IOP Publishing Inverse Problems

Inverse Problems **23** (2007) 507–522 doi:10.1088/0266-5611/23/2/004

On the use of transmission eigenvalues to estimate the index of refraction from far field data*

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Received 9 November 2006, in final form 9 January 2007
Published 31 January 2007
Online at stacks.iop.org/IP/23/507

Abstract
We consider the scattering of time harmonic electromagnetic plane waves by a bounded inhomogeneous medium and show that under certain assumptions a lower bound on the index of refraction can be obtained from a knowledge of the smallest transmission eigenvalue corresponding to the medium. It is then shown by numerical examples that this eigenvalue can be determined from a knowledge of the far field pattern of the scattered wave, thus providing a practical method for estimating the index of refraction from far field data.

1. Introduction

In this paper, we are concerned with the problem of obtaining qualitative estimates of the index of refraction of an inhomogeneous medium from a knowledge of the far field pattern of the scattered time harmonic electromagnetic wave corresponding to plane waves as incident fields. We will assume that qualitative methods in inverse scattering theory [2] have already been used to determine the support D of the inhomogeneous medium and hence the problem is to determine an estimate of the index of refraction $n(x)$ under the assumption that the support D of $m := 1 - n$ is known. Under the assumption that the medium is absorbing, i.e. $\operatorname{Im} n > 0$, one such estimate is provided in [6] where the location of the eigenvalues of the far field operator at a fixed frequency provides a lower bound for the quantity

$$\int_D \frac{|m|^2}{\operatorname{Im} m}\,\mathrm{d}x. \tag{1}$$

Here, following theorem 7 of [7], we first show that, under the assumption that $\operatorname{Im} n = 0$ and $n(x) > 1$ for $x \in D$, a lower bound for the supremum of $n(x)$ for $x \in D$ can be determined from a knowledge of the first transmission eigenvalue $k_0 = k_0(D, n)$ and the first Dirichlet

* This research was supported in part by the US Air Force Office of Scientific Research under grant FA9550-05-1-0127.

0266-5611/07/020507+16$30.00

eigenvalue $\lambda_0 = \lambda_0(D)$. Since D is assumed to be known, so is $\lambda_0(D)$. Furthermore, we show that the transmission eigenvalues can be obtained from the far field operator provided that this operator is known for an appropriate range of frequencies. For a discussion of transmission eigenvalues in inverse scattering theory, we refer the reader to [2, 5, 7] and, for the case of a spherically stratified medium, [12].

Our paper proceeds as follows. In section 2, we consider scattering by an infinite cylinder and establish the lower bound for $n(x)$ stated in the above paragraph. In particular if $n(x) > 1$ for $x \in D$ then

$$\sup_D n > \frac{\lambda_0(D)}{k^2} \tag{2}$$

where k is a transmission eigenvalue. If $n < 1$ it is shown that all transmission eigenvalues must be larger than $\lambda_0(D)$. For small n, these estimates in general yield no information on n. In particular, if the Born approximation is assumed then there are no transmission eigenvalues at all [7]! Hence, we also provide bounds for $\sup_D n$ that, although cruder than (2), yield limited information in the case of small n. In general, the estimates we obtain provide potentially useful lower bounds for $\sup_D n$ only in the case of a 'strong' scatterer. In section 3, we extend these results to the case of the scattering of electromagnetic waves by a bounded inhomogeneous medium in $\mathbb{R}^3$ under the assumption that the index of refraction is constant. Finally, in section 4, we provide numerical examples showing the practicality of our results. A rather surprising result of this investigation is that a lower bound for the supremum of $\operatorname{Re} n(x)$ for $x \in D$ can be determined even when the condition that $\operatorname{Im} n = 0$ is no longer satisfied.

A preliminary study such as the one presented here obviously raises more questions than it answers. In particular, the question of whether or not transmission eigenvalue exists when $n(x)$ is not spherically symmetric remains an open question [7]. Furthermore, in the case of Maxwell's equations, it would be highly desirable to obtain an estimate of the form (2) for the variable index of refraction.

2. The scalar case

We consider the scattering of a time harmonic electromagnetic plane wave by an inhomogeneous infinite cylinder with cross section D such that the electric field $E = (0, 0, u\,\mathrm{e}^{-\mathrm{i}\omega t})$ is polarized parallel to the axis of the cylinder. Then, assuming that the index of refraction is independent of the coordinate $e_z = (0, 0, 1)$ and factoring out the time harmonic term $\mathrm{e}^{-\mathrm{i}\omega t}$, $u = u(x)$ satisfies [2, 5]

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in} \quad \mathbb{R}^2 \tag{3}$$

$$u(x) = \mathrm{e}^{\mathrm{i}kx\cdot d} + u^s(x) \qquad \text{in} \quad \mathbb{R}^2 \tag{4}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right) = 0 \tag{5}$$

where $x \in \mathbb{R}^2$, $r = |x|$, $k > 0$ is the wave number, d is a vector on the unit circle Ω in $\mathbb{R}^2$ and the Sommerfeld radiation condition (5) is assumed to hold uniformly with respect to $\hat{x} = x/|x|$. The index of refraction $n = n(x)$ is assumed to be a piecewise continuously differentiable function with (possible) jumps along piecewise smooth curves such that $\operatorname{Re}(n) > 0$, $\operatorname{Im}(n) \geqslant 0$ and $m := 1 - n$ has compact support $\overline{D}$ where the complement of D is connected and $\overline{D}$ has smooth boundary ∂D with unit outward normal ν. Without loss of generality, we assume that D contains the origin.

The existence of a unique solution to (3)–(5) can be established by either variational methods or the use of integral equations [2, 5]. It can then be shown [2, 5] that the scattered field u^s has the asymptotic behaviour

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O(r^{-3/2}) \tag{6}$$

as $r \to \infty$ uniformly in $\hat{x}$ where u_∞ is the *far field pattern*.

In this paper, we are concerned with the *inverse scattering problem* of determining D and $n(x)$ from a knowledge of $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \Omega$ and fixed k (we will also discuss the case of limited aperture data, see section 4). To this end, D can be determined by using the *linear sampling method* to solve the *far field equation*

$$\int_\Omega u_\infty(\hat{x}, d) g(d)\, \mathrm{d}s(d) = \Phi_\infty(\hat{x}, z) \tag{7}$$

where Φ_∞ is the far field pattern of the radiating fundamental solution

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - z|) \tag{8}$$

and $H_0^{(1)}$ denotes a Hankel function of the first kind of order zero. For details we refer the reader to [2, 5]. In particular, from [2, 5] we have that the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, \mathrm{d}s(d) \tag{9}$$

is injective with dense range provided k is not a *transmission eigenvalue*, i.e. a value of k for which the (homogeneous) *interior transmission problem*

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in} \quad D \tag{10}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \quad D \tag{11}$$

$$w = v \qquad \text{on} \quad \partial D \tag{12}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on} \quad \partial D \tag{13}$$

has a nontrivial solution w, v where $w, v \in L^2(D)$ and $w - v \in H_0^2(D)$ [14]. Here, the Sobolev spaces $H_0^1(D)$ and $H_0^2(D)$ are defined, respectively, by

$$H_0^1(D) := \{u \in H^1(D) : u = 0 \quad \text{on} \quad \partial D\}$$

$$H_0^2(D) := \left\{u \in H^2(D) : u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial D\right\}$$

where the boundary values are assumed in the sense of the trace theorem. In particular, to apply the linear sampling method it is necessary to assume that k is not a transmission eigenvalue.

Having found D by the linear sampling method, we now want to find $n(x)$ and the purpose of this paper is to provide a partial resolution of this problem. In particular, instead of avoiding the transmission eigenvalues as in the linear sampling method, we will show that in certain circumstances a lower bound for $n(x)$ can be obtained from a knowledge of the smallest transmission eigenvalue (assuming such eigenvalues exist, see [7] for the current situation in this regard). To this end, we note that, since D contains the origin, the linear sampling method can be expected to fail when k is a transmission eigenvalue and in particular the norm of the (regularized) solution to

$$(Fg)(\hat{x}) = 1 \tag{14}$$

should be large for such values of k. This should provide us with a method for determining the smallest transmission eigenvalue from the far field data and in section 4 we will provide some numerical evidence that this is indeed the case. Hence, we now turn our attention to deriving a relationship between the coefficient $n(x)$ in (3) and the smallest transmission eigenvalue. We first show that transmission eigenvalues can only occur when $n(x)$ is real valued (see theorem 8.12 in [5]).

Lemma 2.1. *If* $\mathrm{Im}(n) > 0$ *in a ball* $B_r \subset D$ *then there are no transmission eigenvalues.*

Proof. From Green's formula and the boundary conditions we have that

$$\int_D |\nabla w|^2\,\mathrm{d}y - \int_D k^2 n|w|^2\,\mathrm{d}y = \int_{\partial D} \overline{w} \cdot \frac{\partial w}{\partial \nu}\,\mathrm{d}y = \int_D |\nabla v|^2\,\mathrm{d}y - \int_D k^2|v|^2\,\mathrm{d}y.$$

Hence,

$$\int_D \mathrm{Im}(n)|w|^2\,\mathrm{d}y = 0 \qquad \text{in} \quad D. \tag{15}$$

If $\mathrm{Im}(n) > 0$ in B_r then (15) and the unique continuation principle imply that $w \equiv 0$ in D. From the boundary conditions and the integral representation formula v also vanishes in D. □

We now distinguish between two cases, namely $n > 1$ and $0 < n < 1$. The following result is from [7] (see also [14]).

Theorem 2.2. *Suppose that* $\mathrm{Im}(n) = 0$ *and, for some* $\delta > 0$, $n - 1 \geqslant \delta$ *in* D*. Then,*

$$\sup_D n > \frac{\lambda_0(D)}{k^2} \tag{16}$$

where k *is a transmission eigenvalue and* $\lambda_0(D)$ *is the first Dirichlet eigenvalue of* $-\Delta$ *on* D*.*

Proof. Let k be a transmission eigenvalue and $u = w - v$ where (w, v) is a solution to (10)–(13). In particular, we have that u satisfies

$$\Delta u + k^2 u = k^2(1-n)w \qquad \text{in} \quad D \tag{17}$$

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial D. \tag{18}$$

Using the equation satisfied by w we have that u satisfies the fourth-order equation [14]

$$(\Delta + k^2 n)\frac{1}{n-1}(\Delta + k^2)u = 0 \tag{19}$$

together with the zero boundary conditions (18). Taking the $L^2(D)$ inner product of both sides of (19) with $\varphi \in H_0^2(D)$, integrating by parts, and using the zero boundary values for φ we have that

$$\begin{aligned} 0 &= \int_D (\Delta + k^2 n)\frac{1}{n-1}(\Delta + k^2)u\,\overline{\varphi}\,\mathrm{d}x \\ &= \int_D (\Delta\overline{\varphi} + k^2 n\overline{\varphi})\frac{1}{n-1}(\Delta u + k^2 u)\,\mathrm{d}x. \end{aligned} \tag{20}$$

Thus, the eigenvalue problem (18), (19) is equivalent to finding a function $u \in H_0^2(D)$ such that

$$\int_D (\Delta\overline{\varphi} + k^2 n\overline{\varphi})\frac{1}{n-1}(\Delta u + k^2 u)\,\mathrm{d}x = 0 \qquad \text{for all} \quad \varphi \in H_0^2(D). \tag{21}$$

Taking $\varphi = u$ in (21), using Green's theorem and the zero boundary value for u we obtain that

$$0 = \int_D (\Delta\overline{u} + k^2 n\overline{u}) \frac{1}{n-1} (\Delta u + k^2 u)\,\mathrm{d}x$$
$$= \int_D \frac{1}{n-1} |(\Delta u + k^2 nu)|^2\,\mathrm{d}x + k^2 \int_D (|\nabla u|^2 - k^2 n|u|^2)\,\mathrm{d}x. \tag{22}$$

Since $n - 1 \geqslant \delta > 0$, if

$$\int_D (|\nabla u|^2 - k^2 n|u|^2)\,\mathrm{d}x \geqslant 0 \tag{23}$$

then $\Delta u + k^2 nu = 0$ in D which together with the fact $u \in H_0^2(D)$ implies that $u = 0$. Consequently $w = v = 0$, which means that k is not a transmission eigenvalue. But, since [3]

$$\inf_{u\in H_0^2(D)} \frac{(\nabla u, \nabla u)_{L^2(D)}}{(u, u)_{L^2(D)}} \geqslant \inf_{u\in H_0^1(D)} \frac{(\nabla u, \nabla u)_{L^2(D)}}{(u, u)_{L^2(D)}} = \lambda_0(D), \tag{24}$$

we have that

$$\int_D (|\nabla u|^2 - k^2 n|u|^2)\,\mathrm{d}x \geqslant \|u\|^2_{L^2(D)} \big(\lambda_0(D) - k^2 \sup_D(n)\big).$$

Thus, (23) is satisfied whenever $k^2 \leqslant \frac{\lambda_0(D)}{\sup_D(n)}$. Hence, we have shown that any transmission eigenvalue k satisfies $k^2 > \frac{\lambda_0(D)}{\text{ess}\,\sup_D(n)}$ which gives that $\sup_D(n) > \frac{\lambda_0(D)}{k^2}$ with k being a transmission eigenvalue. □

Remark 2.1. From theorem 2.2 it follows that if $n \geqslant \delta > 0$ in D and k_0 is the smallest transmission eigenvalue, then $\frac{\lambda_0(D)}{k_0^2}$ provides a lower bound for $\sup_D(n)$.

Theorem 2.3. *Suppose that* $\mathrm{Im}(n) = 0$, $n \geqslant n_0 > 0$ *and* $1 - n \geqslant \delta > 0$ *in* D. *Then, if* k *is a transmission eigenvalue,* $k^2 > \lambda_0(D)$ *where* $\lambda_0(D)$ *is the first Dirichlet eigenvalue of* $-\Delta$ *on* D.

Proof. Proceeding in the same way as in the first part of the proof of theorem 2.2, we obtain that $u = w - v \in H_0^2(D)$ where (w, v) solves (10)–(13) satisfies

$$\int_D (\Delta\overline{u} + k^2 n\overline{u}) \frac{1}{1-n} (\Delta u + k^2 u)\,\mathrm{d}x = 0. \tag{25}$$

Now, integrating by parts and using the zero boundary values for u we obtain

$$0 = \int_D (\Delta\overline{u} + k^2 n\overline{u}) \frac{1}{1-n} (\Delta u + k^2 u)\,\mathrm{d}x$$
$$= \int_D \frac{1}{1-n} |(\Delta u + k^2 u)|^2\,\mathrm{d}x + k^2 \int_D (|\nabla u|^2 - k^2 |u|^2)\,\mathrm{d}x. \tag{26}$$

From (24), since $1 - n \geqslant \delta > 0$, we see that as long as $k^2 \leqslant \lambda_0(D)$, the second term of (26) is nonnegative which implies that $u = 0$ and consequently $w = v = 0$. Hence k, such that $k^2 \leqslant \lambda_0(D)$, cannot be a transmission eigenvalue which proves the theorem. □

As numerical examples in section 4 show, for small n the above estimates provide no information on n. Hence, we now obtain an estimate for n which complements the estimate obtained above. To this end, let $m := 1 - n$ and without loss of generality assume $m > 0$ (otherwise replace m by $-m$ in the definition of $L^2_m(D)$). Define

$$L^2_m(D) := \left\{ u : u(x)\text{ measurable}, \int_D m(x)|u(x)|^2\,\mathrm{d}x < \infty \right\}$$

and let $\|\cdot\|_{L^2_m(D)}$ be the corresponding weighted L^2_m-norm. Furthermore, define

$$H := \operatorname{span}\{J_n(kr)\,\mathrm{e}^{\mathrm{i}n\theta} : n = 0, \pm 1, \pm 2, \ldots\},$$

let $\overline{H}$ be the closure of H in $L^2_m(D)$, and $T_m : L^2_m(D) \to L^2_m(D)$ be defined by

$$(T_m f)(x) := \frac{\mathrm{i}}{4}\int_D H_0^{(1)}(k|x-y|)m(y)f(y)\,\mathrm{d}y$$

where $H_0^{(1)}$ denotes the Hankel function of the first kind of order zero. From theorem 8.16 of [5] (applied to the case of $\mathbb{R}^2$ instead of $\mathbb{R}^3$) we see that if k is a transmission eigenvalue then $k^2\|T_m\| > 1$ where $\|\cdot\|$ is the operator norm in $L^2_m(D)$. To compute $\|T_m\|$, we will first need an estimate for the absolute value of $H_0^{(1)}(kr)$ where $r = |x-y|$.

Lemma 2.4. *Let d be the diameter of D, $k > 0$, $x, y \in D$ and $r = |x-y|$. Then*

$$\left|H_0^{(1)}(kr)\right| \leqslant C - \frac{2}{\pi}\log\frac{r}{d} \tag{27}$$

where $C := \frac{2\gamma}{\pi} + \frac{2}{\pi}\left|\log\frac{kd}{2}\right| + \frac{2}{\sqrt{3}} + 1$ and γ is Euler's constant.

Proof. From [15], p 96, we have that

$$Y_0(kr) = \frac{2}{\pi}\left[\gamma + \log\frac{kd}{2} + \log\frac{r}{d}\right]J_0(kr) - \frac{4}{\pi}\sum_1^\infty \frac{(-1)^n}{n}J_{2n}(kr)$$

where J_n and Y_n are Bessel functions of the first and second kinds, respectively. From [3], pp 57, 295, we have that

$$\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$

and

$$J_0^2(kr) + 2\sum_{n=1}^\infty J_n^2(kr) = 1$$

and hence from Schwarz's inequality we see that

$$|Y_0(kr)| \leqslant \frac{2\gamma}{\pi} + \frac{2}{\pi}\left|\log\frac{kd}{2}\right| + \frac{2}{\sqrt{3}} - \frac{2}{\pi}\log\frac{r}{d}.$$

Since $H_0^{(1)}(kr) = J_0(kr) + \mathrm{i}Y_0(kr)$ and $|J_0(kr)| \leqslant 1$, the lemma follows. □

Theorem 2.5. *Suppose that* $\operatorname{Im}(n) = 0$ *and let $m := 1 - n$. Then if k is a transmission eigenvalue and $M := \sup_D |m|$,*

$$M > \frac{1}{Ak^2d^2} \tag{28}$$

where $A = \pi/4(C^2 + C + 1/\pi)^{1/2}$ and C is defined in lemma 2.4.

Proof. We need to compute an upper bound for $\|T_m\|$. To this end, we have

$$\begin{aligned}
|(T_m f)(x)|^2 &= \left|\frac{\mathrm{i}}{4}\int_D H_0^{(1)}(k|x-y|)m(y)f(y)\,\mathrm{d}y\right|^2 \\
&\leqslant \frac{1}{4}\int_D \left|H_0^{(1)}(k|x-y|)\right|^2 m(y)\,\mathrm{d}y \int_D m(y)|f(y)|^2\,\mathrm{d}y \\
&\leqslant \frac{M\|f\|^2}{4}\int_0^d\int_0^{2\pi}\left(C^2 - 2C\log\frac{r}{d} + \frac{2}{\pi}\left(\log\frac{r}{d}\right)^2\right) r\,\mathrm{d}r\,\mathrm{d}\theta
\end{aligned}$$

where we have made use of lemma 2.4. Then since

$$\int_0^{\mathrm{d}} r \log \frac{r}{d}\, \mathrm{d}r = -\frac{d^2}{4}$$

and

$$\int_0^{\mathrm{d}} r \left(\log \frac{r}{d}\right)^2 \mathrm{d}r = \frac{d^2}{4}$$

we have that

$$|(T_m f)(x)|^2 \leqslant \frac{\pi M d^2}{4} \|f\|^2 (C^2 + C + 1/\pi).$$

Hence,

$$\begin{aligned} \|T_m f\|^2 = \int_D m |T_m f|^2 \,\mathrm{d}x &\leqslant \frac{\pi M d^2}{4} \sup_D |T_m f(x)| \\ &\leqslant \frac{\pi^2 M^2 d^4}{16} (C^2 + C + 1/\pi) \|f\|^2, \end{aligned} \tag{29}$$

i.e.,

$$\|T_m\| \leqslant A M d^2.$$

The theorem now follows from the observation proceeding lemma 2.4. □

3. The vector case

We now consider the scattering of a time harmonic electromagnetic plane wave by a homogeneous medium $D \subset \mathbb{R}^3$ of compact support such that the complement of D is connected and $\overline{D}$ has smooth boundary ∂D with unit outward normal ν. Factoring out the time harmonic term $\mathrm{e}^{-\mathrm{i}\omega t}$ and eliminating the magnetic field from the Maxwell's equations, we have that the total electric field E satisfies

$$\operatorname{curl}\operatorname{curl} E - k^2 n E = 0 \qquad \text{in} \quad \mathbb{R}^3 \tag{30}$$

$$E(x) = E^i(x) + E^s(x) \qquad \text{in} \quad \mathbb{R}^3 \tag{31}$$

$$\lim_{r\to\infty} (\operatorname{curl} E^s \times x - \mathrm{i}kr E^s) = 0 \tag{32}$$

where $k > 0$ is the wave number and n is the index of refraction of the scattering object D (assumed to be a constant in D and equal to 1 in $\mathbb{R}^3 \backslash D$) such that $\mathrm{Re}(n) > 0$ and $\mathrm{Im}(n) \geqslant 0$. The Silver–Müller radiation condition (32) is satisfied uniformly in $\hat{x} = x/|x|$, where $r = |x|$, $x \in \mathbb{R}^3$, and the incident field E^i is given by

$$E^i(x) := \frac{\mathrm{i}}{k} \operatorname{curl}\operatorname{curl} p\, \mathrm{e}^{\mathrm{i}kx\cdot d} = \mathrm{i}k(d \times p) \times d\, \mathrm{e}^{\mathrm{i}kx\cdot d} \tag{33}$$

where d is now a vector on the unit sphere Ω giving the direction of propagation and p is the (constant) polarization vector. In [13], it is shown that (30)–(32) has a unique solution in $H(\mathrm{curl}, B_R)$ for any ball B_R of radius R. Moreover, the scattered electric field E^s has the asymptotic behaviour [5]

$$E^s(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|} \left\{ E_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right) \right\}$$

as $|x| \to \infty$, where E_∞ is a tangential vector field defined on Ω and is known as the *electric far field pattern*. Note that $E_\infty(\hat{x}, d, p)$ depends linearly on the polarization p.

As in the previous section, the *inverse scattering problem* we are interested in is to determine D and $n(x)$ from a knowledge of $E_\infty(\hat{x}, d, p)$ for all $\hat{x}, d \in \Omega$ and three linearly independent polarizations $p_1, p_2, p_3 \in \mathbb{R}^3$. Our goal is to extend the ideas presented in the scalar case to the vector case for solving the inverse scattering problem. We define *the electric far field operator* $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, \mathrm{d}s(d), \qquad \hat{x} \in \Omega, \tag{34}$$

for $g \in L^2_t(\Omega)$, where $L^2_t(\Omega)$ is the space of square integrable tangential vector-valued functions defined on the unit sphere Ω. Note that F depends linearly on g. As in the scalar case, D can be determined by the linear sampling method [9, 10] which is based on the behaviour of the (regularized) solution to the far field equation

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q) \qquad \hat{x} \in \Omega \quad \text{and} \quad z, q \in \mathbb{R}^3 \tag{35}$$

where

$$E_{e,\infty}(\hat{x}, z, q) = \frac{\mathrm{i}k}{4\pi}(\hat{x} \times q) \times \hat{x}\, \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z} \tag{36}$$

is the far field pattern of the electric field E_e of an electric dipole located at z with polarization q. In particular, E_e is defined by

$$E_e(x, z, q) := \frac{\mathrm{i}}{k}\mathrm{curl}_x\, \mathrm{curl}_x\, q\, \Phi(x, z) \tag{37}$$

where Φ is the fundamental solution of Helmholtz equation in $\mathbb{R}^3$ defined by

$$\Phi(x, z) := \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|x-z|}}{|x - z|}, \qquad x \neq z \quad \text{and} \quad x, z \in \mathbb{R}^3.$$

We shall need to use the following spaces in our analysis

$$H(\mathrm{curl}, D) := \{u \in L^2(D)^3 : \mathrm{curl}\, u \in L^2(D)^3\}$$
$$H_0(\mathrm{curl}, D) := \{u \in H(\mathrm{curl}, D) : \nu \times u = 0 \text{ on } \partial D\}$$
$$H_0(\mathrm{curl}\, 0, D) := \{u \in H_0(\mathrm{curl}, D) : \mathrm{curl}\, u = 0 \text{ on } \partial D\}$$
$$H(\mathrm{div}\, 0, D) := \{u \in L^2(D)^3 : \mathrm{div}\, u = 0\}$$
$$\mathcal{X}(D) := H_0(\mathrm{curl}, D) \cap H(\mathrm{div}\, 0, D)$$

equipped with the obvious scalar product and define

$$\mathcal{U}(D) := \{u \in H(\mathrm{curl}, D) : \mathrm{curl}\, u \in H(\mathrm{curl}, D)\}$$
$$\mathcal{U}_0(D) := \{u \in H_0(\mathrm{curl}, D) : \mathrm{curl}\, u \in H_0(\mathrm{curl}, D)\}$$

equipped with the scalar product

$$(u, v)_{\mathcal{U}(D)} = (u, v)_{L^2(D)} + (\mathrm{curl}\, u, \mathrm{curl}\, v)_{L^2(D)} + (\mathrm{curl}\,\mathrm{curl}\, u, \mathrm{curl}\,\mathrm{curl}\, v)_{L^2(D)}.$$

As in the scalar case, in order to apply the linear sampling methods it is necessary that the operator F is injective with dense range which holds provided that k is not a *transmission eigenvalue* [5], i.e. a value of k for which the interior transmission problem

$$\mathrm{curl}\,\mathrm{curl}\, E - k^2 n E = 0 \qquad \text{in} \quad D \tag{38}$$
$$\mathrm{curl}\,\mathrm{curl}\, E_0 - k^2 E_0 = 0 \qquad \text{in} \quad D \tag{39}$$
$$\nu \times E = \nu \times E_0 \qquad \text{on} \quad \partial D \tag{40}$$
$$\nu \times \mathrm{curl}\, E = \nu \times \mathrm{curl}\, E_0 \qquad \text{on} \quad \partial D \tag{41}$$

has a nontrivial solution, E, E_0, where $E, E_0 \in L^2(D)$ and $E - E_0 \in \mathcal{U}_0(D)$.

As in the scalar case, we expect that the norm of the (regularized) solution to

$$(Fg)(\hat{x}) = (\hat{x} \times q) \times \hat{x} \tag{42}$$

should be large if k is a transmission eigenvalue, thus providing us with a method for determining transmission eigenvalues from far field data. We remark that in the same way as in the proof of lemma 2.1 (see also theorem 9.8 in [5]) one can show that if $\mathrm{Im}(n) > 0$ in D then there are no transmission eigenvalues. Hence, we assume that $\mathrm{Im}(n) = 0$ in D. In the following, we establish a relationship between the transmission eigenvalues and the index of refraction n provided that D is known.

Theorem 3.1. *Suppose that n is a real constant in D and $n > 1$. Then $n > \frac{\lambda_0(D)}{k_0^2}$ where k_0 is the first transmission eigenvalue and $\lambda_0(D)$ is the first Maxwell eigenvalue of the curl curl operator on D.*

Proof. Let k be a transmission eigenvalue and let E, E_0 be a nonzero solution to (38)–(41). The difference $W = E - E_0$ satisfies

$$\operatorname{curl}\operatorname{curl} W - k^2 W = k^2(n-1)E \qquad \text{in} \quad D \tag{43}$$

together with the boundary conditions

$$\nu \times W = 0 \qquad \text{and} \qquad \nu \times \operatorname{curl} W = 0 \qquad \text{on} \quad \partial D. \tag{44}$$

By using the equation satisfied by E we obtain that W satisfies the fourth-order equation [9]

$$(\operatorname{curl}\operatorname{curl} - k^2 n)\frac{1}{n-1}(\operatorname{curl}\operatorname{curl} - k^2)W = 0. \tag{45}$$

Taking the $L^2(D)$ scalar product of (45) with $\Psi \in \mathcal{U}_0(D)$, integrating by parts and using the zero boundary conditions for Ψ we obtain that (45) with the boundary conditions (44) are equivalent to finding $W \in \mathcal{U}_0(D)$ satisfying

$$\int_D (\operatorname{curl}\operatorname{curl}\overline{\Psi} - k^2 n\overline{\Psi})\frac{1}{n-1}(\operatorname{curl}\operatorname{curl} W - k^2 W)\,\mathrm{d}x = 0 \quad \text{for all} \quad \Psi \in \mathcal{U}_0(D). \tag{46}$$

Now taking $\Psi = W$ in (46), integrating by parts and using the zero boundary conditions for W we can rewrite (46) as

$$\begin{aligned} 0 &= \int_D (\operatorname{curl}\operatorname{curl}\overline{W} - k^2 n\overline{W})\frac{1}{n-1}(\operatorname{curl}\operatorname{curl} W - k^2 W)\,\mathrm{d}x \\ &= \int_D \frac{1}{n-1}|\operatorname{curl}\operatorname{curl} W - k^2 n W|^2\,\mathrm{d}x + k^2\int_D (|\operatorname{curl} W|^2 - k^2 n|W|^2)\,\mathrm{d}x. \end{aligned} \tag{47}$$

From the condition that $n - 1 > 0$ we have that k is not a transmission eigenvalue provided

$$\int_D (|\operatorname{curl} W|^2 - k^2 n|W|^2)\,\mathrm{d}x \geqslant 0 \tag{48}$$

since $\operatorname{curl}\operatorname{curl} W - k^2 n W = 0$ together with the fact that $W \in \mathcal{U}_0(D)$ imply that $W = 0$ and consequently $E = E_0 = 0$. But

$$\int_D (|\operatorname{curl} W|^2 - k^2 n|W|^2)\,\mathrm{d}x = \|W\|^2_{L^2(D)}\left(\frac{(\operatorname{curl} W, \operatorname{curl} W)_{L^2(D)}}{(W, W)_{L^2(D)}} - k^2 n\right). \tag{49}$$

We recall that $W \in \mathcal{U}_0(D)$ satisfies $\operatorname{div} W = 0$ since n is constant, whence W is also in $\mathcal{X}(D)$. Hence, we have that

$$\frac{(\operatorname{curl} W, \operatorname{curl} W)_{L^2(D)}}{(W, W)_{L^2(D)}} \geqslant \inf_{U \in \mathcal{X}(D)} \frac{(\operatorname{curl} U, \operatorname{curl} U)_{L^2(D)}}{(U, U)_{L^2(D)}} = \lambda_0(D) \tag{50}$$

where $\lambda_0(D)$ is the first eigenvalue of the curl curl operator in $\mathcal{X}(D)$ [1]. Thus, (48) is satisfied whenever $k^2 \geqslant \frac{\lambda_0(D)}{n}$ whence any transmission eigenvalue k must satisfy $k^2 > \frac{\lambda_0(D)}{n}$. Hence $\frac{\lambda_0(D)}{k_0^2}$, where k_0 is the first transmission eigenvalue, provides a lower bound for n. □

Theorem 3.2. *Suppose that n is a real constant and* $0 < n < 1$. *Then if k is a transmission eigenvalue,* $k^2 > \lambda_0(D)$ *where* $\lambda_0(D)$ *is the first Maxwell eigenvalue of the curl curl operator on* D.

Proof. Proceeding in the same way as in the first part of the proof of theorem 3.1, we obtain that $W = E - E_0 \in \mathcal{U}_0(D)$ satisfies

$$\int_D (\text{curl}\,\text{curl}\,\overline{W} - k^2 n\overline{W})\frac{1}{n-1}(\text{curl}\,\text{curl}\,W - k^2 W)\,\mathrm{d}x = 0. \tag{51}$$

Integrating by parts and using the zero boundary conditions for W we obtain

$$\int_D \frac{1}{1-n}|\text{curl}\,\text{curl}\,W - k^2 n W|^2\,\mathrm{d}x + k^2\int_D (|\text{curl}\,W|^2 - k^2|W|^2)\,\mathrm{d}x = 0. \tag{52}$$

From the assumption that n is a real constant such that $1 - n > 0$ and making use of (50), we observe that values of $k > 0$ such that $k^2 \leqslant \lambda_0(D)$, where $\lambda_0(D)$ is the first eigenvalue of the curl curl operator in $\mathcal{X}(D)$, cannot be transmission eigenvalues since both terms in (52) are nonnegative. Hence, all transmission eigenvalues k satisfy $k^2 > \lambda_0(D)$. □

We now obtain another estimate for n using an integral equation approach. To this end, using the integral equation on p 252 of [5] and arguing as in the scalar case (cf theorem 8.16 of [5]), we see that if n is a constant, $m := 1 - n$ and k is a transmission eigenvalue then $|m|k^2\|T\| > 1$ where the operator $T : (L^2(D))^3 \to (L^2(D))^3$ is given by

$$(Tf)(x) := \int_D \Phi(x, y) f(y)\,\mathrm{d}y$$

and

$$\Phi(x, y) := \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|}, \qquad x \neq y.$$

In particular, if E, E_0 is a solution of the interior transmission problem (38)–(41) then E satisfies the integral equation

$$E + mk^2 PTE = 0$$

where $P : (L^2(D))^3 \to H^\perp$ is a projector operator and H is the set of electric fields of electromagnetic Herglotz pairs (see definition 6.29 of [5]). Note that since n is a constant we do not need to resort to the weighted space $\left(L^2_m(D)\right)^3$.

Theorem 3.3. *Suppose that n is a real constant, d is the diameter of D and* $m := 1 - n$. *Then if k is a transmission eigenvalue*

$$|m| > \frac{2\sqrt{6}}{k^2 d^2}. \tag{53}$$

Proof. By the above discussion, we only need to compute the norm of T in $(L^2(D))^3$. To this end, we have that

$$|(Tf)(x)|^2 = \left|\int_D \Phi(x, y) f(y)\,\mathrm{d}y\right|^2 \leqslant \frac{1}{(4\pi)^2}\int_D \frac{\mathrm{d}y}{|x-y|^2}\int_D |f|^2\,\mathrm{d}y. \tag{54}$$

But

$$\int_D \frac{\mathrm{d}y}{|x-y|^2} \leqslant \int_{|x-y|<d} \frac{1}{|x-y|^2}\,\mathrm{d}y = \int_0^d \int_0^{2\pi} \sin\theta\, d\theta\, d\pi\, \mathrm{d}r = 4\pi d \tag{55}$$

and hence $|(Tf)(x)|^2 \leqslant \frac{d}{4\pi}\|f\|^2$. We now have that

$$\|Tf\|^2 = \int_D |Tf|^2\,\mathrm{d}x \leqslant \frac{d^4}{24}\|f\|^2, \tag{56}$$

i.e., $\|T\| \leqslant d^2/2\sqrt{6}$ and the theorem follows. □

We note that similar estimates as those in theorem 3.3 can be obtained for variable index of refraction by using the results of Kirsch [11].

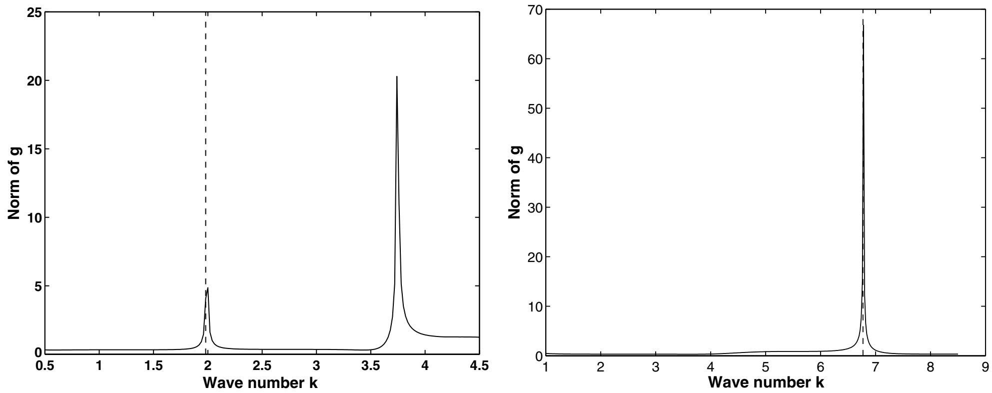
4. Numerical tests

We shall present some simple tests in two dimensions for the Helmholtz equation. We have demonstrated in the past [2] that the linear sampling method can determine, in a qualitative way, the support of the scatterer. Once the shape of the scatter is approximated it is then easy to determine an approximation to the interior Dirichlet eigenvalue $\lambda_0(D)$ and an approximation to the diameter of D. To apply estimate (16) or (28) it remains to find the smallest transmission eigenvalue k_0^2 of D. To this end, we propose to use (14) as described in section 2. However, given that the Herglotz wavefunction computed using the regularized solution of (14) is only an approximation of the appropriate component of the solution of the interior transmission problem, it is not clear that success is assured. It could be that regularization and approximation will obscure the eigenvalues. So, we shall concentrate on the problem of retrieving the transmission eigenvalues from far field data over a range of frequencies and assume that D is precisely known.

We shall use a cubic finite-element code to predict the far field pattern of a given scatterer for many incident directions d and observation directions $\hat{x}$ (precisely, 61 directions for each k uniformly distributed on the unit circle) and a range of wave numbers k. The same finite-element grid is used for all wave numbers (the grid is suitable for the highest wave number) and a perfectly matched layer (PML) of fixed width and parameters is used to truncate the domain. This limits the range of wave numbers at the top end (grid size) and bottom (PML accuracy). Once the approximate far field pattern is known (with roughly 1% noise added as in [4]), we can solve (14) using Tikhonov regularization and the Morozov discrepancy principle as in [4] for each wave number, and try to determine the transmission eigenvalues from peaks in a graph of the L^2-norm of g against k.

Our first example, in which the scatterer is a circle, allows us to analytically compare computed transmission eigenvalues with the results of numerical experiments and verify that the procedure works in this very special case. It also allows us to probe the accuracy of the reconstruction for a wide variety of index of refraction n and investigate the effect of absorption on the estimate.

In the second example, we consider a rectangular scatterer for which the transmission eigenvalues are not known and show that, again, reasonable candidates for the transmission eigenvalues can be determined by a frequency sweep of the solution of (14). The last example is a more complicated L-shaped scatterer.



(a) (b)

Figure 1. The first eigenvalue can be detected from the far field pattern. In the left panel, we show a graph of $\|g\|_{L^2(\Omega)}$ against k for the circle with $n = 16$ using far field data computed using the finite-element method. The left most peak is a good candidate for the lowest transmission eigenvalue and is confirmed using the exact value determined by the determinant criterion (57) and marked as a dashed line. The right panel shows the same result for $n = 4$ where the lowest transmission eigenvalue has increased markedly.

4.1. Circle

In this section, we choose D to be the circle of diameter $d = 1$ in which case $\lambda_0(D) \approx 23$. For a circle with constant $n(x)$, the first transmission eigenvalue is the lowest positive value of $\lambda_0(D) = k^2$ for which

$$\det \begin{pmatrix} J_0(kd/2) & J_0(k\sqrt{n}d/2) \\ -J_1(kd/2) & -\sqrt{n}J_1(k\sqrt{n}d/2) \end{pmatrix} = 0. \tag{57}$$

When $n = 16$ and $d = 1$ this gives an estimate for the lowest transmission eigenvalue of $k_0 = 1.99$. In figure 1, we show a plot of $\|g\|_{L^2(\Omega)}$ against k where g approximately satisfies (14) and is computed as described above for 201 wave numbers equally distributed in $[0.5, 4.5]$ (data from the finite-element solver are used). It is clear that candidates for transmission eigenvalues are visible as peaks in the plot. We also superimpose the true value of k. The match between the eigenvalue computed by the far field pattern and the 'exact' value is very good. In figure 1(b) this is repeated for $n = 4$ where the lowest transmission eigenvalue is $k = 6.77$ computed from the above determinant. This example suggests that the far field equation (14) does provide a means of detecting transmission eigenvalues.

In figure 2, we now present the results of using analytically determined transmission eigenvalues (computed using a numerical approximation of the roots of (57)) to estimate n. This removes error due to the finite-element solver and focuses on the properties of (16) and (28). We show three curves each estimating a lower bound for n. One is computed using (16), the second using (28) and the third is a composite curve taking the maximum of the two estimates under the *a priori* assumption that $n > 1$. Estimate (28) does not give a good lower bound (for example when the exact value is $n = 3$, the lower bound from (28) is $n > 1.0012$). Estimate (16) improves as n increases but ultimately gives an estimated value roughly $1/3$ of the exact value of n.

Although not covered by our theory, it is interesting to consider how absorption effects the algorithm. In figure 3, we plot the norm of g against k when $n = 16 + \mathrm{i}$ using data from

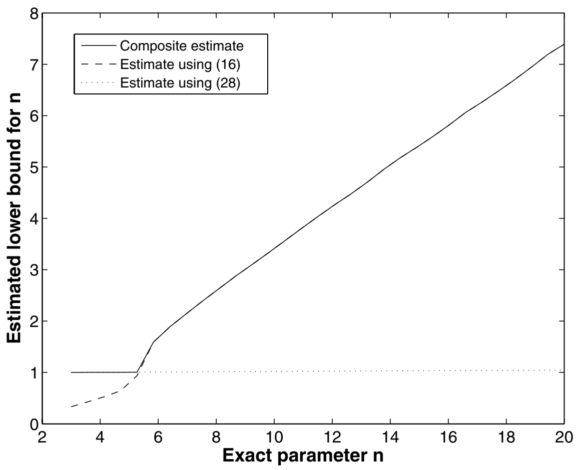


Figure 2. Use of the two estimates of this paper to provide a lower bound for n in the case of a circle. Here, the exact value of n is varied from $n = 3$ to $n = 20$. For each n, the lowest transmission eigenvalue is computed from (57) and the two estimates (16) and (28) evaluated. The composite curve is the maximum of the two estimates assuming $n > 1$. Clearly for low n, neither estimate works well whereas for larger values of n, estimate (16) gives an increasing lower bound that underestimates the true value of n by approximately $1/3$.

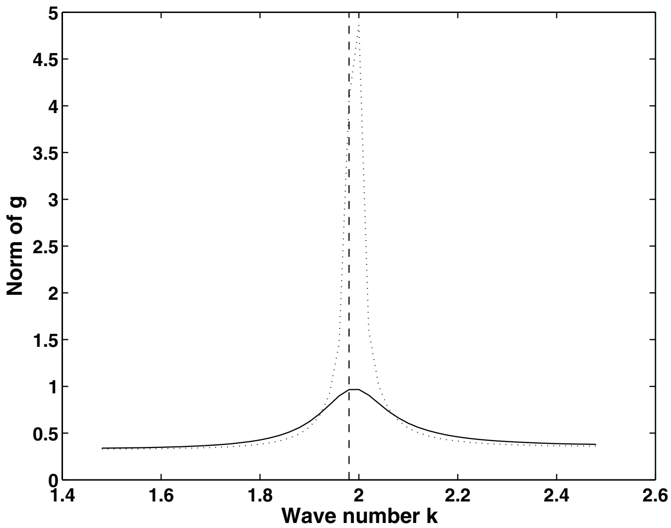


Figure 3. The solid line shows L^2-norm of g plotted against k for the circle with $d = 1$ using $n = 16 + \mathrm{i}$. The dotted line reproduces a portion of the curve from figure 1(a) for comparison and the dashed line gives the exact value of the first transmission eigenvalue when $n = 16$ (with absorption, no eigenvalues exist).

the finite-element solver. The peak at $k = 2$ is now decreased in amplitude compared to figure 1(a) but the presence of an eigenvalue at roughly $k = 2$ is still clearly visible suggesting that the algorithm might be used to estimate a lower bound for the real part of n even for an absorbing medium.

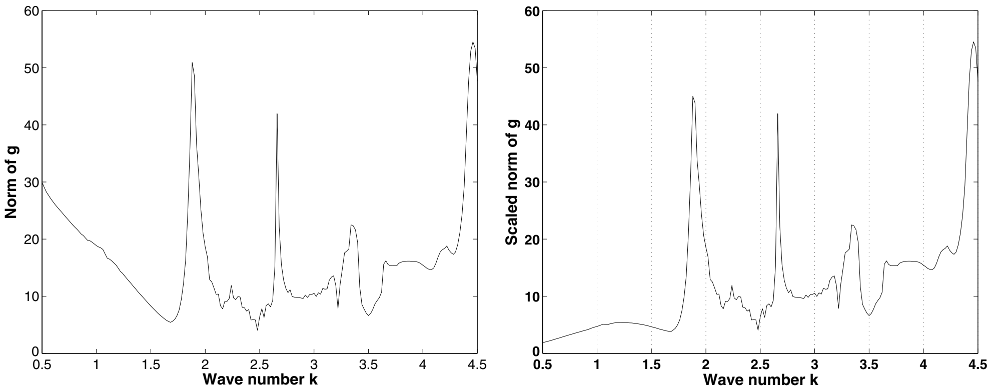


Figure 4. In the left panel, we show $\|g\|_{L^2(\Omega)}$ against k for the rectangle with $n = 16$ using far field data computed using the finite-element method. The large value of the norm of g for small k is likely due to the low frequency decay of the magnitude of the far field pattern. Once this is scaled out, a good candidate for the first transmission eigenvalue at $k_0 = 1.88$ is revealed in the right panel.

4.2. Rectangle

Next, we consider scattering from the rectangle $[-a/2, a/2] \times [-b/2, b/2]$ with $a = 1$ and $b = 0.8$. The lowest Dirichlet eigenvalue can easily be calculated analytically to obtain

$$\lambda_0 = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \approx 25.3.$$

The far field patterns are computed using the finite-element method for $n = 16$ in D and the norm of g is computed as outlined at the start of this section. The results are shown in figure 4. In the left panel, we show a simple plot of the norm of g against k. The norm is at first large and decreases as k increases. This is most likely not indicative of a transmission eigenvalue below $k = 0.5$. From the Born approximation (or the Lippman–Schwinger equation), we know that for small $k\alpha$ (where α is a representative radius of the object) the magnitude of the far field pattern decreases proportionally to $(k\alpha)^2$. Hence, for small $k\alpha$ the least-squares norm of g will increase as $1/(k\alpha)^2$. If we scale the norm of g by a multiplicative factor of $\min((k\alpha)^2, 1)$ (with $\alpha = 1/2$) we can scale away this behaviour to obtain the scaled result in the right-hand panel of figure 4. We see that the first transmission eigenvalue is predicted to be the similar to that of the circle: $k_0 = 1.88$. Hence, we estimate

$$n \geqslant \frac{25.3}{3.5} \approx 7.1.$$

4.3. An L-shaped scatterer

We repeat the determination of a lower bound for n for the L shaped scatterer shown in figure 5 and $n = 16$. The lowest Dirichlet eigenvalue for this domain can easily be computed using adaptive finite elements, but an accurate estimate is also given in [8]: $\lambda_0 = 9.658\ldots$. In figure 6, we plot the norm of g against k and the scaled norm of g against k (as discussed in the previous section). We expect that the norm of g is large for small k and that this can

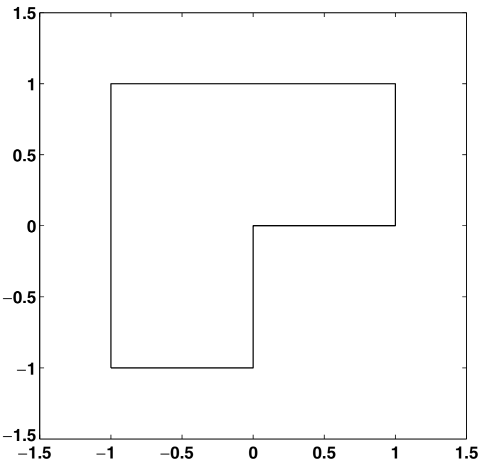


Figure 5. The L-shaped scatterer used in this study (from [8]).

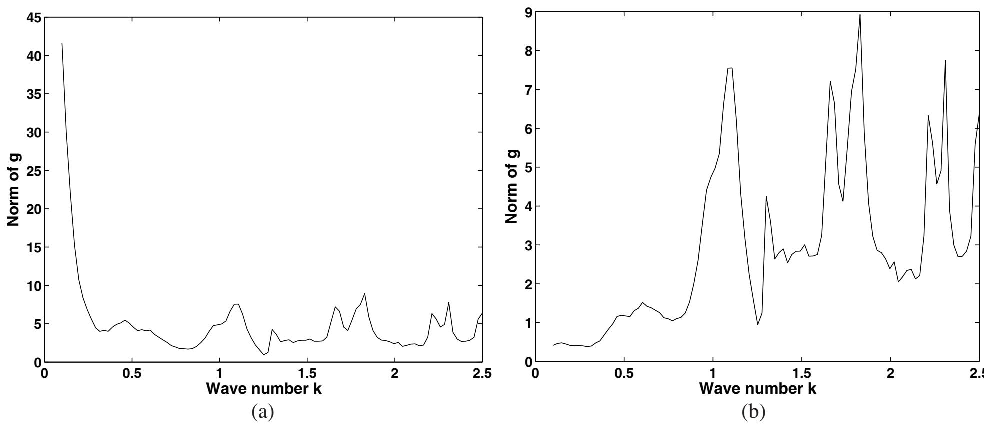


Figure 6. In the left panel, we show $\|g\|_{L^2(\Omega)}$ against k for the L-shape with $n = 16$ using far field data computed using the finite-element method. As in the case of figure 4, the large value of the norm of g for small k is due to the low frequency decay of the magnitude of the far field pattern. Once this is scaled out, a good candidate for the first transmission eigenvalue at $k_0 = 1.09$ is revealed in the right panel.

be scaled away using the procedure from the previous section revealing an estimate for the lowest transmission eigenvalue of $k_0 \approx 1.09$. This gives a lower bound for n of 8.1.

5. Conclusion

We have numerically demonstrated that transmission eigenvalues corresponding to an inhomogeneous medium of compact support D and real index of refraction can be determined from the far field pattern of the scattered field. If the linear sampling method is used to determine the support of D, the first transmission eigenvalue then provides a lower bound for the index of refraction n. Numerical evidence surprisingly suggests that even if the medium is absorbing a lower bound for the real part of n can be obtained in some circumstances.

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