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The linear sampling method for anisotropic media

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Abstract

We consider the inverse scattering problem of determining the support of an anisotropic inhomogeneous medium from a knowledge of the incident and scattered time harmonic acoustic wave at fixed frequency. To this end, we extend the linear sampling method from the isotropic case to the case of anisotropic medium. In the case when the coefficients are real we also show that the set of transmission eigenvalues forms a discrete set.

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1. Introduction

In recent years, there has been considerable interest in the inverse scattering problem for anisotropic medium, particularly in the case of acoustic waves and electromagnetic waves in an orthotropic medium [3,5,7–9,11,12]. This interest is motivated by the fact that all real materials are anisotropic, at least slightly, and some quite a bit. At the same time it has been shown that it is not possible to uniquely determine the constitutive parameters of an anisotropic medium from far-field data but only the support of the medium in a homogeneous background [9,12]. Further complications arise in the behavior of the anisotropic material near the boundary where radically different mathematical techniques are needed depending on whether or not the constitutive parameters vary smoothly across the boundary [5,9].

Due to the lack of uniqueness in determining the constitutive parameters, traditional methods for solving the inverse scattering problem based on the use of weak scattering approximations or nonlinear optimization techniques are problematic. On the other hand, since the support is uniquely

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determined, the recently developed *linear sampling method* for determining the support of an isotropic medium from far-field data [1,11] seems ideally suited to solving the inverse scattering problem for anisotropic medium. In the case when the matrix A that describes the physical properties of the medium is real and constant (and hence does not vary continuously across the boundary) this extension was done in [3] whereas the case of complex valued inhomogeneities that vary smoothly across the boundary (i.e. $A = I$ on the boundary) were treated in [5,7]. The purpose of this paper is to extend the results of [3] to the case of (possibly) complex valued inhomogeneous anisotropic medium that does not vary smoothly across the boundary (i.e. $A \neq I$ on the boundary), a situation which is almost always the case in any realistic physical situation. For the sake of simplicity we will carry out our analysis in $\mathbb{R}^3$ although our results remain valid in $\mathbb{R}^n$ for any integer $n \geqslant 2$.

The linear sampling method for anisotropic medium that we will present in this paper for solving the inverse scattering problem is based on an analysis of a boundary value problem called the *interior transmission problem* (ITP). Due to a lack of uniqueness of a solution to this boundary value problem when A and n are real valued, particular problems occur in this case which leads to the problem of *transmission eigenvalues* [3,4]. In particular, if the wave number k is a transmission eigenvalue the linear sampling method for solving the inverse scattering problem fails. On the other hand, if it can be shown that the transmission eigenvalues form a discrete set, then one can at least assert that the linear sampling method is generically valid when A and n are real valued. The case when $A = I$ on the boundary was considered in [4] and in this paper we will give appropriate conditions on A and n that ensure the set of transmission eigenvalue is discrete when $A \neq I$ on the boundary.

2. The direct and inverse scattering problems for an anisotropic medium

Let $D \subset \mathbb{R}^3$ be a nonempty, open and bounded set having a C^2-boundary ∂D with unit outward normal ν. Moreover, we assume that the exterior domain $\mathbb{R}^3 \setminus \bar{D}$ is connected. Let A be a 3×3 matrix-valued function whose entries a_{jk}, $j = 1,2,3$, $k = 1,2,3$ are continuously differentiable complex-valued functions in $\bar{D}$ such that A is symmetric and satisfies $\bar{\xi} \cdot \mathscr{I}m(A)\xi \leqslant 0$ and $\bar{\xi} \cdot \mathscr{R}e(A)\xi \geqslant \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^3$ and $x \in \bar{D}$ where γ is a positive constant. Note that due to the symmetry of A, $\mathscr{I}m(\bar{\xi} \cdot A\xi) = \bar{\xi} \cdot \mathscr{I}m(A)\xi$ and $\mathscr{R}e(\bar{\xi} \cdot A\xi) = \bar{\xi} \cdot \mathscr{R}e(A)\xi$. For a function $u \in C^1(\bar{D})$ we define the conormal derivative by

$$\frac{\partial u}{\partial \nu_A}(x) := \nu(x) \cdot A(x)\nabla u(x), \quad x \in \partial D.$$

We can now formulate the direct scattering problem for an anisotropic medium. In particular, let $k > 0$ be the wave number and $n \in C(\bar{D})$ such that $\mathscr{I}m(n) \geqslant 0$. Then letting H^k denote the usual Sobolev space we want to find functions $w \in H^1(D)$, $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ such that

$$\text{(i)}\ \operatorname{div} A\nabla w + k^2 n w = 0 \quad \text{in } D,$$
$$\text{(ii)}\ \Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D},$$
$$\text{(iii)}\ w - u = f \quad \text{on } \partial D,$$

$$\text{(iv)}\ \frac{\partial w}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = h \quad \text{on } \partial D,$$

$$\text{(v)}\ \lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0, \tag{1}$$

where $f := \mathrm{e}^{\mathrm{i}kx\cdot d}$ and $h := (\partial/\partial\nu)\mathrm{e}^{\mathrm{i}kx\cdot d}$, $d \in \Omega := \{x\colon |x| = 1\}$, $r = |x|$, the boundary conditions are assumed in the sense of the trace operator, and the radiation condition (1)(v) holds uniformly with respect to $\hat{x} = x/|x|$.

More generally, we consider (1) with $f \in H^{1/2}(\partial D)$ and $h \in H^{-1/2}(\partial D)$ arbitrary and in the sequel will refer to this more general problem as the *transmission problem* (TP). The existence of a unique solution to (TP) has been established by Hähner in [9]. Moreover, he has proved that this solution depends continuously on the boundary data in the sense that the following estimate holds

$$\|w\|_{H^1(D)} + \|u\|_{H^1(B\setminus\bar{D})} \leqslant C(\|f\|_{H^{1/2}(\partial D)} + \|h\|_{H^{-1/2}(\partial D)}), \tag{2}$$

where B is a ball containing D and $C = C(B)$ is a positive constant.

Since u satisfies the radiation condition (1) (v) we can conclude (see [2]) that u has the asymptotic behavior

$$u(x) = \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\, u_\infty(\hat{x}, d) + \mathrm{O}\left(\frac{1}{r^2}\right), \quad r \to \infty,$$

where $u_\infty(\hat{x}, d)$ is the *far-field pattern* of the scattered field u. The inverse scattering problem we are concerned with is to determine D from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}$, $d \in \Omega$. The fact that D is uniquely determined from u_∞ has been established in [9,12] (we remind the reader that A is *not* uniquely determined by u_∞ [8]). Our approach for solving this inverse scattering problem is the linear sampling method as described in [1] for the case of isotropic medium. In particular, we will look for a (regularized) solution $g \in L^2(\Omega)$ of the *far-field equation*

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, \mathrm{d}s(d) = \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}, \tag{3}$$

where $y \in \mathbb{R}^3$ is an artificially introduced parameter point, and $F : L^2(\Omega) \to L^2(\Omega)$ is called the *far-field operator*. It is easily verified (c.f. [2]) that (3) is solvable if and only if $y \in D$ and $v, w \in C^2(D) \cap C^1(\bar{D})$ is a solution of the *interior transmission problem*

$$\text{(i)}\ \operatorname{div} A\nabla w + k^2 n w = 0 \quad \text{in } D,$$

$$\text{(ii)}\ \Delta v + k^2 v = 0 \quad \text{in } D,$$

$$\text{(iii)}\ w - v = \Phi(\cdot, y) \quad \text{on } \partial D,$$

$$\text{(iv)}\ \frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = \frac{\partial \Phi(\cdot, y)}{\partial \nu} \quad \text{on } \partial D, \tag{4}$$

where $\Phi(x, y) := \mathrm{e}^{\mathrm{i}k|x-y|}/|x-y|$, such that v is a *Herglotz wave function*, i.e., a solution v_g of the Helmholtz equation in $\mathbb{R}^3$ of the form

$$v_g(x) = \int_\Omega \mathrm{e}^{\mathrm{i}kx\cdot d} g(d)\, \mathrm{d}s(d), \quad x \in \mathbb{R}^3.$$

Values of k for which a nontrivial solution to the homogeneous interior transmission problem ($\Phi = 0$) exists are called *transmission eigenvalues*.

Our analysis of the interior transmission problem in the next section will lead to showing in Section 4 that, roughly speaking, D can be characterized as the set of points $y \in \mathbb{R}^3$ where an (arbitrarily good) approximation of (3) remains bounded. This approach for solving the inverse scattering problem is called the *linear sampling method* and the rest of the paper is devoted to the analysis and justification of this method.

3. The interior transmission problem

Let the domain $D \subset \mathbb{R}^3$, the matrix-valued function A and the function n satisfy the assumptions of the TP stated in the previous section.

The *interior transmission problem* associated with TP, which in the sequel will be referred to as ITP, is given $f \in H^{1/2}(\partial D)$ and $h \in H^{-1/2}(\partial D)$, find two functions $w \in H^1(D)$ and $v \in H^1(D)$ satisfying

$$\begin{array}{lll} \text{(i)} & \operatorname{div} A\nabla w + k^2 n w = 0 & \text{in } D, \\ \text{(ii)} & \Delta v + k^2 v = 0 & \text{in } D, \\ \text{(iii)} & w - v = f & \text{on } \partial D, \\ \text{(iv)} & \partial_{\nu_A} w - \partial_\nu v = h & \text{on } \partial D. \end{array} \tag{5}$$

Note that for simplicity we use the notations $\partial_{\nu_A} := \partial/\partial\nu_A$ and $\partial_\nu := \partial/\partial\nu$.

We begin by establishing the uniqueness of a solution to (ITP).

Theorem 3.1. *If either $\mathscr{I}m(n) > 0$ or $\mathscr{I}m(\bar{\xi} \cdot A\xi) < 0$ in a neighborhood B_{x_0} of a point $x_0 \in D$, then ITP has at most one solution.*

Proof. Let us consider the homogeneous problem (i.e., $f = h = 0$). Applying the divergence theorem to $\bar{w}$ and $A\nabla w$, making use of the boundary condition and applying Green's theorem for $\bar{v}$ and v we obtain

$$\int_D \nabla\bar{w} \cdot A\nabla w \,\mathrm{d}y - \int_D k^2 n|w|^2 \,\mathrm{d}y = \int_{\partial D} \bar{w} \cdot \partial_{\nu_A} w \,\mathrm{d}y = \int_D |\nabla v|^2 \,\mathrm{d}y - \int_D k^2 |v|^2 \,\mathrm{d}y.$$

Hence

$$\mathscr{I}m\left(\int_D \nabla\bar{w} \cdot A\nabla w \,\mathrm{d}y\right) = 0 \quad \text{and} \quad \mathscr{I}m\left(\int_D n|w|^2 \,\mathrm{d}y\right) = 0. \tag{6}$$

If $\mathscr{I}m(n) > 0$ in B_{x_0}, then the second equality of (6) and the unique continuation principle (c.f. [10, Theorem 17.26]) imply that $w \equiv 0$ in D. In the case of $\mathscr{I}m(\bar{\xi} \cdot A\xi) < 0$ in B_{x_0}, from the first equality of (6) we obtain that $\nabla w \equiv 0$ in B_{x_0} and from (5)(i) $w \equiv 0$ in B_{x_0}, and hence $w \equiv 0$ in D. From the boundary conditions and the integral representation formula v also vanishes in D. □

Next we study the solvability of ITP. To this end, we formulate the *modified interior transmission problem* (MITP) which later will be seen as a compact perturbation of our original ITP: given D,

A as above, a continuous positive real-valued function $m \in C(\bar{D})$, functions $\ell_1 \in L^2(D)$, $\ell_2 \in L^2(D)$, $f \in H^{1/2}(\partial D)$, and $h \in H^{-1/2}(\partial D)$, find $w \in H^1(D)$ and $v \in H^1(D)$ satisfying

$$\begin{array}{lll} \text{(i)} & \operatorname{div} A\nabla w - mw = \ell_1 & \text{in } D, \\ \text{(ii)} & \Delta v - v = \ell_2 & \text{in } D, \\ \text{(iii)} & w - v = f & \text{on } \partial D, \\ \text{(iv)} & \partial_{\nu_A} w - \partial_\nu v = h & \text{on } \partial D. \end{array} \tag{7}$$

Definition 3.2. A strong solution to the interior transmission problem (7) is a pair $(w, v) \in H^1(D) \times H^1(D)$ satisfying (7)(i) and (7)(ii) and in the sense of distributions and satisfying (7)(iii) and (7)(iv) in the sense of the trace operator.

We will now reformulate (7) as a variational problem. Let

$$W(D) = \{\mathbf{v} \in L^2(D)^3:\ \operatorname{div} \mathbf{v} \in L^2(D) \text{ and } \operatorname{curl} \mathbf{v} = 0\} \tag{8}$$

equipped with the natural norm

$$\|\mathbf{v}\|_W = (\|\mathbf{v}\|_{L^2}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2}^2)^{1/2}$$

and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$. Then the duality identity

$$\langle \varphi, \psi \cdot \nu \rangle = \int_D \varphi \operatorname{div} \psi \, \mathrm{d}x + \int_D \nabla \varphi \cdot \psi \, \mathrm{d}x \tag{9}$$

for $(\varphi, \psi) \in H^1(D) \times W(D)$ will be of particular interest in the sequel.

We now introduce the sesquilinear form $\mathscr{A}$ defined on $\{H^1(D) \times W(D)\}^2$ by

$$\begin{aligned} \mathscr{A}(U, V) = & \int_D A\nabla w \cdot \nabla \bar{\varphi} \, \mathrm{d}x + \int_D m w \bar{\varphi} \, \mathrm{d}x + \int_D \operatorname{div} \mathbf{v} \operatorname{div} \bar{\psi} \, \mathrm{d}x + \int_D \mathbf{v} \cdot \bar{\psi} \, \mathrm{d}x \\ & - \langle w, \bar{\psi} \cdot \nu \rangle - \langle \bar{\varphi}, \mathbf{v} \cdot \nu \rangle, \end{aligned} \tag{10}$$

where $U = (w, \mathbf{v})$ and $V = (\varphi, \psi)$ are in $H^1(D) \times W(D)$. We also introduce for $V = (\varphi, \psi) \in H^1(D) \times W(D)$ the antilinear form

$$L(V) = \int_D (\ell_1 \bar{\varphi} + \ell_2 \operatorname{div} \bar{\psi}) \, \mathrm{d}x + \langle \bar{\varphi}, h \rangle - \langle f, \bar{\psi} \cdot \nu \rangle. \tag{11}$$

The variational formulation of the problem (7) is

$$\begin{aligned} & \text{Seek } U \in H^1(D) \times W(D) \text{ such that} \\ & \mathscr{A}(U, V) = L(V), \quad \forall V \in H^1(D) \times W(D). \end{aligned} \tag{12}$$

The following theorem proves the equivalence between the two problems (7) and (12).

Theorem 3.3. *Problem* (7) *has a unique strong solution* (w, v) *if and only if problem* (12) *has a unique solution* $U \in H^1(D) \times W(D)$. *Moreover, if* (w, v) *is the unique strong solution to* (7) *then* $U = (w, \nabla v)$ *is the unique strong solution to* (12). *Conversely, if* U *is the unique solution to* (12) *then the unique strong solution* (w, v) *to* (7) *is such that* $U = (w, \nabla v)$.

Proof. Let us first prove the equivalence between the existence of a strong solution (w,v) to (7) and the existence of a solution U to (12).

(a) $\exists(w,v) \Rightarrow \exists U$: Let (w,v) be a strong solution to (7). We set $\mathbf{v} = \nabla v$. From (7)(ii) we see that if $\operatorname{div}\mathbf{v} = v + l_2 \in L^2(D)$ then $\mathbf{v} \in W(D)$. Taking the L^2 scalar product of (7)(ii) with $\operatorname{div}\psi$ for some $\psi \in W(D)$ and using (9) shows that

$$\int_D \operatorname{div}\mathbf{v}\,\operatorname{div}\bar{\psi}\,\mathrm{d}x + \int_D \mathbf{v}\cdot\bar{\psi}\,\mathrm{d}x - \langle v, \bar{\psi}\cdot v\rangle = \int_D \ell_2 \operatorname{div}\bar{\psi}\,\mathrm{d}x.$$

Hence, by (7)(iii)

$$\int_D \operatorname{div}\mathbf{v}\,\operatorname{div}\bar{\psi}\,\mathrm{d}x + \int_D \mathbf{v}\cdot\bar{\psi}\,\mathrm{d}x - \langle w, \bar{\psi}\cdot v\rangle$$
$$= -\langle f, \bar{\psi}\cdot v\rangle + \int_D \ell_2 \operatorname{div}\bar{\psi}\,\mathrm{d}x. \tag{13}$$

We now take the L^2 scalar product of (7)(i) with φ in $H^1(D)$ and integrate by parts. Using the boundary condition (7)(iv), this shows that

$$\int_D A\nabla w\cdot\nabla\bar{\varphi}\,\mathrm{d}x + \int_D mw\bar{\varphi}\,\mathrm{d}x - \langle\bar{\varphi}, \mathbf{v}\cdot v\rangle = \langle\bar{\varphi}, h\rangle + \int_D \ell_1\bar{\varphi}\,\mathrm{d}x. \tag{14}$$

Adding (13) and (14) together shows that $U = (w,\mathbf{v})$ is a solution to (12).

(b) $\exists U \Rightarrow \exists(w,v)$: We set $U = (w,\mathbf{v}) \in H^1 \times W(D)$. Since $\operatorname{curl}\mathbf{v} = 0$ and D is simply connected, we deduce the existence of a function $v \in H^1(D)$ such that $\mathbf{v} = \nabla v$ where v is determined up to an additive constant. As we shall see later, this constant can be adjusted so that (w,v) is a strong solution to (7). Obviously, if U satisfies (12) then $(w,\mathbf{v})$ satisfies (13) and (14) for all $(\varphi,\psi) \in H^1(D)\times W(D)$. One can easily see from (14) that the pair (w,v) satisfies

$$\begin{aligned} &\operatorname{div} A\nabla w - mw = \ell_1 \quad \text{in } D,\\ &\partial_{v_A} w - \partial_v v = h \qquad \text{on } \partial D. \end{aligned} \tag{15}$$

On the other hand, substituting for $\mathbf{v}$ in (13) and using the duality identity (9) in the second integral shows that

$$\int_D (\Delta v - v)\operatorname{div}\bar{\psi}\,\mathrm{d}x + \langle v - w, \bar{\psi}\cdot v\rangle = -\langle f, \bar{\psi}\cdot v\rangle + \int_D \ell_2 \operatorname{div}\bar{\psi}\,\mathrm{d}x \tag{16}$$

for all ψ in $W(D)$.

Now consider a function $\phi \in L_0^2(D) = \{\phi \in L^2(D)\colon \int_D \phi\,\mathrm{d}x = 0\}$ and let $\chi \in H^1(D)$ be a solution to

$$\begin{aligned} &\Delta\chi = \bar{\phi} \quad \text{in } D,\\ &\partial_v\chi = 0 \quad \text{on } \partial D. \end{aligned} \tag{17}$$

Taking $\psi = \nabla\chi$ in (16) ($\Rightarrow \operatorname{div}\bar{\psi} = \phi$ in D and $\bar{\psi}\cdot v = 0$ on ∂D) shows that

$$\int_D (\Delta v - v - \ell_2)\phi\,\mathrm{d}x = 0 \quad \forall\phi \in L_0^2(D)$$

which implies the existence of a constant c_1 such that

$$\Delta v - v - \ell_2 = c_1 \quad \text{in } D. \tag{18}$$

We now take $\phi \in L^2_0(\partial D)$ and let $\chi \in H^1(D)$ be the solution to

$$\begin{aligned} &\Delta \chi = 0 \quad \text{in } D, \\ &\partial_\nu \chi = \bar{\phi} \quad \text{on } \partial D. \end{aligned} \tag{19}$$

Taking $\psi = \nabla \chi$ in (13) ($\Rightarrow \operatorname{div} \bar{\psi} = 0$ in D and $\bar{\psi} \cdot \nu = \phi$ on ∂D) shows that

$$\int_{\partial D} (v - w + f)\phi \, \mathrm{d}\gamma = 0 \quad \forall \phi \in L^2_0(\partial D)$$

which implies the existence of a constant c_2 such that

$$v - w + f = c_2 \quad \text{on } \partial D. \tag{20}$$

Substituting (18) and (20) into (16) and using (9) now shows that

$$(c_1 - c_2) \int_D \operatorname{div} \bar{\psi} \, \mathrm{d}x = 0 \quad \forall \psi \in W(D)$$

which implies $c_1 = c_2 = c$ (take $\psi = \nabla \chi$ where $\chi \in H^1_0(D)$ and $\Delta \chi = 1$ in D). Eqs. (15), (18) and (20) show that $(w, v - c)$ is a strong solution to (7).

We now consider the uniqueness equivalence.

(c) Uniqueness of $(w, v) \Rightarrow$ Uniqueness of U: Assume that problem (7) has a unique strong solution and consider two solutions $U_1 = (w_1, \mathbf{v}_1)$ and $U_2 = (w_2, \mathbf{v}_2)$ to (12). From step (b) we deduce the existence of v_1 and v_2 in $H^1(D)$ such that $\mathbf{v}_1 = \nabla v_1$ and $\mathbf{v}_2 = \nabla v_2$ and (w_1, v_1) and (w_2, v_2) are strong solutions to (7). Hence, $(w_1, v_1) = (w_2, v_2)$ and $(w_1, \mathbf{v}_1) = (w_2, \mathbf{v}_2)$.

(d) Uniqueness of $U \Rightarrow$ Uniqueness of (w, v): Assume that problem (12) has a unique solution and consider two strong solutions (w_1, v_1) and (w_2, v_2) to (7). We deduce from step (a) that $(w_1, \nabla v_1)$ and $(w_2, \nabla v_2)$ are two solutions to (12). Hence $w_1 = w_2$ and $v = v_1 - v_2$ is a function in $H^1(D)$ that satisfies

$$\begin{aligned} &\Delta v - v = 0 \quad \text{in } D, \\ &v = \partial_\nu v = 0 \quad \text{on } \partial D \end{aligned}$$

which implies $v = 0$. $\square$

Theorem 3.4. *Assume that there exists a constant $\gamma > 1$ such that for $x \in D$,*

$$\mathscr{Re}(\bar{\xi} \cdot A(x)\xi) \geqslant \gamma |\xi|^2 \quad \forall \xi \in \mathbb{C}^3 \quad \text{and} \quad m(x) \geqslant \gamma. \tag{21}$$

Then problem (12) has a unique solution $U = (w, \mathbf{v}) \in H^1(D) \times W(D)$. This solution satisfies the a priori estimate

$$\|w\|_{H^1(D)} + \|\mathbf{v}\|_W \leqslant 2c \frac{\gamma + 1}{\gamma - 1} (\|\ell_1\|_{L^2(D)} + \|\ell_2\|_{L^2(D)} + \|f\|_{H^{1/2}(\partial D)} + \|h\|_{H^{-1/2}(\partial D)}), \tag{22}$$

where the constant c is independent of ℓ_1, ℓ_2, f, h and γ.

Proof. Classical trace theorems and Schwarz's inequality ensure the continuity of the antilinear form L on $H^1(D)\times W(D)$ and the existence of a constant c independent of ℓ_1, ℓ_2, f and h such that

$$\|L\| \leqslant c(\|\ell_1\|_{L^2} + \|\ell_2\|_{L^2} + \|f\|_{H^{1/2}} + \|h\|_{H^{-1/2}}). \tag{23}$$

On the other hand, if $U=(u,\mathbf{v})\in H^1(D)\times W(D)$ then, by assumption (21),

$$|\mathscr{A}(U,U)| \geqslant \gamma\|w\|^2_{H^1} + \|\mathbf{v}\|^2_W - 2\mathscr{R}e(\langle \bar{w},\mathbf{v}\rangle). \tag{24}$$

According to the duality identity (9), one has by Schwarz's inequality that

$$|\langle \bar{w},\mathbf{v}\rangle| \leqslant \|w\|_{H^1}\|\mathbf{v}\|_W$$

and therefore

$$|\mathscr{A}(U,U)| \geqslant \gamma\|w\|^2_{H^1} + \|\mathbf{v}\|^2_W - 2\|w\|_{H^1}\|\mathbf{v}\|_W.$$

Using the identity

$$\gamma x^2 + y^2 - 2xy = \frac{\gamma+1}{2}\left(x - \frac{2}{\gamma+1}\,y\right)^2 + \frac{\gamma-1}{2}\,x^2 + \frac{\gamma-1}{\gamma+1}\,y^2,$$

we conclude that

$$|\mathscr{A}(U,U)| \geqslant \frac{\gamma-1}{\gamma+1}\,(\|\mathbf{v}\|^2_W + \|w\|^2_{H^1})$$

and thus $\mathscr{A}$ is coercive. The continuity of $\mathscr{A}$ follows easily from Schwarz's inequality and classical trace theorems. Theorem 3.4 is therefore a direct consequence of Lax–Milgram theorem applied to (12). □

Theorem 3.5. *Under the assumptions of Theorem* 3.4, *problem* (7) *has a unique strong solution* (w,v) *that satisfies*

$$\|w\|_{H^1(D)} + \|v\|_{H^1(D)} \leqslant c\,\frac{\gamma+1}{\gamma-1}\,(\|\ell_1\|_{L^2(D)} + \|\ell_2\|_{L^2(D)} + \|f\|_{H^{1/2}(\partial D)} + \|h\|_{H^{-1/2}(\partial D)}), \tag{25}$$

where the constant c is independent of ℓ_1, ℓ_2, f, h *and* γ.

Proof. The existence and uniqueness of a strong solution follows from Theorems 3.3 and 3.4. The a priori estimate (25) can be obtained directly from (7) but can be also deduced from (22) as follows. Theorem 3.3 tells us that $(w,\nabla v)$ is the unique solution to (12). Hence, according to (22)

$$\|w\|_{H^1} + \|\nabla v\|_{L^2} \leqslant c_1\,\frac{\gamma+1}{\gamma-1}(\|\ell_1\|_{L^2} + \|\ell_2|_{L^2} + \|f\|_{H^{1/2}} + \|h\|_{H^{-1/2}}).$$

But from the Poincaré inequality,

$$\|v\|_{H^1(D)} \leqslant c_2(\|\nabla v\|_{L^2(D)} + \|v\|_{L^2(\partial D)}),$$

and using the boundary condition (7)(iii) and the trace theorem one deduces that

$$\|v\|_{H^1(D)} \leqslant c_2(\|\nabla v\|_{L^2(D)} + \|w\|_{H^1(D)} + \|f\|_{L^2(\partial D)})$$

for some positive constant c_2. The constants c_1 and c_2 can then be adjusted so that (25) holds. □

Theorem 3.6. *Assume that either $\mathscr{I}m(n) > 0$ or $\mathscr{I}m(\bar{\xi} \cdot A\xi) < 0$ in a neighborhood B_{x_0} of a point $x_0 \in D$ and that there exists a constant $\gamma > 1$ such that for almost every $x \in D$,*

$$\mathscr{R}e(\bar{\xi} \cdot A(x)\xi) \geqslant \gamma|\xi|^2 \quad \forall \xi \in \mathbb{C}^3. \tag{26}$$

Then (ITP) *has a unique solution $(w, v) \in H^1(D) \times H^1(D)$. This solution satisfies the a priori estimate*

$$\|w\|_{H^1(D)} + \|v\|_{H^1(D)} \leqslant C(\|f\|_{H^{1/2}(\partial D)} + \|h\|_{H^{-1/2}(\partial D)}), \tag{27}$$

where the constant C is independent of f and h.

Proof. Let us set

$$\mathscr{X}(D) = \{(w, v) \in H^1(D) \times H^1(D)\colon \ \operatorname{div} A\nabla w \in L^2(D) \text{ and } \Delta v \in L^2(D)\}$$

and consider the operator $\mathscr{G}$ from $\mathscr{X}(D)$ into $L^2(D) \times L^2(D) \times H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ defined by

$$\mathscr{G}(w, v) = (\operatorname{div} A\nabla w - mw, \Delta v - v, (w - v)|_{\partial D}, (\partial_{\nu_A} w - \partial_\nu v)|_{\partial D}), \tag{28}$$

where $m \in C(\bar{D})$ and $m > 1$. Theorem 3.5 shows that the inverse of $\mathscr{G}$ exists and is continuous. Since $\mathscr{G}$ is continuous, we deduce that $\mathscr{G}$ is a bijective operator. Now consider the operator $\mathscr{T}$ from $\mathscr{X}(D)$ into $L^2(D) \times L^2(D) \times H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ defined by

$$\mathscr{T}(w, v) = ((k^2 n + m)w, (k^2 + 1)v, 0, 0).$$

By the compact embedding of $H^1(D)$ into $L^2(D)$, the operator $\mathscr{T}$ is compact. Hence $\mathscr{G} + \mathscr{T}$ is a Fredholm operator of index one. Theorem 3.1 shows that $\mathscr{G}+\mathscr{T}$ is injective and therefore we deduce the existence and the continuity of $(\mathscr{G} + \mathscr{T})^{-1}$, which means in particular the existence of a unique solution to ITP satisfying the a priori estimate (27). □

In general we cannot conclude the solvability of the ITP if A and n do not satisfy the assumptions of the previous theorem. However, we can assert that the set of transmission eigenvalues is discrete.

Theorem 3.7. *Assume that $\mathscr{I}m(n) = 0$ and $\mathscr{I}m(A) = 0$ in D and that there exists a constant $\gamma > 1$ such that for almost every $x \in D$,*

$$(\bar{\xi} \cdot A(x)\xi) \geqslant \gamma|\xi|^2 \quad \forall \xi \in \mathbb{R}^3 \quad \text{and } n(x) \geqslant \gamma. \tag{29}$$

Then the set of the values of $k \in \mathbb{C}$ for which ITP *does not have a unique solution is discrete.*

Proof. Consider the operator $\mathscr{G}$ defined by (28) with $m = n$ and the operator $\mathscr{T}$ from $\mathscr{X}(D)$ into $L^2(D) \times L^2(D) \times H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ defined by

$$\mathscr{T}(w, v) = (nw, v, 0, 0).$$

We want to prove that the operator $\mathscr{G} + (k^2 + 1)\mathscr{T}$ is invertible for all $k \in \mathbb{C}\backslash S$ where S is a discrete subset of $\mathbb{C}$. Since $\mathscr{G}$ is bijective (Theorem 3.5), this is equivalent to showing that $(I + (k^2 + 1)\mathscr{G}^{-1}\mathscr{T})^{-1}$ exists, where I is the identity operator from $\mathscr{X}(D)$ into $\mathscr{X}(D)$. The fact that this operator exists except for a discrete set of k values follows immediately from the theory of compact operators. □

4. The linear sampling method

Now we turn our attention to the scattering problem (1). The *inverse problem* we are interested in is, given the far-field pattern u_∞ of the scattered field u corresponding to the incident plane wave $u^i(x) = e^{ikx\cdot d}$ with incident direction $d \in \Omega$ and a single wave number k, to determine the shape of the anisotropic inhomogeneous penetrable scatterer D. The linear sampling method for solving this inverse problem looks for a solution $g = g(\cdot, y) \in L^2(\Omega)$ of the linear far-field equation (3) for various sampling points $y \in \mathbb{R}^3$. Note that $e^{-ik\hat{x}\cdot y} := \Phi_\infty(\hat{x}, y)$ is the far-field pattern of the fundamental solution $\Phi(x, y)$ to the Helmholtz equation.

We denote by $\mathscr{B}$ the bounded linear operator from $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ onto $L^2(\Omega)$ which maps $(f, h) \in H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ into the far field $u_\infty \in L^2(\Omega)$ of the solution (w, u) of TP corresponding to the boundary data (f, h).

Theorem 4.1. *The set $\mathscr{B}(H^{1/2}(\partial D) \times H^{-1/2}(\partial D))$ is dense in $L^2(\Omega)$.*

Proof. We consider the dual (or transpose) operator $\mathscr{B}^\top : L^2(\Omega) \to H^{-1/2}(\partial D) \times H^{1/2}(\partial D)$ mapping a function g into $(\tilde{f}, \tilde{h})$ such that

$$\langle \mathscr{B}(f, h), g\rangle_{L^2(\Omega)\times L^2(\Omega)} = \langle f, \tilde{f}\rangle_{H^{1/2}(\partial D)\times H^{-1/2}(\partial D)} + \langle h, \tilde{h}\rangle_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)}$$

and let $(\tilde{w}, \tilde{u})$ be the unique solution of (TP) with $(f, h) := (v_g|_{\partial D}, \frac{\partial v_g}{\partial \nu}|_{\partial D})$, where v_g is the Herglotz wave function with kernel g. Then from [2, Theorem 2.5] we have

$$\langle \mathscr{B}(f, h), g\rangle = \int_\Omega u_\infty(\hat{x}) g(\hat{x})\, ds(\hat{x}) = \int_{\partial D} \left(u(y) \frac{\partial v_g(y)}{\partial \nu} - v_g(y) \frac{\partial u(y)}{\partial \nu} \right) ds(y).$$

Since u and $\tilde{u}$ are solutions of the Helmholtz equation in $\mathbb{R}^3 \backslash \bar{D}$ satisfying the Sommerfeld radiation condition then

$$\int_{\partial D} \left[u(y) \frac{\partial \tilde{u}(y)}{\partial \nu} - \tilde{u}(y) \frac{\partial u(y)}{\partial \nu} \right] ds(y) = 0$$

and using the transmission conditions on the boundary we can write

$$\begin{aligned} &\langle \mathscr{B}(f, h), g\rangle_{L^2(\Omega)\times L^2(\Omega)} \\ &\quad = \int_{\partial D} \left[u(y) \left(\frac{\partial v_g(y)}{\partial \nu} + \frac{\partial \tilde{u}(y)}{\partial \nu} \right) - (v_g(y) + \tilde{u}(y)) \frac{\partial u(y)}{\partial \nu} \right] ds(y) \\ &\quad = \int_{\partial D} \left(u(y) \frac{\partial \tilde{w}(y)}{\partial \nu_A} - \tilde{w}(y) \frac{\partial u(y)}{\partial \nu} \right) ds(y) \\ &\quad = \int_{\partial D} \left[(w(y) - f(y)) \frac{\partial \tilde{w}(y)}{\partial \nu_A} - \tilde{w}(y) \left(\frac{\partial w(y)}{\partial \nu_A} - h(y) \right) \right] ds(y). \end{aligned}$$

Finally, an application of the divergence theorem for w and $\tilde{w}$ yields

$$\langle \mathscr{B}(f, h), g\rangle_{L^2(\Omega)\times L^2(\Omega)} = \int_{\partial D} \left[f(y) \left(-\frac{\partial \tilde{w}(y)}{\partial \nu_A} \right) + \tilde{w}(y) h(y) \right] ds(y),$$

and hence the dual operator $\mathscr{B}^\top$ can be characterized as

$$\mathscr{B}^\top g = \left(-\frac{\partial \tilde{w}}{\partial v_A}\bigg|_{\partial D}, \tilde{w}|_{\partial D} \right).$$

In the following, we want to show that the operator $\mathscr{B}^\top$ is injective. To this end let $\mathscr{B}^\top g \equiv 0$ and $g \in L^2(\Omega)$. This implies that $\tilde{w} \equiv 0$ and $\partial\tilde{w}/\partial v_A \equiv 0$ on the boundary ∂D. Therefore, $\tilde{u}$ satisfies the Helmholtz equation in $\mathbb{R}^3 \backslash \bar{D}$, the Sommerfeld radiation condition and $\tilde{u} = -v_g$ and $\partial\tilde{u}/\partial v = -\partial v_g/\partial v$ on the boundary. Thus, setting $\tilde{u} \equiv -v_g$ in D we have that $\tilde{u}$ can be extended to an entire solution of the Helmholtz equation satisfying the radiation condition. This is only possible if $\tilde{u}$ vanishes which implies that v_g vanishes also, and thus $g \equiv 0$. Now the range of $\mathscr{B}$ can be characterized as

$$(\text{range}\,\mathscr{B})^{\text{a}} := \{g \in L^2(\Omega)\colon \langle g, \psi\rangle = 0 \quad \text{for all } \psi \in \text{ range}\,\mathscr{B}\} = \text{kern}\,\mathscr{B}^\top,$$

where $(\cdot)^{\text{a}}$ denotes the annihilator set. Injectivity of $\mathscr{B}^\top$ implies that the range of $\mathscr{B}$ is dense in $L^2(\Omega)$. This ends the proof. □

Since $\mathscr{B}$ is bounded, we also have that

$$(\text{kern}\,\mathscr{B}) = (\text{range}\,\mathscr{B}^\top)^{\text{a}} := \left\{ (f_0, h_0)\colon \int_{\partial D} \left(-f_0 \frac{\partial \tilde{w}}{\partial v_A} + h_0 \tilde{w} \right) \mathrm{d}s = 0 \right\},$$

where $\tilde{w}$ is as in the proof of Theorem 4.1. Hence, using Green's formulas we see that the pairs $(w|_{\partial D}, \partial w/\partial v_A|_{\partial D})$, where $w \in H^1(D)$ is a solution of $\nabla \cdot A\nabla w + k^2 n w = 0$ in D, are in the kernel of $\mathscr{B}$. So $\mathscr{B}$ is not injective.

Letting $\bar{H}$ denote the closure in $H^1(D)$ of all Herglotz wave functions with kernel $g \in L^2(\Omega)$, we define the subset $H(\partial D)$ of $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ by

$$H(\partial D) = \left\{ \left(v|_{\partial D}, \frac{\partial v}{\partial v}\bigg|_{\partial D} \right) : v \in \bar{H} \right\}.$$

Note that $\bar{H} = \{v \in H^1(D)\colon \Delta v + k^2 v = 0\}$ since Herglotz wave functions approximate any H^1 solution to the Helmholtz equation in D [6].

Lemma 4.2. *$H(\partial D)$ is a closed subset of $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$.*

Proof. Consider $(f, h) \in \overline{H(\partial D)}$. There exists a sequence $(v_n, \partial_v v_n)$ converging to (f, h) in $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ where $v_n \in \bar{H}$. Since the sequence $(v_n, \partial_v v_n)$ is bounded in $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$, by considering v_n to be the solution of an impedance boundary value problem as in [6] we can deduce that (v_n) is bounded in $H^1(D)$. From this it follows that a subsequence (still denoted by (v_n)) converges weakly in $H^1(D)$ to a function v which is clearly in $\bar{H}$. From the weak continuity of the trace operator we deduce that $(v_n, \partial_v v_n)$ converges weakly in $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ to $(v, \partial_v v)$ and by the uniqueness of the limit $(f, h) = (v, \partial_v v)$. Hence $(f, h) \in H(\partial D)$. This completes the proof. □

From the above lemma, $H(\partial D)$ equipped with the induced norm from $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ is a Banach space. Let $\mathscr{B}_0$ denote the restriction of the above operator $\mathscr{B}$ to $H(\partial D)$.

Theorem 4.3. *Assume that k is not an eigenvalue of the interior transmission problem (ITP). Then the operator $\mathscr{B}_0 : H(\partial D) \to L^2(\Omega)$ is injective, compact and has dense range.*

Proof. Let $\mathscr{B}_0(f,h) = 0$ for $(f,h) \in H(\partial D)$ and let (w,u) be the solution to TP corresponding to this boundary data. Then the radiating solution to the Helmholtz equation in the exterior of D has zero far-field pattern, whence $u \equiv 0$ for $x \in \mathbb{R}^3 \setminus \bar{D}$. This implies that w satisfies

$$\nabla \cdot A \nabla w + k^2 n w = 0 \quad \text{in } D, \quad w = f \quad \text{and} \quad \frac{\partial w}{\partial v} = h \text{ on } \partial D.$$

From the definition of $H(\partial D)$, f,h are traces on ∂D of an $H^1(D)$ solution v to the Helmholtz equation and its normal derivative, respectively. Therefore (w,v) solves the homogeneous (ITP) and from our assumption $w \equiv 0$ and $v \equiv 0$ in D, whence $f = h \equiv 0$.

Compactness is a simple consequence of the fact that $\mathscr{B}_0$ can be seen as a composition of the continuous solution operator of TP with the compact operator which maps a radiating solution to its far field (see [2]).

It remains to show that the set $\mathscr{B}_0(H(\partial D))$ is dense in $L^2(\Omega)$. To this end, it is sufficient to show that the range of $\mathscr{B}$ is contained in the range of $\mathscr{B}_0$ since the range of $\mathscr{B}$ is dense in $L^2(\Omega)$ (Theorem 4.1). Let u_∞ be in range of $\mathscr{B}$, that is u_∞ is the far field of the radiating part u of a solution (w_0,u) to TP. Let (w,v) be the unique solution to ITP with the boundary data $(u|_{\partial D}, \partial u/\partial v|_{\partial D})$. Hence (w,u) is the solution of TP with boundary data $(v|_{\partial D}, \partial v/\partial v|_{\partial D}) \in H(\partial D)$ and has a far field that coincides with u_∞, whence $\mathscr{B}_0(v|_{\partial D}, \partial v/\partial v|_{\partial D}) = u_\infty$. Note that we have showed that any $(f,h) \in H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ can be written as $(f,h) = (v|_{\partial D}, \partial v/\partial v|_{\partial D}) + (f_0,h_0)$ where $(f_0,h_0) \in \operatorname{kern} \mathscr{B}$ and $v \in \bar{H}$. □

Corollary 4.4. *Assume that $u_\infty \in L^2(\Omega)$ is in the range of $\mathscr{B}_0$. Then for every $\varepsilon > 0$ there exists a $g_\varepsilon \in L^2(\Omega)$ such that $\mathscr{H} g_\varepsilon := (v_{g_\varepsilon}|_{\partial D}, \partial v_{g_\varepsilon}/\partial v|_{\partial D})$* satisfies

$$\|\mathscr{B}_0(\mathscr{H} g_\varepsilon) - u_\infty\|_{L^2(\Omega)} \leqslant \varepsilon, \tag{30}$$

where v_{g_ε} is the Herglotz wave function with kernel g_ε.

Proof. The proof is a straight forward application of the definition of the space $H(\partial D)$ and the continuity of the trace operator and the operator $\mathscr{B}_0$. □

Turning to our main goal, that is the study of the solvability of the far-field equation (3), we rewrite (3) in terms of the operator $\mathscr{B}_0$ as

$$\mathscr{B}_0(-\mathscr{H} g) = \Phi_\infty(\hat{x}, y), \tag{31}$$

where $\mathscr{H} g$ denotes the traces $(v_g|_{\partial D}, \partial v_g/\partial v|_{\partial D})$ for v_g, a Herglotz wave function with kernel g. As we remarked in Section 2, (31) has a solution g if and only if the solution (w,v) of the interior transmission problem (4) is such that v is a Herglotz wave function with kernel g and Φ_∞ is in the range of $\mathscr{B}_0$. In general this is not true. However, we can construct an approximate solution as follows.

We first assume that $y \in D$. Let $(w, \Phi(\cdot, y))$ be the solution of the transmission problem (TP) satisfying the boundary condition $(v|_{\partial D}, \partial v/\partial v|_{\partial D})$ and having the far field $\Phi_\infty(\cdot, y)$, where (w,v) is the solution of ITP with $(f,h):=(\Phi(\cdot,y)|_{\partial D},\partial\Phi(\cdot,y)/\partial\nu|_{\partial D})$. Hence, $\Phi_\infty(\cdot,y)$ is in the range of $\mathscr{B}_0$. From Corollary 4.4 we can find a $g_\varepsilon(\cdot,y)$ such that

$$\|\mathscr{B}_0(\mathscr{H}g_\varepsilon(\cdot,y))+\Phi_\infty(\cdot,y)\|_{L^2(\Omega)}\leqslant\varepsilon \tag{32}$$

for an arbitrary small ε. We now want to show that if y approaches the boundary from the interior of D then the kernel $g_\varepsilon(\cdot,y)$ and the corresponding Herglotz wave function blow up in the appropriate norms. To this end, assume that k is not a transmission eigenvalue (which implies that there exists a unique solution to ITP) and choose a sequence of points $y_j\in D$ such that

$$y_j=y^*-\frac{R}{j}\nu(y^*),\quad j=1,2,\ldots$$

with sufficiently small R, where $y^*\in\partial D$ and $\nu(y^*)$ is the outwards normal vector at y^*. We denote by w_j,v_j the solution of ITP corresponding to $(f,h):=(\Phi(\cdot,y_j)|_{\partial D},\partial\Phi(\cdot,y_j)/\partial\nu|_{\partial D})$. As $j\to\infty$ the points y_j approach the boundary point y^* and therefore $\|\Phi(\cdot,y_j)\|_{H^{1/2}(\partial D)}\to\infty$. From the trace theorem and by using the boundary conditions we can write

$$\|w_j\|_{H^1(D)}+\|v_j\|_{H^1(D)}\geqslant\|w_j-v_j\|_{H^{1/2}(\partial D)}=\|\Phi(\cdot,y_j)\|_{H^{1/2}(\partial D)}. \tag{33}$$

In particular we show that relation (33) implies that $\lim_{j\to\infty}\|v_j\|_{H^1(D)}=\infty$. Assume on the contrary that

$$\|v_j\|_{H^1(D)}\leqslant\bar{C},\quad j=1,2,\ldots$$

for some positive constant $\bar{C}$. From the trace theorem we have

$$\|v_j\|_{H^{1/2}(\partial D)}\leqslant\bar{C}\quad\text{and}\quad\left\|\frac{\partial v_j}{\partial\nu}\right\|_{H^{1/2}(\partial D)}\leqslant\bar{C},\quad j=1,2,\ldots\ .$$

We recall that for every j the pair $(w_j,\Phi(\cdot,y_j))$ is the solution of TP with $(f,g):=(v_j|_{\partial D},\partial v_j/\partial\nu|_{\partial D})$. The estimate (2) for a solution of (TP) implies

$$\|w_j\|_{H^1(D)}+\|\Phi(\cdot,y_j)\|_{H^1(B\backslash\bar{D})}\leqslant C\left(\|v_j\|_{H^{1/2}(\partial D)}+\left\|\frac{\partial v_j}{\partial\nu}\right\|_{H^{-1/2}(\partial D)}\right)\leqslant 2C\bar{C},$$

which contradicts the fact that $\|\Phi(\cdot,y_j)\|_{H^1(B\backslash\bar{D})}$ does not remain bounded as $y_j\to y^*\in\partial D$. So we have that

$$\lim_{j\to\infty}\|v_j\|_{H^1(D)}=\infty.$$

Now, since for every $j=1,2,\ldots$ the corresponding Herglotz wave functions $v_{g_\varepsilon}(\cdot,y_j)$ satisfying (32) approximates the solution v_j of ITP in the $H^1(D)$ norm, we conclude that

$$\lim_{j\to\infty}\|v_{g_\varepsilon}(\cdot,z_j)\|_{H^1(D)}=\infty$$

and hence

$$\lim_{j\to\infty}\|g_\varepsilon(\cdot,z_j)\|_{L^2(\Omega)}=\infty.$$

Next we again assume that k is not a transmission eigenvalue and consider $y\in\mathbb{R}^3\backslash\bar{D}$. For these points $\Phi_\infty(\cdot,y)$ does not belong to the range of the operator $\mathscr{B}_0$ because $\Phi(\cdot,y)$ is not an H^1

solution to the Helmholtz equation in the exterior of D. But, from Theorem 4.3, using Tikhonov regularization, we can construct a regularized solution of the equation

$$\mathscr{B}_0(f,h) = -\Phi_\infty(\cdot,y). \tag{34}$$

In particular, if $(f_y^\alpha, h_y^\alpha) = (v^\alpha(\cdot,y)|_{\partial D}, \partial v^\alpha(\cdot,y)/\partial v|_{\partial D}) \in H(\partial D)$ with $v^\alpha(\cdot,y) \in \bar{H}$ is a regularized solution of (34) corresponding to the regularization parameter α chosen by a regular regularization strategy (e.g., the Morozov discrepancy principle [2]), we have

$$\|\mathscr{B}_0(f_y^\alpha, h_y^\alpha) + \Phi_\infty(\cdot,y)\|_{L^2(\Omega)} \leqslant \delta \tag{35}$$

for an arbitrary small but fixed $\delta > 0$, and

$$\lim_{\alpha\to 0}(\|f_y^\alpha\|_{H^{1/2}(\partial D)} + \|h_y^\alpha\|_{H^{-1/2}(\partial D)}) = \infty. \tag{36}$$

Note that in this case we have that $\alpha \to 0$ as $\delta \to 0$. Using Corollary 4.4, for every α and $\varepsilon > 0$ we can find a Herglotz wave function $v_{g_{\alpha,\varepsilon}}(\cdot,y)$ with kernel $g_{\alpha,\varepsilon}(\cdot,y) \in L^2(\Omega)$ such that

$$\|\mathscr{B}_0(\mathscr{H}_{g_{\alpha,\varepsilon}}(\cdot,y)) - \mathscr{B}_0(f_y^\alpha, h_y^\beta)\|_{L^2(\Omega)} \leqslant \varepsilon \tag{37}$$

and thus

$$\|\mathscr{B}_0(\mathscr{H}_{g_{\alpha,\varepsilon}}(\cdot,y)) + \Phi_\infty(\cdot,y)\|_{L^2(\Omega)} \leqslant \delta + \varepsilon. \tag{38}$$

Moreover, we know that the Herglotz wave function $v_{g_{\alpha,\varepsilon}}(\cdot,y)$ approximates $v^\alpha(\cdot,y)$ in $H^1(D)$. Hence, the continuity of the trace operator yields

$$\|\mathscr{H}_{g_{\alpha,\varepsilon}}(\cdot,y) - (f_y^\alpha, h_y^\beta)\|_{H(\partial D)} \leqslant C\|v_{g_{\alpha,\varepsilon}}(\cdot,y) - v^\alpha(\cdot,y)\|_{H^1(D)} < \varepsilon. \tag{39}$$

Finally, (36) and (39) imply

$$\lim_{\alpha\to 0}\|\mathscr{H}_{g_{\alpha,\varepsilon}}(\cdot,y)\|_{H(\partial D)} = \infty \quad \text{and} \quad \lim_{\alpha\to 0}\|v_{g_{\alpha,\varepsilon}}(\cdot,y)\|_{H^1(D)} = \infty$$

and hence

$$\lim_{\alpha\to 0}\|g_{\alpha,\varepsilon}\|_{L^2(\Omega)} = \infty.$$

We summarize these results in the following main theorem.

Theorem 4.5. *Let the symmetric matrix-valued function $A = (a_{j,k})_{3\times 3}$, $a_{j,k} \in C^1(\bar{D})$, satisfy $\bar{\xi} \cdot \mathscr{I}m(A)\xi \leqslant 0$ and $\bar{\xi} \cdot \mathscr{R}e(A)\xi \geqslant \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^3$ and $x \in \bar{D}$ with the constant $\gamma > 1$, and $n \in C(\bar{D})$ such that $\mathscr{I}m(n) \geqslant 0$, where D is a connected and bounded set having a C^2-boundary ∂D. Assume that k is not a transmission eigenvalue. Then, if F is the far field operator* (3) *corresponding to the transmission problem* (1), *we have that*

(1) *If $y \in D$ then for every $\varepsilon > 0$ there exists a solution $g^\varepsilon(\cdot,y) \in L^2(\Omega)$ satisfying the inequality*

$$\|Fg_\varepsilon(\cdot,y) - \Phi_\infty(\cdot,y)\|_{L^2(\Omega)} < \varepsilon.$$

Moreover, this solution satisfies

$$\lim_{y\to\partial D}\|g_\varepsilon(\cdot,y)\|_{L^2(\Omega)} = \infty \quad \textit{and} \quad \lim_{y\to\partial D}\|v_{g_\varepsilon}(\cdot,y)\|_{H^1(D)} = \infty,$$

where v_{g_ε} is the Herglotz wave function with kernel g_ε, and

(2) *If* $y \in \mathbb{R}^2 \setminus \bar{D}$ *then for every* $\varepsilon > 0$ *and* $\delta > 0$ *there exists a solution* $g_{\varepsilon,\delta}(\cdot, y) \in L^2(\Omega)$ *of the inequality*

$$\|Fg_{\varepsilon,\delta}(\cdot, y) - \Phi_\infty(\cdot, y)\|_{L^2(\Omega)} < \varepsilon + \delta$$

such that

$$\lim_{\delta \to 0} \|g_{\varepsilon,\delta}(\cdot, y)\|_{L^2(\Omega)} = \infty \quad \textit{and} \quad \lim_{\delta \to 0} \|v_{g_{\varepsilon,\delta}}(\cdot, y)\|_{H^1(D)} = \infty,$$

where $v_{g_{\varepsilon,\delta}}$ *is the Herglotz wave function with kernel* $g_{\varepsilon,\delta}$.

The importance of Theorem 4.5 in solving the inverse scattering problem of determining the support D of the inhomogeneity from the far-field pattern is now clear from our discussion in Section 2. In particular, by using regularization methods to solve the far-field equation $Fg = \Phi_\infty(\cdot, y)$ for y on an appropriate grid containing D, an approximation to $g(\cdot, y)$ can be obtained and hence ∂D can be determined by those points where $\|g(\cdot, y)\|_{L^2(\Omega)}$ becomes unbounded (c.f. [1]).

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