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Transmission Eigenvalues and the Riemann Zeta Function in Scattering Theory for Automorphic Forms on Fuchsian Groups of Type I

Fioralba CAKONI **Sagun CHANILLO**
Department of Mathematics, Rutgers University, Piscataway, NJ 08854*, USA*
E-mail: *fc*292*@math.rutgers.edu* *chanillo@math.rutgers.edu*

To Carlos Kenig in friendship and admiration

Abstract We introduce the concept of transmission eigenvalues in scattering theory for automorphic forms on fundamental domains generated by discrete groups acting on the hyperbolic upper half complex plane. In particular, we consider Fuchsian groups of type I. Transmission eigenvalues are related to those eigen-parameters for which one can send an incident wave that produces no scattering. The notion of transmission eigenvalues, or non-scattering energies, is well studied in the Euclidean geometry, where in some cases these eigenvalues appear as zeros of the scattering matrix. As opposed to scattering poles, in hyperbolic geometry such a connection between zeros of the scattering matrix and non-scattering energies is not studied, and the goal of this paper is to do just this for particular arithmetic groups. For such groups, using existing deep results from analytic number theory, we reveal that the zeros of the scattering matrix, consequently non-scattering energies, are directly expressed in terms of the zeros of the Riemann zeta function. Weyl's asymptotic laws are provided for the eigenvalues in those cases along with estimates on their location in the complex plane.

Keywords Transmission eigenvalues, Fuchsian groups, Riemann zeta function, scattering theory

MR(2010) Subject Classification 11F72, 11M36, 20H10, 34L25, 78A46

1 Introduction

Spectral properties of operators associated with scattering phenomena carry essential information about the scattering media. The theory of scattering resonances is a rich and beautiful part of scattering theory, and although the notion of resonances is intrinsically dynamical, an elegant mathematical formulation comes from considering them as the poles of meromorphic extension of the scattering operator [24, 27] (we refer the reader to the upcoming monograph [9] for an account on the vast literature on the subject). These scattering poles capture physical information by identifying the rate of oscillations with the real part of a pole and the rate of decay with its imaginary part. The transmission eigenvalue problem is also inherent to the scattering for inhomogeneous media, and hence it plays an important role in understanding the corresponding inverse problem. Transmission eigenvalues are related to those wave numbers for which one is able to construct an incident field that does not scatter by a given media. For

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non-absorbing media, real transmission eigenvalues exist [4], and can be determined from the scattering data [3, 22] thus providing estimates on the constitutive material properties of the scattering object. For the past 20 years, transmission eigenvalues corresponding to compactly supported potentials in Euclidean geometry have been the subject of extensive research, and we refer the reader to [3] for a comprehensive survey of the theory of transmission eigenvalues. At the partial differential equations level, transmission eigenvalues form the spectrum of a non-selfadjoint compact operator, which under some appropriate assumptions is proven to have infinitely many real and complex eigenvalues, whereas at the scattering theory level, there is a profound relationship between transmission eigenvalues and the scattering operator.

1.1 Non-scattering Frequencies, Transmission Eigenvalues and the Scattering Operator in $\mathbb{R}^n$

Here we introduce the concept of transmission eigenvalues for the scattering by an inhomogeneity supported in a bounded Euclidean region in $\mathbb{R}^n$. Consider the scattering of an incident wave v of monochromatic radiation with frequency ω, which satisfies the Helmholtz equation

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^n \tag{1.1}$$

by an inhomogeneity supported in the bounded region D having piece wise smooth boundary ∂D with refractive index $1+m$, where m is a complex valued L^∞ function supported in $\overline{D}$ such that $\Re(1+m) > 0$ and $\Im(m) > 0$ (the refractive index of the background is normalized to one). Here $k = \omega/c_0$ is the wave number, c_0 is the constant background sound speed. The total field $u \in H^2_{\text{loc}}(\mathbb{R}^n)$ which is decomposed as

$$u = u^s + v \tag{1.2}$$

satisfies

$$\Delta u + k^2(1+m(x))u = 0 \quad \text{in } \mathbb{R}^n \tag{1.3}$$

with the scattered field u^s satisfying the outgoing Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s \right) = 0 \tag{1.4}$$

which holds uniformly with respect to $\hat{x} := x/|x|$, $r = |x|$ [5, 27]. In the sequel we will refer to the incident field v in the decomposition (1.3) as the v part. The scattered field $u^s \in H^2_{\text{loc}}(\mathbb{R}^n)$ satisfies

$$\Delta u^s + k^2(1+m)u^s = -k^2 m v \quad \text{in } \mathbb{R}^n \tag{1.5}$$

and it is known [5, 27] to have the asymptotic behavior

$$u^s(x) = \gamma(n,k)\frac{\mathrm{e}^{\mathrm{i}kr}}{r^{\frac{n-1}{2}}} u^\infty(\hat{x}) + O\left(\frac{1}{r^{\frac{n+1}{2}}}\right), \quad r \to \infty, \tag{1.6}$$

where the constant $\gamma(n,k) = \left(\frac{2\pi}{\mathrm{i}k}\right)^{\frac{n-1}{2}}$. The function $u^\infty(\hat{x})$ defined on the unit sphere S^{n-1} is called the *far field pattern*. In particular, we consider free waves as incident field $v := v_g$, otherwise known as Herglotz wave functions of the form

$$v_g(x) = \int_{S^{n-1}} g(\hat{y})\mathrm{e}^{\mathrm{i}kx\cdot\hat{y}}\, ds \tag{1.7}$$

where $\hat{y} = y/|y|$ and $g \in L^2(S^{n-1})$ is referred to as the kernel. It is known that such a representation of v_g is equivalent to satisfying in addition to (1.1)

$$\|v_g\|_{B^*} := \sup_{R>0} \frac{1}{\sqrt{R}} \|v_g\|_{L^2(B_R)} < \infty$$

where B_R is the ball of radius R centered at the origin. Hence the Herglotz wave functions can be characterized as the free space solution to the Helmholtz equation whose Fourier transform belongs to the Besov space $B^{-1/2}_{2,\infty}$. Every v_g in the space of Herglotz wave functions (closed space in the B^* topology) can be uniquely decompose as $v_g := u_g - u^s_g$ where the total field u_g is solution of (1.3) and the outgoing scattered field u^s_g which solves (1.5) and (1.4). The *scattering operator* (matrix) as defined by Lax and Phillips in [24] maps $v_g \mapsto u_g$ and for k such that $\Im(k) \geq 0$ is an isomorphism in the above respective spaces equipped with the B^* topology. A heuristic argument for the latter can be given using the Lipmann–Schwinger equation for the solution of (1.5) in terms of the compact k-analytic integral operator $T(k) : L^2(B_R) \to L^2(B_R)$ (below given explicitly only in $\mathbb{R}^3$ to fix the idea)

$$(I - T(k))u = v_g, \quad T(k)u := \int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|} m(y)u(y)\, dy. \tag{1.8}$$

A fix point theorem argument implies that for $|k|$ small enough $I - T(k)$ is invertible, and hence by the Analytic Fredholm Theory [5] we have that $u_g := (I - T(k))^{-1} v_g$ is meromorphic on $k \in \mathbb{C}$. Furthermore, for k such that $\Im(k) \geq 0$ uniqueness of the scattering problem implies that u_g is analytic thus its poles are on the lower-half complex plane.

In the sequel, of particular interest to us will be the "incoming-to-outgoing" mapping $v_g \mapsto u^s_g := u_g - v_g$. We shall characterize this in terms of the far field asymptotic behavior. To this end let u^∞_g denote the far field pattern of the scattered field u^s_g corresponding to the incident field v_g. The compact linear operator $\mathcal{S}^+(k) : L^2(S^{n-1}) \to L^2(S^{n-1})$ defined by

$$\mathcal{S}^+(k) : g \mapsto u^\infty_g \tag{1.9}$$

is called the *relative scattering operator* [27], or otherwise referred to as the *far field operator* [3, 5]. Clearly from (1.7)

$$u^\infty_g(\hat{x}) = (\mathcal{S}^+(k)g)(\hat{x}) = \int_{S^{n-1}} u^\infty(\hat{x}; \hat{y}, k) g(\hat{y})\, ds$$

where $u^\infty(\hat{x}; \hat{y}, k)$ is the far field pattern of the scattered field due to an incident plane wave $v := \mathrm{e}^{\mathrm{i}kx\cdot\hat{y}}$ in the $\hat{y}$ direction. The scattering operator can also have a characterization in terms of the asymptotic behavior of fields as $\mathcal{S}(k) : L^2(S^{n-1}) \to L^2(S^{n-1})$

$$\mathcal{S}(k) := I + \mathcal{S}^+(k).$$

If $\Im(n) = 0$ then $\mathcal{S}^+(k)$ is normal and $\mathcal{S}(k)$ is unitary for real $k > 0$, which is not the case when $\Im(n) > 0$ is a subset of D of non-zero measure. Both are analytic operator valued functions of k in the upper half complex plane. The scattering poles are the poles of the meromorphic extension of $\mathcal{S}(k)$ in the lower half complex plane. In general $\mathcal{S}(\overline{k}, \overline{m}) = [\mathcal{S}^*(k, m)]^{-1}$ holds true for the scattering operator.

Now we are ready to introduce the transmission eigenvalue problem. An application of Rellich's lemma implies that the incident field v_g with $g \in$ Kernel $\mathcal{S}^+(k)$ does not scatterer by the medium with contrast m. Straightforward calculation reveals that the kernel of $\mathcal{S}^+(k)$ contains $g \in L^2(S^{n-1})$ such that, if v_g is the corresponding Herglotz function, $v := v_g|_D$ and u satisfy the *transmission eigenvalue problem*

$$\begin{cases} \Delta u + k^2(1+m(x))u = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ u = v & \text{on } \partial D, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & \text{on } \partial D. \end{cases} \tag{1.10}$$

A value of $k \in \mathbb{C}$ is said to be a *transmission eigenvalue* if (1.10) has non-trivial solutions $u \in L^2(D)$, $v \in L^2(D)$, such that $u - v \in H_0^2(D)$. We call the pair (u, v) the corresponding eigenfunction. In general at a transmission eigenvalue, the part v of the corresponding eigenfunction does not take the form of a Herglotz function, thus the kernel of the relative scattering (or far field) operator is in general empty (hence the operator is injective see [3, 5]). In fact, in [1] it is proven that if D contains a corner, then this v can not be extended as a solution of the Helmholtz equation outside D. Thus the set of *non-scattering wave numbers* $k \in \mathbb{C}$ for which the Kern $\mathcal{S}^+(k) \neq \emptyset$ is a subset (possibly empty) of the transmission eigenvalues.

Obviously (1.5) (or equivalently (1.8) will define an outgoing scattered field in $u^s \in H^2_{\text{loc}}(\mathbb{R})$ and its far field pattern is $u^\infty \in L^2(S^{n-1})$ for incident field

$$v \in H_{\text{inc}}(D) := \{v \in L^2(D) : \Delta v + k^2 v = 0, \text{ in the distributional sense}\}.$$

$H_{\text{inc}}(D)$ is a Hilbert space and densely contains the Herglotz wave functions v_g [3]. Thus $\mathcal{G}(k) : H_{\text{inc}}(D) \to L^2(S^{n-1})$ mapping $v \mapsto u^\infty$ is a compact linear operator, and k is a transmission eigenvalue if and only if the Kern $\mathcal{G}(k)$ is nontrivial (in fact the part v of the corresponding eigenfunction belongs to Kern$(\mathcal{G})$). Evidently the following relation holds

$$\mathcal{S}^+(k)g = \mathcal{G}(k)\mathcal{H}g, \quad \text{where } \mathcal{H} : g \mapsto v_g|_D, \ \overline{\mathcal{H}(L^2(S^{n-1}))} = H_{\text{inc}}(D).$$

Hence at a transmission eigenvalue one can construct a v_g that produces arbitrarily small scattered field u_g^s.

In the Euclidean geometry $\mathbb{R}^n$, however, there is a special configuration for which the transmission eigenvalues and non-scattering frequencies coincide, and this is exactly the counterpart of what the case is in our scattering problem in hyperbolic geometry. When D is a ball of radius a centered at the origin and $m := m(r)$, $r = |x|$, is radial function, the part v of a transmission eigenfunction is indeed a Herglotz function and hence transmission eigenvalues coincide with the values of $k \in \mathbb{C}$ for which Kern $\mathcal{S}^+(k) \neq \emptyset$. To see explicitly what the transmission eigenvalues are in this case, we consider $D := B_1 \in \mathbb{R}^3$, and use as incident field the Herglotz wave function

$$v = j_\ell(k|x|)Y_\ell(\hat{x}),$$

where j_ℓ is the spherical Bessel function (i.e., solution of the Bessel ODE regular at $r = 0$) and Y_ℓ is a spherical harmonic of order $\ell \in \mathbb{N}$. Straightforward calculations by separation of variables [5] lead to the following expression for the scattered field

$$u^s(x) := \frac{C(k; m, \ell)}{W(k; m, \ell)} h_\ell^{(1)}(k|x|)Y_n\ell(\hat{x}), \quad \text{with } u^\infty(x) := \frac{C(k; m, \ell)}{W(k; m, \ell)} \frac{1}{k} Y_\ell(\hat{x}),$$

where $h_\ell^{(1)}(r)$ is the Hankel function of the first kind of order ℓ and

$$C(k;m,\ell) = \mathrm{Det}\begin{pmatrix} y_\ell(1) & -j_\ell(k) \\ y'_\ell(1) & -kj'_\ell(k) \end{pmatrix}, \quad W(k;m,\ell) = \mathrm{Det}\begin{pmatrix} y_\ell(1) & -h_\ell^{(1)}(k) \\ y'_\ell(1) & -kh_\ell^{(1)'}(k) \end{pmatrix} \tag{1.11}$$

with y_ℓ (which depends on k and m) the solution to

$$y'' + \frac{2}{r} + \left(k^2(1+m(r)) - \frac{\ell(\ell+1)}{r^2}\right)y = 0$$

which as $r \to 0$ behaves like $j_\ell(kr)$, i.e.,

$$\lim_{r\to 0} r^{-\ell} y_\ell(r) = \frac{\sqrt{\pi}k^\ell}{2^{\ell+1}\Gamma(\ell+3/2)}.$$

Thus transmission eigenvalues are those values of $k \in \mathbb{C}$ such that $C(k;m,\ell) = 0$. Note that $C(k;m,\ell)$ are entire functions of k. Also we remark that scattering poles are $k \in \mathbb{C}$ for which $W(k;m,\ell) = 0$. If k is a zero of $C(k;m,\ell)$ (i.e., a transmission eigenvalue) then the part v of the corresponding eigenfunction is $v = j_\ell(k|x|)Y_\ell(\hat{x})$. Furthermore note that all transmission eigenvalues for this spherically stratified media are obtained this away by moving $\ell \in \mathbb{N}$. Note that $C(k;m,\ell)$ are entire functions of k and, except for exceptional cases, have infinitely many real zeros and infinitely many complex zeros [6, 7]. As for the location of transmission eigenvalues, for smooth $m(r)$ such that $m(1) \neq 0$ they all lie in strip around the real axis [34] whereas if $m(1) = 0$ they lie in a parabolic region and do not approach the real axis [8, 33]. If only spherical symmetric incident fields are considered, i.e., $v = j_0(k|x|)$, then the meromorphic function

$$\mathcal{S}^+(s) = \frac{C(k;m,0)}{kW(k;m,0)} \tag{1.12}$$

is the counterpart of the scattering matrix for the non-Euclidean geometry we shall consider in this paper. Thus in this spherically stratified case there is direct connection between transmission eigenvalues, non-scattering frequencies and the zeros of the scattering operator.

There is a vast of literature on the study of the (non-selfadjoint) transmission eigenvalue problem (1.10) at the PDE level. Under the assumption that $m(x)$ does not change sign in a neighborhood of the boundary and for smooth enough m the completeness of the eigenfunctions is proven in [29] and Weyl laws are proven in [23, 34]. The existence of real transmission eigenvalues with monotonicity properties are proven in [4] for L^∞ non-changing sign real valued m. We remark that when the scattering object D is non-penetrable, then the role of transmission eigenvalues is played by the corresponding interior eigenvalues. For instance if the Dirichlet condition is assumed on the boundary ∂D of the scatterer D then non-scattering frequencies are related to the Dirichlet eigenvalues for $-\Delta$ in D and exactly the same discussion as above can be carried out. This interplay between the outgoing scattering field and the interior eigenvalue problem is referred to as inside-outside duality.

In [10] a connection was made between the harmonic analysis of automorphic functions with respect to the group $SL_2(\mathbb{R})$ of real matrices of determinant one and the scattering theory for non-Euclidean wave equations and Selberg's pioneering work on spectral theory for compact and finite-area Riemann surfaces. This work was redone and further developed for non-compact hyperbolic domains of finite area in [25] (see also [26]), leading to more recent development of the scattering theory for hyperbolic surfaces of infinite area (for the latter we refer the reader to the comprehensive monograph by Borthwick [2]). On the other hand, limited to automorphic forms with respect to Fuchsian groups of the first kind that have only cusps at infinity, the spectral theory and the study of the corresponding scattering matrix has a profound connection to fundamental results from analytic number theory (see the book by Iwaniec [19]). The concern has always been with the poles of the scattering matrix in this scattering theory of automorphic forms. No attempt as far as we are aware has been made to understand the counterpart of transmission eigenvalues in this framework. This paper tries to remedy this in the particular case of Fuchsian groups of the first kind.

In the next section we introduce Fuchsian groups and the scattering matrix associated with the Laplace–Beltrami operator for the Riemannian metric of the upper half plane on the fundamental domains generated by discrete groups. An important part of our analysis relies on the availability of explicit expressions for the scattering matrix, hence here we provide two examples of such calculations. Section 3 introduces transmission eigenvalues in this configuration connecting them to the zeros of the scattering matrix, which in turn, for particular discrete groups, relate to the zeros of the Riemann zeta function. Such a connection allows us to derive Weyl's asymptotic laws and describe the location of transmission eigenvalues. In the last section we make a connection between our analysis of transmission eigenvalues with the non-scattering energies in the Euclidean case discussed above. We furthermore discuss some related open questions in analytic number theory and arithmetic groups as well as possible research prospects in this direction.

2 Fuchsian Groups of the First Kind and the Scattering Matrix

Let us start by recalling some basics concepts on the hyperbolic plane and the groups acting discontinuously on the hyperbolic plane. In general, on a Riemannian manifold with metric given by the tensor (g_{ij}), the Laplace operator, often referred to as the Laplace–Beltrami operator, takes the form

$$\Delta_g u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial u}{\partial x_j} \right),$$

where $g := \det(g_{ij})$ and g^{ij} are the components of the inverse of the metric tensor such that $g_{ij} g^{ij} = \delta_{ij}$, with δ_{ij} being the Kronecker delta. A model of the hyperbolic plane is $\mathbb{H} := \{(x, y) : y > 0\}$, $z = x + \mathrm{i}y$, the upper half complex plane or Poincaré plane. It is a Riemannian manifold with the complete metric

$$ds^2 = y^{-2}(dx^2 + dy^2), \quad g_{ij} = \frac{\delta_{ij}}{y^2}, \quad g^{ij} = y^2 \delta_{ij}.$$

Hence the Laplace–Beltrami operator in this case is

$$\Delta_{\mathbb{H}} u := y^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Our governing equation of wave propagation on the hyperbolic plane that replaces the Helmholtz equation (1.1) on the Euclidean plane is

$$\Delta_{\mathbb{H}} u + s(1 - s)u = 0 \quad \text{or} \quad y^2 \Delta u + s(1 - s)u = 0. \tag{2.1}$$

Obviously the functions y^s and y^{1-s}, $s \in \mathbb{C}$ satisfy (2.1) (for our purpose we refer to them here as wave packets). They are invariant solutions under $z \mapsto z+1$ and play the same role as the plane waves $\mathrm{e}^{\pm ikx\cdot d}$ with respect to the Helmholtz equation in the Euclidean case. We may view y^s as an outgoing solution using an appropriate Sommerfeld condition. The $\mathbb{H}$ gradient is given by

$$\nabla_{\mathbb{H}} u = \left(g^{ij} \frac{\partial u}{\partial x_j} \right)_i = y^2 \nabla u.$$

Thus the unit upward at (x, y) is $y(0, 1) = y\vec{j} := \nu$, since $\langle \nu, \nu \rangle_{\mathbb{H}} = \frac{y^2}{y^2} \vec{j} \cdot \vec{j} = 1$ and hence

$$\left. \frac{\partial u}{\partial \nu} \right|_{\mathbb{H}} = \langle \nu, \nabla_{\mathbb{H}} u \rangle_{\mathbb{H}} = \frac{1}{y^2} \cdot y^2 \cdot y \frac{\partial u}{\partial y} = y \frac{\partial u}{\partial y}.$$

Similarly to the Sommerfeld radiation condition in the Euclidean case (1.4) we have that $u := y^s$ satisfies

$$\left. \frac{\partial u}{\partial \nu} \right|_{\mathbb{H}} - su = y \frac{\partial u}{\partial y} - su = 0.$$

So if $\Im(s) > 0$, $\Im(1-s) < 0$ thus y^s is outgoing (away from the cusp) and y^{1-s} is incoming (toward the cusp). Note that the terminology used in the Euclidean geometry is in reference to the scattering medium, accordingly y^s will be called incoming (traveling toward the scattering medium like $\mathrm{e}^{ikx\cdot d}$ in $\mathbb{R}^n$) whereas y^{1-s} outgoing (traveling away from the scattering medium like $\mathrm{e}^{-ikx\cdot d}$ in $\mathbb{R}^n$). Similar to the Euclidean case, it is possible to develop a scattering theory on hyperbolic surfaces and define the relative scattering operator in terms of outgoing y^s and incoming y^{1-s} asymptotic behavior of solutions to (2.1) at the cusps of fundamental domains with respect to an isometry group. For a comprehensive discussion on the scattering theory for hyperbolic surfaces we refer the reader to [2].

Given the large isometry group of $\mathbb{H}$, a natural way to obtain a hyperbolic surface is by a quotient $\Gamma \setminus \mathbb{H}$, for some discrete subgroup Γ of the group of real 2×2 matrices of determinant one $\mathrm{SL}_2(\mathbb{R})$. In particular here we are concerned with Fuchsian groups. To precisely define Fuchsian groups, we recall Möbius transformations which are fractional linear functions

$$gz = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R},\ ad - bc = 1, \tag{2.2}$$

and the group $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) \setminus (\pm 1)$ (± 1 stands for $\pm$ identity 2×2 matrix since g determines the matrix up to sign) of all Möbius transformations acting on the whole compactified complex plane $\hat{\mathbb{C}}$. A Fuchsian group Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ that acts discontinuously on $\mathbb{H}$ (i.e., the orbit $\Gamma z := \{\gamma z : \gamma \in \Gamma\}$ of any $z \in \mathbb{H}$ has no limit point in $\mathbb{H}$). In fact Poincaré has proved that for a subgroup of $\mathrm{SL}_2(\mathbb{R})$ to be discrete is equivalent to acting discontinuously on $\mathbb{H}$, if considered as a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. The stability group $\Gamma_z = \{\gamma \in \Gamma : \gamma z = z\}$ of a Fuchsian group is cyclic. For the purpose of this study, we assume that the Fuchsian group Γ is of the first kind, i.e., every point $(x_0, 0)$ on the x-axis is a limit of the orbit Γz for some $z \in \mathbb{H}$. A Fuchsian group can be visualized by its fundamental domain $F := \Gamma/\mathbb{H}$, that is F is a domain in $\mathbb{H}$, whose distinct points of are not equivalent (different modulo Γ) and such that any orbit of Γ contains points in the closure of F in the $\hat{\mathbb{C}}$ topology. A Fuchsian group is of the first kind if and only if it has a fundamental domain of finite volume [19, Sec. 2.2]. Thus, we further restrict ourselves to first kind Fuchsian groups that are non co-compact, which means that the closure in $\hat{\mathbb{C}}$ of the fundamental domain is not compact. The fundamental domain $\Gamma/\mathbb{H} = F$ of finite volume has only cusps at infinity if it is non-compact. The cusps are formed by the two sides of F meeting at a vertex in $\hat{\mathbb{R}}$ (extended reals) orthogonally to $\hat{\mathbb{R}}$ (see also [2, Sec. 2.4]). To summarize our assumptions, the groups Γ we are dealing with here are *Fuchsian of the first kind and non co-compact.* There are many ways to describe the fundamental domain $F := \Gamma/\mathbb{H}$. Most useful to us is the Ford fundamental domain, where the fundamental domain is given by boundaries that are geodesics on the Riemann surface $\Gamma/\mathbb{H}$. The polygon bounded by geodesics in this case is referred to as the standard polygon. More specifically, set

$$F_\infty = \{z \in \mathbb{H} : \beta < x < \beta + 1\}, \quad z = x + \mathrm{i}y,$$

where we have arranged that ∞ is a cusp. Then define

$$F = \{z \in F_\infty : \Im(z) > \Im(\gamma z), \text{ for all } \gamma \in \Gamma, \gamma \in \Gamma_\infty\}$$

where Γ_∞ is generated by $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, $n \in \mathbb{N}$ the stabilizer group of the cusp at ∞. F is called the Ford fundamental domain. Recall that the stabilizer group of the cusp $\mathbf{a}$ is

$$\Gamma_a = \{\gamma \in \Gamma, \ \gamma\mathbf{a} = \mathbf{a}\}.$$

We provide as an example the classical case of the modular group $\mathrm{SL}_2(\mathbb{Z})$, which will play a special role in the sequel. The modular group $\mathrm{SL}_2(\mathbb{Z})$ is the group of 2×2 matrices with integer entries of determinant one with its fundamental domain $F = \{z = x + \mathrm{i}y : |x| < 1/2, |z| > 1\}$ (see Fig. 1 left panel). For this fundamental domain i is an elliptic vertex of order 2, $\xi = (1 + \mathrm{i}\sqrt{3})/2$ is an elliptic vertex of order 3, and ∞ is the only cusp up to equivalence (see [2, Chap. 2] or [19, Sec. 1.5]). Now consider for example $\gamma_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, which acts according to $z \to -\frac{1}{z}$. Applying γ_1 to F we get an equivalent fundamental domain to F by periodicity, $F_1 = \gamma_1 F$ (see Figure 1 right panel). Thus the images of F under the modular group Γ tesselate $\mathbb{H}$. Hence when we consider functions f on fundamental domains F we need them to be automorphic, i.e.,

$$f(\gamma z) = f(z), \quad \text{for all } \gamma \in \Gamma \text{ and } z \in F. \tag{2.3}$$

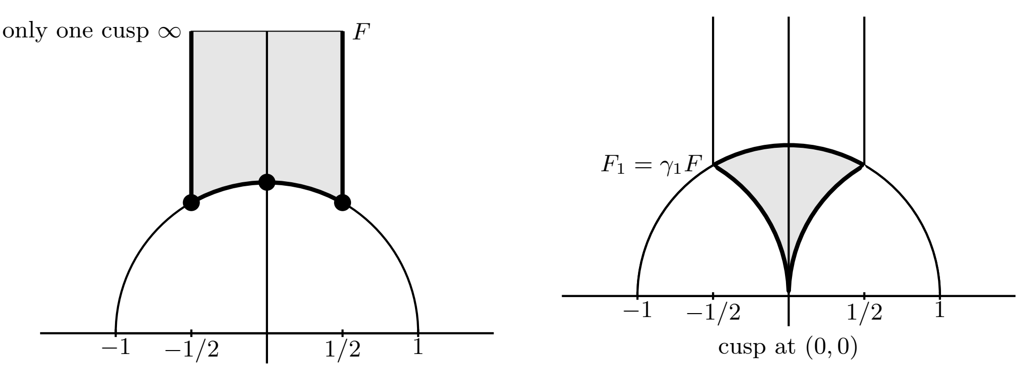

Figure 1 Left panel: Shaded region depicts Ford fundamental domain F for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Black dots indicate 3 elliptic points. Right panel: Shaded region depicts an equivalent fundamental domain F_1 to F after applying γ_1.

Definition 2.1 *Given a cusp* $\mathbf{a}$*, the scaling matrix* $\sigma_{\mathbf{a}} \in \mathrm{SL}_2(\mathbb{R})$ *is defined such that*

$$\sigma_{\mathbf{a}}\infty = \mathbf{a}.$$

Note that $\sigma_{\mathbf{a}}$ need not to belong to Γ and $\sigma_{\mathbf{a}}$ need not be unique, as $\sigma_{\mathbf{a}}\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) = \mathbf{a}$.

To do analysis of (2.1) in F we need to periodify y^s on F. For the subgroup Γ_∞ and functions p invariant under Γ_∞, like y^s and y^{1-s} are (under $z \to z+n$) one can form the Poincaré series (which here plays the same role of Fourier series in the Euclidean case)

$$E_{\mathbf{a}}(z \setminus p) = \sum_{\gamma \in \Gamma_{\mathbf{a}} \setminus \Gamma} p\left(\sigma_{\mathbf{a}}^{-1}\gamma z\right),$$

where we note that $p(z)$ is periodic in x with period 1 thanks to the invariance property. As shown in [19, Sec. 3.4], the periodification of y^s is performed on any Γ, thus taking $p(z) = y^s$ in the Poincaré series leads to the Eisenstein series. Hence, for the given fundamental domain $\Gamma \setminus \mathbb{H}$ and $\Re(s) > 1$, at each cusp $\mathbf{a}$ there is a solution $E_{\mathbf{a}}(\sigma_{\mathbf{b}} z, s)$ of (2.1), the Eisenstein series, such that as $y \to \infty$ within the cusp $\mathbf{a}$

$$E_{\mathbf{a}}(\sigma_{\mathbf{b}} z, s) = \delta_{\mathbf{ab}} y^s + \varphi_{\mathbf{ab}}(s) y^{1-s} + O((1 + y^{-\Re(s)})\mathrm{e}^{-2\pi y}) \tag{2.4}$$

uniformly in $z \in \mathbb{H}$, where $\sigma_{\mathbf{b}}\infty = \mathbf{b}$ and $\delta_{\mathbf{ab}}$ is the Kronecker delta, vanishing when $\mathbf{a}, \mathbf{b}$ are inequivalent cusps. In a similar manner as for the relative scattering operator $\mathcal{S}^+$ in the Euclidean geometry, we define here the *scattering matrix* by

$$\Phi(s) := (\varphi_{\mathbf{ab}}(s)), \quad \text{where } \mathbf{a} \text{ and } \mathbf{b} \text{ run over all cusps} \tag{2.5}$$

(this is a rare situation when the "incoming-to-outgoing" relative scattering operator is a matrix). The scattering matrix has a meromorphic continuation to $s \in \mathbb{C}$. As we will see later, $\Phi(s)$ in our framework plays the same role as $C(k; m, \ell)/W(k; m, \ell)$ given by (1.11) in the spherically stratified Euclidean geometry, and its zeros will correspond to non-scattering waves sent from and observed at the same cusp.

2.1 The Scattering Matrix

In this section we provide explicit calculations of the scattering matrix $\Phi(s)$ for general discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$, which will be used in the next section to characterize transmission eigenvalues in this framework. To this end, we must make (2.4) more specific. An application of the Poisson summation formula and computation of Fourier transformations ensuing from Poisson summation leads one to the formula below for the solutions of the wave equation in $\Gamma \setminus \mathbb{H}$ (see [19, Theorem 3.4] for the proof).

Theorem 2.2 *Let $E_{\mathbf{a}}(\sigma_{\mathbf{b}} z, s)$ for the given cusp $\mathbf{a}$ solve (2.1) in $\Gamma \setminus \mathbb{H}$ as above. Then*

$$E_{\mathbf{a}}(\sigma_{\mathbf{b}} z, s) = \delta_{\mathbf{ab}} y^s + \varphi_{\mathbf{ab}}(s) y^{1-s} + \sum_{n \neq 0} \varphi_{\mathbf{ab}}(n, s) W_s(nz),$$

where $\sigma_{\mathbf{b}}\infty = \mathbf{b}$ and $\delta_{\mathbf{ab}}$ is the Kronecker delta vanishing when $\mathbf{a}, \mathbf{b}$ are inequivalent cusps. Furthermore

$$\varphi_{\mathbf{ab}}(s) = \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{c > 0} c^{-2s} S_{\mathbf{ab}}(0, 0, c). \tag{2.6}$$

Here $\Gamma(s) := \int_0^\infty \mathrm{e}^{-x} x^{s-1}$ is the Euler gamma function, $S_{\mathbf{ab}}(0,0,c)$ are the completely degenerate Kloosterman sums, and $W_s(z)$ are the Whittaker functions that vanish as z tends to the cusp $\mathbf{a}$ (given by [19, (1.26)]).

From (2.6) we can see easily that the scattering matrix is symmetric, i.e.

$$\varphi_{\mathbf{ab}}(s) = \varphi_{\mathbf{ba}}(s), \quad \text{for } \Re(s) > 1$$

and by analytic continuation for all $s \in \mathbb{C}$.

When $\mathbf{a} = \mathbf{b}$, the leading term of the wave packet or Eisenstein series is given by

$$y^s + \varphi_{\mathbf{aa}}(s) y^{1-s}.$$

Thus the absence of back scattering at cusps $\mathbf{a}$ is completely determined by understanding the zeros of $\varphi_{\mathbf{aa}}(s)$, i.e., the zeros of the diagonal terms of the scattering matrix $\Phi(s) = (\varphi_{\mathbf{ab}}(s))_{p\times p}$ where p is the number of inequivalent cusps. Note that by symmetry we have that $\varphi_{\mathbf{ba}}(s) = \varphi_{\mathbf{ba}}(s)$. By using the Maass–Selberg relations one obtains that the scattering matrix $(\varphi_{\mathbf{ab}}(s))$ can be continued to the entire complex plane and the following theorem can be proven (see [19, Theorem 6.9] where the exposition has additional technical innovations).

Theorem 2.3 *The functions $\varphi_{\mathbf{ab}}(s)$ are holomorphic in $\Re(s) \geq 1/2$ except for a finite number of simple poles in the segment $(1/2, 1]$. If $s = s_j$ is a pole of $\varphi_{\mathbf{ab}}(s)$, then it is also a pole of $\varphi_{\mathbf{aa}}(s)$. The residue of $\varphi_{\mathbf{aa}}(s)$ at $s = s_j > 1/2$ is real and positive.*

In addition, one has the functional relation

$$\Phi(s)\Phi(1-s) = I_{p\times p}$$

where $\Phi(s) := (\varphi_{\mathbf{ab}}(s))$ is the scattering matrix and p the number of inequivalent cusps. For s with $\Re(s) = 1/2$ the scattering matrix is unitary, i.e., $\Phi(s)\overline{\Phi(s)^\top} = I$. For s real the scattering matrix is Hermitian.

To calculate explicitly the scattering matrix $\Phi(s)$ we use (2.6), hence we need the notion of Kloosterman sums.

We start with the simple case of the *modular group* $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. In this classical case Kloosterman in [21] introduced the sums, that now carry his name, in connection with an improvement of the Hardy–Littlewood–Ramanujan circle method (see [17, Chap. 20]).

Definition 2.4 *The Kloosterman sums for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, denoted by $S(m,n,N)$, $N \in \mathbb{N}$, $N > 0$, $m, n \in \mathbb{Z}$ are defined by*

$$S(m,n,N) = \sum_{hh^* \equiv 1 (\mathrm{mod} N)} \mathrm{e}^{2\pi \mathrm{i}\left(\frac{mh}{N} + \frac{nh^*}{N}\right)}. \tag{2.7}$$

In the particular case when $n = 0$, we get the Ramanujan sum

$$S(m,0,N) = c_N(m) = \sum_{h:(h,N)=1} \mathrm{e}^{2\pi \mathrm{i} \frac{mh}{N}}.$$

We refer the reader to [13, p. 308] for various properties of Ramanujan sums. The completely degenerate Kloosterman sums $S(0,0,N)$ is what plays a role in the formula for the scattering matrix. As explained above, since the fundamental domain $F := \mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ has only one cusp at infinity (see Figure 1(a)), the scattering matrix is a scalar. In particular, one has via equation [19, (3.21), p. 60] the formula for the scattering matrix for F

$$\varphi_{\infty\infty}(s) = \pi^{1/2} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{N=1}^{\infty} \frac{S(0,0,N)}{N^{2s}}. \tag{2.8}$$

Proposition 2.5 *For $F := \mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$*

$$\varphi_{\infty\infty} = \pi^{1/2} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)},$$

where $\zeta(s)$ is the Riemann zeta function.

Proof It suffices to show that

$$\sum_{N=1}^{\infty} \frac{S(0,0,N)}{N^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)}. \tag{2.9}$$

To this end from Definition 2.4 we have that

$$S(0,0,N) = \boldsymbol{\varphi}(N) = \#\left\{m \,|m < N,\ (m,N) = 1\right\},$$

i.e., $\boldsymbol{\varphi}(N)$ is Euler's totient function giving the number of positive integers less than N that are co-prime to N. Thus the left hand side of (2.9) becomes

$$S(0,0,N) = \sum_{N=1}^{\infty} \frac{\boldsymbol{\varphi}(N)}{N^{2s}}. \tag{2.10}$$

But by a standard fact stated in [13, Theorem 271]

$$\boldsymbol{\varphi}(N) = \sum_{d/N} \boldsymbol{\mu}\left(\frac{N}{d}\right) d, \tag{2.11}$$

where $\boldsymbol{\mu}(\cdot)$ is the number-theoretic Möbius function. We also have that for $\Re(s) > 1/2$ [18, (1.7), p. 4]

$$\frac{1}{\zeta(2s)} = \sum_{m=1}^{\infty} \frac{\boldsymbol{\mu}(m)}{m^{2s}}.$$

Thus for $\Re(s) > 1/2$ we can write

$$\frac{\zeta(2s-1)}{\zeta(2s)} = \sum_{m\geq 1} \sum_{k \geq 1} \frac{\boldsymbol{\mu}(m)}{m^{2s}} \frac{1}{k^{2s-1}}. \tag{2.12}$$

Setting $N = mk$ (2.12) becomes

$$\frac{\zeta(2s-1)}{\zeta(2s)} = \sum_{N\geq 1} \frac{1}{N^{2s}} \left(\sum_{d/N} d\boldsymbol{\mu}\left(\frac{N}{d}\right)\right)$$

and so by (2.11)

$$\frac{\zeta(2s-1)}{\zeta(2s)} = \sum_{N=1}^{\infty} \frac{\boldsymbol{\varphi}(N)}{N^{2s}}. \tag{2.13}$$

Hence comparing (2.10) and (2.13) establishes (2.9) for $\Re(s) > 1/2$. For $s \in \mathbb{C}$ the result follows by analytic continuation and the functional relation for the Riemann zeta function. This ends the proof. □

Before we proceed to derive formulae for other tractable classes of discrete subgroups Γ, we need to describe how to define Kloosterman sums associated with general discrete groups of $\mathrm{SL}_2(\mathbb{R})$. This construction is carried out in [19, Chap. 2]. All we do here is just list the salient points of that construction, referring the reader to the excellent exposition in [19]. As described at the beginning of this section, our discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ consists of unimodular 2×2

matrices. But for general Γ the entry "c" is a real number. Thus, the various "Dirichlet series" that arise (if one can even call them that) will no longer have the property that multiplying any two of them will lead to a series whose terms come from the entries of the matrices of the discrete subgroups Γ. This makes the analysis of determining scattering matrices associated to general discrete groups difficult. Given any discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ one has a double coset decomposition stated in the theorem below.

Theorem 2.6 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be cusps for* $\Gamma/\mathbb{H}$. *We then have a disjoint union*

$$\sigma_{\mathbf{a}}^{-1}\Gamma\sigma_{\mathbf{b}} = \delta_{\mathbf{ab}}\Omega_\infty \cup \bigcup_{c>0}\ \bigcup_{d\,(\mathrm{mod}\,c)} \Omega_{d/c}$$

where $\Omega_\infty = B\omega_\infty B$,

$$B = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad n \in \mathbb{Z}, \quad \omega_\infty = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

and

$$\Omega_{d/c} = B \begin{pmatrix} * & * \\ c & d \end{pmatrix} B,$$

$\delta_{\mathbf{ab}}$ *is the Kronecker delta, i.e.,* $\delta_{\mathbf{ab}} = 0$ *for inequivalent cusps and* $\delta_{\mathbf{aa}} = 1$.

For the proof we refer the reader to [19, Theorem 2.7]. The key point here is that coset classes are fixed by c, d, the elements of the bottom row of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and for fixed c there is a further break up determined by elements d associated to the fixed c. In the modular case, the Kloosterman sums were defined by taking $c = N$ and letting d run over precisely the coset representatives associated with $c = N$. Thus we may define for general Γ [19, (2.23)] in analogy,

$$S(m,n,c) = \sum_{\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in B\backslash\sigma_{\mathbf{a}}^{-1}\Gamma\sigma_{\mathbf{b}}/B} \mathrm{e}^{2\pi\mathrm{i}}\left(\frac{md}{c} + \frac{na}{c}\right), \tag{2.14}$$

where the summation is taken over the representatives of the double coset for fixed c, that is a, d are changing for fixed c.

In general the following bounds for completely degenerate Kloosterman sums are proven in [19, Prop. 2.8]

$$S(0,0,c) \le c_{\mathbf{ab}}c^2, \tag{2.15}$$

and the superior bound on average $\sum_{c\le X} \frac{S(0,0,c)}{c} \le c_{\mathbf{ab}}^{-1}X$, where $c_{\mathbf{ab}} = \min\{c_{\mathbf{a}}, c_{\mathbf{a}}\}$ and

$$c_{\mathbf{a}} = \min\left\{ c > 0 \,\middle|\, \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathbf{a}}^{-1}\Gamma\sigma_{\mathbf{b}} \right\}.$$

However these bounds are not enough to get refined asymptotics for the zeros of $\varphi_{\mathbf{ab}}$.

We can tackle the aforementioned complication with Kloosterman sums in the particular case of *congruence subgroups.*

Definition 2.7 *The principal congruence subgroup* $\Gamma(N)$ *of level* N *is defined by*

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{array}{l} a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \\ a \equiv d \equiv 1\,(\mathrm{mod}\ N), b \equiv c \equiv 0\,(\mathrm{mod}\ N) \end{array} \right\}.$$

The Hecke congruence subgroups $\Gamma_0(N)$ *and* $\Gamma_1(N)$ *of level* N *are defined as*

$$\Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}),\ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

and

$$\Gamma_1(N) = \left\{ \gamma \in SL_2(\mathbb{Z}),\ \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\},$$

respectively.

One has the following inclusions as subgroups

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z}),$$

and each is a finite index subgroup of the succeeding one. The index of $\Gamma(N)$ is given by the formula

$$[SL_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p/N} \left(1 - \frac{1}{p^2}\right)$$

and $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$. All these facts are proved in [32]. Note that $\gamma \in \Gamma(N)$ for $N > 3$ satisfies $|\mathrm{Trace}(\gamma)| \geq 2$, so $\Gamma(N) \setminus \mathbb{H}$ has no elliptic points and hence, viewing $\Gamma(N) \setminus \mathbb{H}$ as a manifold in $\mathbb{R}^3$, it is a smooth manifold. We caution the reader that $\Gamma(N) \setminus \mathbb{H}$ does not embed isometrically in $\mathbb{R}^3$ as the Gauss curvature of $\Gamma(N) \setminus \mathbb{H}$ is -1 and thus the Efimov–Hilbert theorem precludes such an isometric embedding into $\mathbb{R}^3$ as $\Gamma(N) \setminus \mathbb{H}$ is also complete. We already introduced the concept of fundamental domains and their description for Fuchsian groups at the beginning of this section. A picture of the Ford fundamental domain F for $\Gamma(2)$ is displayed in Figure 2.

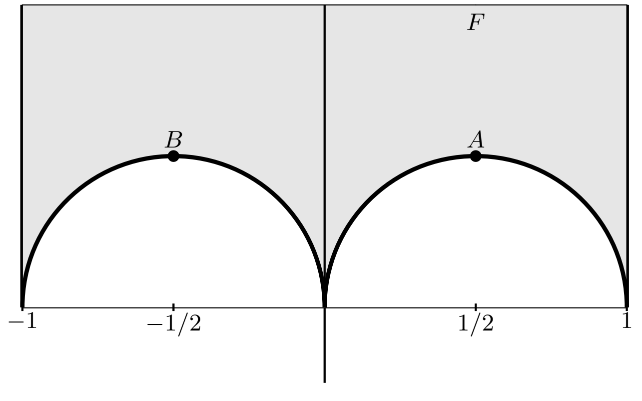

Figure 2 Ford fundamental domain F for $\Gamma(2)$. The circles $|1 \pm 2z| = 1$ seen here are isometric circles, and $\Gamma(2) \setminus \mathbb{H}$ is seen to have 4 cusps at $z = 0$, $z = \pm 1$ and $z = \infty$, but $z = \pm 1$ are equivalent. A, B are elliptic points of order 2.

Since there are three non-equivalent cusps in the fundamental domain for $\Gamma(2)$, the corresponding scattering matrix $(\varphi_{\mathbf{ab}}(s))$ is 3×3. For more on fundamental domains the reader can consult [20] or [19, Chapter 2]. We have the following formula for the scattering matrix for congruence subgroups $\Gamma(N)$ due to [14, 16] and [19, Chapter 2].

Theorem 2.8 *For the cusp* $\mathbf{a} = \infty$ *and* $\Gamma = \Gamma(N)$ *we have that*

$$\varphi_{\infty\infty}(s) = \pi^{1/2} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} \frac{\boldsymbol{\varphi}(N)}{N^{4s}} \prod_{p/N} \left(1 - \frac{1}{p^{2s}}\right)^{-1}, \tag{2.16}$$

where $\boldsymbol{\varphi}(N)$ *is Euler's totient function. Furthermore the scattering matrix for* $\Gamma_0(p)$*,* p *prime is*

$$\Phi(s) = \begin{pmatrix} \varphi_{\infty\infty} & \varphi_{\infty 0} \\ \varphi_{0\infty} & \varphi_{00} \end{pmatrix} = \psi(s) N_p(s),$$

where

$$\psi(s) = \pi^{1/2} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}$$

and

$$N_p(s) = \left(p^{2s} - 1\right)^{-1} \begin{pmatrix} p-1 & p^s - p^{1-s} \\ p^s - p^{1-s} & p-1 \end{pmatrix}.$$

For N *square free, the scattering matrix for* $\Gamma_0(N)$ *becomes*

$$\varphi_{\mathbf{ab}} = \psi(s) \bigotimes_{p/N} N_p(s).$$

We shall be content to establish (2.16) when N is a prime power and thus establish yet another link between transmission eigenvalues (to be defined in the next section) and the zeros of the Riemann zeta function. Recall that this link for the modular group $\mathrm{SL}_2(\mathbb{Z})$ has been established earlier. Thus, we now derive (2.16) when $N = p^a$, p prime and $a > 0$.

Proof (Proof of (2.16) in Theorem 2.8 for $N = p^a$, p prime, $a > 0$) We wish to show that, for $\Gamma = \Gamma(p^a)$, $a > 0$, p prime, one has

$$\sum_{c>0} \frac{S(0,0,c)}{c^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)} \frac{\boldsymbol{\varphi}(N)}{N^{4s}} \prod_{p/N} \left(1 - \frac{1}{p^{2s}}\right)^{-1}. \tag{2.17}$$

This will yield the formula for $\varphi_{\infty\infty}$. Note we can take $\sigma_{\mathbf{a}} = I$ since $\mathbf{a} = \infty$. We also have $c \equiv 0 \pmod N$, and since

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N), \quad a, b, c, d \in \mathbb{Z}, \quad c > 0,$$

$ad - bc = 1$, $b \equiv 0 \equiv c \pmod N$, we obtain

$$ad \equiv 1 \pmod{N^2}. \tag{2.18}$$

Next since $\gamma \in \Gamma(N)$ we also have

$$a \equiv d \equiv 1 \pmod N. \tag{2.19}$$

Now since $\mathbf{a} = \mathbf{b} = \infty$ and $\sigma_{\mathbf{a}} = I$, then

$$S(0,0,c) = \#\left\{ d \;\middle|\; \begin{pmatrix} * & * \\ c & d \end{pmatrix} \text{ is a representative of the double coset } B \setminus \Gamma / B \right\}.$$

Next set $c = kN^2$. Then by (2.18) and (2.19) we have that

$$S(0,0,c) = \frac{\varphi(kN^2)}{N}.$$

Inserting this in the left-hand side of (2.17) yields

$$\sum_{c>0} \frac{S(0,0,c)}{c^{2s}} = \frac{1}{N^{4s+1}} \sum_{k=1}^{\infty} \frac{\varphi(kN^2)}{k^{2s}}. \tag{2.20}$$

We rewrite the left-hand side of (2.20) as

$$\frac{1}{N^{4s+1}}\left(\sum_{(k,N)=1}^{\infty} \frac{\varphi(kN^2)}{k^{2s}} + \sum_{(k,N)>1}^{\infty} \frac{\varphi(kN^2)}{k^{2s}} \right). \tag{2.21}$$

For the first sum in (2.21), using the basic fact about Euler's totient function (see [13, Theorem 60])

$$\varphi(mn) = \varphi(m)\varphi(n) \quad \text{ if } (m,n) = 1, \tag{2.22}$$

we obtain that the first sum in (2.21) becomes

$$\frac{1}{N^{4s+1}} \varphi(kN^2) \sum_{(k,N)=1}^{\infty} \frac{\varphi(k)}{k^{2s}}.$$

For the second sum in (2.21), we notice that if $N = p^a$ and $k = p_1^{a_1} \cdots p_j^{a_j} p^b$ by straight forward calculation, writing $kN^2 = p_1^{a_1} \cdots p_j^{a_j} p^{b+2a}$, using property (2.22) of the Euler totient function and the fact that

$$\varphi(p^b) = p^b - p^{b-1} = p^b(1 - 1/p) \tag{2.23}$$

(since $p^b - p^{b-1}$ is the number of numbers m co-prime to p^b such that $1 \leq m < p^b$), one has

$$\varphi(kN^2) = \varphi(k)p^{2a}.$$

Thus (2.21) for $N = p^a$ (which we now assume for the rest of calculations) takes the form

$$\begin{aligned} \sum_{c>0} \frac{S(0,0,c)}{c^{2s}} &= \frac{1}{p^{(4s+1)a}} \left[\varphi(p^{2a}) \sum_{(k,p^a)=1}^{\infty} \frac{\varphi(k)}{k^{2s}} + p^{2a} \sum_{(k,p^a)>1}^{\infty} \frac{\varphi(k)}{k^{2s}} \right] \\ &= \frac{1}{p^{(4s+1)a}} \left[(\varphi(p^{2a}) - p^{2a}) \sum_{(k,p^a)=1}^{\infty} \frac{\varphi(k)}{k^{2s}} + p^{2a} \frac{\zeta(2s-1)}{\zeta(2s)} \right] \end{aligned} \tag{2.24}$$

where in the second equality we have used (2.13). Now we note that, as already mentioned above,

$$\varphi(p^{2a}) - p^{2a} = p^{2a}\left(1 - \frac{1}{p} - 1\right) = -\frac{p^{2a}}{p}.$$

Thus (2.24) on factoring p^{2a} yields

$$\sum_{c>0} \frac{S(0,0,c)}{c^{2s}} = \frac{p^a}{p^{4sa}} \left[-\frac{1}{p} \sum_{(k,p^a)=1}^{\infty} \frac{\varphi(k)}{k^{2s}} + \frac{\zeta(2s-1)}{\zeta(2s)} \right]. \tag{2.25}$$

Next we multiply and divide the first sum in the right-hand side of (2.25) by

$$1 + \sum_{k\geq 1} \frac{\varphi(p^k)}{p^{2ks}} = 1 + \left(1 - \frac{1}{p}\right) \frac{1}{p^{(2s-1)}} \left(1 - \frac{1}{p^{2s-1}}\right)^{-1},$$

where we have used (2.23) and the sum of a geometric series. Thus (2.25) becomes

$$\sum_{c>0} \frac{S(0,0,c)}{c^{2s}} = \frac{p^a}{p^{4sa}} \frac{\zeta(2s-1)}{\zeta(2s)} \left[1 - \frac{1}{p}\left(1 + \left(1-\frac{1}{p}\right)\frac{1}{p^{2s-1}}\left(1 - \frac{1}{p^{2s-1}}\right)^{-1}\right)^{-1}\right]. \tag{2.26}$$

Since the expression in the square brackets simplifies in a routine manner to $(1-1/p)/(1-1/p^{2s})$ we obtain

$$\sum_{c>0} \frac{S(0,0,c)}{c^{2s}} = \frac{p^a}{p^{4sa}}\left(1-\frac{1}{p}\right)\frac{\zeta(2s-1)}{\zeta(2s)}\left(1-\frac{1}{p^{2s}}\right)^{-1}$$

Using again (2.23) for $\boldsymbol{\varphi}(p^a)$ finally yields (2.16) for $N = p^a$, p prime and $a > 0$, and the proof is finished. □

3 Transmission Eigenvalues, Zeros of the Scattering Matrix and the Zeros of Riemann ζ Function

We now have all the ingredients to define and study the non-scattering waves in connection to the zeros of the scattering matrix $\Phi(s) := (\varphi_{\mathbf{ab}})$. As already mentioned in the previous section, in [19, Chap. 6] using Maass–Selberg relations $E_{\mathbf{a}}(\sigma_{\mathbf{b}}z, s)$ is continued to *all* $s \in \mathbb{C}$ and the asymptotic relation (2.4) holds where the last contribution comes from the Whittaker functions

$$\sum \varphi_{\mathbf{ab}}(n,s)W_s(nz).$$

Each $\varphi_{\mathbf{ab}}(s)$ is a meromorphic function and so is $E_{\mathbf{a}}(\sigma_{\mathbf{b}}z, s)$ as a function of s. Moreover, in general from the Fredholm theory (see [19, Appendix] and its use in [19, Chap. 6]) we can conclude that

$$\varphi_{\mathbf{ab}}(s) = \frac{f(s)}{g(s)}$$

where $f(s)$ and $g(s)$ are entire functions of *order* ≤ 2. The Fourier inversion formula

$$f(z) = \sum_j \langle f, u_j\rangle u_j + \sum_{\mathbf{a}} \frac{1}{4\pi}\int_{-\infty}^{\infty} \left\langle f, E_{\mathbf{a}}\left(\cdot, \frac{1}{2} + \mathrm{i}r\right)\right\rangle E_{\mathbf{a}}\left(z, \frac{1}{2}+\mathrm{i}r\right) dr,$$

proven in [19, Theorem 7.3], shows that $E_{\mathbf{a}}(\cdot, s)$ for $s = \frac{1}{2} + \mathrm{i}r$ are the wave packets in our framework, the analog of $\mathrm{e}^{\mathrm{i}kx}$, $k \leftrightarrow \frac{1}{2} + \mathrm{i}r$. However here there is also a discrete contribution from cusps form

$$\sum_j \langle f, u_j\rangle u_j.$$

3.1 The Transmission Eigenvalue Problem and Non-scattering Waves

We are given the Riemannian manifold $\mathbb{H}$, the subgroup Γ and the fundamental domain $F = \Gamma \setminus \mathbb{H}$. The non-identically zero total field u satisfies the equation

$$y^2\Delta u + s(1-s)u = 0, \quad z = (x,y) \in F.$$

On the boundary of F we impose periodic boundary conditions

$$u(\gamma z) = u(z), \quad z \in \partial F,\ \gamma \in \Gamma,$$

$$\frac{\partial u}{\partial \nu}(\gamma z) = \frac{\partial u}{\partial \nu}(z), \quad z \in \partial F,\ \gamma \in \Gamma.$$

For an example see Figure 3 where $\gamma A = B$ and $\gamma = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Note that the scattering in this structure results from the interaction of the incident wave y^s with the boundary of $\Gamma/\mathbb{H}$ which in turn results from the discrete group Γ. This should be compared with the scattering problem in $\mathbb{R}^n$ where, under smoothness assumptions on m in $\mathbb{R}^n$, (1.3) can be written as the Helmholtz equation for the Laplace–Beltrami operator associated with an appropriate metric g_m determined by $m(x)$. Hence the scattering in this case occurs due to the interaction of the incident plane wave $e^{ikx\cdot d}$ with the perturbation of the Euclidean metric given by g_m.

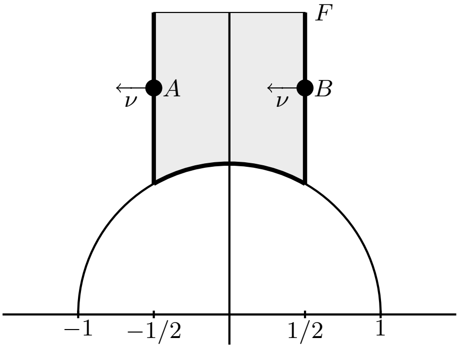

Figure 3 Values of u and its normal derivative at A, B coincide.

Definition 3.1 *A* transmission parameter *for the cusp* $\mathbf{a}$ *is* $s \in \mathbb{C}$ *such that*

$$\varphi_{\mathbf{aa}}(s) = 0.$$

Such s *is said to be a transmission parameter for the* transmission eigenvalue

$$s(1-s).$$

Note that we do *not* define transmission eigenvalues for *inequivalent* cusps $\mathbf{a}, \mathbf{b}$. An observer shining a light beam and sitting at cusp $\mathbf{a}$ will observe "far-field" back-scattering determined by $\varphi_{\mathbf{ab}} y^{1-s}$ (note that $\sum \varphi_{\mathbf{ab}}(n,s) W_s(nz) \to 0$ as $y \to \infty$). Thus for y^s, the light shining at frequency s from the cusp $\mathbf{a}$, the back-scattered wave y^{1-s} is absent. Thus the light is completely transmitted and *fails* to see the periodic boundary condition imposed by Γ that leads to the scattering term $\varphi_{\mathbf{ab}}$. However, note that for such s an observer sitting at an inequivalent cusp $\mathbf{a}$ may or may not see any scattered wave, since it is not known if $\varphi_{\mathbf{aa}}$ and $\varphi_{\mathbf{ab}}$ have common zeros, (see Section 4).

The rest of the paper is dedicated to locating the zeros of $\varphi_{\infty\infty}(s)$ that correspond to transmission parameters associated with the cusp ∞ and consequently the corresponding transmission eigenvalues $s(1-s)$. We will consider the two special cases of Γ, namely the modular groups $\mathrm{SL}_2(\mathbb{Z})$ and congruence groups $\Gamma(N)$, for which we have explicit calculations of the scattering matrix in Section 2.1.

Our goal is to obtain a Weyl law for transmission eigenvalues. To accomplish this we recall

Theorem 3.2 (Riemann and Von Mangoldt) *Let* B_T *be the box*

$$B_T := \{s = x + \mathrm{i}y : \ 0 < x < 1, \, -T < y < T\}, \quad T \geq 2,$$

and let $N(T)$ denotes the number of zeros of the Riemann zeta function $\zeta(s)$ in B. Then

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O\left(\log T\right). \tag{3.1}$$

For a sketch of the proof see [18, Chap. 9].

Proposition 3.3 *Let λ denote a transmission eigenvalue for $\Gamma\setminus\mathbb{H}$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ or any of the discrete groups considered in Theorem 2.8. Let*

$$N_\lambda(A) = |\{\lambda : |\lambda| < A\}|.$$

Then we have the Weyl law,

$$N_\lambda(A) \sim \frac{\sqrt{A}}{2\pi}\log\frac{A}{\pi^2\mathrm{e}^2} + O(\log A), \quad A \to \infty.$$

Moreover, the entire scattering matrix vanishes at such λ.

Proof The scattering matrices for the groups Γ considered in Theorem 2.8 all have the form

$$\Phi(s) = \psi(s)M(s)$$

where M is some matrix and where $\psi(s)$ is the same for all Γ and given by

$$\psi(s) = \pi^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}.$$

The matrix entries for $M(s)$ vanish at finitely many points s. Furthermore, $\Gamma(s)\zeta(2s)$ have poles only at $s = 0, \frac{1}{2}$. Thus $\psi(s)$ and hence the entire scattering matrix vanishes exactly at the zeros of $\zeta(2s-1)$. Let s_0 be a zero of $\zeta(u)$. Then, setting $2s-1 = s_0$, we see that

$$\lambda = s(1-s) = \frac{1-s_0^2}{4}.$$

Thus

$$|s_0| = |4\lambda - 1|^{1/2}.$$

It follows, if $|\lambda| < A$ as $A \to \infty$, $|s_0| \sim 2\sqrt{A}$. Applying Theorem 3.2, we conclude

$$N_\lambda(A) \sim \frac{\sqrt{A}}{2\pi}\log\frac{A}{\pi^2\mathrm{e}^2} + O(\log A), \quad A \to \infty. \qquad \square$$

Unfortunately, such a precise asymptotic expression is not available for general groups Γ, as we have a poor knowledge of the Kloosterman sums defined by (2.14) that appear in the formula for the scattering matrix. For example diagonal entries have terms

$$\sum_{c>0}\frac{S(0,0,c)}{c^{2s}}$$

(here c may not even be a natural number for general Γ). The bound in (2.15) is too crude to say much. However, we can use the Poisson–Jensen formula to estimate the number of the zeros of the scattering matrix inside a circle. To this end, we recall a standard result concerning the zeros of an entire function which is a consequence of Poisson–Jensen formula (for the proof see e.g. [30])

Lemma 3.4 *If $f(s)$ is an entire function of order 2, i.e., for any $\epsilon > 0$, $|f(s)| \le C\mathrm{e}^{M|s|^{2+\epsilon}}$ for some constant $C > 0$, then*

$$\#\left\{s : \textit{such that } f(s) = 0 \textit{ and } |s| < R\right\} \le CR^{2+\epsilon}$$

and, if s_n is such that $f(s_n) = 0$,

$$\sum_{n\in\mathbb{N}} \frac{1}{|s_n|^{2+\eta}} < +\infty \quad \textit{for any } \eta > 0.$$

Proposition 3.5 *Let λ_n denote the transmission eigenvalues for a diagonal entry $\varphi_{\mathbf{aa}}(s)$ of the scattering matrix. That is $\lambda_n = s_n(1 - s_n)$ and $\varphi_{\mathbf{aa}}(s_n) = 0$. Then, since $\varphi_{\mathbf{aa}}(s)$ has at most order 2 we have*

$$\#\left\{\lambda_n \textit{ such that } |\lambda_n| < R\right\} \leq CR^{1+\epsilon} \quad \textit{for any } \epsilon > 0.$$

and

$$\sum_{n\in\mathbb{N}} \frac{1}{|\lambda_n|^{1+\eta}} < +\infty \quad \textit{for any } \eta > 0.$$

Proof We know from [19, Chaps. 6, 10] that $\varphi_{\mathbf{aa}}(s)$ is a meromorphic function of order at most 2. That is

$$\varphi_{\mathbf{aa}}(s) = \frac{f(s)}{g(s)}$$

with $f(s)$, $g(s)$ entire and $|f(s)| \leq c_1 \mathrm{e}^{c_2|s|^{2+\epsilon}}$. Since $\lambda_n = s_n(1 - s_n)$, $|\lambda_n| \sim |s_n|^2 < R^2$ for $|s_n| < R$, as $R \to \infty$. Thus

$$\#\left\{\lambda_n \text{ such that } |\lambda_n| < R\right\} \sim \#\{s_n \text{ such that } f(s_n) = 0, |s_n| < R^{1/2}\}.$$

Applying Lemma 3.4, we easily get the desired conclusion of this proposition. □

Remark 3.6 Let us define a density for transmission eigenvalues. For λ a transmission eigenvalue, define

$$\rho = \limsup_{R\to\infty} \frac{\log|\{\lambda : |\lambda| < R\}|}{\log R}.$$

From Proposition 3.3, for Γ an arithmetic group as in Theorem 2.8, via the Riemann and Von Mangoldt Theorem, we see that $\rho = \frac{1}{2} + \epsilon$ for any $\epsilon > 0$. For general Γ, from Proposition 3.5, $\rho \leq 1 + \epsilon$ for any $\epsilon > 0$. This difference arises due to the growth order of the entries of the scattering matrix which is meromorphic of order 1 for Γ as in Theorem 2.8 and of order 2 in general. Thus the value of ρ is also tied in with the existence of cusp forms which relies on the growth order of $\varphi_{\mathbf{aa}}(s)$.

Let λ_j denote the eigenvalue of a cusp form. That is

$$\lambda_j = \frac{1}{4} + t_j^2.$$

Define

$$N_\Gamma(T) = \left|\left\{j : |t_j| < T,\, \lambda_j = \frac{1}{4} + t_j^2\right\}\right|.$$

We now wish to make a link between the density of transmission eigenvalues as defined in Remark 3.6 and the work of Phillips and Sarnak [28] on the existence of cusp forms. Essentially we want to show that if cusp forms are absent or are sparse, then the density of transmission eigenvalues is much more than for those discrete groups which have an abundance of cusp forms. The work of Phillips and Sarnak seems to suggest that cusp forms are rather rare for generic discrete groups Γ and so for generic discrete groups there should be more transmission eigenvalues than the arithmetic groups considered in Theorem 2.8.

Proposition 3.7 *Assume for some $\epsilon > 0$, we have $N_\Gamma(T) = O(T^{2-\epsilon}), T \to \infty$. Then the growth order of the entries of the scattering matrix $\Phi(s)$ is 2.*

Proof We proceed by contradiction and assume the growth order of the scattering matrix is $\rho = 2 - \delta,\ \delta > 0$. Next, we note that the scattering matrix $\Phi(s)$ is unitary when $s = \frac{1}{2} + \mathrm{i}t$. For the sequel we restrict our attention to $s = \frac{1}{2} + \mathrm{i}t$. Set,

$$\phi(s) = \det \Phi(s).$$

We need to consider for $s = \frac{1}{2} + \mathrm{i}t$, the integral

$$M_\Gamma(T) = \int_{-T}^{T} \frac{\phi'(s)}{\phi(s)} dt.$$

Since $\Phi(s)$ is unitary, $|\det \Phi(s)| = 1$ and so the denominator of the integrand in $M_\Gamma(T)$ does not vanish. Next since $\Phi(s)$ is unitary for $s = \frac{1}{2} + \mathrm{i}t$, we have

$$\phi(s) = \mathrm{e}^{\mathrm{i}f(t)} \tag{3.2}$$

with $f(t)$ real. Now assume by contradiction that the entries of the scattering matrix, have order ρ. This means that

$$|f(t)| \le c|t|^\rho. \tag{3.3}$$

Hence using (3.2)–(3.3) we have immediately,

$$M_\Gamma(T) = \left| \int_{T}^{T} \frac{\phi'(s)}{\phi(s)} dt \right| = |\log \phi(T) - \log \phi(-T)| = |f(T) - f(-T)| \le cT^\rho.$$

Now, by [19, Corollary 11.2], as a consequence of Selberg's trace formula we have

$$N_\Gamma(T) + M_\Gamma(T) \sim T^2, \quad \text{as } T \to \infty.$$

Thus, under the hypothesis that the order of the entries of the scattering matrix is at most ρ, and the hypothesis $N_\Gamma(T) = O(T^{2-\epsilon})$,

$$cT^{2-\epsilon} + T^\rho \sim T^2, \quad \text{as } T \to \infty.$$

If $\rho < 2$ we have a contradiction, which proves the proposition. □

Thus if Selberg's conjecture is not true for generic discrete groups Γ, that is such groups have no cusp forms, or cusp forms exist but their number $N_\Gamma(T)$ grow no faster that $T^{2-\epsilon}$, then we expect to have $\rho = 1$ in Remark 3.6, that is we expect to have more transmission eigenvalues for such discrete groups with sparse cusp forms.

3.2 Transmission Eigenvalues and the Riemann Hypothesis

We only focus on the modular groups $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ (the same situation occurs for congruent groups $\Gamma = \Gamma(N)$) where transmission parameters s are the zeros of

$$\psi(s) = \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}.$$

The *trivial zeros* appear at the poles of the denominator, i.e., $s = 0$ and $s = 1/2$. These trivial zeros yield $\lambda = 0$ and $\lambda = 1/4$ as the only real transmission eigenvalues.

The *non-trivial zeros* are the zeros of $\zeta(2s-1)$ such that $\Im(2s-1) \neq 0$. The Riemann hypothesis states that complex zeros of the Riemann zeta function have real part equal to 1/2, i.e., if $\zeta(u) = 0$ then $u = \frac{1}{2} \pm it$. This implies that the transmission parameters s have the form $s = \frac{3}{4} \pm \mathrm{i}t$, giving for the transmission eigenvalues $\lambda = \frac{3}{16} + t^2 - \mathrm{i}\frac{t}{2}$. Setting $x = \frac{3}{16} + t^2$ and $y = -\frac{t}{2}$, it yields that the real part x and the imaginary part y of the transmission eigenvalues satisfy

$$x = \frac{3}{16} + 4y^2.$$

Remark 3.8 The *Riemann Hypothesis* is equivalent to the statement that *all* transmission eigenvalues lie on the parabola $x = \frac{3}{16} + 4y^2$ except the finitely many trivial ones.

It is a known fact that all zeros of $\zeta(u)$ lie in the strip $0 < \Re(u) < 1$. Thus if $u = 1 + \mathrm{i}t$ we obtain for $s = \frac{1+u}{2}$ that $s = \frac{3}{2} \pm \mathrm{i}\frac{t}{2}$ implying that the real part x and the imaginary part y of $s(1-s)$ satisfy

$$x = -\frac{3}{4} + \frac{y^2}{4}.$$

Thus all transmission eigenvalues lie *inside* this parabola (see Figure 4). In fact from Hardy's theorem, which states that infinitely many zeros of the Riemann zeta function lie on $\Re(u) = 1/2$, to be precise 40% of the zeros (see e.g. [18, Part 2] for the proof), we have that infinitely many transmission eigenvalues lie on the parabola $x = \frac{3}{16} + 4y^2$. More refined estimates can be made using zero free regions of the Riemann zeta function based on the deep work of Vinogradov.

Remark 3.9 The transmission eigenvalues for the case of congruence groups $\Gamma(N)$ are located precisely at the same position as the transmission eigenvalues for the case of modular groups $\mathrm{SL}_2(\mathbb{Z})$. Theorem 2.8 shows that, although the scattering matrix $\Phi(s)$ in the case of $\Gamma(N)$ is now a 2×2 matrix, its zeros are solely determined by the function $\psi(s)$ which has exactly the same expression as the (scalar) scattering matrix $\varphi_{\infty\infty}(s)$ in the case of $\mathrm{SL}_2(\mathbb{Z})$ studied in this subsection.

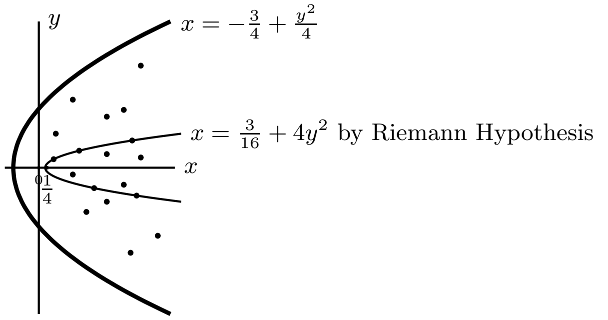

Figure 4 Black dots indicate possible location of transmission eigenvalues. There are infinitely many transmission eigenvalues on the inner parabola and all lie inside the outer parabola. If the Riemann Hypothesis is true, all transmission eigenvalues lie on the inner parabola. There are no real transmission eigenvalues except for the trivial ones, 0 and 1/4.

4 Remarks and Prospects

For a fundamental domain $F := \Gamma/\mathbb{H}$, we define transmission eigenvalues associated with a cusp $\mathbf{a}$ as the values $\lambda \in \mathbb{C}$ for which the transmission parameter s, $\lambda = s(1-s)$, is such that the incident wave y^s sent from the cusps $\mathbf{a}$ does not produce any back-scattering as seen by an observer at the same cusp $\mathbf{a}$. The wave packet y^s, with s being a transmission parameter, propagates through without seeing any boundaries of the fundamental domain generated by the group Γ (which are Riemannian geodesics). In this sense, transmission eigenvalues correspond to non-scattering energies for the wave equation on the Riemannian manifold on $\mathbb{H}$ generated by a given discrete group Γ. In inverse scattering, such a situation is referred to as invisibility [11]. Thus the notion of transmission eigenvalues in our framework is similar to the notion of transmission eigenvalues in the Euclidean space $\mathbb{R}^n$ for a spherically stratified scattering media with contrast m supported on the closed unit ball B_1, as is described in the Introduction. The former turns out to be the zeros of the diagonal terms of the (meromorphic in s) scattering matrix (2.5)

$$\Phi(s) = (\varphi_{\mathbf{ab}}(s))$$

(which for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ becomes exactly the scalar function $\varphi_{\mathbf{aa}}(s)$), whereas the latter are the zeros of the meromorphic function (1.12)

$$\mathcal{S}^+(s) = \frac{C(s; m, 0)}{sW(s; m, 0)}$$

(the relative scattering operator or the far field operator). As for a comparison of the location of transmission eigenvalues in both cases, it is shown here that the former lies inside a parabolic region for modular groups $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and congruent groups $\Gamma = \Gamma(N)$, whereas the latter also lies inside a parabolic region if the refractive index $1 + m(r)$ is continuous in $\mathbb{R}^n$ [33] (see also [15]), and in a strip around the real axis if $m(1) \neq 0$ (i.e., $1 + m(r)$ has jump across $r = 1$). The fascinating fact in our configuration is that for modular groups $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and congruent groups $\Gamma = \Gamma(N)$, these transmission eigenvalues are explicitly expressed in terms of the zeros of the Riemann zeta function. This leads to a precise location of transmission eigenvalues if the Riemann Hypothesis is true.

In the case of multiple inequivalent cusps, the question arises if at a transmission parameter s, the incident wave y^s sent away from the cusp $\mathbf{a}$ produces no scattering waves if observed at an inequivalent cusp $\mathbf{b}$, in other words, whether the Riemannian geodesics given by the boundaries of the fundamental domain are invisible at every inequivalent cusps. Mathematically this question amounts to whether the off-diagonal terms $\varphi_{\mathbf{ab}}(s)$ in the scattering matrix share the same zeros with the diagonal terms $\varphi_{\mathbf{aa}}(s)$. This is indeed the case for our explicit example of congruence groups $\Gamma(N)$ where there are two inequivalent cusps. Theorem 2.8 states that the zeros of the scattering matrix all arise from the scalar function $\psi(s)$ and the matrix $N(s)$ contributes nothing. Here the entire scattering matrix vanishes at a transmission parameter s, hence one has complete invisibility from/at each cusp. In fact, as pointed out in Remark 3.9, the zeros of the entire scattering matrix for these arithmetic groups coincide exactly with the zeros of the scalar scattering matrix for the standard case of modular groups $\mathrm{SL}_2(\mathbb{Z})$. The question now is: what is the situation for general Fuchsian groups of Type I? For general Type I Fuchsian groups as consequence of the Maass–Selberg relation in [19, Theorem 6.9], it is shown that the set of poles of the off-diagonal terms are contained in the set of the poles of the diagonal terms of the scattering matrix, but unfortunately the proof does not work for the zeros. Thus, is it just a property of special arithmetic groups that off diagonal zeros coincide with the diagonal zeros? We believe these are interesting open questions in analytic number theory. Another question is if the density ρ defined in Remark 3.6 is continuous or upper semi-continuous when Γ is deformed in Teichmüller space in the sense of Phillips–Sarnak [28]. It is also unclear if ρ is a discrete subset of $\left[\frac{1}{2}, 1\right]$ or all of it.

For Fuchsian groups of Type II, the scattering theory is developed in a comprehensive manner in the monograph by Borthwick [2]. Again the concerns there are only with poles of the scattering matrix. Hence, it is desirable to introduce the notion of non-scattering energies and transmission eigenvalues in this framework and connect them to the zeros of the scattering matrix. Fundamental domains generated by Fuchsian groups of Type II exhibit both cusps and funnels. Thus the scattering matrix is no longer literally a matrix. The contribution from funnels gives rise to pseudo-differential operators instead, very much like the scattering operator in the general Euclidean case.

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