Hence $\partial D_{I_1} = \partial D_{I_2} = \partial D_I$. Next

$$(\lambda_1 - \lambda_2)u_1 = 0 \qquad \text{on } \partial D_I,$$

and again one can conclude that $\lambda_1 = \lambda_2$ as in Theorem 4.7. $\square$

Having proved the uniqueness results, we now turn our attention to finding an approximation to D and λ. Our reconstruction algorithm is based on solving the far field equation

$$Fg = \Phi_\infty(\cdot,\, z) \qquad z \in \mathbb{R}^2$$

where $\Phi_\infty(\hat{x}, z)$ is the far field pattern of the fundamental solution (see Section 4.3). The far field equation can be written as

$$-(BHg) = \Phi_\infty(\cdot,\, z) \qquad z \in \mathbb{R}^2$$

where $B : H^{\frac{1}{2}}(\partial D_D) \times H^{-\frac{1}{2}}(\partial D_I) \to L^2[0,\, 2\pi]$ maps the boundary data (f, h) to the far field pattern u_∞ of the radiating solution u to the corresponding exterior mixed boundary value problem (8.6)–(8.9) and $H : L^2[0,\, 2\pi] \to H^{\frac{1}{2}}(\partial D_D) \times H^{-\frac{1}{2}}(\partial D_I)$ is defined by

$$(Hg)(x) = \begin{cases} v_g(x) & x \in \partial D_D \\ \dfrac{\partial v_g(x)}{\partial \nu} + i\lambda(x) v_g(x) & x \in \partial D_I \end{cases}$$

with v_g being the Herglotz wave function with kernel g.

Lemma 8.12. *Any pair $(f, h) \in H^{\frac{1}{2}}(\partial D_D) \times H^{-\frac{1}{2}}(\partial D_I)$ can be approximated in $H^{\frac{1}{2}}(\partial D_D) \times H^{-\frac{1}{2}}(\partial D_I)$ by Hg.*

Proof. Let u be the unique solution to (8.10)–(8.12) with boundary data (f, h). Then the result of this lemma is a consequence of Lemma 6.19 applied to this u and the trace Theorems 1.36 and 5.5. □

Lemma 8.13. *The bounded linear operator $B : H^{\frac{1}{2}}(\partial D_D) \times H^{-\frac{1}{2}}(\partial D_I) \to L^2[0,\, 2\pi]$ is compact, injective and has dense range.*

Proof. The proof proceeds as the proof of Theorem 4.8 making use of Theorem 8.5 and Theorem 8.8. □

Using Lemma 8.12 and Lemma 8.13 we can now prove in a similar way as Theorem 4.11 the following result:

Theorem 8.14. *Assume that $\partial D_I \neq \emptyset$ and $\lambda \neq 0$. Let u_∞ be the far field pattern corresponding to the scattering problem (8.1)–(8.4) with associated far field operator F.*

1. *If $z \in D$ then for every $\epsilon > 0$ there exists $g_z^\epsilon := g_z \in L^2[0, 2\pi]$ satisfying the inequality*
$$\|Fg_z - \Phi_\infty(\cdot, z)\|_{L^2[0,\, 2\pi]} < \epsilon$$
such that
$$\lim_{z \to \partial D} \|g_z\|_{L^2[0,\, 2\pi]} = \infty$$
and
$$\lim_{z \to \partial D} \|v_{g_z}\|_{H^1(D)} = \infty$$
where v_{g_z} is the Herglotz wave function with kernel g_z.

2. *If $z \notin D$ then for every $\epsilon > 0$ and $\delta > 0$ there exists $g_z^{\epsilon,\delta} := g_z \in L^2[0, 2\pi]$ satisfying the inequality*

$$||Fg_z - \Phi_\infty(\cdot, z)||_{L^2[0,\,2\pi]} < \epsilon + \delta$$

such that

$$\lim_{\delta \to 0} ||g_z||_{L^2[0,\,2\pi]} = \infty$$

and

$$\lim_{\delta \to 0} ||v_{g_z}||_{H^1(D)} = \infty$$

where v_{g_z} is the Herglotz wave function with kernel g_z.

An approximation to D can be now obtain as the set of points z where $\|g_z\|_{L^2[0,\,2\pi]}$ becomes large with g_z being the approximate solution to the far field equation given by Theorem 8.14.

Having determined D, in a similar way as in Section 4.4, we can now use g_z given by Theorem 8.14 to determine an approximation to the maximum value of λ. In particular, let u_z be the unique solution to

$$\Delta u_z + k^2 u_z = 0 \qquad \text{in} \qquad D \tag{8.26}$$

$$u_z = -\Phi(\cdot,\, z) \qquad \text{on} \qquad \partial D_D \tag{8.27}$$

$$\frac{\partial u_z}{\partial \nu} + i\lambda u_z = -\frac{\partial \Phi(\cdot,\, z)}{\partial \nu} - i\lambda \Phi(\cdot,\, z) \qquad \text{on} \qquad \partial D_I \tag{8.28}$$

where $z \in D$ and $\lambda \in C(\partial D_I)$, $\lambda(x) \geq \lambda_0 > 0$. From the proof of the first part of Theorem 8.14 the following result is valid:

Lemma 8.15. *Assume $\partial D_I \neq \emptyset$ and $\lambda \neq 0$. Let $\epsilon > 0$, $z \in D$ and let u_z be the unique solution of (8.26)–(8.28). Then, there exists a Herglotz wave function v_{g_z} with kernel $g_z \in L^2[0,\, 2\pi]$ such that*

$$\|u_z - v_{g_z}\|_{H^1(D)} \leq \epsilon \tag{8.29}$$

Moreover there exists a positive constant $C > 0$ independent on ϵ such that

$$\|Fg_z - \Phi_\infty(\cdot, z)\|_{L^2[0,\,2\pi]} \leq C\epsilon\ . \tag{8.30}$$

Now define w_z by

$$w_z := u_z + \Phi(\cdot, z)\ . \tag{8.31}$$

In particular,

$$w_{z|_{\partial D_D}} = 0 \qquad \text{and} \qquad \left(\frac{\partial w_z}{\partial \nu} + i\lambda w_z\right)_{|_{\partial D_I}} = 0 \tag{8.32}$$

interpreted in the sense of the trace theorem. Repeating the proof of Theorem 4.13 with minor changes accounting for the boundary conditions (8.32) we have the following result:

Lemma 8.16. *For every* $z_1, z_2 \in D$ *we have that*

$$2\int_{\partial D_I} w_{z_1}\lambda\bar{w}_{z_2}\,ds = -4\pi k\,|\gamma|^2\,J_0(k\,|z_1 - z_2|) - i\left(\overline{u_{z_2}(z_1)} - u_{z_1}(z_2)\right)$$

where $\gamma = e^{i\pi/4}/\sqrt{8\pi k}$ *and* J_0 *is a Bessel function of order zero.*

Assuming D is connected, consider a disk $\Omega_r \subset D$ of radius r contained in D (see Remark 4.14) and define

$$W := \left\{ f \in L^2(\partial D_I) : \begin{array}{c} f = w_z|_{\partial D_I} \text{ with } w_z = u_z + \Phi(\cdot, z), \\ z \in \Omega_r \text{ and } u_z \text{ the solution of (8.26)–(8.28)} \end{array} \right\}.$$

Theorem 8.17. *Assume* $D_I \neq \emptyset$. *Let* $\lambda = \lambda(x)$ *be the surface impedance of the scattering problem (8.1)–(8.4). Then*

$$\max_{x\in\partial D_I} \lambda(x) = \sup_{\substack{z_i\in\Omega_r\\ \alpha_i\in\mathbb{C}}} \frac{\sum\limits_{i,j}\alpha_i\bar{\alpha}_j\left[-4\pi\,|\gamma|^2\,J_0\left(k\,|z_i - z_j|\right) - i\left(\overline{u_{z_j}(z_i)} - u_{z_i}(z_j)\right)\right]}{2\left\|\sum\limits_i \alpha_i\left(u_{z_i}(\cdot) + \Phi(\cdot, z_i)\right)\right\|^2_{L^2(\partial D)}}$$

where the sums are arbitrary finite sums.

Proof. First we show that W is complete in $L^2(\partial D_I)$. To this end let φ be a function in $L^2(\partial D_I)$ such that for every $z \in \Omega_r$

$$\int_{\partial D_I} w_z\varphi\,ds = 0.$$

Using Theorem 8.4, let $v \in H^1(D)$ be the unique solution of the interior mixed boundary value problem

$$\begin{aligned} \Delta v + k^2 v &= 0 \qquad \text{in} \qquad D \\ v &= 0 \qquad \text{on} \qquad \partial D_D \\ \frac{\partial v}{\partial \nu} + i\lambda v &= \varphi \qquad \text{on} \qquad \partial D_I. \end{aligned}$$

Then for every $z \in \Omega_r$, using the boundary conditions and the integral representation formula, we have that

$$
\begin{aligned}
0 = \int\limits_{\partial D_I} w_z \varphi \, ds &= \int\limits_{\partial D_I} w_z \left(\frac{\partial v}{\partial \nu} + i\lambda v \right) ds = \int\limits_{\partial D} w_z \left(\frac{\partial v}{\partial \nu} + i\lambda v \right) ds \\
&= \int\limits_{\partial D} \left(u_z \frac{\partial v}{\partial \nu} + i\lambda u_z v + \Phi(\cdot, z) \frac{\partial v}{\partial \nu} + i\lambda \Phi(\cdot, z) v \right) ds \\
&= \int\limits_{\partial D} \left[u_z \frac{\partial v}{\partial \nu} + v \left(-\frac{\partial u_z}{\partial \nu} - \frac{\partial \Phi(\cdot, z)}{\partial \nu} - i\lambda \Phi(\cdot, z) \right) \right] ds \\
&+ \int\limits_{\partial D} \left(\Phi(\cdot, z) \frac{\partial v}{\partial \nu} + i\lambda v \Phi(\cdot, z) \right) ds = v(z).
\end{aligned}
$$

The unique continuation principle for solutions of the Helmholtz equation now implies that $v(z) = 0$ for all $z \in D$ whence from the trace theorem $\varphi = 0$. Noting that $u_{z_i} + \Phi(\cdot, z_i) = 0$ on ∂D_D, the theorem now follows from the fact that

$$
\max_{x \in \partial D_I} \lambda(x) = \sup_{f \in L^2(\partial D_I)} \frac{1}{||f||^2_{L^2(\partial D_I)}} \int_{\partial D} \lambda \, |f|^2 \, ds \, .
$$

□

Given that D is known (for example by using the far field equation and the linear sampling method as discussed above) by Lemma 8.15 we obtain an approximation to $\max_{x \in \partial D_I} \lambda(x)$ by replacing u_z in Theorem 8.17 by the Herglotz wave function v_{g_z} with kernel g_z being the approximate (regularized) solutions of the far field equation.

In the particular case when the surface impedance is a positive constant $\lambda > 0$ we can set $z_1 = z_2 = z_0$ in Lemma 8.16 where $z_0 \in \Omega_r$ and obtain a simpler formula for λ, namely

$$
\lambda = \frac{-2k\pi|\gamma|^2 - \operatorname{Im}\left(u_{z_0}(z_0)\right)}{\|u_{z_0} + \Phi(\cdot,\, z_0)\|^2_{L^2(\partial D)}} \, . \tag{8.33}
$$

Note that the expression (8.33) can be used as a target signature to detect if an obstacle is coated or not. In particular an object is coated if and only if the denominator in nonzero.

8.3 Numerical Examples

We now present some numerical examples of the above reconstruction algorithm when the surface impedance λ is a constant. As we explained above, an approximation for λ in this case is given by

$$
\frac{-2k\pi|\gamma|^2 - \operatorname{Im}\left(v_{g_z}(z)\right)}{\|v_{g_z}(\cdot) + \Phi(\cdot,\, z)\|^2_{L^2(\partial D)}}, \qquad z = (z_1, z_2) \in D \tag{8.34}
$$

where v_{g_z} is the Herglotz wave function with kernel kernel g_z the solution of the far field equation

$$\int_0^{2\pi} u_\infty(\phi, \theta) g_z(\phi) d\phi = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik(z_1 \cos\theta + z_2 \sin\theta)}. \tag{8.35}$$

We fix the wave number $k = 3$, select a domain D, boundaries ∂D_D and ∂D_I (in some examples $\partial D_D = \emptyset$) and a constant λ. Then using the incident field $e^{ikx\cdot d}$ where $|d| = 1$ we use the finite element method to solve the scattering problem (8.1)–(8.4) and compute the far field pattern. This is obtained as a trigonometric series

$$u_\infty = \sum_{n=-N}^{N} u_{\infty,n} \exp(in\theta).$$

Of course these coefficients are already in error by the discretization error in using the finite element method. However we also add random noise to the Fourier coefficients by setting

$$u_{\infty,a,n} = u_{\infty,n}(1 + \epsilon\chi_n)$$

where ϵ is a parameter and χ_n is given by a random number generator that provides uniformly distributed random numbers in the interval $[-1, 1]$. Thus the input to the inverse solver for computing g is the approximate far field pattern

$$u_{\infty,a} = \sum_{n=-N}^{N} u_{\infty,a,n} \exp(in\theta).$$

The far field equation is then solved using Tikhonov regularization and the Morozov discrepancy principle as described in Chap. 2. In particular, using the above expression for $u_{\infty,a}$, the far field equation (8.35) is rewritten as an ill-conditioned matrix equation for the Fourier coefficients of g which we write in the form

$$A g_z = f_z \tag{8.36}$$

As already noted, this equation needs to be regularized. We start by computing the singular value decomposition of A

$$A = U \Lambda V^*$$

where U and V are unitary and Λ is real diagonal with $\Lambda_{i,i} = \sigma_i$, $1 \le i \le n$. The solution of (8.36) is then equivalent to solving

$$\Lambda V^* g_z = U^* f_z\ . \tag{8.37}$$

Let

$$\rho_z = (\rho_{z,1}, \rho_{z,2}, \cdots \rho_{z,n})^\top = U^* f_z\ .$$

Then the Tikhonov regularization of (8.37) leads to solving

$$\min_{g_z \in \mathbb{R}^n} \|\Lambda V^* g_z - f_z\|_{\ell^2}^2 + \alpha \|g\|_{\ell^2}^2$$

where $\alpha > 0$ is the Tikhonov regularization parameter chosen by using the Morozov discrepancy principle. Defining $u_z = V^* g_z$, we see that the solution to the problem is

$$u_{z,i} = \frac{\sigma_i}{\sigma_i^2 + \alpha} \rho_{z,i}, \qquad 1 \le i \le n,$$

and hence

$$g_z = V u_z \qquad \text{and} \qquad \|g_z\|_{\ell^2} = \|u_z\|_{\ell^2} = \left(\sum_{i=1}^{n} \frac{\sigma_i^2}{(\sigma_i^2 + \alpha)^2} |\rho_{z,i}|^2 \right)^{\frac{1}{2}} .$$

For the presented examples, we compute the far field pattern for 100 incident directions and observation directions equally distributed on the unit circle and add random noise of 1% or 10% to the Fourier coefficients of the far field pattern. We choose the sampling points z on a uniform grid of 101×101 points in the square region $[-5,\ 5]^2$ and compute the corresponding g_z. To visualize the obstacle we plot the level curves of the inverse of the discrete ℓ_2 norm of g_z (note that by the linear sampling method the boundary of the obstacle is characterized as the set of points where the L^2-norm of g starts to become large; see the comments at the end of Section 4.3). Then we compute (8.34) at the sampling points in the disk centered at the origin with radius 0.5 (in our examples this circle is always inside D). Although (8.34) is theoretically a constant, because of the ill-posed nature of the far field equation we evaluated (8.34) at all the grid points z in the disk and exhibit the maximum, the average and the median of the computed values of (8.34). In particular, the average, median and maximum each provides a reasonable approximation to the true impedance.

For our examples we select two scatterers shown in Fig. 8.1 (the kite and the peanut).

The kite. We consider the impedance boundary value problem for the kite described by the equation (the left curve in Fig. 8.1)

$$x(t) = (1.5\sin(t), \cos(t) + 0.65\cos(2t) - 0.65)\,, \qquad 0 \le t \le 2\pi$$

with impedance $\lambda = 2$, $\lambda = 5$ and $\lambda = 9$. In Fig. 8.2 we show two examples of the reconstructed kite (The reconstructions for the other tested cases look similar). In the numerical results for the reconstructed λ shown in Table 8.1 and Table 8.2 we use the exact boundary ∂D when we compute the $L^2(\partial D)$-norm that appears in the denominator of (8.34).

The peanut. Next we consider a peanut described by the equation (the right curve in Fig. 8.1)

$$x(t) = \left(\sqrt{\cos^2(t) + 4\sin^2(t)}\, \cos(t), \sqrt{\cos^2(t) + 4\sin^2(t)}\, \sin(t),\ \ 0 \le t \le 2\pi \right)$$

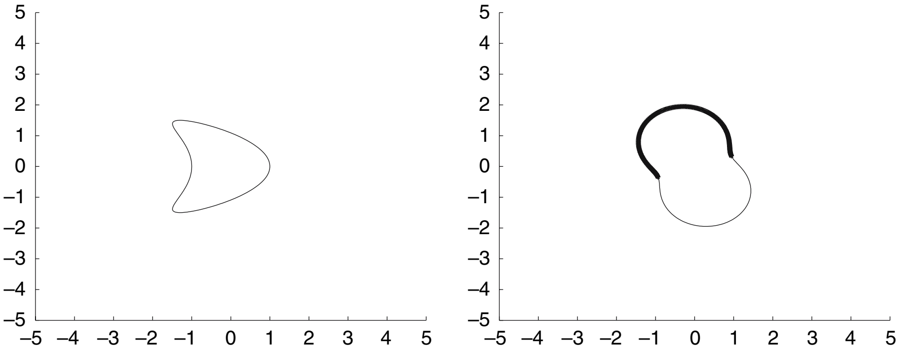

Fig. 8.1. The boundary of the scatterers used in this study: kite/peanut. When a mixed condition is used for the peanut, the thicker portion of the boundary is ∂D_D.[1]

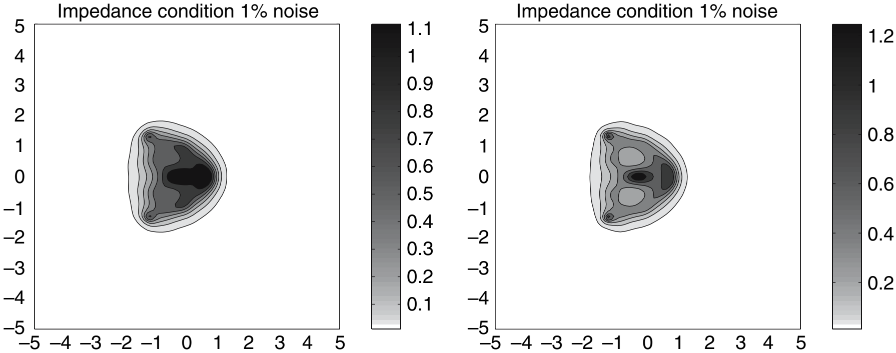

Fig. 8.2. These figures show the reconstruction of a kite with impedance boundary condition with 1% noise: on the left with $\lambda = 5$ and on the right with $\lambda = 9$.[1]

Table 8.1. The reconstruction of the surface impedance λ for the kite with 1% noise.[1]

	Maximum	Average	Median
λ=2	2.050	1.975	1.982
λ=5	4.976	4.679	4.787
λ=9	8.883	8.342	8.403

Table 8.2. The reconstruction of the surface impedance λ for the kite with 10% noise.[1]

	Maximum	Average	Median
λ=2	2.043	1.960	1.957
λ=5	4.858	4.513	4.524
λ=9	9.0328	8.013	7.992

rotated by $\pi/9$. Here we choose the surface impedance $\lambda = 2$ and $\lambda = 5$ and consider the case of a totally coated peanut (i.e. impedance boundary value problem) as well as of a partially coated peanut (i.e. mixed Dirichlet-impedance boundary value problem with ∂D_I being the lower half of the peanut as shown in Fig. 8.1). Two examples of the reconstructed peanut are presented in Fig. 8.3. A natural guess for the boundary of the scatterer is the ellipse shown by dashed line in Fig. 8.4 and we examine the sensitivity of our formula on the approximation of the boundary by using this ellipse for computing $\|v_{g_z} + \Phi(\cdot, z)\|_{L^2(\partial D)}$ in (8.34). The recovered values of λ for our experiments are shown in Table 8.3 and Table 8.4.

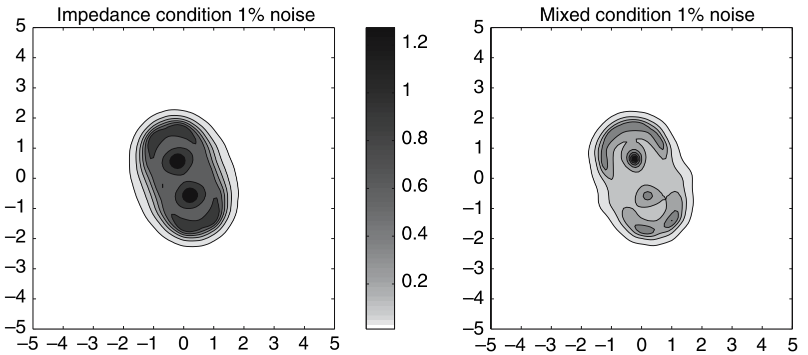

Fig. 8.3. The figure on the left shows the reconstruction of a peanut with impedance boundary condition with $\lambda = 5$. The figure on the right shows the reconstruction of a peanut with mixed condition with $\lambda = 5$ on the impedance part. Both examples are for $k = 3$ with 1% noise.[1]

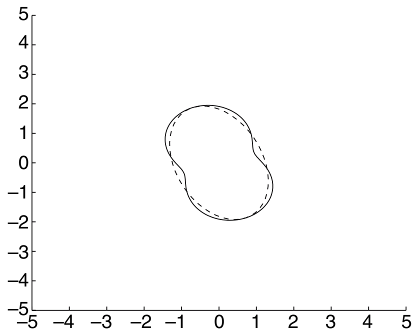

Fig. 8.4. The dash line is the approximated boundary we use for computing $\|v_{g_z} + \Phi(\cdot; z)\|_{L^2(\partial D)}$ in (8.34) in the case of peanut with impedance boundary condition.[1]

Table 8.3. Reconstruction of λ for the peanut with 1% noise.[1]

	Maximum	Average	Median
λ=2 impedance	2.192	1.992	1.979
λ=2 imped., approx. bound.	2.395	1.823	1.886
λ=2 mixed conditions	2.595	2.207	2.257
λ=5 impedance	5.689	4.950	5.181
λ=5 imped., approx. bound.	5.534	4.412	4.501
λ=5 mixed conditions	5.689	4.950	5.180

Table 8.4. Reconstruction of λ for the peanut with 10% noise.[1]

	Maximum	Average	Median
λ=2 impedance	2.297	1.985	1.978
λ=2 imped., approx. bound.	2.301	1.828	1.853
λ=2 mixed conditions	2.681	2.335	2.374
λ=5 impedance	5.335	4.691	4.731
λ=5 imped., approx. bound.	5.806	4.231	4.313
λ=5 mixed conditions	5.893	4.649	4.951

8.4 Scattering by a Partially Coated Dielectric

We now consider the scattering of time harmonic electromagnetic waves by an infinitely long cylindrical orthotropic dielectric partially coated with a very thin layer of a highly conductive material. Let the bounded domain $D \subset \mathbb{R}^2$ be the cross section of the cylinder and assume that the exterior domain $\mathbb{R}^2 \setminus \bar{D}$ is connected and let ν be the unit outward normal to the smooth boundary ∂D. The boundary $\partial D = \overline{\partial D_1} \cap \overline{\partial D_2}$ is split into two parts ∂D_1 and ∂D_2, each an open set relative to ∂D and possibly disconnected. The open arc ∂D_1 corresponds to the uncoated part and ∂D_2 corresponds to the coated part. We assume that the incident electromagnetic field and the constitutive parameters are as described in Section 5.1. In particular the fields inside D and outside D satisfy (5.5) and (5.6) respectively, and on ∂D_1, the uncoated portion of the boundary, we have the transmission condition (5.7). However on the coated portion of the cylinder we have the conductive boundary condition given by

$$\nu \times E^{ext} - \nu \times E^{int} = 0 \quad \text{and} \quad \nu \times H^{ext} - \nu \times H^{int} = \eta(\nu \times E^{ext}) \times \nu \quad (8.38)$$

where the *surface conductivity* $\eta = \eta(x)$ describes the physical properties of the thin highly conductive coating [1, 2]. Assuming that η does not depend on the z-coordinate (we recall that the cylinder axis is assumed to be parallel to the z-direction), on ∂D_2 the transmission conditions (8.38) now become

[1]Reprinted from F.Cakoni and D.Colton, The determination of the surface impedance of a partially coated obstacle from far field data, SIAM J. Appl. Math. 64 (2004), 709-723.

$$v - (u^s + u^i) = -i\eta \frac{\partial}{\partial \nu}(u^s + u^i) \quad \text{and} \quad \frac{\partial v}{\partial \nu_A} - \frac{\partial}{\partial \nu}(u^s + u^i) = 0 \qquad \text{on} \quad \partial D_2 \,.$$

where $\partial v / \partial \nu_A := \nu \cdot A(x) \nabla v$.

The direct scattering problem for a partially coated dielectric can now be formulated as follows: Assume that A, n and D satisfy the assumptions of Sect. 5.1 and $\eta \in C(\overline{\partial D_2})$ satisfies $\eta(x) \geq \eta_0 > 0$ for all $x \in \partial D_2$. Given the incident field u^i satisfying

$$\Delta u^i + k^2 u^i = 0 \qquad \text{in } \mathbb{R}^2,$$

we look for $u^s \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ and $v \in H^1(D)$ such that

$$\nabla \cdot A \nabla v + k^2 n \, v = 0 \qquad \text{in} \quad D \tag{8.39}$$

$$\Delta u^s + k^2 \, u^s = 0 \qquad \text{in} \quad \mathbb{R}^2 \setminus \bar{D} \tag{8.40}$$

$$v - u^s = u^i \qquad \text{on} \quad \partial D_1 \tag{8.41}$$

$$v - u^s = -i\eta \frac{\partial (u^s + u^i)}{\partial \nu} + u^i \qquad \text{on} \quad \partial D_2 \tag{8.42}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \qquad \text{on} \quad \partial D \tag{8.43}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0. \tag{8.44}$$

We start with a brief discussion of the well posedness of the above scattering problem.

Theorem 8.18. *The problem (8.39)–(8.44) has at most one solution.*

Proof. Let $v \in H^1(D)$ and $u^s \in H^1_{loc}(D_e)$ be the solution of (8.39)–(8.44) corresponding to the incident wave $u^i = 0$. Applying Green's first identity in D and $(\mathbb{R}^2 \setminus \bar{D}) \cap \Omega_R$ where (and in what follows) Ω_R is a disk of radius R centered at the origin and containing $\bar{D}$, using the transmission conditions we have that

$$\begin{aligned} &\int_D \left(\nabla \overline{v} \cdot A \nabla v - k^2 n |v|^2 \right) dy + \int_{\Omega_R \setminus \bar{D}} \left(|\nabla u^s|^2 - k^2 |u^s|^2 \right) dy \\ &= \int_{\partial D} \overline{v} \cdot \frac{\partial v}{\partial \nu_A} \, ds - \int_{\partial D} \overline{u^s} \cdot \frac{\partial u^s}{\partial \nu} \, ds + \int_{\partial \Omega_R} \overline{u^s} \cdot \frac{\partial u^s}{\partial \nu} \, ds \\ &= i \int_{\partial D_2} \frac{1}{\eta} |v - u^s|^2 \, ds + \int_{\partial \Omega_R} \overline{u^s} \cdot \frac{\partial u^s}{\partial \nu} \, ds. \end{aligned}$$

Taking the imaginary part of the both sides and using the fact that $\mathrm{Im}(A) \leq 0$, $\mathrm{Im}(n) \geq 0$ and $\eta \geq \eta_0 > 0$ we obtain

$$\operatorname{Im} \int\limits_{\partial\Omega_R} u^s \cdot \frac{\partial \overline{u^s}}{\partial \nu}\, ds \geq 0 \,.$$

Finally, an application of Theorem 3.6 and the unique continuation principle yield, as the proof in Lemma 5.23, $u^s = v = 0$. □

We now rewrite the scattering problem in a variational form. Multiplying the equations in (8.39)–(8.44) with a test function φ and using Green's first identity together with the transmission conditions we obtain that the total field w defined in Ω_R by $w|_D := v$ and $w|_{\Omega_R \setminus \bar{D}} = u^s + u^i$ satisfies

$$\int\limits_D \left(\nabla\overline{\varphi} \cdot A\nabla w - k^2 n\overline{\varphi}\, w\right)\, dy + \int\limits_{\Omega_R\setminus\bar{D}} \left(\nabla\overline{\varphi} \cdot \nabla w - k^2\overline{\varphi}\, w\right)\, dy \tag{8.45}$$

$$- \int\limits_{\partial D_2} \frac{i}{\eta}\, [\overline{\varphi}] \cdot [w]\, ds - \int\limits_{\partial\Omega_R} \overline{\varphi}\, Tw\, ds = - \int\limits_{\partial\Omega_R} \overline{\varphi}\, Tu^i\, ds + \int\limits_{\partial\Omega_R} \overline{\varphi} \frac{\partial u^i}{\partial \nu}\, ds$$

where $T : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{\frac{1}{2}}(\partial\Omega_R)$ is the Dirichlet to Neumann operator and $[w] = w^+|_{\partial D} - w^-|_{\partial D}$ denotes the jump of w across ∂D with w^+ and w^- the traces (in the sense of the trace operator) of $w \in H^1(\Omega_R \setminus \bar{D})$ and $w \in H^1(D)$, respectively. Note that $[w] \in \tilde{H}^{\frac{1}{2}}(\partial D_2)$ since from the transmission conditions $[w]|_{\partial D_1} = 0$.

Hence, the natural variational space for w and φ is $H^1(\Omega_R \setminus \overline{\partial D}_2)$. Note that if $u \in H^1(\Omega_R \setminus \overline{\partial D}_2)$ then $u \in H^1(D)$, $u \in H^1(\Omega_R \setminus \bar{D})$, $[u]|_{\partial D_1} = 0$, and

$$\|u\|^2_{H^1(\Omega_R\setminus\overline{\partial D}_2)} = \|u\|^2_{H^1(D)} + \|u\|^2_{H^1(\Omega_R\setminus\bar{D})} \,.$$

Now, letting

$$a_1(w,\varphi) := \int\limits_D \left(\nabla\overline{\varphi} \cdot A\nabla w + \overline{\varphi}\, w\right)\, dy + \int\limits_{\Omega_R\setminus\bar{D}} \left(\nabla\overline{\varphi} \cdot \nabla w + \overline{\varphi}\, w\right)\, dy$$

$$- \int\limits_{\partial D_2} \frac{i}{\eta}\, [\overline{\varphi}] \cdot [w]\, ds - \int\limits_{\partial\Omega_R} \overline{\varphi}\, T_0 w\, ds \tag{8.46}$$

and

$$a_2(w,\varphi) := - \int\limits_{\Omega_R} (nk^2 + 1)\, \overline{\varphi}\, w\, dy - \int\limits_{\partial\Omega_R} \overline{\varphi}\, (T_0 - T) w\, ds$$

where T_0 is the negative definite part of the Dirichlet to Neumann mapping defined in Theorem 5.20, the variational formulation of the mixed transmission problem reads: find $w \in H^1(\Omega_R \setminus \overline{\partial D}_2)$ such that

$$a_1(w,\varphi) + a_2(w,\varphi) = L(\varphi) \qquad \forall \varphi \in H^1(\Omega_R \setminus \overline{\partial} D_2) \tag{8.47}$$

where $L(\varphi)$ denotes the bounded conjugate linear functional defined by the right hand side of (8.45). We leave as an exercise to the reader to prove that if $w \in H^1(\Omega_R \setminus \overline{\partial D_2})$ solves (8.47), then $v := w|_D$ and $u^s = w|_{\Omega_R \setminus \bar{D}} - u^i$ satisfy (8.39), (8.40) in $\Omega_R \setminus \bar{D}$, the boundary conditions (8.41), (8.42) and (8.43), and $Tu^s = \partial u^s / \partial \nu$ on $\partial \Omega_R$. Exactly in the same way as in Example 5.21 one can show that u^s can be uniquely extended to a solution in $\mathbb{R}^2 \setminus \bar{D}$.
Now using the trace theorem, the Cauchy-Schwarz inequality, the chain of continuous imbeddings

$$\tilde{H}^{\frac{1}{2}}(\partial D_2) \subset H^{\frac{1}{2}}(\partial D_2) \subset L^2(\partial D_2) \subset \tilde{H}^{-\frac{1}{2}}(\partial D_2) \subset H^{-\frac{1}{2}}(\partial D_2),$$

and the assumptions on A, n and η, one can now show in a similar way as in Sect. 5.4 that the sesquilinear form $a_1(\cdot,\, \cdot)$ is bounded and strictly coercive and the sesquilinear form $a_2(\cdot,\, \cdot)$ is bounded and gives rise to a compact linear operator due to the compact imbedding of $H^1(\Omega_R \setminus \overline{\partial D_2})$ in $L^2(\Omega_R)$. Hence, using the Lax-Milgram lemma and Theorem 5.14, the above analysis combined with Theorem 8.18 implies the following result:

Theorem 8.19. *The problem (8.39)–(8.44) has exactly one solution $v \in H^1(D)$ and $u^s \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ that satisfies*

$$\|v\|_{H^1(D)} + \|u^s\|_{H^1(\Omega_R \setminus \overline{D})} \leq C \|u^i\|_{H^1(\Omega_R)}$$

where the positive constant $C > 0$ is independent of u^i but depends on R.

The scattered field u^s again has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\theta) + O(r^{-3/2}), \qquad r \to \infty$$

where the corresponding far field pattern $u_\infty(\cdot)$ depends on the observation direction $\hat{x} := (\cos\,\theta,\, \sin\,\theta)$. In the case of incident plane waves $u^i(x) = e^{ikx\cdot d}$ the interior field v and the scattered field u^s also depend on the incident direction $d := (\cos\,\phi,\, \sin\,\phi)$ and so does the corresponding far field pattern $u_\infty(\cdot) := u_\infty(\cdot,\, \phi)$. The far field pattern in turn defines the corresponding far field operator $F : L^2[0,\, 2\pi] \to L^2[0,\, 2\pi]$ by (6.7).

As will be seen, the *mixed interior transmission problem* associated with the mixed transmission problem (8.39)–(8.44) plays an important role in studying the far field operator. Hence, we now proceed to the discussion of this problem. Consider the Sobolev space

$$\mathbb{H}^1(D, \partial D_2) := \left\{ u \in H^1(D) \quad \text{such that} \quad \frac{\partial u}{\partial \nu} \in L^2(\partial D_2) \right\}$$

equipped with the graph norm

$$\|u\|^2_{\mathbb{H}^1(D,\partial D_2)} := \|u\|^2_{H^1(D)} + \left\| \frac{\partial u}{\partial \nu} \right\|^2_{L^2(\partial D_2)} .$$

Then the *mixed interior transmission problem* corresponding to the mixed transmission problem (8.39)–(8.44) reads: Given $f \in H^{\frac{1}{2}}(\partial D)$, $h \in H^{-\frac{1}{2}}(\partial D)$ and $r \in L^2(\partial D_2)$ find $v \in H^1(D)$ and $w \in \mathbb{H}^1(D, \partial D_2)$ such that

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad \text{in} \quad D \tag{8.48}$$

$$\Delta w + k^2\, w = 0 \qquad \text{in} \quad D \tag{8.49}$$

$$v - w = f|_{\partial D_1} \qquad \text{on} \quad \partial D_1 \tag{8.50}$$

$$v - w = -i\eta\, \frac{\partial w}{\partial \nu} + f|_{\partial D_2} + r \qquad \text{on} \quad \partial D_2 \tag{8.51}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial w}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{8.52}$$

Theorem 8.20. *If either Im$(n) > 0$ or Im$\left(\bar{\xi} \cdot A\,\xi\right) < 0$ at a point $x_0 \in D$, then the mixed interior transmission (8.48)–(8.52) has at most one solution.*

Proof. Let v and w be a solution of the homogeneous mixed interior transmission problem (i.e. $f = h = r = 0$). Applying the divergence theorem to $\overline{v}$ and $A\nabla v$ (see Corollary 5.6), using the boundary condition and applying Green's first identity to $\overline{w}$ and w (see Remark 5.8) we obtain

$$\int\limits_D \nabla\overline{v} \cdot A\nabla v\, dy - \int\limits_D k^2 n|v|^2\, dy = \int\limits_D |\nabla w|^2\, dy - \int\limits_D k^2|w|^2\, dy + \int\limits_{\partial D_2} i\eta \left|\frac{\partial w}{\partial \nu}\right|^2\, ds.$$

Hence

$$\mathrm{Im}\left(\int_D \nabla\overline{v} \cdot A\nabla v\, dy\right) = 0, \quad \mathrm{Im}\left(\int_D n|v|^2\, dy\right) = 0, \quad \text{and} \int\limits_{\partial D_2} \eta \left|\frac{\partial w}{\partial \nu}\right|^2\, ds = 0.$$

The last equation implies that $\partial w/\partial\nu = 0$ on ∂D_2, whence w and v satisfy the homogeneous interior transmission problem (6.12)–(6.15). The result of the theorem now follows from Theorem 6.4. □

The values of k^2 for which the homogeneous mixed interior transmission problem (8.48)–(8.52) has a nontrivial solution are called transmission eigenvalues. From the proof of Theorem 8.20 we have the following result:

Corollary 8.21. *The transmission eigenvalues corresponding to (8.48)–(8.52) form a subset of the transmission eigenvalues corresponding to (6.12)–(6.15) defined in Definition 6.3*

The above corollary justifies the use of the same name for the set of eigenvalues corresponding to both the interior transmission problem and the mixed interior transmission problem.

From the proof of Theorem 8.20 we also see that if the scatterer is fully coated, i.e. $\partial D_2 = \partial D$, then the solution (v, w) of the homogeneous mixed interior transmission problem satisfies

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \ \text{ in } D, \qquad \frac{\partial v}{\partial \nu_A} = 0 \ \text{ on } \partial D$$

and

$$\Delta w + k^2 w = 0 \ \text{ in } D, \qquad \frac{\partial w}{\partial \nu} = 0 \ \text{ on } \partial D\,.$$

From this it follows that if $\partial D_2 = \partial D$ then the uniqueness of the mixed interior transmission problem is guaranteed if at least one of the above homogeneous Neumann problems has only the trivial solution.

The following important result can be shown in the same way as in Theorem 6.2.

Theorem 8.22. *The far field operator F corresponding to the scattering problem (8.39)–(8.44) is injective with dense range if and only if there does not exists a Herglotz wave function v_g such that the pair v, v_g is a solution to the homogeneous mixed interior transmission problem (8.48)–(8.52) with $w = v_g$.*

We shall now discuss the solvability of the mixed interior transmission problem (8.48)–(8.52). We will adapt the variational approach used in Sect. 6.2 for solving (6.12)–(6.15). In order to avoid repetition, we will only sketch the proof, emphasizing the changes due to the boundary terms involving η.

Theorem 8.23. *Assume that k^2 is not a transmission eigenvalue and that there exists a constant $\gamma > 1$ such that*

$$\textit{either} \quad \bar{\xi} \cdot \mathcal{R}e(A)\, \xi \geq \gamma|\xi|^2 \quad \textit{or} \quad \bar{\xi} \cdot \mathcal{R}e(A^{-1})\, \xi \geq \gamma|\xi|^2 \qquad \forall \xi \in \mathbb{C}^2\,.$$

Then the mixed interior transmission problem (8.48)–(8.52) has a unique solution (v, w) which satisfies

$$\|v\|^2_{H^1(D)} + \|w\|^2_{\mathbb{H}^1(D,\partial D_2)} \leq C\left(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)} + \|r\|_{L^2(\partial D_2)}\right).$$

Proof. We first assume that $\bar{\xi} \cdot \mathcal{R}e(A)\, \xi \geq \gamma|\xi|^2$ for some $\gamma > 1$. In the same way as in the proof of Theorem 6.8, we can show that (8.48)–(8.52) is a compact perturbation of the modified mixed interior transmission problem

$$\nabla \cdot A\nabla v - mv = \rho_1 \qquad \text{in} \quad D \tag{8.53}$$
$$\Delta w - w = \rho_2 \qquad \text{in} \quad D \tag{8.54}$$
$$v - w = f|_{\partial D_1} \qquad \text{on} \quad \partial D_1 \tag{8.55}$$
$$v - w = -i\eta\, \frac{\partial w}{\partial \nu} + f|_{\partial D_2} + r \qquad \text{on} \quad \partial D_2 \tag{8.56}$$
$$\frac{\partial v}{\partial \nu_A} - \frac{\partial w}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{8.57}$$

where $m \in C(\overline{D})$ such that $m(x) \geq \gamma$. It is now sufficient to study (8.53)–(8.57) since the result of the theorem will then follow by an application of Theorem 5.14 and the fact that k^2 is not a transmission eigenvalue.

We first reformulate (8.53)–(8.57) as an equivalent variational problem. To this end, let

$$W(D) := \left\{ \mathbf{w} \in \left(L^2(D)\right)^2 : \nabla \cdot \mathbf{w} \in L^2(D),\ \nabla \times \mathbf{w} = 0, \text{ and } \nu \cdot \mathbf{w} \in L^2(\partial D_2) \right\}$$

equipped with the natural inner product

$$(\mathbf{w}_1, \mathbf{w}_2)_W = (\mathbf{w}_1, \mathbf{w}_2)_{L^2(D)} + (\nabla \cdot \mathbf{w}_1, \nabla \cdot \mathbf{w}_2)_{L^2(D)} + (\nu \cdot \mathbf{w}_1, \nu \cdot \mathbf{w}_2)_{L^2(\partial D_2)}$$

and norm

$$\|\mathbf{w}\|_W^2 = \|\mathbf{w}\|_{L^2(D)}^2 + \|\nabla \cdot \mathbf{w}\|_{L^2(D)}^2 + \|\nu \cdot \mathbf{w}\|_{L^2(\partial D_2)}^2 . \tag{8.58}$$

We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$ and recall

$$\langle \varphi, \boldsymbol{\psi} \cdot \nu \rangle = \int\limits_D \varphi \, \nabla \cdot \boldsymbol{\psi} \, dx + \int\limits_D \nabla \varphi \cdot \boldsymbol{\psi} \, dx \, , \tag{8.59}$$

for $(\varphi, \boldsymbol{\psi}) \in H^1(D) \times W(D)$. Then the variational form of (8.53)–(8.57) is: Find $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$ such that

$$\mathcal{A}(U, V) = L(V) \qquad \text{for all } V := (\varphi, \boldsymbol{\psi}) \in H^1(D) \times W(D) \tag{8.60}$$

where the sesquilinear form $\mathcal{A}$ defined on $(H^1(D) \times W(D))^2$ is given by

$$\begin{aligned} \mathcal{A}(U, V) = &\int\limits_D A \nabla v \cdot \nabla \bar{\varphi} \, dx + \int\limits_D m \, v \, \bar{\varphi} \, dx + \int\limits_D \nabla \cdot \mathbf{w} \, \nabla \cdot \bar{\boldsymbol{\psi}} \, dx + \int\limits_D \mathbf{w} \cdot \bar{\boldsymbol{\psi}} \, dx \\ &- i \int\limits_{\partial D_2} \eta \, (\mathbf{w} \cdot \nu) \, (\bar{\boldsymbol{\psi}} \cdot \nu) ds - \left\langle v, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle - \langle \bar{\varphi}, \mathbf{w} \cdot \nu \rangle \end{aligned}$$

and the conjugate linear functional L is given by

$$L(V) = \int\limits_D (\rho_1 \, \bar{\varphi} + \rho_2 \, \nabla \cdot \bar{\boldsymbol{\psi}}) \, dx - i \int\limits_{\partial D_2} \eta \, r \, (\bar{\boldsymbol{\psi}} \cdot \nu) \, ds + \langle \bar{\varphi}, h \rangle - \left\langle f, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle .$$

By proceeding exactly as in the proof of Theorem 6.5 we can establish the equivalence between (8.53)–(8.57) and (8.60). In particular, if (v, w) is the unique solution solution (8.53)–(8.57), then $U = (v, \nabla w)$ is a unique solution to (8.60). Conversely, if U is the unique solution to (8.60) than the unique solution (v, w) to (8.53)–(8.57) is such that $U = (v, \nabla w)$.

Notice that the definitions of $\mathcal{A}$ and L differ from the definitions (6.22) and (6.23) of $\mathcal{A}$ and L corresponding to (6.12)–(6.15) only by an additional

$L^2(\partial D_2)$-inner product term, which appears in the W-norm given by (8.58). Using the trace theorem and Schwarz's inequality one can show that $\mathcal{A}$ and L are bounded in the respective norms. On the other hand, by taking the real and the imaginary parts of $\mathcal{A}(U,U)$, we have from the assumptions on $\mathrm{Re}(A)$, $\mathrm{Im}(A)$ and η that

$$\begin{aligned}|\mathcal{A}(U,U)| \geq \gamma\|v\|^2_{H^1(D)} + \|\mathbf{w}\|^2_{L^2(D)} + \|\nabla\cdot\mathbf{w}\|^2_{L^2(D)} \\ - 2\mathrm{Re}(\langle\bar{v},\, \nu\cdot\mathbf{w}\rangle) + \eta_0\|\nu\cdot\mathbf{w}\|^2_{L^2(\partial D_2)}.\end{aligned}$$

From the duality pairing (8.59) and Schwarz's inequality we have that

$$2\mathrm{Re}(\langle\bar{v},\, \nu\cdot\mathbf{w}\rangle) \leq |\,\langle\bar{v},\, \mathbf{w}\rangle\,| \leq \|v\|_{H^1(D)}\left(\|\mathbf{w}\|^2_{L^2(D)} + \|\nabla\cdot\mathbf{w}\|^2_{L^2(D)}\right)^{\frac{1}{2}}\ .$$

Hence, since $\gamma > 1$, we conclude that

$$|\mathcal{A}(U,U)| \geq \frac{\gamma-1}{\gamma+1}\left(\|v\|^2_{H^1(D)} + \|\mathbf{w}\|^2_{L^2(D)} + \|\nabla\cdot\mathbf{w}\|^2_{L^2(D)}\right) + \eta_0\|\nu\cdot\mathbf{w}\|^2_{L^2(\partial D_2)}$$

which means that $\mathcal{A}$ is coercive, i.e.

$$|\mathcal{A}(U,U)| \geq C\left(\|v\|^2_{H^1(D)} + \|\mathbf{w}\|^2_{W(D)}\right)$$

where $C = \min((\gamma-1)/(\gamma+1),\, \eta_0)$. Therefore from the Lax-Milgram lemma we have that the variational problem (8.60) is uniquely solvable, and hence so is the modified interior transmission problem (8.53)–(8.57). Finally, the uniqueness of a solution to the mixed interior transmission problem and an application of Theorem 5.14 imply that (8.48)–(8.52) has a unique solution (v,w) which satisfies

$$\|v\|_{H^1(D)} + \|w\|_{\mathbb{H}^1(D,\partial D_2)} \leq C\left(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)} + \|r\|_{L^2(\partial D_2)}\right)$$

where $C > 0$ is independent on f, h, r. The case of $\bar{\xi}\cdot\mathcal{R}e(A^{-1})\,\xi$ can be treated in a similar way. □

Another main ingredient which we need to solve the inverse scattering problem for partially coated penetrable obstacles is an approximation property of Herglotz wave functions. In particular we need to show that, if (v,w) is the solution of the mixed interior transmission problem, then w can be approximated by a Herglotz wave function with respect to the $\mathbb{H}^1(D,\partial D_2)$-norm (which is a stronger norm than the $H^1(D)$ used in Lemma 6.19).

Theorem 8.24. *Assume that k^2 is not a transmission eigenvalue and let (w,v) be the solution of the mixed interior transmission problem (8.53)–(8.57). Then for every $\epsilon > 0$ there exists a Herglotz wave function v_{g_ϵ} with kernel $g_\epsilon \in L^2[0,\, 2\pi]$ such that*

$$\|w - v_{g_\epsilon}\|_{\mathbb{H}^1(D,\partial D_2)} \leq \epsilon\,. \tag{8.61}$$

Proof. We proceed in two steps:

1. We first show that the operator $H : L^2[0,\, 2\pi] \to H^{\frac{1}{2}}(\partial D_1) \times L^2(\partial D_2)$ defined by

$$(Hg)(x) := \begin{cases} v_g(x) & x \in \partial D_1 \\ \dfrac{\partial v_g(x)}{\partial \nu} + i v_g(x) & x \in \partial D_2 \end{cases}$$

has dense range, where v_g is a Herglotz wave function written in the form

$$v_g(x) = \int_0^{2\pi} e^{-ik(x_1 \cos\theta + x_2 \sin\theta)} g(\theta) ds(\theta), \qquad x = (x_1, x_2)\,.$$

To this end, according to Lemma 6.16, it suffices to show that the corresponding transpose operator $H^\top : \tilde{H}^{-\frac{1}{2}}(\partial D_1) \times L^2(\partial D_2) \to L^2[0,\, 2\pi]$ defined by

$$\begin{aligned} \langle Hg,\, \phi\rangle_{H^{\frac{1}{2}}(\partial D_1), \tilde{H}^{-\frac{1}{2}}(\partial D_1)} &+ \langle Hg,\, \psi\rangle_{L^2(\partial D_2), L^2(\partial D_2)} \\ &= \langle g,\, H^*(\phi,\, \psi)\rangle_{L^2[0,\, 2\pi], L^2[0,\, 2\pi]}\,, \end{aligned}$$

for $g \in L^2[0,\, 2\pi]$, $\phi \in \tilde{H}^{-\frac{1}{2}}(\partial D_1), \psi \in L^2(\partial D_2)$, is injective, where $\langle \cdot,\, \cdot\rangle$ denotes the duality pairing between the denoted spaces. By interchanging the order of integration one can show that

$$\begin{aligned} H^\top(\phi,\, \psi)(\hat{x}) = &\int_{\partial D} e^{-iky\cdot\hat{x}} \tilde{\phi}(y)\, ds(y) + \int_{\partial D} \frac{\partial e^{-iky\cdot\hat{x}}}{\partial \nu} \tilde{\psi}(y)\, ds(y) \\ &+ i \int_{\partial D} e^{-iky\cdot\hat{x}} \tilde{\psi}(y)\, ds(y) \end{aligned}$$

where $\tilde{\phi} \in H^{-\frac{1}{2}}(\partial D)$ and $\tilde{\psi} \in L^2(\partial D)$ are the extension by zero to the whole boundary ∂D of ϕ and ψ, respectively. Note that from the definition of $\tilde{H}^{-\frac{1}{2}}(\partial D_1)$ in Sect. 8.1 such an extension exists.
Assume now that $H^\top(\phi,\, \psi) = 0$. Since $H^\top(\phi,\, \psi)$ is, up to a constant factor, the far field pattern of the potential

$$\begin{aligned} P(x) = &\int_{\partial D} \Phi(x,y) \tilde{\phi}(y)\, ds(y) + \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu} \tilde{\psi}(y)\, ds(y) \\ &+ i \int_{\partial D} \Phi(x,y) \tilde{\psi}(y)\, ds(y), \end{aligned}$$

which satisfies the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{D}$, from Rellich's lemma we have that $P(x) = 0$ in $\mathbb{R}^2 \setminus \bar{D}$. As $x \to \partial D$ the following jump relations hold

$$P^+ - P^-|_{\partial D_1} = 0 \qquad\qquad P^+ - P^-|_{\partial D_2} = \psi$$

$$\left.\frac{\partial P^+}{\partial \nu} - \frac{\partial P^-}{\partial \nu}\right|_{\partial D_1} = -\phi \qquad\qquad \left.\frac{\partial P^+}{\partial \nu} - \frac{\partial P^-}{\partial \nu}\right|_{\partial D_2} = -i\psi$$

where by the superscript $+$ and $-$ we distinguish the limit obtained by approaching the boundary ∂D from $\mathbb{R}^2 \setminus \bar{D}$ and D, respectively (see [33], page 45, for the jump relations of potentials with L^2 densities, and [85] for the jump relations of the single layer potential with $H^{-\frac{1}{2}}$ density). Using the fact that $P^+ = \partial P^+/\partial \nu = 0$ we see that P satisfies the Helmholtz equation and

$$P^-|_{\partial D_1} = 0 \qquad\qquad \left.\frac{\partial P^-}{\partial \nu} + iP^-\right|_{\partial D_2} = 0$$

where the equalities are understood in the L^2 limit sense. Using Green's first identity and a parallel surface argument one can conclude, as in Theorem 8.2, that $P = 0$ in D whence from the above jump relations $\phi = \psi = 0$.

2. Next, we take $w \in \mathbb{H}^1(D, \partial D_2)$ which satisfies the Helmholtz equation in D. By considering w as the solution of (8.10)–(8.12) with $f := w|_{\partial D_1} \in H^{\frac{1}{2}}(\partial D_1)$, $h := \partial w/\partial \nu + iw|_{\partial D_2} \in L^2(\partial D_2) \subset H^{-\frac{1}{2}}(\partial D_2)$, $\lambda = 1$, $\partial D_D = \partial D_1$ and $\partial D_I = \partial D_2$, the a priori estimate (8.15) yields

$$\|w\|_{H^1(D)} + \left\|\frac{\partial w}{\partial \nu}\right\|_{L^2(\partial D_2)} \leq C\|w\|_{H^{\frac{1}{2}}(\partial D_1)} + C\left\|\frac{\partial w}{\partial \nu} + iw\right\|_{L^2(\partial D_2)} .$$

Since v_g also satisfies the Helmholtz equation in D, we can write

$$\begin{aligned} \|w - v_g\|_{\mathbb{H}^1(D,\partial D_2)} \leq\; & C\|w - v_g\|_{H^{\frac{1}{2}}(\partial D_1)} \\ & + C\left\|\frac{\partial (w - v_g)}{\partial \nu} + i(w - v_g)\right\|_{L^2(\partial D_2)} . \end{aligned} \tag{8.62}$$

From the first part of the proof, given ϵ, we can now find $g_\epsilon \in L^2[0,\, 2\pi]$ that makes the right hand side of the inequality (8.62) less then ϵ. The theorem is now proved.

□

8.5 The Inverse Scattering Problem for a Partially Coated Dielectric

The main goal of this section is the solution of the *inverse scattering problem* for partially coated dielectrics which is formulated as follows: determine *both* D and η from a knowledge of the far field pattern $u_\infty(\theta, \phi)$ for $\theta, \phi \in [0,\, 2\pi]$. As shown in Section 4.5, it suffices to know the far field pattern corresponding to $\theta \in [\theta_0,\, \theta_1] \subset [0,\, 2\pi]$ and $\phi \in [\phi_0,\, \phi_1] \subset [0,\, 2\pi]$. We begin with a uniqueness theorem.

Theorem 8.25. *Let the domains D^1 and D^2 with the boundaries ∂D^1 and ∂D^2 respectively, the matrix valued functions A_1 and A_2, the functions n_1 and n_2, and the functions η_1 and η_2 determined on the portions $\partial D_2^1 \subseteq \partial D^1$ and $\partial D_2^2 \subseteq \partial D^2$, respectively, (either ∂D_2^1 or ∂D_2^2, or both, can be empty sets) satisfy the assumptions of (8.39)–(8.44). Assume that either $\bar{\xi}\cdot Re(A_1)\,\xi \geq \gamma|\xi|^2$ or $\bar{\xi}\cdot Re(A_1^{-1})\,\xi \geq \gamma|\xi|^2$, and either $\bar{\xi}\cdot Re(A_2)\,\xi \geq \gamma|\xi|^2$ or $\bar{\xi}\cdot Re(A_2^{-1})\,\xi \geq \gamma|\xi|^2$ for some $\gamma > 1$. If the far field patterns $u_\infty^1(\theta,\phi)$ corresponding to D^1, A_1, n_1, η_1 and $u_\infty^2(\theta,\phi)$ corresponding D^2, A_2, n_2, η_2 coincide for all $\theta,\phi \in [0,\,2\pi]$ then $D^1 = D^2$.*

Proof. The proof follows the lines of the uniqueness proof for the inverse scattering problem for an orthotropic medium given in Theorem 6.14. The main two ingredients are the well-posedness of the forward problem established in Theorem 8.19 and the well-posedness of the modified mixed interior transmission problem established in Theorem 8.23. Only minor changes are needed in the proof to account for the space $\mathbb{H}^1(D,\partial D_2)\times H^1(D)$ where the solution of the mixed interior transmission problem exists which replaces $H^1(D)\times H^1(D)$ in the proof of Theorem 6.14. In order to avoid repetition, we do not present here the technical details. The proof of this theorem for the case of Maxwell's equations in $\mathbb{R}^3$ can be found in [7]. □

The next question to ask concerns the unique determination of the surface conductivity η. From the above theorem we can now assume that D is known. Furthermore, we require that for an arbitrarily choice of ∂D_2, A and η there exists at least one incident plane wave such that the corresponding total field u satisfies $\partial u/\partial\nu|_{\partial D_0} \neq 0$ where $\partial D_0 \subset \partial D$ is an arbitrary portion of ∂D. In the context of our application this is a reasonable assumption since otherwise the portion of the boundary where $\partial u/\partial\nu = 0$ for all incident plane waves will behave like a perfect conductor, contrary to the assumption that the metallic coating is thin enough for the incident field to penetrate into D.

We say that k^2 is a Neumann eigenvalue if the homogeneous problem

$$\nabla\cdot A\nabla V + k^2 n\,V = 0 \quad \text{in } D, \qquad \frac{\partial V}{\partial\nu_A} = 0 \quad \text{on } \partial D \tag{8.63}$$

has a nontrivial solution. In particular, it is easy to show (the reader can try it as an exercise) that if $\mathrm{Im}(A) < 0$ or $\mathrm{Im}(n) > 0$ at a point $x_0 \in D$ then there are no Neumann eigenvalues. The reader can also show as in Example 5.15 that if $\mathrm{Im}(A) = 0$ and $\mathrm{Im}(n) = 0$ then the Neumann eigenvalues exist and form a discrete set.

We can now prove the following uniqueness result for η.

Theorem 8.26. *Assume that k^2 is not a Neumann eigenvalue. Then under the above assumptions and for fixed D and A the surface conductivity η is uniquely determined from the far field pattern $u_\infty(\theta,\phi)$ for $\theta,\phi \in [0,\,2\pi]$.*

Proof. Let D and A be fixed and suppose there exists $\eta_1 \in C(\overline{\partial D}_2^1)$ and $\eta_2 \in C(\overline{\partial D}_2^2)$ such that the corresponding scattered fields $u^{s,1}$ and $u^{s,2}$, respectively, have the same far field patterns $u^1_\infty(\theta,\phi) = u^2_\infty(\theta,\phi)$ for all $\theta, \phi \in [0, 2\pi]$. Then from Rellich's lemma $u^{s,1} = u^{s,2}$ in $\mathbb{R}^2 \setminus \bar{D}$. Hence from the transmission condition the difference $V = v^1 - v^2$ satisfies

$$\nabla\cdot A\nabla V + k^2 n\, V = 0 \qquad \text{in} \quad D \tag{8.64}$$

$$\frac{\partial V}{\partial \nu_A} = 0 \qquad \text{on} \quad \partial D \tag{8.65}$$

$$V = -i(\tilde{\eta}_1 - \tilde{\eta}_2)\frac{\partial u^1}{\partial \nu} \quad \text{on} \quad \partial D \tag{8.66}$$

where $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are the extension by zero of η_1 and η_2, respectively, to the whole of ∂D and $u^1 = u^{s,1} + u^i$. Since k^2 is not a Neumann eigenvalue, (8.64) and (8.65) imply that $V = 0$ in D and hence (8.66) becomes

$$(\tilde{\eta}_1 - \tilde{\eta}_2)\frac{\partial u^1}{\partial \nu} = 0 \qquad \text{on} \quad \partial D$$

for all incident waves. Since for a given $\partial D_0 \subset \partial D$ there exists at least one incident plane wave such that $\partial u^1/\partial\nu|_{\partial D_0} \neq 0$, the continuity of η_1 and η_2 in $\overline{\partial D}_2^1$ and $\overline{\partial D}_2^2$, respectively, implies that $\tilde{\eta}_1 = \tilde{\eta}_2$. $\square$

As the reader has seen in Chap. 4, Chapter 6 and Sect. 8.1, our method for solving the inverse problem is based on finding an approximate solution to the far field equation

$$Fg = \Phi_\infty(\cdot, z), \qquad z \in \mathbb{R}^2$$

where F is the far field operator corresponding to the scattering problem (8.53)–(8.57). If we consider the operator $B : \mathbb{H}^1(D, \partial D_2) \to L^2[0, 2\pi]$ which takes the incident field u^i satisfying

$$\Delta u^i + k^2 u^i = 0 \qquad \text{in} \;\; D$$

to the far field pattern u_∞ of the solution to (8.39)–(8.44) corresponding to this incident field, then the far field equation can be written as

$$(Bv_g)(\hat{x}) = \Phi_\infty(\hat{x}, z), \qquad z \in \mathbb{R}^2$$

where v_g is the Herglotz wave function with kernel g. Note that the formulation of the scattering problem and Theorem 8.19 remains valid if the incident field u^i is defined as a solution to the Helmhotz equation only in D (or in a neighborhood of ∂D) since the traces of u^i only appear in the boundary conditions. From the well-posedness of (8.39)–(8.44) we see that B is a bounded linear operator. Furthermore, in the same way as in Theorem 6.22, one can show that B is in addition a compact operator. Assuming that k^2 is not a transmission eigenvalue, one can now easily see that the range of B is dense in $L^2[0, 2\pi]$ since it contains the range of F, which from Theorem 8.22 is dense in $L^2[0, 2\pi]$. We next observe that

$$\Phi_\infty(\cdot, z) \in \mathrm{Range}(B) \Longleftrightarrow z \in D \tag{8.67}$$

providing that k^2 is not a transmission eigenvalue. Indeed, if $z \in D$ then the solution u^i of $(Bu^i)(\hat{x}) = \Phi_\infty(\hat{x}, z)$ is $u^i = w_z$ where $w_z \in \mathbb{H}^1(D, \partial D_2)$ and $v_z \in H^1(D)$ is the unique solution of the mixed interior transmission problem

$$\nabla\cdot A\nabla v_z + k^2 n\, v_z = 0 \qquad \text{in} \quad D \tag{8.68}$$

$$\Delta w_z + k^2\, w_z = 0 \qquad \text{in} \quad D \tag{8.69}$$

$$v_z - (w_z + \Phi(\cdot, z)) = 0 \qquad \text{on} \quad \partial D_1 \tag{8.70}$$

$$v_z - (w_z + \Phi(\cdot, z)) = -i\eta\, \frac{\partial}{\partial \nu}(w_z + \Phi(\cdot, z)) \qquad \text{on} \quad \partial D_2 \tag{8.71}$$

$$\frac{\partial v_z}{\partial \nu_A} - \frac{\partial}{\partial \nu}(w_z + \Phi(\cdot, z)) = 0 \qquad \text{on} \quad \partial D\,. \tag{8.72}$$

On the other hand, for $z \in \mathbb{R}^2 \setminus \bar{D}$ the fact that $\Phi(\cdot, z)$ has a singularity at z, together with Rellich's lemma, implies that $\Phi_\infty(\cdot, z)$ is not in the range of B. Notice that since in general the solution w_z of (8.68)–(5.5) is not a Herglotz wave function, the far field equation in general does not have a solution for any $z \in \mathbb{R}^2$. However, for $z \in D$, from Theorem 8.24 we can approximate w_z by a Herglotz function v_g and its kernel g is an approximate solution of the far field equation. Finally, noting that if u^s, v solves (8.39)–(8.44) with $u^i \in \mathbb{H}^1(D, \partial D_2)$ then u^i, v solves the mixed interior transmission problem (8.68)–(8.72) with $\Phi(\cdot, z)$ replaced by u^s and $Bu^i = u_\infty$ where u_∞ is the far field pattern of u^s, one can easily deduce that B is injective, provided that k^2 is not a transmission eigenvalue. The above discussion and the theory of ill-posed problems now imply in the same way as in Theorem 6.24 the following result:

Theorem 8.27. *Assume that k^2 is not a transmission eigenvalue and D, A, n and η satisfy the assumptions in the formulation of the scattering problem (8.39)–(8.44). Then if F is the far field operator corresponding to (8.39)–(8.44), we have that*

1. *If $z \in D$ then for every $\epsilon > 0$ there exists a solution $g_z^\epsilon := g_z \in L^2[0, 2\pi]$ satisfying the inequality*

$$\|Fg_z - \Phi_\infty(\cdot, z)\|_{L^2[0,\,2\pi]} < \epsilon.$$

Moreover this solution satisfies

$$\lim_{z \to \partial D} \|g_z\|_{L^2[0,\,2\pi]} = \infty \quad \textit{and} \quad \lim_{z \to \partial D} \|v_{g_z}\|_{\mathbb{H}^1(D, \partial D_2)} = \infty,$$

where v_{g_z} is the Herglotz wave function with kernel g_z.

2. *If $z \in \mathbb{R}^2 \setminus \bar{D}$ then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g_z^{\epsilon,\delta} := g_z \in L^2[0, 2\pi]$ of the inequality*

$$\|Fg_z - \Phi_\infty(\cdot, z)\|_{L^2[0,\, 2\pi]} < \epsilon + \delta$$

such that

$$\lim_{\delta \to 0} \|g_z\|_{L^2[0,\, 2\pi]} = \infty \quad \textit{and} \quad \lim_{\delta \to 0} \|v_{g_z}\|_{\mathbb{H}^1(D, \partial D_2)} = \infty,$$

where v_{g_z} is the Herglotz wave function with kernel g_z.

The approximate solution g of the far field equation given by Theorem 8.27 (assuming that it can be determined using regularization methods) can be used as in the previous inverse problems considered in Chap. 4, Chapter 6 and Sect. 8.1 to reconstruct an approximation to D. In particular the boundary ∂D of D can be visualized as the set of points z where the L^2-norm of g_z becomes large.

Provided that an approximation to D is obtained as above, our next goal is to use the same g to estimate the maximum of the surface conductivity η. To this end we define W_z by

$$W_z := w_z + \Phi(\cdot, z) \tag{8.73}$$

where (v_z, w_z) satisfy (8.68)–(8.72). In particular, since $w_z \in \mathbb{H}^1(D, \partial D_2)$, $\Delta w_z \in L^2(D)$ and $z \in D$, we have that $W_z|_{\partial D} \in H^{\frac{1}{2}}(\partial D)$, $\partial W_z/\partial \nu|_{\partial D} \in H^{-\frac{1}{2}}(\partial D)$ and $\partial W_z/\partial \nu|_{\partial D_2} \in L^2(\partial D_2)$.

Lemma 8.28. *For every two points z_1 and z_2 in D we have that*

$$-2\int_D \nabla v_{z_1} \cdot Im(A)\nabla \overline{v}_{z_2}\, dx + 2k^2 \int_D Im(n) v_{z_1} \overline{v}_{z_2}\, dx + 2\int_{\partial D_2} \eta(x) \frac{\partial W_{z_1}}{\partial \nu} \frac{\partial \overline{W}_{z_2}}{\partial \nu}\, ds$$
$$= -4k\pi|\gamma|^2 J_0(k|z_1 - z_2|) + i\left(w_{z_1}(z_2) - \overline{w}_{z_2}(z_1)\right).$$

where w_{z_1}, W_{z_1} and w_{z_2}, W_{z_2} are defined by (8.68)–(8.72) and (8.73), respectively, and J_0 is a Bessel function of order zero.

Proof. Let z_1 and z_2 be two points in D and v_{z_1}, w_{z_1}, W_{z_1} and v_{z_2}, w_{z_2}, W_{z_2} the corresponding functions defined by (8.68)–(8.72). Applying the divergence theorem (see Corollary 5.6) to $v_{z_1}, \overline{v}_{z_2}$ and using (8.68)–(8.72) together with the fact that A is symmetric we have that

$$\int_{\partial D} \left(v_{z_1} \frac{\partial \overline{v}_{z_2}}{\partial \nu_{\overline{A}}} - \overline{v}_{z_2} \frac{\partial v_{z_1}}{\partial \nu_A} \right) ds = \int_D \left(\nabla v_{z_1} \cdot \overline{A} \nabla \overline{v}_{z_2} - \nabla \overline{v}_{z_2} \cdot A \nabla v_{z_1} \right) dx$$
$$+ \int_D \left(v_{z_1} \nabla \cdot \overline{A} \nabla \overline{v}_{z_2} - \overline{v}_{z_2} \nabla \cdot A \nabla v_{z_1} \right) dx = -2i \int_D \nabla v_{z_1} \cdot \mathrm{Im}(A) \nabla \overline{v}_{z_2}\, dx$$
$$+2ik^2 \int_D \mathrm{Im}(n) v_{z_1} \overline{v}_{z_2}\, dx\,. \tag{8.74}$$

On the other hand, from the boundary conditions we have

$$\int_{\partial D} \left(v_{z_1} \frac{\partial \overline{v}_{z_2}}{\partial \nu_{\overline{A}}} - \overline{v}_{z_2} \frac{\partial v_{z_1}}{\partial \nu_A} \right) ds$$
$$= \int_{\partial D} \left(W_{z_1} \frac{\partial \overline{W}_{z_2}}{\partial \nu} - \overline{W}_{z_2} \frac{\partial W_{z_1}}{\partial \nu} \right) ds - 2i \int_{\partial D_2} \eta(x) \frac{\partial W_{z_1}}{\partial \nu} \frac{\partial \overline{W}_{z_2}}{\partial \nu}\, ds.$$

Hence

$$-2i \int_D \nabla v_{z_1} \cdot \mathrm{Im}(A) \nabla \overline{v}_{z_2}\, dx + 2ik^2 \int_D \mathrm{Im}(n) v_{z_1} \overline{v}_{z_2}\, dx$$
$$+2i \int_{\partial D_2} \eta(x) \frac{\partial W_{z_1}}{\partial \nu} \frac{\partial \overline{W}_{z_2}}{\partial \nu}\, ds = \int_{\partial D} \left(W_{z_1} \frac{\partial \overline{W}_{z_2}}{\partial \nu} - \overline{W}_{z_2} \frac{\partial W_{z_1}}{\partial \nu} \right) ds$$
$$= \int_{\partial D} \left(\Phi(\cdot, z_1) \frac{\partial \overline{\Phi(\cdot, z_2)}}{\partial \nu} - \overline{\Phi(\cdot, z_2)} \frac{\partial \Phi(\cdot, z_1)}{\partial \nu} \right) ds$$
$$+ \int_{\partial D} \left(w_{z_1} \frac{\partial \overline{\Phi(\cdot, z_2)}}{\partial \nu} - \overline{\Phi(\cdot, z_2)} \frac{\partial w_{z_1}}{\partial \nu} \right) ds$$
$$+ \int_{\partial D} \left(\Phi(\cdot, z_1) \frac{\partial \overline{w}_{z_2}}{\partial \nu} - \overline{w}_{z_2} \frac{\partial \Phi(\cdot, z_1)}{\partial \nu} \right) ds.$$

Green's second identity applied to the radiating solution $\Phi(\cdot, z)$ of the Helmholtz equation in D_e implies that

$$\int_{\partial D} \left(\Phi(\cdot, z_1) \frac{\partial \overline{\Phi(\cdot, z_2)}}{\partial \nu} - \overline{\Phi(\cdot, z_2)} \frac{\partial \Phi(\cdot, z_1)}{\partial \nu} \right) ds = -2ik \int_0^{2\pi} \Phi_\infty(\cdot, z_1) \overline{\Phi_\infty(\cdot, z_2)} ds$$
$$= -2ik \int_0^{2\pi} |\gamma|^2 e^{-ik\hat{x}\cdot z_1} e^{ik\hat{x}\cdot z_2}\, ds = -4ik\pi |\gamma|^2 J_0(k|z_1 - z_2|)$$

and from the representation formula for w_{z_1} and w_{z_2} we now obtain

$$-2i \int_D \nabla v_{z_1} \cdot \mathrm{Im}(A) \nabla \overline{v}_{z_2}\, dx + 2ik^2 \int_D \mathrm{Im}(n) v_{z_1} \overline{v}_{z_2}\, dx$$
$$+2i \int_{\partial D_2} \eta(x) \frac{\partial W_{z_1}}{\partial \nu} \frac{\partial \overline{W}_{z_2}}{\partial \nu}\, ds = -4ik\pi |\gamma|^2 J_0(k|z_1 - z_2|) + \overline{w}_{z_2}(z_1) - w_{z_1}(z_2).$$

Dividing both sides of the above relation by i we have the result. $\square$

Assuming D is connected, consider a ball $\Omega_r \subset D$ of radius r contained in D (see Remark 4.14) and define a subset of $L^2(\partial D_2)$ by

$$\mathcal{V} := \left\{ f \in L^2(\partial D_2) : \quad \begin{array}{c} f = \left.\dfrac{\partial W_z}{\partial \nu}\right|_{\partial D_2} \quad \text{with } W_z = w_z + \Phi(\cdot, z), \\ z \in \Omega_r \text{ and } w_z, v_z \text{ the solution of (8.68)–(8.72)} \end{array} \right\}.$$

Lemma 8.29. *Assume that k^2 is not a transmission eigenvalue. Then $\mathcal{V}$ is complete in $L^2(\partial D_2)$.*

Proof. Let φ be a function in $L^2(\partial D_2)$ such that for every $z \in \Omega_r$

$$\int\limits_{\partial D_2} \frac{\partial W_z}{\partial \nu}\, \varphi \, ds = 0.$$

Since k^2 is not a transmission eigenvalue, we can construct $v \in H^1(D)$ and $w \in \mathbb{H}^1(D, \partial D_2)$ as the unique solution of the mixed interior transmission problem

$$\begin{array}{rccl}
(i) & \nabla \cdot A \nabla v + k^2 n\, v = 0 & \text{in} & D \\
(ii) & \Delta w + k^2\, w = 0 & \text{in} & D \\
(iii) & v - w = 0 & \text{on} & \partial D_1 \\
(iv) & v - w = -i\eta\, \dfrac{\partial w}{\partial \nu} + \varphi & \text{on} & \partial D_2 \\
(v) & \dfrac{\partial v}{\partial \nu_A} - \dfrac{\partial w}{\partial \nu} = 0 & \text{on} & \partial D.
\end{array}$$

Then we have

$$\begin{aligned}
0 = \int\limits_{\partial D_2} \frac{\partial W_z}{\partial \nu}\, \varphi \, ds &= \int\limits_{\partial D} \frac{\partial W_z}{\partial \nu}\, (v - w)\, ds + i \int\limits_{\partial D_2} \eta\, \frac{\partial W_z}{\partial \nu}\, \frac{\partial w}{\partial \nu}\, ds \\
&= \int\limits_{\partial D} \frac{\partial W_z}{\partial \nu}\, v\, ds - \int\limits_{\partial D} \frac{\partial W_z}{\partial \nu}\, w\, ds + i \int\limits_{\partial D_2} \eta\, \frac{\partial W_z}{\partial \nu}\, \frac{\partial w}{\partial \nu}\, ds. \qquad (8.75)
\end{aligned}$$

From the equations for v_z and v, the divergence theorem and the transmission boundary conditions we have

$$\begin{aligned}
\int\limits_{\partial D} \frac{\partial W_z}{\partial \nu}\, v\, ds = \int\limits_{\partial D} \frac{\partial v_z}{\partial \nu_A}\, v\, ds &= \int\limits_{\partial D} \frac{\partial v}{\partial \nu_A}\, v_z\, ds \\
&= \int\limits_{\partial D} \frac{\partial w}{\partial \nu}\, W_z\, ds - i \int\limits_{\partial D_2} \eta\, \frac{\partial W_z}{\partial \nu}\, \frac{\partial w}{\partial \nu}\, ds. \qquad (8.76)
\end{aligned}$$

Finally substituting (8.76) into (8.75) and using the integral representation formula we obtain

$$
\begin{aligned}
0 &= \int_{\partial D} \left(\frac{\partial w}{\partial \nu} W_z - \frac{\partial W_z}{\partial \nu} w \right) ds = \int_{\partial D} \left(\frac{\partial w}{\partial \nu} w_z - \frac{\partial w_z}{\partial \nu} w \right) ds \\
&= \int_{\partial D} \left(\frac{\partial w}{\partial \nu} \Phi(\cdot, z) - \frac{\partial \Phi(\cdot, z)}{\partial \nu} w \right) ds = w(z) \qquad \forall z \in \Omega_r \, . \qquad (8.77)
\end{aligned}
$$

The unique continuation principle for the Helmholtz equation now implies that $w = 0$ in D. Then (c.f. the proof of Theorem 8.2) $v = 0$ and therefore $\varphi = 0$ which proves the lemma. □

Now we are ready to prove the main result of this section.

Theorem 8.30. *Let $\eta \in C(\partial D_2)$ be the surface conductivity of (8.39)–(8.44), assume that $Im(A) = 0$ and $Im(n) = 0$ in D and that k^2 is neither a transmission eigenvalue nor a Neumann eigenvalue. Then*

$$
\begin{aligned}
&\max_{x \in \partial D_2} \eta(x) \qquad (8.78) \\
&= \sup_{\substack{z_i,\, z_j \in \Omega_r \\ \alpha_i \in \mathbb{C}}} \frac{\sum_{i,j} \alpha_i \overline{\alpha_j} \left(-4\pi k |\gamma|^2 J_0(k|z_i - z_j|) + i w_{z_i}(z_j) - i \overline{w}_{z_j}(z_i) \right)}{2 \left\| \sum_i \alpha_i \frac{\partial}{\partial \nu} (w_{z_i}(\cdot) + \Phi(\cdot; z_i)) \right\|^2_{L^2(\partial D_2)}}
\end{aligned}
$$

where w_z is such that (w_z, v_z) satisfies (8.68)–(8.72) and the sums are arbitrary finite sums.

Proof. It is obvious that

$$
\max_{x \in \partial D_2} \eta(x) = \sup_{f \in L^2(\partial D_2)} \frac{1}{\|f\|^2_{L^2(\partial D_2)}} \int_{\partial D_2} \eta(x) |f|^2 ds.
$$

The theorem then follows from Lemma 8.28 and Lemma 8.29 by fixing first z_2 and then z_1 and considering linear combinations of $\partial W_z / \partial \nu$ for different $z \in \Omega_r$. □

Given that D is known, w_z in the right hand side of (8.78) still can not be computed since it depends on the unknown functions η and A. However, from Theorem 8.24, we can use in (8.78) an approximation to w_z given by the Herglotz wave function v_{g_z} with kernel g_z being the (regularized) solution of the far field equation.

In the particular case where the coating is homogeneous, i.e the surface conductivity is a positive constant $\eta > 0$, we can further simplify (8.78). In particular, fix an arbitrary point $z_0 \in \Omega_r$ and consider $z_1 = z_2 = z_0$. Then (8.78) simply becomes

$$\eta = \frac{-2k\pi|\gamma|^2 - \mathrm{Im}\,(w_{z_0}(z_0))}{\left\| \dfrac{\partial}{\partial \nu}(w_{z_0}(\cdot) + \Phi(\cdot\,; z_0)) \right\|^2_{L^2(\partial D_2)}} \,. \tag{8.79}$$

A drawback of both (8.78) and (8.79) is that the extent of the coating ∂D_2 is in general not known. Hence, replacing ∂D_2 by ∂D, these expressions in practice only provide a lower bound for the maximum of η unless it is known a priori that D is completely coated.

8.6 Numerical Examples

We now present some numerical tests of the above inversion scheme using synthetic data. For our examples, in (8.39)–(8.44) we choose $A = (1/4)I$, $n = 1$ and η equal to a constant. The far field data is computed by using a finite element method on a domain that is terminated by a rectangular perfectly matched layer (PML) and the far field equation is solved by the same procedure as described at the end of Sect. 8.1 to compute g [16].

We present some results for an ellipse given by the parametric equations $x = 0.5\cos(s)$ and $y = 0.2\sin(s)$, $s \in [0, 2\pi]$. For the ellipse we consider either a fully coated or partially coated object shown in Fig. 8.5.

We begin by assuming an exact knowledge of the boundary in order to assess the accuracy of (8.79). Having computed g by using regularization methods to solve the far field equation, we approximate (8.79) using the trapezoidal rule with 100 integration points and use $z_0 = (0, 0)$. In Fig 8.6 we show results

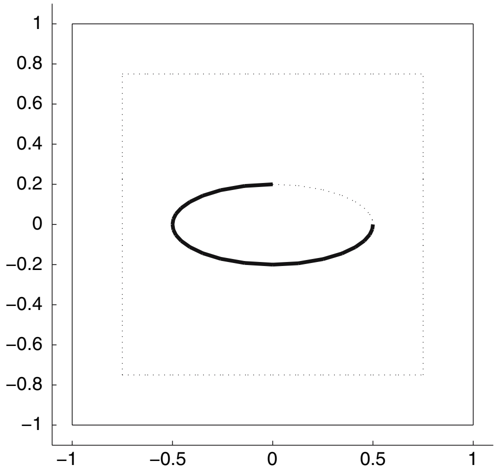

Fig. 8.5. A diagram showing the coated portion of the partially coated ellipse as a thick line. The dotted square is the inner boundary of the PML and the solid square is the boundary of the finite element computational domain.[2]

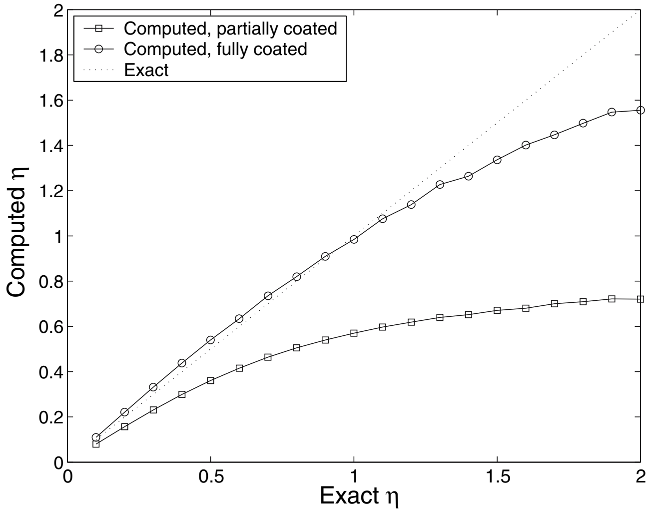

Fig. 8.6. Computation of η using the exact boundary for the fully coated and partially coated ellipse. Clearly in all cases the approximation of η deteriorates for large conductivities.[2]

of reconstruction of a range of conductivities η for the fully coated ellipse and partially coated ellipse. Recall that for the partially coated ellipse (8.79) with ∂D_2 replaced by ∂D provides only a lower bound for η. For each exact η we compute the far field data, add noise and compute an approximation to w_z as discussed before and in Sect. 8.1.

We now wish to investigate the solution of the full inverse problem. We start by using the linear sampling method to approximate the boundary of the scatterer which is based on the behavior of g given by Theorem 8.27. In particular we compute $1/\|g\|$ for z on a uniform grid in the sampling domain. In the upcoming numerical results we have chosen 61 incident directions equally distributed on the unit circle and we sample on a 101×101 grid on the square $[-1, 1] \times [-1, 1]$.

Having computed g by using Tikhonov regularization and the Morozov discrepancy principle to solve the far field equation, for each sample point we have a discrete level set function $1/\|g\|$. Choosing a contour value C then provides a reconstruction of the support of the given scatterer. We extract the edge of the reconstruction and then fit this using a trigonometric polynomial of degree M assuming that the reconstruction is star-like with respect to the origin (for more advanced applications it would be necessary to employ a more elaborate smoothing procedure). Thus for an angle θ the radius of the reconstruction is given by

$$r(\theta) = \mathrm{Re}\left(\sum_{n=-M}^{M} r_n \exp(in\theta)\right)$$

where r is measured from the origin (since in all the examples here the origin is within the scatterer). The coefficients r_n are found using a least squares fit to the boundary identified in the previous step of the algorithm. Once we have a parameterization of the reconstructed boundary we can compute the normal to the boundary and evaluate (8.79) for some choice of z_0 (in the examples always $z_0 = (0,0)$) using the trapezoidal rule with 100 points. This provides our reconstruction of η. The results of the experiments for a fully coated ellipse are shown in Fig. 8.7 and Fig. 8.8. For more details on the choice of the contour value C that provides a good reconstruction of the boundary of the scatterer we refer the reader to [16].

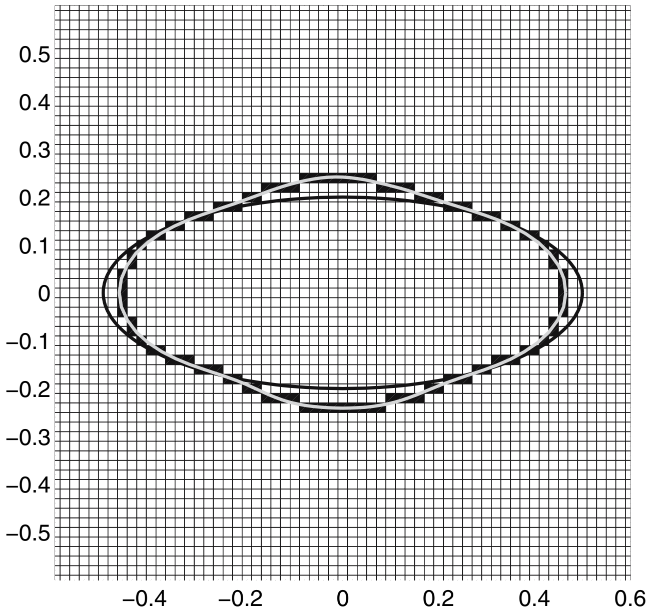

Fig. 8.7. Reconstruction of the fully coated ellipse for $\eta = 1$.[2]

In the case of partially coated ellipse (see Fig. 8.5) the inversion algorithm is unchanged (both the boundary of the scatterer and η are reconstructed). The result of reconstruction of D when $\eta = 1$ is shown in Fig. 8.9 and the results for a range of η shown are in Fig. 8.10. We recall again that for a partially coated obstacle (8.79) only provides a lower bound for η (i.e. ∂D_2 is replaced by ∂D).

[2]Reprinted from F.Cakoni, D.Colton and P.Monk, The determination of the surface conductivity of a partially coated dielectric, SIAM J. Appl. Math. 65 (2005), 767-789.

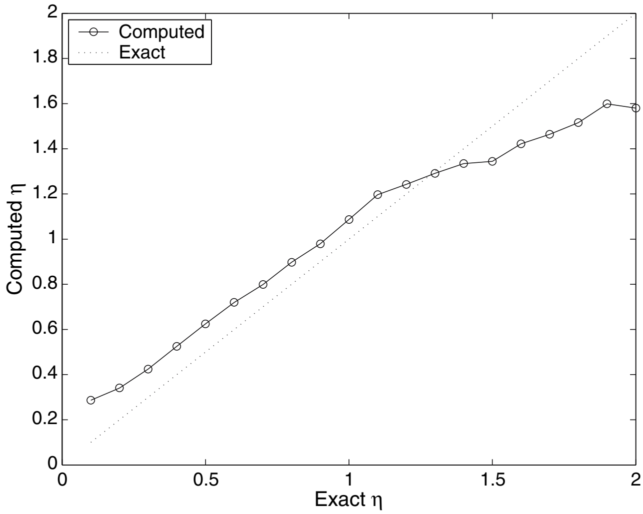

Fig. 8.8. Determination of a range of η for the (reconstructed) fully coated ellipse. For each exact η we apply the reconstruction algorithm using a range of cutoffs and plot the corresponding reconstruction. An exact reconstruction would lie on the dotted line.[2]

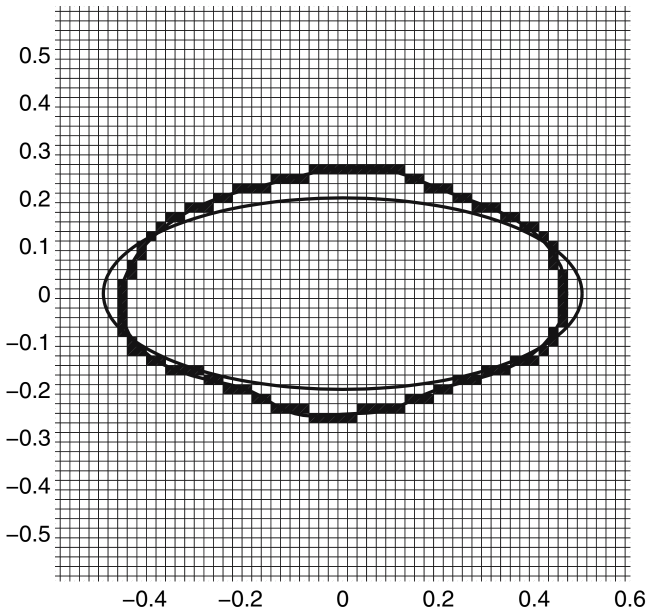

Fig. 8.9. Reconstruction of the partially coated ellipse for $\eta = 1$.[2]

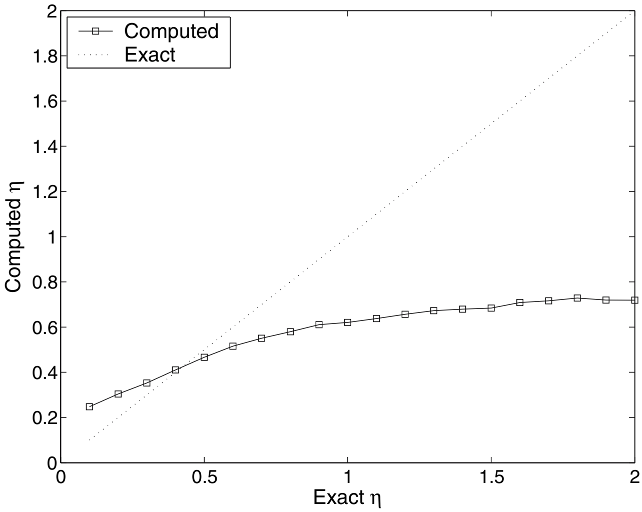

Fig. 8.10. Determination of a range of η for the (reconstructed) partially coated ellipse.[2]

8.7 Scattering by Cracks

In the last sections of this chapter we will discuss the scattering of a time harmonic electromagnetic plane wave by an infinite cylinder having an open arc in $\mathbb{R}^2$ as cross section. We assume that the cylinder is a perfect conductor that is (possibly) coated on one side by a material with (constant) surface impedance λ. This leads to a (possibly) mixed boundary value problem for the Helmholtz equation defined in the exterior of an open arc in $\mathbb{R}^2$. Our aim is to establish the existence and uniqueness of a solution to this scattering problem and to then use this knowledge to study the inverse scattering problem of determining the shape of the open arc (or "crack") from a knowledge of the far field pattern of the scattered field [9].

The inverse scattering problem for cracks was initiated by Kress [74] (see also [76, 78, 86]). In particular, Kress considered the inverse scattering problem for a perfectly conducting crack and used Newton's method to reconstruct the shape of the crack from a knowledge of the far field pattern corresponding to a single incident wave. Kirsch and Ritter [72] used the factorization method (see Chapter 7) to reconstruct the shape of the open arc from a knowledge of the far field pattern assuming a Dirichlet or Neuman boundary condition.

Let $\Gamma \subset \mathbb{R}^2$ be a smooth, open, nonintersecting arc. More precisely, we consider $\Gamma \subset \partial D$ to be a portion of a smooth curve ∂D that encloses a region D in $\mathbb{R}^2$. We choose the unit normal ν on Γ to coincide with the outward normal to ∂D. The scattering of a time harmonic incident wave u^i by a thin infinitely long cylindrical perfect conductor leads to the problem of determining u satisfying

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \bar{\Gamma} \tag{8.80}$$

$$u^{\pm} = 0 \qquad \text{on} \qquad \Gamma\,, \tag{8.81}$$

where $u^{\pm}(x) = \lim_{h \to 0^+} u(x \pm h\nu)$ for $x \in \Gamma$. The total field u is decomposed as $u = u^s + u^i$ where u^i is an entire solution of the Helmholtz equation, u^s is the scattered field which is required to satisfy the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \tag{8.82}$$

uniformly in $\hat{x} = x/|x|$ with $r = |x|$. In particular the incident field can again be a plane wave given by $u^i(x) = e^{ikx \cdot d}$, $|d| = 1$.
In the case where one side of the thin cylindrical obstacle Γ is coated by a material with constant surface impedance $\lambda > 0$, we obtain the following mixed crack problem for the total field $u = u^s + u^i$:

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \bar{\Gamma} \tag{8.83}$$

$$u^{-} = 0 \qquad \text{on} \qquad \Gamma, \tag{8.84}$$

$$\frac{\partial u^+}{\partial \nu} + i\lambda u^+ = 0 \qquad \text{on} \qquad \Gamma, \tag{8.85}$$

where again $\partial u^{\pm}(x)/\partial \nu = \lim_{h \to 0^+} \nu \cdot \nabla u(x \pm h\nu)$ for $x \in \Gamma$ and u^s satisfies the Sommerfeld radiation condition (8.82).

Recalling the Sobolev spaces $H^1_{loc}(\mathbb{R}^2 \setminus \bar{\Gamma})$, $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$ from Sect. 8.1 and Sect. 8.4, we observe that the above scattering problems are particular cases of the following more general boundary value problems in the exterior of Γ:
Dirichlet crack problem: Given $f \in H^{\frac{1}{2}}(\Gamma)$ find $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{\Gamma})$ such that

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \bar{\Gamma} \tag{8.86}$$

$$u^{\pm} = f \qquad \text{on} \qquad \Gamma \tag{8.87}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0\,. \tag{8.88}$$

Mixed crack problem: Given $f \in H^{\frac{1}{2}}(\Gamma)$ and $h \in H^{-\frac{1}{2}}(\Gamma)$ find $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{\Gamma})$ such that

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \bar{\Gamma} \tag{8.89}$$

$$u^{-} = f \qquad \text{on} \qquad \Gamma \tag{8.90}$$

$$\frac{\partial u^+}{\partial \nu} + i\lambda u^+ = h \qquad \text{on} \qquad \Gamma \tag{8.91}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0\,. \tag{8.92}$$

Note that the boundary conditions in the both problems are assumed in the sense of the trace theorems. In particular $u^+|_{\Gamma}$ is the restriction to Γ of the

trace $u \in H^{\frac{1}{2}}(\partial D)$ of $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ while $u^-|_\Gamma$ is the restriction to Γ of the trace $u \in H^{\frac{1}{2}}(\partial D)$ of $u \in H^1(D)$. Since $\nabla u \in L^2_{loc}(\mathbb{R}^2)$, the same comment is valid for $\partial u^\pm/\partial\nu$ where $\partial u/\partial\nu \in H^{-\frac{1}{2}}(\partial D)$ is interpreted in the sense of Theorem 5.5.
It is easy to see that the scattered field u^s in the scattering problem for a perfect conductor and for a partially coated perfect conductor satisfies the Dirichlet crack problem with $f = -u^i|_\Gamma$ and the mixed crack problem with $f = -u^i|_\Gamma$ and $h = -\partial u^i/\partial\nu - i\lambda u^i|_\Gamma$, respectively.

We now define $[u] := u^+ - u^-|_\Gamma$ and $\left[\dfrac{\partial u}{\partial \nu}\right] := \left.\dfrac{\partial u^+}{\partial\nu} - \dfrac{\partial u^-}{\partial\nu}\right|_\Gamma$, the jump of u and $\dfrac{\partial u}{\partial\nu}$, respectively, across the crack Γ.

Lemma 8.31. *If u is a solution of the Diriclet crack problem (8.86)–(8.88) or the mixed crack problem (8.89)–(8.92) then $[u] \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ and $\left[\dfrac{\partial u}{\partial\nu}\right] \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$.*

Proof. . Let $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{\Gamma})$ be a solution to (8.86)–(8.88) or (8.89)–(8.92). Then from the trace theorem and Theorem 5.5 $[u] \in H^{\frac{1}{2}}(\partial D)$ and $[\partial u/\partial\nu] \in H^{-\frac{1}{2}}(\partial D)$. But the solution u of the Helmholtz equation is such that $u \in C^\infty$ away from Γ, whence $[u] = [\partial u/\partial\nu] = 0$ on $\partial D \setminus \bar{\Gamma}$. Hence by definition (see Sect. 8.1) $[u] \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ and $[\partial u/\partial\nu] \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$. □

We first establish uniqueness for the problems (8.86)–(8.88) and (8.89)–(8.92).

Theorem 8.32. *The Diriclet crack problem (8.86)–(8.88) and the mixed crack problem (8.89)–(8.92) have at most one solution.*

Proof. Denote by Ω_R a sufficiently large ball with radius R containing $\overline{D}$. Let u be a solution to the homogeneous Dirichlet or mixed crack problem, i.e. u satisfies (8.86)–(8.88) with $f = 0$ or (8.89)–(8.92) with $f = h = 0$. Obviously, $u \in H^1(\Omega_R \setminus \overline{D}) \cup H^1(D)$ satisfies the Helmholtz equation in $\Omega_R \setminus \overline{D}$ and D and from the above lemma u satisfies the following transmission conditions on the complementary part $\partial D \setminus \bar{\Gamma}$ of ∂D:

$$u^+ = u^- \qquad \text{and} \qquad \frac{\partial u^+}{\partial\nu} = \frac{\partial u^-}{\partial\nu} \qquad \text{on } \partial D \setminus \bar{\Gamma}\,. \tag{8.93}$$

By an application of Green's first identity for u and $\overline{u}$ in D and $\Omega_R \setminus \overline{D}$ and using the transmission conditions (8.93) we see that

$$\begin{aligned}\int_{\partial\Omega_R} u\frac{\partial\overline{u}}{\partial\nu}ds = \int_{\Omega_R\setminus\overline{D}} |\nabla u|^2dx + \int_D |\nabla u|^2 dx - k^2 \int_{\Omega_R\setminus\overline{D}} |u|^2dx - k^2\int_D |u|^2dx \\ + \int_\Gamma u^+\frac{\partial\overline{u}^+}{\partial\nu}ds - \int_\Gamma u^-\frac{\partial\overline{u}^-}{\partial\nu}ds\,.\end{aligned} \tag{8.94}$$

For problem (8.86)–(8.88) the boundary condition (8.87) implies

$$\int_{\Gamma} u^+ \frac{\partial \overline{u}^+}{\partial \nu} ds = \int_{\Gamma} u^- \frac{\partial \overline{u}^-}{\partial \nu} ds = 0,$$

while for problem (8.89)–(8.88), since $\lambda > 0$, the boundary conditions (8.91) and (8.90) imply

$$\int_{\Gamma} u^+ \frac{\partial \overline{u}^+}{\partial \nu} ds - \int_{\Gamma} u^- \frac{\partial \overline{u}^-}{\partial \nu} ds = i\lambda \int_{\Gamma} |u^+|^2 ds \, .$$

Hence for the both problems we can conclude that

$$\mathrm{Im} \int_{\partial \Omega_R} u \frac{\partial \overline{u}}{\partial \nu} ds \geq 0,$$

whence from Theorem 3.6 and the unique continuation principle we obtain that $u = 0$ in $\mathbb{R}^2 \setminus \bar{\Gamma}$. □

To prove the existence of a solution to the above crack problems we will use an integral equation approach. In Chap. 3 the reader has already been introduced to the use of integral equations of the second kind to solve boundary value problems. Here we will employ a *first kind* integral equation approach which is based on applying the Lax-Milgram lemma to boundary integral operators [85]. In this sense the method of first kind integral equations is similar to variational methods.

We start with the representation formula (see Remark 5.8)

$$u(x) = \int_{\partial D} \left(\frac{\partial u(y)}{\partial \nu_y} \Phi(x,y) - u(y) \frac{\partial}{\partial \nu_y} \Phi(x,y) \right) ds_y, \quad x \in D \tag{8.95}$$

$$u(x) = \int_{\partial D} \left(u(y) \frac{\partial}{\partial \nu_y} \Phi(x,y) - \frac{\partial u(y)}{\partial \nu_y} \Phi(x,y) ds_y \right) ds_y, \quad x \in \mathbb{R}^2 \setminus \bar{D}$$

where $\Phi(\cdot, \cdot)$ is again the fundamental solution to the Helmholtz equation defined by

$$\Phi(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|) \tag{8.96}$$

with $H_0^{(1)}$ being a Hankel function of the first kind of order zero. By making use of the known jump relations of the single- and double- layer potentials across the boundary ∂D (see Sect. 7.1) and by eliminating the integrals over $\partial D \setminus \bar{\Gamma}$, from (8.93) we obtain

$$\frac{1}{2} \left(u^- + u^+ \right) = -S_\Gamma \left[\frac{\partial u}{\partial \nu} \right] + K_\Gamma[u] \qquad \text{on } \Gamma \tag{8.97}$$

$$\frac{1}{2} \left(\frac{\partial u^-}{\partial \nu} + \frac{\partial u^+}{\partial \nu} \right) = -K'_\Gamma \left[\frac{\partial u}{\partial \nu} \right] + T_\Gamma[u] \qquad \text{on } \Gamma, \tag{8.98}$$

where, S, K, K', T are the boundary integral operators

$$S : H^{-\frac{1}{2}}(\partial D) \longrightarrow H^{\frac{1}{2}}(\partial D) \qquad K : H^{\frac{1}{2}}(\partial D) \longrightarrow H^{\frac{1}{2}}(\partial D)$$
$$K' : H^{-\frac{1}{2}}(\partial D) \longrightarrow H^{-\frac{1}{2}}(\partial D) \qquad T : H^{\frac{1}{2}}(\partial D) \longrightarrow H^{-\frac{1}{2}}(\partial D),$$

defined by (7.3), (7.4), (7.5) and (7.6), respectively, and $S_\Gamma, K_\Gamma, K'_\Gamma, T_\Gamma$ are the corresponding operators restricted to Γ defined by

$$(S_\Gamma \psi)(x) := \int_\Gamma \psi(y)\Phi(x,y)ds_y \qquad \psi \in \tilde{H}^{-\frac{1}{2}}(\Gamma), \qquad x \in \Gamma$$

$$(K_\Gamma \psi)(x) := \int_\Gamma \psi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y \qquad \psi \in \tilde{H}^{\frac{1}{2}}(\Gamma), \qquad x \in \Gamma$$

$$(K'_\Gamma \psi(x)) := \int_\Gamma \psi(y)\frac{\partial}{\partial \nu_x}\Phi(x,y)ds_y \qquad \psi \in \tilde{H}^{-\frac{1}{2}}(\Gamma), \qquad x \in \Gamma$$

$$(T_\Gamma \psi)(x) := \frac{\partial}{\partial \nu_x}\int_\Gamma \psi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y \qquad \psi \in \tilde{H}^{-\frac{1}{2}}(\Gamma), \qquad x \in \Gamma.$$

Recalling that functions in $\tilde{H}^{\frac{1}{2}}(\Gamma)$ and $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ can be extended by zero to functions in $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$, respectively, the above restricted operators are well defined. Moreover, they have the following mapping properties:

$$S_\Gamma : \tilde{H}^{-\frac{1}{2}}(\Gamma) \longrightarrow H^{\frac{1}{2}}(\Gamma) \qquad K_\Gamma : \tilde{H}^{\frac{1}{2}}(\Gamma) \longrightarrow H^{\frac{1}{2}}(\Gamma)$$
$$K'_\Gamma : \tilde{H}^{-\frac{1}{2}}(\Gamma) \longrightarrow H^{-\frac{1}{2}}(\Gamma) \qquad T_\Gamma : \tilde{H}^{\frac{1}{2}}(\Gamma) \longrightarrow H^{-\frac{1}{2}}(\Gamma).$$

In the case of the Dirichlet crack problem, since $[u] = 0$ and $u^+ = u^- = f$, the relation (8.97) gives the following first kind integral equation for the unknown jump of the normal derivative of the solution across Γ:

$$S_\Gamma \left[\frac{\partial u}{\partial \nu}\right] = -f \,. \tag{8.99}$$

In the case of the mixed crack problem, the unknowns are both $[u] \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ and $\left[\frac{\partial u}{\partial \nu}\right] \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$. Using the boundary conditions (8.90) and (8.91), together with the relations (8.97) and (8.98), we obtain the following integral equation of the first kind for the unknowns $[u]$ and $\left[\frac{\partial u}{\partial \nu}\right]$:

$$\begin{pmatrix} S_\Gamma & -K_\Gamma + I \\ K'_\Gamma - I & -T_\Gamma - i\lambda I \end{pmatrix} \begin{pmatrix} \left[\frac{\partial u}{\partial \nu}\right] \\ [u] \end{pmatrix} = \begin{pmatrix} -f \\ i\lambda f - h \end{pmatrix} . \tag{8.100}$$

We let A_Γ denote the matrix operator in (8.100) and note that A_Γ is a continuous mapping from $\tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$.

Lemma 8.33. *The operator $S_\Gamma : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is invertible with bounded inverse.*

Proof. From Theorem 7.3 we have that the bounded linear operator $S_i : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$, defined by (7.3) with k replaced by i in the fundamental solution, satisfies

$$(S_i\psi, \psi) \geq C\|\psi\|^2_{H^{-\frac{1}{2}}(\partial D)} \qquad \text{for } \psi \in H^{-\frac{1}{2}}(\partial D)$$

where $(\cdot,\cdot)$ denotes the conjugated duality pairing between $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$ defined by Defintion 7.1. Furthermore the operator $S_c = S - S_i$ is compact from $H^{-\frac{1}{2}}(\partial D)$ to $H^{\frac{1}{2}}(\partial D)$. Since for any $\psi \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$ its extension by zero $\tilde{\psi}$ is in $H^{-\frac{1}{2}}(\partial D)$, we have that for $\psi \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$

$$(S_{i\Gamma}\psi, \psi) = \left(S_i\tilde{\psi}, \tilde{\psi}\right) \geq C\|\tilde{\psi}\|^2_{H^{-\frac{1}{2}}(\partial D)} = C\|\psi\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma)}$$

and $S_{c\Gamma}$ is compact from $\tilde{H}^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$ where $S_{i\Gamma}, S_{c\Gamma} : \tilde{H}^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ are the corresponding restrictions of S_i and S_c.
Applying the Lax Milgram lemma to the bounded and coercive sesquilinear form

$$a(\psi, \phi) := (S_{i\Gamma}\psi, \phi), \qquad \phi, \psi \in \tilde{H}^{-\frac{1}{2}}(\partial D)$$

we conclude that $S_{i\Gamma}^{-1} : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ exists and is bounded. Since S_c is compact, an application of Theorem 5.14 to $S_\Gamma = S_{i\Gamma} + S_{c\Gamma} : \tilde{H}^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ gives that the injectivity of S_Γ implies that S_Γ is invertible with bounded inverse. Hence it remains to show that S_Γ is injective. To this end let $\alpha \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$ be such that $S_\Gamma\alpha = 0$. Define the potential

$$u(x) = -\int_\Gamma \alpha(y)\Phi(x, y)\, ds_y = -\int_{\partial D} \tilde{\alpha}(y)\Phi(x, y)\, ds_y \qquad x \in \mathbb{R}^2 \setminus \bar{\Gamma}$$

where $\tilde{\alpha} \in H^{-\frac{1}{2}}(\partial D)$ is the extension by zero of α. This potential satisfies the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{\Gamma}$ and the Sommerfeld radiation condition, and moreover $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{\Gamma})$. Note that from the jump relations for single layer potentials we have that $\tilde{\alpha} = [\partial u/\partial\nu]$ on ∂D. Furthermore, the continuity of S across ∂D and the fact that $S_\Gamma\alpha = S\tilde{\alpha} = 0$ imply that $u^\pm|_\Gamma = -S\tilde{\alpha} = 0$. Hence u satisfies the homogeneous Dirichlet crack problem and from Theorem 8.32 $u = 0$ in $\mathbb{R}^2 \setminus \bar{\Gamma}$ whence $\tilde{\alpha} = [\partial u/\partial\nu] = 0$. This proves that S_Γ is injective. $\square$

Lemma 8.34. *The operator $A_\Gamma : \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ is invertible with bounded inverse.*

Proof. The proof follows that of Lemma 8.33. Let $\tilde{\zeta} = (\tilde{\phi}, \tilde{\psi}) \in H^{-\frac{1}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ be the extension by zero to ∂D of $\zeta = (\phi, \psi) \in \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$.

From Theorem 7.3 and Theorem 7.5 we have that $S = S_i + S_c$ and $T = T_i + T_c$ where

$$S_c : H^{-\frac{1}{2}}(\partial D) \longrightarrow H^{\frac{1}{2}}(\partial D), \qquad T_c : H^{\frac{1}{2}}(\partial D) \longrightarrow H^{-\frac{1}{2}}(\partial D)$$

are compact and

$$\left(S_i\tilde{\phi}, \tilde{\phi}\right) \geq C\|\tilde{\phi}\|^2_{H^{-\frac{1}{2}}(\partial D)} \qquad \text{for} \qquad \tilde{\phi} \in H^{-\frac{1}{2}}(\partial D) \tag{8.101}$$

$$\left(-T_0\tilde{\psi}, \tilde{\psi}\right) \geq C\|\tilde{\psi}\|^2_{H^{\frac{1}{2}}(\partial D)} \qquad \text{for} \qquad \tilde{\psi} \in H^{\frac{1}{2}}(\partial D) \tag{8.102}$$

where $(\cdot,\cdot)$ denotes the conjugated duality pairing between $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$ defined by Defintion 7.1. Let K_0 and K_0' be the operators corresponding to the Laplace operator, i.e. defined as K and K' with kernel $\Phi(x,y)$ replaced by $\Phi_0(x,y) = -\frac{1}{2\pi}\ln|x-y|$. Then $K_c = K - K_0$ and $K_c' = K' - K_0'$ are compact since they have continuous kernels [75]. It is easy to show that K_0 and K_0' are adjoint since their kernels are real, i.e.

$$\left(K_0\tilde{\psi}, \tilde{\phi}\right) = \left(\tilde{\psi}, K_0'\tilde{\phi}\right) \quad \text{for } \tilde{\phi} \in H^{-\frac{1}{2}}(\partial D) \text{ and } \tilde{\psi} \in H^{\frac{1}{2}}(\partial D)\,. \tag{8.103}$$

Collecting together all the compact terms we can write $A = (A_0 + A_c)$ where

$$A_0\zeta = \begin{pmatrix} S_0\tilde{\phi} + (-K_0 + I)\tilde{\psi} \\ \\ (K_0' - I)\tilde{\phi} - (T_0 + 2i\lambda I)\tilde{\psi} \end{pmatrix} \quad \text{and} \quad A_c\zeta = \begin{pmatrix} S_c\tilde{\phi} - K_c\tilde{\psi} \\ \\ K_c'\tilde{\phi} - T_c\tilde{\psi} \end{pmatrix}.$$

In this decomposition $A_c : H^{-\frac{1}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ is compact. Furthermore, we have that

$$\begin{aligned} \left(A_0\tilde{\zeta}, \tilde{\zeta}\right) &= \left(S_0\tilde{\phi}, \tilde{\phi}\right) + \left(-K_0\tilde{\psi}, \tilde{\phi}\right) + \left(\tilde{\psi}, \tilde{\phi}\right) + \left(K_0'\tilde{\phi}, \tilde{\psi}\right) \\ &\quad - \left(\tilde{\phi}, \tilde{\psi}\right) - \left(T_0\tilde{\psi}, \tilde{\psi}\right) - i\lambda\left(\tilde{\psi}, \tilde{\psi}\right). \end{aligned} \tag{8.104}$$

Taking the real part of (8.104), from (8.101) and (8.102), we obtain

$$\mathrm{Re}\left[\left(S_0\tilde{\phi}, \tilde{\phi}\right) - \left(T_0\tilde{\psi}, \tilde{\psi}\right)\right] \geq C\left(\|\tilde{\phi}\|^2_{H^{-\frac{1}{2}}(\partial D)} + \|\tilde{\psi}\|^2_{H^{\frac{1}{2}}(\partial D)}\right) \tag{8.105}$$

and (8.103) implies that

$$\begin{aligned} &\mathrm{Re}\left[\left(-K_0\tilde{\psi}, \tilde{\phi}\right) + \left(K_0'\tilde{\phi}, \tilde{\psi}\right)\right] = \mathrm{Re}\left[-\left(\tilde{\psi}, K_0'\tilde{\phi}\right) + \left(K_0'\tilde{\phi}, \tilde{\psi}\right)\right] \\ &= \mathrm{Re}\left[-\overline{\left(K_0'\tilde{\phi}, \tilde{\psi}\right)} + \left(K_0'\tilde{\phi}, \tilde{\psi})\right)\right] = 0. \end{aligned} \tag{8.106}$$

Finally

$$\mathrm{Re}\left[\left(\tilde{\psi},\tilde{\phi}\right)-\left(\tilde{\phi},\tilde{\psi}\right)-i\lambda\left(\tilde{\psi},\tilde{\psi}\right)\right]=0\,. \tag{8.107}$$

Combining (8.105), (8.106) and (8.107) we now have that

$$\left|\left(A_0\tilde{\zeta},\tilde{\zeta}\right)\right|\geq \mathrm{Re}\left(A_0\tilde{\zeta},\tilde{\zeta}\right)\geq C\|\tilde{\zeta}\|^2 \text{ for } \tilde{\zeta}\in H^{-\frac{1}{2}}(\partial D)\times\tilde{H}^{\frac{1}{2}}(\partial D)\,. \tag{8.108}$$

Recalling that $\tilde{\zeta}$ is the extension by zero of $\zeta=(\phi,\psi)\in\tilde{H}^{-\frac{1}{2}}(\Gamma)\times\tilde{H}^{\frac{1}{2}}(\Gamma)$, we can rewrite (8.108) as

$$|(A_{0\Gamma}\zeta,\zeta)|\geq C\|\zeta\|^2 \text{ for } \zeta\in\tilde{H}^{-\frac{1}{2}}(\Gamma)\times\tilde{H}^{\frac{1}{2}}(\Gamma)$$

where $A_{0,\Gamma}$ is the restriction to Γ of A_0 defined for $\zeta\in\tilde{H}^{-\frac{1}{2}}(\Gamma)\times\tilde{H}^{\frac{1}{2}}(\Gamma)$. The corresponding restriction $A_{c\Gamma}:\tilde{H}^{-\frac{1}{2}}(\Gamma)\times\tilde{H}^{\frac{1}{2}}(\Gamma)\to H^{\frac{1}{2}}(\Gamma)\times H^{-\frac{1}{2}}(\Gamma)$ of A_c clearly remains compact. Hence, the Lax-Milgram lemma together with Theorem 5.14 imply in the same way as in Lemma 8.33 that A_Γ is invertible with bounded inverse if and only if A_Γ injective.

We now show that A_Γ is injective. To this end, let $\zeta=(\alpha,\,\beta)\in\tilde{H}^{-\frac{1}{2}}(\Gamma)\times\tilde{H}^{\frac{1}{2}}(\Gamma)$ be such that $A_\Gamma\zeta=0$ and let $\tilde{\zeta}=(\tilde{\alpha},\,\tilde{\beta})\in H^{-\frac{1}{2}}(\partial D)\times\tilde{H}^{\frac{1}{2}}(\partial D)$ be its extension by zero. Define the potential

$$u(x)=-\int_\Gamma\alpha(y)\Phi(x,y)ds_y+\int_\Gamma\beta(y)\frac{\partial}{\partial\nu_y}\Phi(x,y)ds_y\quad x\in\mathbb{R}^2\setminus\bar{\Gamma}\,. \tag{8.109}$$

This potential is well defined in $\mathbb{R}^2\setminus\bar{\Gamma}$ since the densities α and β can be extended by zero to functions in $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$, respectively. Moreover $u\in H^1_{loc}(\mathbb{R}^2\setminus\bar{\Gamma})$ satisfies the Helmholtz equation in $\mathbb{R}^2\setminus\bar{\Gamma}$ and the Sommerfeld radiation condition. One can easily show that $\alpha=[\partial u/\partial\nu]$ and $\beta=[u]$. In particular, the jump relations of the single- and double- layer potentials and the first equation of $A_\Gamma\zeta=0$ imply

$$u^-|_\Gamma=-S\left[\frac{\partial u}{\partial\nu}\right]+K[u]-[u]=0\,. \tag{8.110}$$

We also have that

$$\left.\frac{\partial u^+}{\partial\nu}\right|_\Gamma=-K'\left[\frac{\partial u}{\partial\nu}\right]+T[u]+\left[\frac{\partial u}{\partial\nu}\right]$$

and from the fact that $u^+=[u]$ on Γ (8.110) and the second equation of $A_\Gamma\zeta=0$ we have that

$$\left.\frac{\partial u^+}{\partial\nu}+i\lambda u^+\right|_\Gamma=-K'\left[\frac{\partial u}{\partial\nu}\right]+\left[\frac{\partial u}{\partial\nu}\right]+T[u]+i\lambda[u]=0\,. \tag{8.111}$$

Hence u defined by (8.109) is a solution of the mixed crack problem with zero boundary data and from the uniqueness Theorem 8.32 $u=0$ in $\mathbb{R}^2\setminus\bar{\Gamma}$ and hence $\zeta=([\partial u/\partial\nu]\,,[u])=0$.

$\square$

Theorem 8.35. *The Dirichlet crack problem (8.86)–(8.88) has a unique solution. This solution satisfies the a priori estimate*

$$\|u\|_{H^1(\Omega_R\setminus\bar\Gamma)} \leq C\|f\|_{H^{\frac{1}{2}}(\Gamma)} \tag{8.112}$$

where Ω_R is a disk of radius R containing $\bar\Gamma$ and the positive constant C depends on R but not on f.

Proof. Uniqueness is proved in Theorem 8.32. The solution of (8.86)–(8.88) is given by

$$u(x) = -\int_\Gamma \left[\frac{\partial u(y)}{\partial \nu}\right] \Phi(x,y)ds_y, \quad x \in \mathbb{R}^2 \setminus \bar\Gamma$$

where $[\partial u/\partial\nu]$ is the unique solution of (8.99) given by Lemma 8.33. The estimate (8.112) is a consequence of the continuity of S_Γ^{-1} from $H^{\frac{1}{2}}(\Gamma)$ to $\tilde H^{-\frac{1}{2}}(\Gamma)$ and the continuity of the single layer potential from $\tilde H^{-\frac{1}{2}}(\Gamma)$ to $H^1_{loc}(\mathbb{R}^2 \setminus \bar\Gamma)$. □

Theorem 8.36. *The mixed crack problem (8.89)–(8.92) has a unique solution. This solution satisfies the estimate*

$$\|u\|_{H^1(\Omega_R\setminus\bar\Gamma)} \leq C(\|f\|_{H^{\frac{1}{2}}(\Gamma)} + \|h\|_{H^{-\frac{1}{2}}(\Gamma)}) \tag{8.113}$$

where Ω_R is a disk of radius R containing $\bar\Gamma$ and the positive constant C depends on R but not on f and h.

Proof. Uniqueness is proved in Theorem 8.32. The solution of (8.89)–(8.92) is given by

$$u(x) = -\int_\Gamma \left[\frac{\partial u(y)}{\partial \nu_y}\right] \Phi(x,y)ds_y + \int_\Gamma [u(y)] \frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y \quad x \in \mathbb{R}^2 \setminus \bar\Gamma,$$

where $\left(\left[\frac{\partial u}{\partial \nu}\right], [u]\right)$ is the unique solution of (8.100) given by Lemma 8.34. The estimate (8.113) is a consequence of the continuity of A_Γ^{-1} from $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ to $\tilde H^{-\frac{1}{2}}(\Gamma) \times \tilde H^{\frac{1}{2}}(\Gamma)$, the continuity of the single layer potential from $\tilde H^{-\frac{1}{2}}(\Gamma)$ to $H^1_{loc}(\mathbb{R}^2 \setminus \bar\Gamma)$ and the continuity of the double layer potential from $\tilde H^{\frac{1}{2}}(\Gamma)$ to $H^1_{loc}(\mathbb{R}^2 \setminus \bar\Gamma)$. □

Remark 8.37. More generally, one can consider the Dirichlet crack problem with boundary data having a jump across Γ, namely $u^\pm = f^\pm$ on Γ, where both f^+ and f^- are in $H^{\frac{1}{2}}(\Gamma)$. In this case the right hand side of the integral equation (8.99) will be replaced by $-(f^+ + f^-)/2$.

We end our discussion on direct scattering problems for cracks with a remark on the regularity of solutions. It is in fact known that the solution of the crack problem with Dirichlet boundary conditions has a singularity near a crack tip no matter how smooth the boundary data is. In particular, the solution does not belong to $H^{\frac{3}{2}}(\mathbb{R}^2 \setminus \bar{\Gamma})$ due to the fact that the solution has a singularity of the form $r^{\frac{1}{2}}\phi(\theta)$, where (r, θ) are the polar coordinates centered at the crack tip. In the case of the crack problem with mixed boundary conditions one would expect a stronger singular behavior of the solution near the tips. Indeed, for this case the solution of the mixed crack problem with smooth boundary data belongs to $H^{\frac{5}{4}-\epsilon}(\mathbb{R}^2 \setminus$
$bar\Gamma)$ for all $\epsilon > 0$ but not to $H^{\frac{5}{4}}(\mathbb{R}^2 \setminus \bar{\Gamma})$ due to the presence of a term of the form $r^{\frac{1}{4}+i\eta}\phi(\theta)$ in the asymptotic expansion of the solution in a neighborhood of the crack tip where η is a real number. A complete investigation of crack singularities can be found in [40].

8.8 The Inverse Scattering Problem for Cracks

We now turn our attention to the inverse scattering problem for cracks. To this end, we recall that approximation properties of Herglotz wave functions are a fundamental ingredient of the linear sampling method for solving the inverse problem. Hence, we first show that traces on Γ of the solution to crack problems can be approximated by the corresponding traces of Herglotz wave functions. More precisely, let v_g be a Herglotz wave function written in the form

$$v_g(x) = \int_0^{2\pi} g(\phi) e^{-ik(x_1 \cos\phi + x_2 \sin\phi)}\, d\phi, \qquad x = (x_1, x_2) \in \mathbb{R}^2$$

and consider the operator $H : L^2[0, 2\pi] \to H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ defined by

$$(Hg)(x) := \begin{cases} v_g^- & \text{on} \quad \Gamma \\ \dfrac{\partial v_g^+}{\partial \nu} + i\lambda v_g^+ & \text{on} \quad \Gamma \end{cases} \tag{8.114}$$

Theorem 8.38. *The range of $H : L^2[0, 2\pi] \to H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ is dense.*

Proof. From Corollary 6.17, we only need to show that the transpose operator $H^\top : \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma) \to L^2[0, 2\pi]$ is injective. In order to characterize the transpose operator we recall that $H^\top$ is defined by

$$\langle Hg, (\alpha, \beta)\rangle = \langle g, H^\top(\alpha, \beta)\rangle \tag{8.115}$$

for $g \in L^2[0, 2\pi]$ and $(\alpha, \beta) \in \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$. Note that the left hand side of (8.115) is the duality pairing between $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ and $\tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$

while the right hand side is the $L^2[0,\,2\pi]$-inner product without conjugation. One can easily see from (8.115) by changing the order of integration that

$$H^\top(\alpha,\beta)(\phi) := \int_\Gamma \alpha(x)e^{-ikx\cdot d}ds_x + i\lambda\int_\Gamma \beta(x)e^{-ikx\cdot d}ds_x \\ + \int_\Gamma \beta(x)\frac{\partial}{\partial\nu_x}e^{-ikx\cdot d}ds_x, \quad \phi\in[0,\,2\pi]$$

where $d=(\cos\,\phi,\sin\,\phi)$. Hence $\gamma H^\top(\alpha,\beta)$ coincides with the far field pattern of the potential

$$\gamma^{-1}V(z) := \int_\Gamma \alpha(x)\Phi(z,x)ds_x + i\lambda\int_\Gamma \beta(x)\Phi(z,x)ds_x \\ + \int_\Gamma \beta(x)\frac{\partial}{\partial\nu_x}\Phi(z,x)ds_x, \quad z\in\mathbb{R}^2\setminus\bar\Gamma$$

where $\gamma = \dfrac{e^{i\pi/4}}{\sqrt{8\pi k}}$. Note that V is well defined in $\mathbb{R}^2\setminus\bar\Gamma$ since the densities α and β can be extended by zero to functions in $H^{-\frac12}(\partial D)$ and $H^{\frac12}(\partial D)$, respectively. Moreover $V\in H^1_{loc}(\mathbb{R}^2\setminus\bar\Gamma)$ satisfies the Helmholtz equation in $\mathbb{R}^2\setminus\bar\Gamma$ and the Sommerfeld radiation condition. Now assume that $H^\top(\alpha,\beta)=0$. This means that the far field pattern of V is zero and from Rellich's lemma and the unique continuation principle we conclude that $V=0$ in $\mathbb{R}^2\setminus\bar\Gamma$. Using the jump relations across ∂D for the single- and double- layer potentials with α and β defined to be zero on $\partial D\setminus\bar\Gamma$ we now obtain

$$\beta = [V]_\Gamma$$
$$\alpha + i\lambda\beta = -\left[\frac{\partial V}{\partial\nu}\right]_\Gamma$$

and hence $\alpha=\beta=0$. Thus $H^\top$ is injective and the theorem is proven. □

As a special case of the above theorem we obtain:

Theorem 8.39. *Every function in $H^{\frac12}(\Gamma)$ can be approximated by the trace of a Herglotz wave function $v_g|_\Gamma$ on Γ with respect to the $H^{\frac12}(\Gamma)$ norm.*

Assuming the incident field $u^i(x) = e^{ikx\cdot d}$ is a plane wave with incident direction $d = (\cos\phi,\,\sin\phi)$, the *inverse problem* we now consider is to determine the shape of the crack Γ from a knowledge of the far field pattern $u_\infty(\cdot,\,\phi)$, $\phi\in[0,\,2\pi]$, of the scattered field $u^s(\cdot,\,\phi)$. The scattered field is either the solution of the Dirichlet crack problem (8.86)–(8.88) with $f=-e^{ikx\cdot d}|_\Gamma$ or of the mixed crack problem (8.89)–(8.92) with $f=-e^{ikx\cdot d}|_\Gamma$

and $h = -\left(\frac{\partial}{\partial\nu} + i\lambda\right) e^{ikx\cdot d}|_\Gamma$. In either case, the far field pattern is defined by the asymptotic expansion of the scattered field

$$u^s(x,\,\phi) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\theta,\phi) + O(r^{-3/2}), \qquad r = |x| \to \infty\,.$$

Theorem 8.40. *Assume Γ_1 and Γ_2 are two perfectly conducting or partially coated cracks with surface impedance λ_1 and λ_2 such that the far-field patterns $u^1_\infty(\theta,\phi)$ and $u^2_\infty(\theta,\phi)$ coincide for all incidence angles $\phi \in [0,\,2\pi]$ and for all observation angles $\theta \in [0,\,2\pi]$. Then $\Gamma_1 = \Gamma_2$.*

Proof. Let $G := \mathbb{R}^2 \setminus (\bar{\Gamma}_1 \cup \bar{\Gamma}_2)$ and $x_0 \in G$. Using Lemma 4.4 and the well-posedeness of the forward crack problems one can show as in Theorem 4.5 that the scattered fields w^s_1 and w^s_2 corresponding to the incident field $u^i = -\Phi(\cdot, x_0)$ (i.e. w^s_j, $j = 1, 2$ satisfy (8.86)–(8.88) with $f = -\Phi(\cdot, x_0)|_{\Gamma_j}$, or (8.89)–(8.92) with $f = -\Phi(\cdot, x_0)|_{\Gamma_j}$ and $h = -\left(\frac{\partial}{\partial\nu} + i\lambda\right)\Phi(\cdot, x_0)|_{\Gamma_j}$) coincide in G.
Now assume that $\Gamma_1 \neq \Gamma_2$. Then without loss of generality there exists $x^* \in \Gamma_1$ such that $x^* \notin \Gamma_2$. We can choose a sequence $\{x_n\}$ from G such that $x_n \to x^*$ as $n \to \infty$ and $x_n \notin \bar{\Gamma}_2$. Hence we have that $w^s_{n,1} = w^s_{n,2}$ in G where $w^s_{n,1}$ and $w^s_{n,2}$ are as above with x_0 replaced by x_n. Consider $w^s_n = w^s_{n,2}$ as the scattered wave corresponding to Γ_2. From the boundary data $(w^s_n)^- = -\Phi(\cdot, x_n)$ on Γ_2 and from (8.112) or (8.113) we have that $\|w^s_n\|_{H^1(\Omega_R\setminus\bar{\Gamma}_2)}$ is uniformly bounded with respect to n, whence from the trace theorem $\|w^s_n\|_{H^{\frac{1}{2}}(\Omega_r(x^*)\cap\Gamma_1)}$ is uniformly bounded with respect to n, where $\Omega_r(x^*)$ is a small neighborhood centered at x^* not intersecting Γ_2. On the other hand, considering $w^s_n = w^s_{n,1}$ as the scattered wave corresponding to Γ_1, from the boundary conditions $(w^s_n)^- = -\Phi(\cdot, x_n)$ on Γ_1 we have $\|w^s_n\|_{H^{\frac{1}{2}}(\Omega_r(x^*))\cap\Gamma_1)} \to \infty$ as $n \to \infty$ since $\|\Phi(\cdot, x_n)\|_{H^{\frac{1}{2}}(\Omega_r(x^*)\cap\Gamma_1)} \to \infty$ as $n \to \infty$. This is a contradiction. Therefore $\Gamma_1 = \Gamma_2$. □

To solve the inverse problem we will use the linear sampling method which is based on a study of the far field equation

$$Fg = \Phi^L_\infty \tag{8.116}$$

where $F : L^2[0,\,2\pi] \to L^2[0,\,2\pi]$ is the far field operator defined by

$$(Fg)(\theta) := \int_0^{2\pi} u_\infty(\theta,\phi) g(\phi) d\phi$$

and Φ^L_∞ is a function to be defined shortly. In particular, due to the fact that the scattering object has an empty interior, we need to modify the linear sampling method previously developed for obstacles with non empty interior.

Assume for the moment that the crack is partially coated and define the operator $B : H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \to L^2[0,\, 2\pi]$ which maps the boundary data (f, h) to the far field pattern of the solution to the corresponding scattering problem (8.89)–(8.92). By superposition we have the relation

$$Fg = -BHg$$

where Hg is defined by (8.114) with the Herglotz wave function v_g now written as

$$v_g(x) = \int_0^{2\pi} g(\phi) e^{ikx \cdot d}\, d\phi\ .$$

We now define the compact operator $\mathcal{F} : \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma) \longrightarrow L^2[0,\, 2\pi]$ by

$$\mathcal{F}(\alpha, \beta)(\theta) = \gamma \int_\Gamma \alpha(y) e^{-ik\hat{x}\cdot y}\, ds_y + \gamma \int_\Gamma \beta(y) \frac{\partial}{\partial \nu_y} e^{-ik\hat{x}\cdot y}\, ds_y\ , \tag{8.117}$$

where $\hat{x} = (\cos\theta, \sin\theta)$ and $\gamma = e^{i\pi/4}/\sqrt{8\pi k}$, and observe that for a given pair $(\alpha, \beta) \in \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$, the function $\mathcal{F}(\alpha, \beta)(\hat{x})$ is the far field pattern of the radiating solution $P(\alpha, \beta)(x)$ of the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{\Gamma}$ where the potential P is defined by

$$P(\alpha, \beta)(x) := \int_\Gamma \alpha(y) \Phi(x, y) ds_y + \int_\Gamma \beta(y) \frac{\partial}{\partial \nu_y} \Phi(x, y) ds_y\ . \tag{8.118}$$

Proceeding as in the proof of Theorem 8.38, by using the jump relations across ∂D for the single- and double- layer potential with densities extended by zero to ∂D we obtain that $\alpha := -\left[\partial P/\partial \nu\right]_\Gamma$ and $\beta := [P]_\Gamma$. Moreover P satisfies

$$\begin{pmatrix} P^-(\alpha, \beta)|_\Gamma \\ \\ \left(\dfrac{\partial}{\partial \nu} + i\lambda\right) P^+(\alpha, \beta)|_\Gamma \end{pmatrix} = M \begin{pmatrix} \alpha \\ \\ \beta \end{pmatrix} \tag{8.119}$$

where the operator $M : \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ is given by

$$\begin{pmatrix} S_\Gamma & K_\Gamma - I \\ \\ K'_\Gamma - I + i\lambda S_\Gamma & T_\Gamma + i\lambda(I + K_\Gamma) \end{pmatrix}.$$

The operator M is related to the operator A_Γ given in (8.100) by the relation $M = \begin{pmatrix} I & 0 \\ i\lambda k I & I \end{pmatrix} A_\Gamma \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, whence $M^{-1} : H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \to \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$ exists and is bounded. In particular, we have that

$$\mathcal{F}(\alpha, \beta) = BM(\alpha, \beta)\ . \tag{8.120}$$

In the case of the Dirichlet crack problem (8.86)–(8.88), by proceeding exactly as above, we have $\mathcal{F}_D(\alpha) = BS_\Gamma(\alpha)$ where $\alpha \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$, $B : H^{\frac{1}{2}}(\Gamma) \to L^2[0,\, 2\pi]$, $\mathcal{F}_D : \tilde{H}^{-\frac{1}{2}}(\Gamma) \to L^2[0,\, 2\pi]$ is defined by

$$\mathcal{F}_D(\alpha)(\theta) := \gamma \int_\Gamma \alpha(y) e^{-ik\hat{x}\cdot y}\, ds_y \,. \tag{8.121}$$

and S_Γ is given by (8.99).

Lemma 8.41. *The operator $\mathcal{F} : \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma) \longrightarrow L^2[0,\, 2\pi]$ defined by (8.117) is injective and has dense range.*

Proof. Injectivity follows from the fact that $\mathcal{F}(\alpha, \beta)$ is the far field pattern of $P(\alpha, \beta)$ for $(\alpha, \beta) \in \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$ given by (8.118). Hence $\mathcal{F}(\alpha, \beta) = 0$ implies $P(\alpha, \beta) = 0$ and so $\alpha := -\left[\partial P/\partial \nu\right]_\Gamma = 0$ and $\beta := [P]_\Gamma = 0$. We now note that the transpose operator $\mathcal{F}^\top : L^2[0,\, 2\pi] \to H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ is given by

$$\gamma^{-1} \mathcal{F}^\top g(y) := \begin{cases} v_g^-(y) & \\ \dfrac{\partial v_g^+(y)}{\partial \nu_y} & y \in \Gamma \end{cases} \tag{8.122}$$

where $v_g(y) = \int_0^{2\pi} g(\phi) e^{-ik\hat{x}\cdot y} d\phi$, $\hat{x} = (\cos\phi, \sin\phi)$. From Corollary 6.17, it is enough to show that $\mathcal{F}^\top$ is injective. But $\mathcal{F}^\top g = 0$ implies that there exists a Herglotz wave function v_g such that $v_g|_\Gamma = 0$ and $\left.\dfrac{\partial v_g}{\partial \nu}\right|_\Gamma = 0$ (note that the limit of v_g and its normal derivative from both sides of the crack is the same). From the representation formula (8.95) and the analyticity of v_g, we now have that $v_g = 0$ in $\mathbb{R}^2$ and therefore $g = 0$. This proves the lemma. □

We obtain a similar result for the operator $\mathcal{F}_D$ corresponding to the Dirichlet crack problem. But in this case $\mathcal{F}_D$ has dense range only under certain restrictions. More precisely the following result holds.

Lemma 8.42. *The operator $\mathcal{F}_D : \tilde{H}^{-\frac{1}{2}}(\Gamma) \to L^2[0,\, 2\pi]$ defined by (8.121) is injective. The range of $\mathcal{F}_D$ is dense in $L^2[0,\, 2\pi]$ if and only if there does not exist a Herglotz wave function which vanishes on Γ.*

Proof. The injectivity can be proved in the same way as in the Lemma 8.41 if one replaces the potential V by the single layer potential.

The dual operator $\mathcal{F}_D^\top : L^2[0,\, 2\pi] \to H^{\frac{1}{2}}(\Gamma)$ in this case coincides with $v_g|_\Gamma$. Hence $\mathcal{F}_D^\top$ is injective if and only if here does not exist a Herglotz wave function which vanishes on Γ. □

In polar coordinates $x = (r, \theta)$ the functions

$$u_n(x) = J_n(kr)\cos n\theta, \qquad v_n(x) = J_n(kr)\sin n\theta, \qquad n = 0, 1, \cdots,$$

where J_n denotes a Bessel function of order n provide examples of Herglotz wave functions. Therefore, by Lemma 8.42, for any straight line segment the range $\mathcal{F}_D$ (and consequently the range of the far field operator) is not dense. The same is true for circular arcs with radius R such that kR is a zero of one of the Bessel functions J_n.

From the above analysis we can factorize the far field operator corresponding to the mixed crack problem as

$$(Fg) = -\mathcal{F}M^{-1}Hg, \qquad g \in L^2[0,\, 2\pi]\,. \tag{8.123}$$

and the far field operator corresponding to the Dirichlet crack problem as

$$(Fg) = -\mathcal{F}_D S_\Gamma^{-1}(v_g|_\Gamma), \qquad g \in L^2[0,\, 2\pi]\,. \tag{8.124}$$

The following lemma will help us to choose an appropriate right hand side of the far field equation (8.116).

Lemma 8.43. *For any smooth non intersecting arc L and two functions $\alpha_L \in \tilde{H}^{-\frac{1}{2}}(L)$, $\beta_L \in \tilde{H}^{\frac{1}{2}}(L)$ we define $\Phi_\infty^L \in L^2[0,\, 2\pi]$ by*

$$\Phi_\infty^L(\theta) := \gamma \int\limits_L \alpha_L(y) e^{-ik\hat{x}\cdot y} ds_y + \gamma \int\limits_L \beta_L(y) \frac{\partial}{\partial \nu_y} e^{-ik\hat{x}\cdot y} ds_y \tag{8.125}$$

$\hat{x} = (\cos\theta, \sin\theta)$*. Then, $\Phi_\infty^L \in \mathrm{R}(\mathcal{F})$ if and only if $L \subset \Gamma$, where $\mathcal{F}$ is given by (8.117)*

Proof. First assume that $L \subset \Gamma$. Then since $\tilde{H}^{\pm\frac{1}{2}}(L) \subset \tilde{H}^{\pm\frac{1}{2}}(\Gamma)$ it follows directly from the definition of $\mathcal{F}$ that $\Phi_\infty^L \in \mathrm{R}(\mathcal{F})$.
Now let $L \not\subset \Gamma$ and assume, on the contrary, that $\Phi_\infty^L \in \mathrm{R}(\mathcal{F})$, i.e. there exists $\alpha \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$ and $\beta \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ such that

$$\Phi_\infty^L(\theta) = \gamma \int\limits_\Gamma \alpha(y) e^{-ik\hat{x}\cdot y} ds_y + \gamma \int\limits_\Gamma \beta(y) \frac{\partial}{\partial \nu_y} e^{-ik\hat{x}\cdot y} ds_y\,.$$

Then by Rellich's lemma and the unique continuation principle we have that the potentials

$$\Phi^L(x) = \int\limits_L \alpha_L(y)\Phi(x,y) ds_y + \int\limits_L \beta_L(y) \frac{\partial}{\partial \nu_y}\Phi(x,y) ds_y \qquad x \in \mathbb{R}^2 \setminus \bar{L}$$

$$P(x) = \int\limits_\Gamma \alpha(y)\Phi(x,y) ds_y + \int\limits_\Gamma \beta(y) \frac{\partial}{\partial \nu_y}\Phi(x,y) ds_y \qquad x \in \mathbb{R}^2 \setminus \bar{\Gamma}$$

coincide in $\mathbb{R}^2 \setminus (\bar{\Gamma} \cup \bar{L})$. Now let $x_0 \in L$, $x_0 \notin \Gamma$, and let $\Omega_\epsilon(x_0)$ be a small ball with center at x_0 such that $\Omega_\epsilon(x_0) \cap \Gamma = \emptyset$. Hence P is analytic in $\Omega_\epsilon(x_0)$ while Φ^L has a singularity at x_0 which is a contradiction. Hence $\Phi_\infty^L \notin \mathrm{R}(\mathcal{F})$. $\square$

Remark 8.44. The statement and proof of Lemma 8.43 remain valid for the operator $\mathcal{F}_D$ given by (8.121) if we set $\beta_L = 0$ in (8.125).

Now let us denote by $\mathcal{L}$ the set of open nonintersecting smooth arcs and look for a solution $g \in L^2[0,\, 2\pi]$ of the far field equation

$$-Fg = \mathcal{F}M^{-1}Hg = \Phi_\infty^L \qquad \text{for } L \in \mathcal{L} \tag{8.126}$$

where Φ_∞^L is given by (8.125) and F is the far field operator corresponding to the mixed crack problem. If $L \subset \Gamma$ then the corresponding (α_L, β_L) is in $\tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$. Since $M(\alpha_L, \beta_L) \in H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ then from Theorem 8.38 for every $\epsilon > 0$ there exists a $g_L^\epsilon \in L^2[0,\, 2\pi]$ such that

$$\|M(\alpha_L, \beta_L) - Hg_L^\epsilon\|_{H^{\frac{1}{2}}(\Gamma)\times H^{-\frac{1}{2}}(\Gamma)} \le \epsilon$$

whence from the continuity of M^{-1}

$$\|(\alpha_L, \beta_L) - M^{-1}Hg_L^\epsilon\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma)\times \tilde{H}^{\frac{1}{2}}(\Gamma)} \le C\epsilon \tag{8.127}$$

with a positive constant C. Finally (8.123), the continuity of $\mathcal{F}$ and the fact that $\mathcal{F}(\alpha_L, \beta_L) = \Phi_\infty^L$ imply that

$$\|Fg_L^\epsilon + \Phi_\infty^L\|_{L^2[0,\, 2\pi]} \le \tilde{C}\epsilon\ . \tag{8.128}$$

Next, we assume that $L \not\subset \Gamma$. In this case Φ_∞^L does not belong to the range of $\mathcal{F}$. But, from Theorem 8.41 and the fact that $\mathcal{F}$ is compact, by using Tikhonov regularization we can construct a regularized solution of

$$\mathcal{F}(\alpha, \beta) = \Phi_\infty^L\ . \tag{8.129}$$

In particular, if $(\alpha_L^\rho, \beta_L^\rho) \in \tilde{H}^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{\frac{1}{2}}(\Gamma)$ is the regularized solution of (8.129) corresponding to the regularization parameter ρ (chosen by a regular regularization strategy e.g. the Morozov discrepancy principle), we have for a given $\delta > 0$

$$\|\mathcal{F}(\alpha_L^\rho, \beta_L^\rho) - \Phi_\infty^L\|_{L^2[0,\, 2\pi]} < \delta\ , \tag{8.130}$$

and

$$\lim_{\rho\to 0} \|(\alpha_L^\rho, \beta_L^\rho)\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma)\times \tilde{H}^{\frac{1}{2}}(\Gamma)} = \infty\ . \tag{8.131}$$

The above considerations for (α_L, β_L) can now be applied to $(\alpha_L^\rho, \beta_L^\rho)$. In particular, let $g_L^{\epsilon,\rho} \in L^2[0,\, 2\pi]$ be such that

$$\|M(\alpha_L^\rho, \beta_L^\rho) - Hg_L^{\epsilon,\rho}\|_{H^{\frac{1}{2}}(\Gamma)\times H^{-\frac{1}{2}}(\Gamma)} \le \epsilon$$

and

$$\|(\alpha_L^\rho, \beta_L^\rho) - M^{-1} H g_L^{\epsilon,\rho}\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma)\times \tilde{H}^{\frac{1}{2}}(\Gamma)} \leq C\epsilon \,. \tag{8.132}$$

Combining (8.130) and (8.132) we obtain that for every $\epsilon > 0$ and $\delta > 0$ there exists a $g_{\epsilon,\rho}^L \in L^2[0,\, 2\pi]$ such that

$$\|F g_L^{\epsilon,\rho} + \Phi_\infty^L\|_{L^2[0,\,2\pi]} \leq \epsilon + \delta \,. \tag{8.133}$$

Furthermore, from (8.131) and the boundness of M and M^{-1}, we have that

$$\lim_{\rho\to 0} \|H g_L^{\epsilon,\rho}\|_{H^{\frac{1}{2}}(\Gamma)\times H^{-\frac{1}{2}}(\Gamma)} = \infty \quad \text{and} \quad \lim_{\rho\to 0} \|v_{g_L^{\epsilon,\rho}}\|_{H^1(\Omega_R)} = \infty$$

where $v_{g_L^{\epsilon,\rho}}$ is the Herglotz wave function with kernel $g_L^{\epsilon,\rho}$ and

$$\lim_{\rho\to 0} \|g_L^{\epsilon,\rho}\|_{L^2[0,\,2\pi]} = \infty \,.$$

We summarize these results in the following theorem, noting that for $L \in \mathcal{L}$ we have that $\rho \to 0$ as $\delta \to 0$.

Theorem 8.45. *Assume that Γ is a nonintersecting smooth open arc. If F is the far field operator corresponding to the scattering problem (8.83)–(8.85) and (8.82), then the following is true:*

1. *If $L \subset \Gamma$ then for every $\epsilon > 0$ there exists a solution $g_L^\epsilon := g_L \in L^2[0,\, 2\pi]$ of the inequality*
$$\|F g_L + \Phi_\infty^L\|_{L^2[0,\,2\pi]} \leq \epsilon.$$
2. *If $L \not\subset \Gamma$ then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g_L^{\epsilon,\delta} + g_L \in L^2[0,\, 2\pi]$ of the inequality*
$$\|F g_L + \Phi_\infty^L\|_{L^2[0,\,2\pi]} \leq \epsilon + \delta$$
such that
$$\lim_{\delta\to 0} \|g_L\|_{L^2[0,\,2\pi]} = \infty \quad \textit{and} \quad \lim_{\delta\to 0} \|v_{g_L}\|_{H^1(\Omega_R)} = \infty$$
where v_{g_L} is the Herglotz wave function with kernel g_L and Ω_R is a large enough disc of radius R.

Remark 8.46. The statement and proof of Theorem 8.45 remain valid in the case when F is the far field operator correspondig to the Dirichlet crack if we set $\beta_L = 0$ in the definition of Φ_∞^L and assume that there does not exist a Herglotz wave function that vanishes on Γ.

In particular, if $L \subset \Gamma$ we can find a bounded solution to the far field equation (8.126) with discrepancy ϵ whereas if $L \not\subset \Gamma$ then there exists solutions of the far field equation with discrepancy $\epsilon + \delta$ with arbitrary large norm in the limit as $\delta \to 0$. For numerical purposes we need to replace Φ_∞^L in the far field equation (8.126) by an expression independent of L. To this end, assuming that there does not exist a Herglotz wave function which vanishes on L, we can conclude from Lemma 8.42 that the class of potentials of the form

$$\int_L \alpha(y) e^{-ik\hat{x}\cdot y}\, ds_y, \qquad \alpha \in \tilde{H}^{-\frac{1}{2}}(L) \tag{8.134}$$

is dense in $L^2[0,\, 2\pi]$ and hence for numerical purposes we can replace Φ_∞^L in (8.126) by an expression of the form (8.134). Finally, we note that as L degenerates to a point z with α_L an appropriate delta sequence we have that the integral in (8.134) approaches $-\gamma e^{-ik\hat{x}\cdot z}$. Hence, it is reasonable to replace Φ_∞^L by $-\Phi_\infty$ where $\Phi_\infty(\hat{x}, z) := \gamma e^{-ik\hat{x}\cdot z}$ when numerically solving the far field equation (8.126) .

8.9 Numerical Examples

As we explained in the last paragraph of the previous section, in order to determine the shape of a crack we compute a regularized solution to the far field equation

$$\int_0^{2\pi} u_\infty(\theta, \phi) g(\phi)\, d\phi = \gamma e^{-ik\hat{x}\cdot z} \qquad \hat{x} = (\cos\phi, \sin\phi),\ z \in \mathbb{R}^2$$

where u_∞ is the far field data of the scattering problem. This is the same far field equation we have used in all the inverse problems presented in this chapter, which emphasizes one of the advantages of the linear sampling method, namely it does not make use of any a priori information on the geometry of the scattering object.

To solve the far field equation we apply the same procedure as in Sect. 8.3. In all our examples we use synthetic data corrupted with random noise. We show reconstruction examples for four different cracks all of which are subject to the Dirichlet boundary condition.

1. The curve given by the parametric equation (Fig. 8.11, top left)

$$\Gamma := \left\{ \varrho(s) = \left(2 \sin \frac{s}{2}, \sin s\right) : \frac{\pi}{4} \le s \le \frac{7\pi}{4} \right\} .$$

2. The line given by the parametric equation (Fig. 8.11, top right)

$$\Gamma := \{ \varrho(s) = (-2 + s, 2s) : -1 \le s \le 1 \} .$$

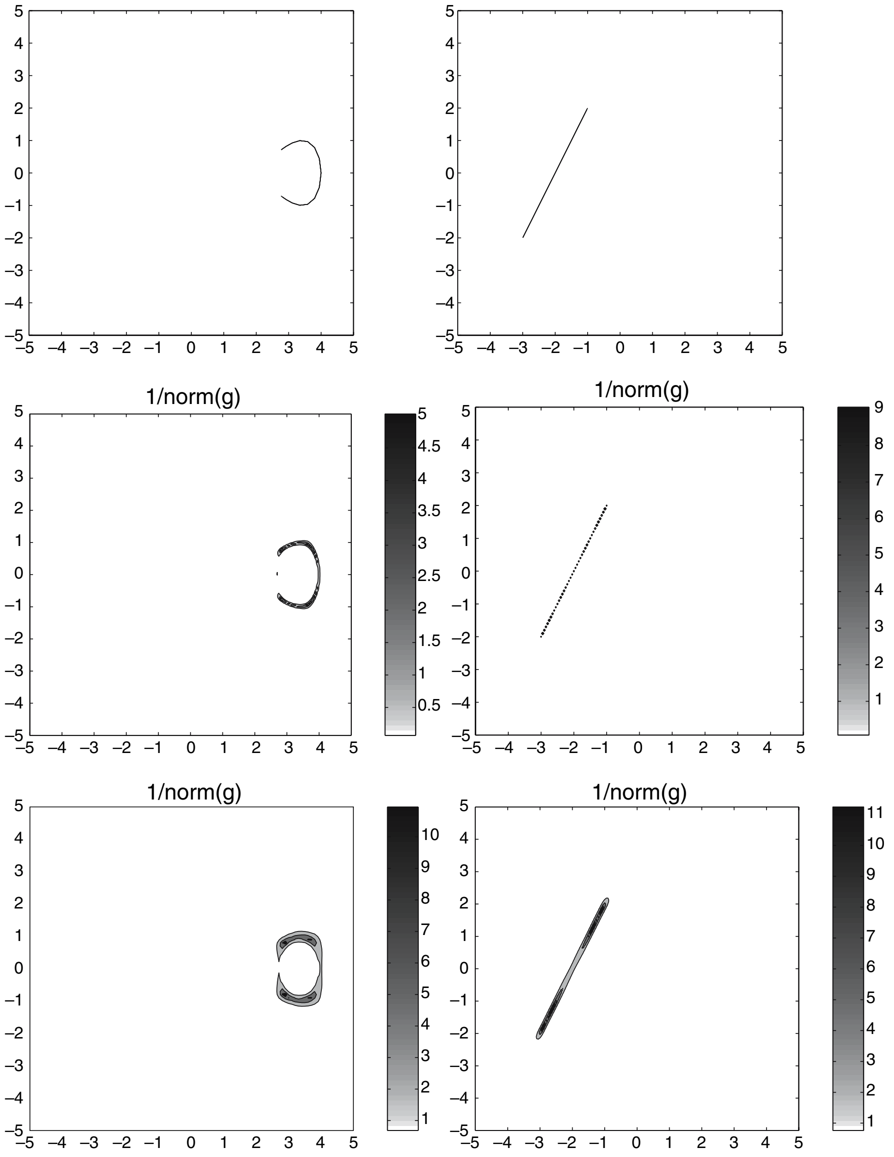

Fig. 8.11. The true object (*top*), reconstruction with 0.5% noise (*middle*) and with 5% noise (*bottom*). The wave number is $k = 3$.[3]

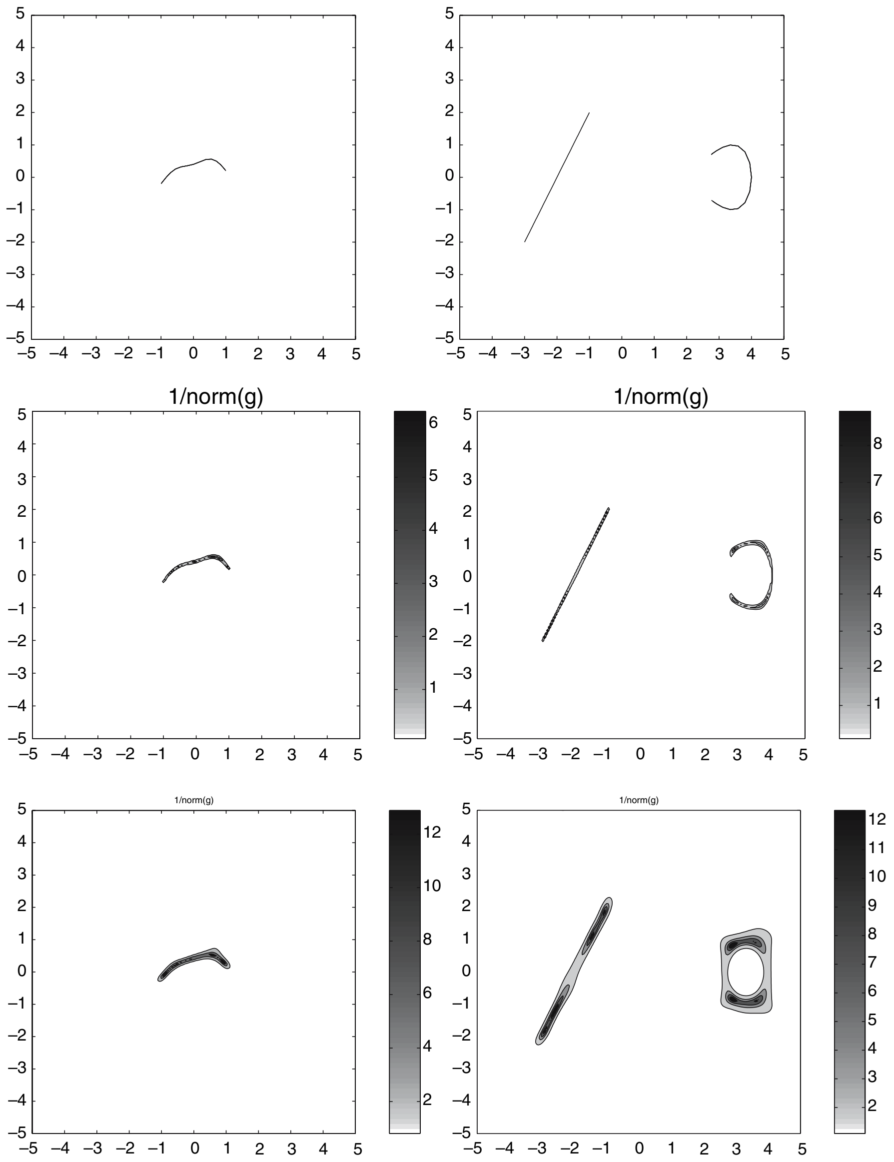

Fig. 8.12. The true object (*top*), reconstruction with 0.5% noise (*middle*) and with 5% noise (*bottom*). The wave number is $k = 3$.[3]

[3] Reprinted from F.Cakoni and D.Colton, The linear sampling method for cracks, Inverse Problems 19 (2003), 279-295.

3. The curve given by the parametric equation (Fig. 8.12, top left)

$$\Gamma := \left\{ \varrho(s) = \left(s, 0.5 \cos \frac{\pi s}{2} + 0.2 \sin \frac{\pi s}{2} - 0.1 \cos \frac{3\pi s}{2} \right) : -1 \le s \le 1 \right\} .$$

4. Two disconnected curves described as in example 2 and 3 (Fig. 8.12, top right).

In all our examples $k = 3$ and the far field data is given for 32 incident directions and 32 observation directions equally distributed on the unit circle.

9

A Glimpse at Maxwell's Equations

In the previous chapters we have used the scattering of electromagnetic waves by an infinite cylinder as our model, thus reducing the three dimensional Maxwell system to a two dimensional scalar equation. In this last chapter we want to briefly indicate the modifications needed in order to treat three dimensional electromagnetic scattering problems. In view of the introductory nature of our book, our presentation will be brief and for details we will refer to Chapter 14 of [87] and the forthcoming monograph [88].

There are two basic problems that arise in treating three dimensional electromagnetic scattering problems. The first of these problems is that the formulation of the direct scattering problem must be done in function spaces that are more complicated that the ones used for two dimensional problems. The second problem follows from the first in that, due to more complicated function spaces, the mathematical techniques used to study both the direct and inverse problems become rather sophisticated. Nevertheless, the logical scheme one must follow in order to obtain the desired theorems is basically the same as that followed in the two dimensional case.

We first consider the scattering of electromagnetic waves by a (possibly) partially coated obstacle D in $\mathbb{R}^3$. We assume that D is a bounded region with smooth boundary ∂D such that $D_e := \mathbb{R}^3 \setminus \bar{D}$ is connected. We assume that the boundary ∂D is split into two disjoint parts ∂D_D and ∂D_I where ∂D_D and ∂D_I are disjoint, relatively open subsets (possibly disconnected) of ∂D and let ν denote the unit outward normal to ∂D. We allow the possibility that either ∂D_D or ∂D_I is the empty set. The direct scattering problem we are interested in is to determine an electromagnetic field E, H such that

$$\begin{aligned} \operatorname{curl} E - ikH = 0 \\ \operatorname{curl} H + ikE = 0 \end{aligned} \tag{9.1}$$

for $x \in D_e$ and

$$\nu \times E = 0 \quad \text{on } \partial D_D \tag{9.2}$$

$$\nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu = 0 \quad \text{on } \partial D_I \tag{9.3}$$

where $\lambda > 0$ is the surface impedance which, for the sake of simplicity, is assumed to be a (possibly different) constant on each connected subset of ∂D_I. Note that the case of a perfect conductor corresponds to the case when $\partial D_I = \emptyset$ and the case of an imperfect conductor corresponds to the case when $\partial D_D = \emptyset$. We introduce the incident fields

$$\begin{aligned} E^i(x) :&= \frac{i}{k} \operatorname{curl}\operatorname{curl} pe^{ikx\cdot d} \\ &= ik(d \times p) \times de^{ikx\cdot d} \end{aligned} \tag{9.4}$$

$$\begin{aligned} H^i(x) :&= \operatorname{curl} pe^{ikx\cdot d} \\ &= ikd \times pe^{ikx\cdot d} \end{aligned} \tag{9.5}$$

where $k > 0$ is the wave number, $d \in \mathbb{R}^3$ is a unit vector giving the direction of propagation and $p \in \mathbb{R}^3$ is the polarization vector. Finally, the scattered field E^s, H^s defined by

$$\begin{aligned} E &= E^i + E^s \\ H &= H^i + H^s \end{aligned} \tag{9.6}$$

is required to satisfy the *Silver-Müller radiation condition*

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{9.7}$$

uniformly in $\hat{x} = x/|x|$ where $r = |x|$.

The scattering problem (9.1) – (9.7) is a special case of the exterior mixed boundary value problem

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \quad \text{in } D_e \tag{9.8}$$

$$\nu \times E = f \quad \text{on } \partial D_D \tag{9.9}$$

$$\nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu = h \quad \text{on } \partial D_I \tag{9.10}$$

$$\lim_{r\to\infty} (H \times x - rE) = 0 \tag{9.11}$$

for prescribed functions of f and h with $H = \frac{1}{ik} \operatorname{curl} E$. The first problem that needs to be addressed is under what conditions on f and h does there exist a unique solution to (9.8) – (9.11). To this end we define

$$X(D, \partial D_I) := \left\{ u \in H(\text{curl}, D) : \nu \times u|_{\partial D_I} \in L^2_t(\partial D_I) \right\}$$

equipped with the norm

$$||u||^2_{X(D,\partial D)} := ||u||^2_{H(\text{curl},D)} + ||\nu \times u||^2_{L^2(\partial D_I)}$$

where

$$H(\text{curl}, D) := \left\{ u \in \left(L^2(D)\right)^3 : \operatorname{curl} u \in \left(L^2(D)\right)^3 \right\}$$

$$L_t^2(\partial D_I) := \left\{ u \in \left(L^2(\partial D_I)\right)^3 : \nu \times u = 0 \quad \text{on } \partial D_I \right\}$$

with norms

$$||u||^2_{H(\text{curl},D)} := ||u||_{(L^2(D))^3} + ||\text{curl}\, u||_{(L^2(D))^3}$$

$$||u||_{L_t^2(\partial D_I)} = ||u||_{(L^2(\partial D_I))^3}$$

respectively. As in Chapter 3, we can also define the spaces $X_{loc}(D_e, \partial D_I)$ and $H_{loc}(\text{curl}, D_e)$. Finally, we introduce the trace space of $X(D, \partial D_I)$ on the complementary part ∂D_D by

$$Y(\partial D_D) := \left\{ f \in \left(H^{-1/2}(\partial D_D)\right)^3 : \text{There exists } u \in H_0(\text{curl}, \Omega_R) \right.$$
$$\left. \text{such that } \nu \times u|_{\partial D_I} \in L_t^2(\partial D_I) \text{ and } f = \nu \times u|_{\partial D_D} \right\}$$

where $D \subset \Omega_R = \{x : |x| < R\}$ and

$$H_0(\text{curl}, \Omega_R) := \left\{ u \in H(\text{curl}, \Omega_R) : \nu \times u|_{\partial \Omega_R} = 0 \right\}.$$

The trace space is equipped with the norm

$$||f||^2_{Y(\partial D_D)} := \inf \left\{ ||u||^2_{H(\text{curl},\Omega_R)} + ||\nu \times u||^2_{L^2(\partial D_I)} \right\}$$

where the minimum is taken over all functions $u \in H_0(\text{curl}, \Omega_R)$ such that $\nu \times u|_{\partial D_I} \in L_t^2(\partial D_I)$ and $f = \nu \times u|_{\partial D_D}$ (for details see [87]). We now have the following theorem [15]:

Theorem 9.1. *Given $f \in Y(\partial D_D)$ and $h \in L_t^2(\partial D_I)$ there exists a unique solution $E \in X_{loc}(D_e, \partial D_I)$ to (9.8)-(9.11) such that*

$$||E||_{X(D_e \cap \Omega_R, \partial D_I)} \leq C(||f||_{Y(\partial D_D)} + ||h||_{L^2(\partial D_I)})$$

for some positive constant C depending on R but not on f and h.

We now turn our attention to the inverse problem of determining D and λ from a knowledge of the far field data of the electric field. In particular, from [33] it is known that the solution E^s, H^s to (9.1) – (9.7) has the asymptotic behavior

$$\begin{aligned} E^s(x) &= \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right) \right\} \\ H^s(x) &= \frac{e^{ik|x|}}{|x|} \left\{ H_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right) \right\} \end{aligned} \tag{9.12}$$

as $|x| \to \infty$ where $E_\infty(\cdot, d, p)$ and $H_\infty(\cdot, d, p)$ are tangential vector fields defined on the unit sphere S^2 and are known as the electric and magnetic far field patterns, respectively. Our aim is to determine λ and D from $E_\infty(\hat{x}, d, p)$ without any a priori assumption or knowledge of Γ_D, Γ_I and λ. The solution of this inverse scattering problem is unique and this can be proved following the approach described in Theorem 7.1 of [33] (where only the well-posedness of the direct scattering problem is required).

The derivation of the linear sampling method for the vector case now under consideration follows the same approach as the scalar case discussed in Section 8.2. In particular, we begin by defining the *far field operator* $F : L^2_t(S^2) \to L^2_t(S^2)$ by

$$(Fg)(\hat{x}) := \int_{S^2} E_\infty\left(\hat{x}, d, g(d)\right)\, ds(d) \tag{9.13}$$

and define the *far field equation* by

$$Fg = E_{e,\infty}(\hat{x}, z, q) \tag{9.14}$$

where $E_{e,\infty}$ is the electric far field pattern of the *electric dipole*

$$\begin{aligned} E_e(x, z, q) &:= \frac{i}{k}\,\mathrm{curl}_x\,\mathrm{curl}_x\, q\,\Phi(x, z) \\ H_e(x, z, q) &:= \mathrm{curl}_x\, q\,\Phi(x, z) \end{aligned} \tag{9.15}$$

where $q \in \mathbb{R}^3$ is a constant vector and Φ is the fundamental solution of the Helmholtz equation given by

$$\Phi(x, z) := \frac{e^{ik|x-z|}}{4\pi\,|x - z|}\,. \tag{9.16}$$

We can explicitly compute $E_{e,\infty}$, arriving at

$$E_{e,\infty}(\hat{x}, z, q) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x} e^{-ik\hat{x}\cdot z}\,. \tag{9.17}$$

Note that the far field operator given by (9.13) is linear since $E_\infty(\hat{x}, d, p)$ depends linearly on the polarization p.

We now return to the exterior mixed boundary value problem (9.8) – (9.11) and introduce the linear operator $B : Y(\partial D_D) \times L^2_t(\partial D_I) \to L^2_t(S^2)$ mapping the boundary data (f, h) onto the electric far field pattern E_∞. In [15] it is shown that this operator is injective, compact and has dense range in $L^2_t(S^2)$. By using B it is now possible to write the far field equation as

$$-(B\,\Lambda\, E_g)(\hat{x}) = \frac{1}{ik} E_{e,\infty}(\hat{x}, z, q) \tag{9.18}$$

where Λ is the trace operator corresponding to the mixed boundary condition, i.e. $\Lambda\, u := \nu \times u|_{\partial D_D}$ on ∂D_D and $\Lambda\, u := \nu \times \mathrm{curl}\, u - i\lambda(\nu \times u) \times \nu|_{\partial D_I}$ on ∂D_I, and E_g is the electric field of the electromagnetic Herglotz pair with kernel $g \in L^2_t(S^2)$ defined by

$$\begin{aligned} E_g(x) &:= \int_{S^2} e^{ikx\cdot d} g(d)\, ds(d) \\ H_g(x) &:= \frac{1}{ik} \operatorname{curl} E_g(x)\,. \end{aligned} \tag{9.19}$$

We note that $E_{e,\infty}(\hat{x}, z, q)$ is in the range of B if and only if $z \in D$ [15].

Finally, we consider the interior mixed boundary value problem

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \quad \text{in } D \tag{9.20}$$

$$\nu \times E = f \quad \text{on } \partial D_D \tag{9.21}$$

$$\nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu = h \quad \text{on } \partial D_I \tag{9.22}$$

where $f \in Y(\partial D_D)$, $h \in L^2(\partial D_I)$. It is shown in [15] that if $\partial D_I \neq \emptyset$ then there exists a unique solution to (9.20) – (9.22) in $X(D, \partial D_I)$ and that the following theorem is valid:

Theorem 9.2. *Assume that $\partial D_I \neq \emptyset$. Then the solution E of the interior mixed boundary value problem* (9.20) – (9.22) *can be approximated in $X(D, \partial D_I)$ by the electric field of an electromagnetic Herglotz pair.*

The factorization (9.18) together with Theorem 9.2 now allows us to prove the following theorem [15]:

Theorem 9.3. *Assume that $\partial D_I \neq \emptyset$. Then if F is the far field operator corresponding to the scattering problem* (9.1) – (9.7) *we have that*

1. *if $z \in D$ then for every $\epsilon > 0$ there is a function $g_z^\epsilon := g_z \in L^2_t(S^2)$ satisfying the inequality*

$$||Fg_z - E_{e,\infty}(\cdot, z, q)||_{L^2_t(S^2)} < \epsilon$$

such that

$$\lim_{z\to\partial D} ||g_z||_{L^2_t(S^2)} = \infty$$

and

$$\lim_{z\to\partial D} ||E_{g_z}||_{X(D,\partial D_I)} = \infty$$

where E_{g_z} is the electric field of the electromagnetic Herglotz pair with kernel g_z, and

2. *if $z \in D_e$ then for every $\epsilon > 0$ and $\delta > 0$ there exists $g_z^{\epsilon,\delta} := g_z \in L^2_t(S^2)$ satisfying the inequality*

$$||Fg_z - E_{e,\infty}(\cdot, z, q)||_{L^2_t(S^2)} < \epsilon + \delta$$

such that

$$\lim_{\delta\to 0} ||g_z||_{L^2_t(S^2)} = \infty$$

and

$$\lim_{\delta\to 0} ||E_{g_z}||_{X(D,\partial D_I)} = \infty$$

where E_{g_z} is the electric field of the electromagnetic Herglotz pair with kernel g_z.

Theorem 9.3 is also valid for the case of a perfect conductor (i.e. $\partial D_I = \emptyset$) provided we modify the far field operator F in an appropriate manner [10]. For numerical examples demonstrating the use of Theorem 9.3 in reconstructing D, see [15, 23] and [27]. By a method analogous to that of Section 4.4 for the scalar case, the function g_z can also be used to determine the surface impedance λ [11]. The case of mixed boundary value problems for screens was examined in [17].

We next examine the case of Maxwell's equations in an inhomogeneous anisotropic medium (which, of course, includes the isotropic medium as a special case). We again assume that $D \subset \mathbb{R}^3$ is a bounded domain with connected complement such that its boundary ∂D is in class C^2 with unit outward normal ν. Let N be a 3×3 symmetric matrix whose entries are piecewise continuous complex valued functions in $\mathbb{R}^3$ such that N is the identity matrix outside D. We further assume that there exists a positive constant $\gamma > 0$ such that

$$\mathrm{Re}\left(\bar{\xi}\cdot N(x)\xi\right) \geq \gamma|\xi|^2$$

for every $\xi \in \mathbb{C}^3$ where N is continuous and

$$\mathrm{Im}\left(\bar{\xi}\cdot N(x)\xi\right) > 0$$

for every $\xi \in \mathbb{C}^3 \setminus \{\emptyset\}$ and points $x \in D$ where N is continuous. Finally, we assume that $N - I$ is invertible and $\mathrm{Re}(N - I)^{-1}$ is uniformly positive definite in D (partial results for the case when this is not true can be found in [97]).

Now consider the scattering of the time harmonic incident field (9.4), (9.5) by an anisotropic inhomogeneous medium D with refractive index N satisfying the above assumptions. Then the mathematical formulation of the scattering of a time harmonic plane wave by an anisotropic medium is to find $E \in H_{loc}(\mathrm{curl}, \mathbb{R}^3)$ such that

$$\mathrm{curl}\,\mathrm{curl}\,E - k^2 N E = 0 \tag{9.23}$$

$$E = E^s + E^i \tag{9.24}$$

$$\lim_{r\to\infty} (\mathrm{curl}\,E^s \times x - ikrE^s) = 0\,. \tag{9.25}$$

A proof of the existence of a unique solution to (9.23) – (9.25) can be found in [87]. It can again be shown that E^s has the asymptotic behavior given in (9.12). Unfortunately, in general the electric far field pattern E_∞ does not uniquely determine N (although it does in the case when the medium is isotropic, i.e. $N(x) = n(x)I$, where n is a scalar [36, 56]). However E_∞ does uniquely determine D [7] and a derivation of the linear sampling method for determining D from E_∞ can be found in [52]. Numerical examples using this approach for determining D when the medium is isotropic can be found in [54]. Finally, a treatment of the factorization method for the case of electromagnetic waves in an isotropic medium is given in [69].

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