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Domain sensitivity analysis of the elastic far-field patterns in scattering from nonsmooth obstacles

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Abstract

We consider elastic scattering problems described by the Dirichlet or the Neumann boundary value problem for the elastodynamic equation in the exterior of a 2D bounded domain or in the exterior of a crack. The boundary of the domain is assumed to have a finite set of corner points where the scattered wave may have singular behaviour. The paper is concerned with the sensitivity of the far scattered field with respect to small perturbations of the shape of the scatterer. Using a modification of the method of adjoint problems (K. Dems, Z. Mróz, Internat. J. Solids Structures 20 (1984) 527–552) we obtain a representation for the shape derivative which is well suited for a numerical realization with boundary element methods and which shows in some cases directly the influence of the singularities of the solution on the sensitivity of the far-field patterns.
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1. Formulation of the problem

The mathematical modelling of the scattering of time-harmonic elastic waves from an obstacle Ω_i, surrounded by a homogeneous elastic medium with density $\rho = 1$ and Lamé parameters μ and λ satisfying $\lambda + 2\mu > 0$, and $\mu > 0$, leads to an exterior boundary value problem for the Navier equations

$$\Delta^*\mathbf{u} + \omega^2\mathbf{u} := \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \omega^2\mathbf{u} = 0 \quad \text{in } \Omega := \mathbb{R}^2\backslash\Omega_i. \tag{1}$$

Here, the total elastic wave $\mathbf{u} = \mathbf{u}^i + \mathbf{u}^s$ is decomposed into a given time-harmonic incident wave $\mathbf{u}^i$ with the frequency ω and the unknown scattered wave $\mathbf{u}^s$.

Let us assume that the exterior domain $\Omega := \mathbb{R}^2\backslash\overline{\Omega}_i$ is of one of the following two types:

(B) Ω_i is a bounded domain with a piecewise smooth boundary Γ. We denote by $S = \{P_1,\dots,P_Q\}$ the finite set of boundary points, such that $\Gamma\backslash S$ is smooth. Furthermore, we assume that Ω is locally diffeomorphic in the neighbourhood of every corner point P_q to an infinite cone C_q with the opening angle $\varphi_q^0 \notin \{0, 2\pi\}$. The unit normal vector $\mathbf{n} = (n_1, n_2)$ on Γ is directed towards Ω_i.

(C) $\Omega_i = \Gamma$ is a crack, i.e., a piecewise smooth curve with a finite set $S = \{P_1,\dots,P_Q\}$ consisting of two crack tips and of several interior corner points. The corner points must satisfy the angle condition $\omega_q^0 \neq \{0, 2\pi\}$. The direction of the unit normal vector $\mathbf{n}$ on Γ is chosen arbitrarily but is fixed along the crack.

The scattered wave $\mathbf{u}^s$ is requested to satisfy the Dirichlet boundary conditions

$$\mathbf{u}^s(\mathbf{x}) = -\mathbf{u}^i(\mathbf{x}) \quad \text{for } \mathbf{x}\in\Gamma, \tag{2}$$

or the Neumann boundary conditions

$$\mathbf{T}_{\mathbf{n}(\mathbf{x})}\mathbf{u}^s(\mathbf{x}) = -\mathbf{T}_{\mathbf{n}(\mathbf{x})}\mathbf{u}^i(\mathbf{x}) \quad \text{for } \mathbf{x}\in\Gamma. \tag{3}$$

Here $\mathbf{T}_\mathbf{n}$ denotes the matrix surface traction operator defined by

$$(\mathbf{T}_{\mathbf{n}(\mathbf{x})})_{i,j} := \lambda\mathbf{n}_i(\mathbf{x})\frac{\partial}{\partial x_j} + \mu\mathbf{n}_j(\mathbf{x})\frac{\partial}{\partial x_i} + \mu\delta_{i,j}\nabla_x\cdot\mathbf{n}(\mathbf{x}), \quad i,j = 1,2. \tag{4}$$

The Neumann boundary conditions can be given in terms of the elastic stress tensor, namely $\mathbf{T}_\mathbf{n}\mathbf{u} = \sigma(\mathbf{u})\cdot\mathbf{n}$, where

$$\sigma_{i,j}(\mathbf{u}) := \lambda\delta_{ij}\nabla\cdot\mathbf{u} + \mu\left(\frac{\partial\mathbf{u}_i}{\partial x_j} + \frac{\partial\mathbf{u}_j}{\partial x_i}\right),$$

for $i, j = 1, 2$.

For crack problems (C) the boundary conditions have to be posed on both sides of the crack, i.e.,

$$\mathbf{u}^s_\pm(\mathbf{x}) = -\mathbf{u}^i(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma, \quad \text{or}$$
$$\mathbf{T}_{\mathbf{n}(\mathbf{x})}\mathbf{u}^s_\pm(\mathbf{x}) = -\mathbf{T}_{\mathbf{n}(\mathbf{x})}\mathbf{u}^i(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma, \tag{5}$$

where

$$\mathbf{u}^s_\pm(\mathbf{x}) = \lim_{h\to 0+} \mathbf{u}^s(\mathbf{x} \pm h\mathbf{n}), \tag{6}$$
$$\mathbf{T}_{\mathbf{n}(\mathbf{x})}\mathbf{u}^s_\pm(\mathbf{x}) = \lim_{h\to 0+} \sigma\big(\mathbf{u}^s(\mathbf{x} \pm h\mathbf{n})\big) \cdot \mathbf{n}. \tag{7}$$

The boundary conditions describe the physical scattering properties of the obstacle: the Dirichlet conditions model a rigid elastic unpenetrable scatterer while the Neumann conditions model a cavity.

In addition, the scattered field $\mathbf{u}^s$ is required to satisfy the Kupradze radiation condition [13]

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \mathbf{u}^s_p}{\partial r} - ik_p\mathbf{u}^s_p\right) = 0, \qquad \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \mathbf{u}^s_s}{\partial r} - ik_s\mathbf{u}^s_s\right) = 0,$$
$$r = |\mathbf{x}|, \tag{8}$$

uniformly in all directions $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, where $\mathbf{u}^s = \mathbf{u}^s_p + \mathbf{u}^s_s$ indicates the Helmholtz decomposition of $\mathbf{u}^s$ into longitudinal or P-wave $\mathbf{u}^s_p$ ($\nabla \times \mathbf{u}^s_p \equiv 0$) and transversal or S-wave $\mathbf{u}^s_s$ ($\nabla \cdot \mathbf{u}^s_s \equiv 0$) (see, e.g., [6]). Longitudinal and transversal waves are solution to the vectorial Helmholtz equation with the wave number k_p and k_s, respectively, given by

$$k_p^2 = \frac{\omega^2}{\lambda + 2\mu}, \qquad k_s^2 = \frac{\omega^2}{\mu}.$$

In the following, we will consider incident fields elastic plane waves $\mathbf{u}^i := \mathbf{u}^i_p + \mathbf{u}^i_s$ in the form of a linear combination of longitudinal plane waves

$$\mathbf{u}^i_p(\mathbf{x}; \hat{\mathbf{d}}) = \hat{\mathbf{d}} e^{-ik_p \mathbf{x}\cdot\hat{\mathbf{d}}}$$

and transversal plane waves

$$\mathbf{u}^i_s(\mathbf{x}; \hat{\mathbf{d}}) = \hat{\mathbf{b}} e^{-ik_s \mathbf{x}\cdot\hat{\mathbf{d}}},$$

where the unit vector $\hat{\mathbf{d}} \in \mathbb{R}^2$ denotes the direction of propagation and the unit vector $\hat{\mathbf{b}} \in \mathbb{R}^2$ is a polarization vector such that $\hat{\mathbf{b}} \perp \hat{\mathbf{d}}$.

Let B_R be the open ball with radius R centred at 0 and $S_R = \partial B_R$. It is known [9,22] that there exists a unique solution $\mathbf{u}^s(\mathbf{x}) = \mathbf{u}^s_p(\mathbf{x}) + \mathbf{u}^s_s(\mathbf{x})$ of the direct problem (1)–(8), which belongs to $(H^1(\Omega \cap B_R))^3$ and satisfies the following identity for large enough R:

$$\mathbf{u}^s(\mathbf{x}) = \int_{S_R} \big\{ \big[\mathbf{T}_{\mathbf{n}(\mathbf{y})}\Phi(\mathbf{x},\mathbf{y})\big]^\top \mathbf{u}^s(\mathbf{y}) - \Phi(\mathbf{x},\mathbf{y})\mathbf{T}_{\mathbf{n}(\mathbf{y})}\mathbf{u}^s(\mathbf{y}) \big\}\, ds_{\mathbf{y}}. \tag{9}$$

Here the normal vector $\mathbf{n}$ is pointed outward, and $\Phi(\mathbf{x},\mathbf{y})$ is the fundamental solution tensor to the Navier equations given by

$$\Phi(\mathbf{x},\mathbf{y}) := \frac{i}{4\mu} H_0^{(1)}\big(k_s|\mathbf{x}-\mathbf{y}|\big)\mathbf{I} + \frac{i}{4\omega^2}\nabla_x \otimes \nabla_x\big[H_0^{(1)}\big(k_s|\mathbf{x}-\mathbf{y}|\big) - H_0^{(1)}\big(k_p|\mathbf{x}-\mathbf{y}|\big)\big]$$

in terms of the identity matrix $\mathbf{I}$ and the Hankel function $H_0^{(1)}$ of order zero of the first kind.

Applying the radiation condition (8) and the asymptotic behaviour of the fundamental solution Φ to the integral identity (9), we obtain the following behaviour of the scattered wave $\mathbf{u}^s$ as $r \to \infty$ and uniformly in $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ (see [6,8,12]):

$$\mathbf{u}_p^s(\mathbf{x}) = \frac{1}{\lambda+2\mu}\frac{e^{i\pi/4}}{\sqrt{8\pi k_p}}\frac{e^{ik_p r}}{\sqrt{r}}\mathcal{F}_p(\Gamma)(\hat{\mathbf{x}}) + o\left(\frac{1}{\sqrt{r}}\right), \tag{10}$$

$$\mathbf{u}_s^s(\mathbf{x}) = \frac{1}{\mu}\frac{e^{i\pi/4}}{\sqrt{8\pi k_s}}\frac{e^{ik_p r}}{\sqrt{r}}\mathcal{F}_s(\Gamma)(\hat{\mathbf{x}}) + o\left(\frac{1}{\sqrt{r}}\right). \tag{11}$$

The vector functions $\mathcal{F}_p(\Gamma)$, $\mathcal{F}_s(\Gamma)$, defined on the unit circle S_1, are known as the far-field patterns or the scattering amplitudes of the longitudinal part $\mathbf{u}_p^s$ and the transversal part $\mathbf{u}_s^s$, respectively. From the existence and the uniqueness of the solution of the direct scattering problem follows that the far-field patterns are uniquely determined by the boundary Γ. The far-field patterns are given by the following integral formulae:

$$\mathcal{F}_p(\Gamma)(\hat{\mathbf{x}}) = \int_{S_R}\Big\{\big[\mathbf{T}_{\mathbf{n}(\mathbf{y})}\hat{\mathbf{x}}\otimes\hat{\mathbf{x}}e^{-ik_p\hat{\mathbf{x}}\cdot\mathbf{y}}\big]^\top\mathbf{u}^s(\mathbf{y}) - \hat{\mathbf{x}}\otimes\hat{\mathbf{x}}e^{-ik_p\hat{\mathbf{x}}\cdot\mathbf{y}}\mathbf{T}_{\mathbf{n}(\mathbf{y})}\mathbf{u}^s(\mathbf{y})\Big\}\,ds_{\mathbf{y}} \tag{12}$$

and

$$\mathcal{F}_s(\Gamma)(\hat{\mathbf{x}}) = \int_{S_R}\Big\{\big[\mathbf{T}_{\mathbf{n}(\mathbf{y})}[\mathbf{I}-\hat{\mathbf{x}}\otimes\hat{\mathbf{x}}]e^{-ik_s\hat{\mathbf{x}}\cdot\mathbf{y}}\big]^\top\mathbf{u}^s(\mathbf{y}) - [\mathbf{I}-\hat{\mathbf{x}}\otimes\hat{\mathbf{x}}]e^{-ik_s\hat{\mathbf{x}}\cdot\mathbf{y}}\mathbf{T}_{\mathbf{n}(\mathbf{y})}\mathbf{u}^s(\mathbf{y})\Big\}\,ds_{\mathbf{y}}. \tag{13}$$

We note that the longitudinal far-field pattern $\mathcal{F}_p$ is normal to S_1 whereas the transversal far-field pattern $\mathcal{F}_s$ is tangential to S_1. Although the boundary of the domain representing the scattering object is nonsmooth, the integral representations (11)–(12) show that the far-field patterns are analytic functions.

Our main concern in this work is to study how the perturbation of the domain influences the far-field operators $\Gamma \mapsto \mathcal{F}_p(\Gamma)$ and $\Gamma \mapsto \mathcal{F}_s(\Gamma)$. This is performed using the material derivative approach, which is well known in the form sensitivity analysis of bounded elastic bodies [4,5,20]. We derive formulae for the Gâteaux derivative of the elastic far-field pattern with respect to an admissible class of domain perturbations and show that the derivative depends in fact only on the perturbation of the boundary.

2. Exterior boundary value problems for the Navier equations

In this section we formulate the existence and regularity results for solutions of the exterior boundary value problem

$$\Delta^*\mathbf{u}(\mathbf{x}) + \omega^2\mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad \text{in } \Omega, \tag{14}$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{on } \Gamma, \quad \text{or} \tag{15}$$

$$\mathbf{T}_{\mathbf{n}(\mathbf{x})}\mathbf{u}(\mathbf{x}) = \mathbf{h}(\mathbf{x}) \quad \text{on } \Gamma, \tag{16}$$

which satisfy the Kupradze radiation condition (8) at infinity.

To this end we introduce weighted Sobolev spaces which take into account the singular behaviour of the functions near the singular points $P_q \in S$ and at infinity.

Definition 2.1. Let $\Omega = \mathbb{R}^2 \backslash \Omega_i$ with Ω_i being of type (B) or (C) as defined in Section 1. We choose for every singular point $P_q \in S$ a cut-off function $\eta_q \in C_0^\infty(\mathbb{R}^2)$ with support in a neighbourhood of P_q and set $\eta_0 = 1 - \sum_{q=1}^{Q} \eta_q$. For $d \in \mathbb{N}_0$, $\vec{\beta} = (\beta_1, \ldots, \beta_Q) \in \mathbb{R}^Q$ we define the space $V^d_{\vec{\beta},\gamma}(\Omega)$ of all generalized functions which have the finite norm

$$\|u\|_{V^d_{\vec{\beta},\gamma}(\Omega)} := \left\| \left(1 + |\mathbf{x}|^2\right)^{-\gamma/2} (\eta_0 u) \right\|_{H^d(\Omega)} + \sum_{q=1}^{\Omega} \sum_{|p| \leqslant d} \left\| r_q^{\beta_q - d + |p|} D^p (\eta_q u) \right\|_{L_2(\Omega)}, \tag{17}$$

where $r_q = \operatorname{dist}(\mathbf{x}, P_q)$. For $d = 1, 2, \ldots,$ we denote by $V^{d-1/2}_{\vec{\beta}}(\Gamma)$ the space of traces on $\Gamma \backslash S$ of functions in $V^d_{\vec{\beta},\gamma}(\Omega)$. Furthermore we set $\mathbf{V}^d_{\vec{\beta},\gamma}(\Omega) = [V^d_{\vec{\beta},\gamma}(\Omega)]^2$ and $\mathbf{V}^d_{\vec{\beta},\gamma}(\Gamma) = [V^d_{\vec{\beta},\gamma}(\Gamma)]^2$.

The behaviour of the solution $\mathbf{u}$ of the boundary value problem (14)–(16) in a neighbourhood of the singular point $P_q \in S$ can be described with the help of solutions of a homogeneous boundary value problem for the Lamé operator Δ^* in the infinite sector with opening angle φ_q^0. It can be shown [11,16] that the set of all solutions of this homogeneous problem has a basis consisting of the so-called "power solutions"

$$\mathbf{v}_j^{l,k}(r_q, \varphi_q) = r_q^{\alpha_j} \sum_{s=0}^{l} \frac{1}{s!} (\log r_q)^s \omega_j^{l-s,k}(\varphi_q), \tag{18}$$

where (r_q, φ_q) are polar coordinates with origin in P_q and $\{\omega_j^{l,k}\colon 1 \leqslant k \leqslant m_g(\alpha_j),\ 0 \leqslant l \leqslant \kappa_{k,j} - 1\}$ is a canonical system of Jordan chains of some linear operator pencil $\mathcal{A}_q$ corresponding to the eigenvalue α_j. Here $m_g(\alpha_j)$ is the geometric multiplicity of α_j and $\kappa_{k,j}$ is the length of the kth Jordan chain. The spectrum of $\mathcal{A}_q$ will be denoted by $\Sigma(\mathcal{A}_q)$.

Let us define $a_q := \min\{\Re\alpha_j\}$, where the minimum is taken over all eigenvalues α_j of $\mathcal{A}_q$ with a positive real part. Note that in both Dirichlet and Neumann case $a_q \geqslant 1/2$. Then we denote by a_0 the smallest of a_q for every singular point $P_q \in S$.

Theorem 2.1 (Existence and regularity result). *Let $d \in \mathbb{N}_0$, $\gamma > 1$ and $\vec{\beta} = (\beta_1, \ldots, \beta_Q) \in \mathbb{R}^Q$ with*

$$d + 1 - \beta_q \in (0, a_q) \quad \text{for } P_q \in S.$$

Suppose that $\mathbf{f} \in V^d_{\vec{\beta},-\gamma}(\Omega)$, $\mathbf{g} \in V^{d+3/2}_{\vec{\beta}}(\Gamma)$ and $\mathbf{h} \in V^{d+1/2}_{\vec{\beta}}(\Gamma)$. Then there exist a unique solution $\mathbf{u} \in V^{d+2}_{\vec{\beta},\gamma}(\Omega)$ of (14)–(16) *and the following a priori estimate is valid*:

$$\|\mathbf{u}\|_{V^{d+2}_{\vec{\beta},\gamma}(\Omega)} \leqslant c\Big\{\|\mathbf{f}\|_{V^d_{\vec{\beta},-\gamma}(\Omega)} + c_D\|\mathbf{g}\|_{V^{d+3/2}_{\vec{\beta}}(\Gamma)} + c_N\|\mathbf{h}\|_{V^{d+1/2}_{\vec{\beta}}(\Gamma)}\Big\}. \tag{19}$$

Here, c is a positive constant, $c_D = 1$, $c_N = 0$ in the case of the Dirichlet problem and $c_D = 0$, $c_N = 1$ in the case of the Neumann problem.

Proof. With the help of a priori estimates for solutions of boundary value problems in the exterior of a smooth domain [21] and in bounded domains with corners [10,11,16], we can prove by means of a partition of unity that

$$\|\mathbf{u}\|_{V^{d+2}_{\vec{\beta},\gamma}(\Omega)} \leqslant c\Big\{\|\mathbf{f}\|_{V^d_{\vec{\beta},-\gamma}(\Omega)} + c_D\|\mathbf{g}\|_{V^{d+3/2}_{\vec{\beta}}(\Gamma)} + c_N\|\mathbf{h}\|_{V^{d+1/2}_{\vec{\beta}}(\Gamma)} + \|\mathbf{u}\|_{V^{d+1}_{\vec{\beta},\gamma}(\Omega\cap B_R)}\Big\} \tag{20}$$

with some real constant c and some positive R. The last norm on the right-hand side of (20) can be omitted as in [1, Lemma III, 3.10] because the kernel of the problem is trivial. In case of a smooth domain Theorem 2.1 reduces to the result proved in [21]. □

Theorem 2.2 (Asymptotic behaviour near singular points). *Suppose that $\mathbf{u}$ satisfies the homogeneous elastic Navier equations and homogeneous boundary conditions in the neighbourhood of the singular point $P_q \in S$. Let $\alpha_1, \ldots, \alpha_N$ be all eigenvalues of $\mathcal{A}_q$ with $0 < \Re\alpha_j < 1$ and let $\mathbf{v}_j^{l,k}$ be the corresponding singular functions defined by* (18). *Then $\mathbf{u}$ behaves in the vicinity of P_q asymptotically as*

$$\mathbf{u}(r_q, \varphi_q) = \sum_{j=1}^{N} \sum_{k=1}^{m_g(\alpha_j)} \sum_{l=0}^{\kappa_{k,j}-1} K_{j,k,l} \mathbf{v}_j^{l,k}(r_q, \varphi_q) + O(r_q), \tag{21}$$

with $K_{j,k,l} \in \mathbb{R}$.

Proof. The assertion is a simple application of the results from the theory of general elliptic problems in domains with corners [10,11,16]. Explicit formulae for the singular functions $\mathbf{S}_j$ are available in the literature [17]. □

3. Domain sensitivity of elastic fields

3.1. Description of the domain perturbation

In order to describe the shape sensitivity of exterior boundary value problems, i.e., the influence of the shape of the domain on the solution, we introduce a family of perturbed domains Ω_ε, $\varepsilon \in [0, \varepsilon_0]$, as the image of a fixed domain Ω under a family of diffeomorphisms

$$\left\{\Psi_\varepsilon = I + \varepsilon\Psi \in \left[C^{d+2}(\overline{\Omega})\right]^2;\ \varepsilon \in [0, \varepsilon_0]\right\}, \quad d \in \mathbb{N}_0. \tag{22}$$

Thus we have

$$\Omega_\varepsilon := \Psi_\varepsilon(\Omega), \qquad \Gamma_\varepsilon := \Psi_\varepsilon(\Gamma), \qquad S_\varepsilon := \Psi_\varepsilon(S)$$

with S_ε being the set of singular points of Γ_ε. Since we are interested in the perturbation of the boundary Γ we can assume that

$$\exists R: \quad \Psi_\varepsilon(\mathbf{x}) = \mathbf{x} \quad \forall |\mathbf{x}| > R. \tag{23}$$

We consider the following exterior Dirichlet or Neumann boundary value problem:

$$\begin{aligned}
&\Delta^* \mathbf{u}_\varepsilon^s(\mathbf{x}_\varepsilon) + \omega^2 \mathbf{u}_\varepsilon^s(\mathbf{x}_\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon,\\
&\mathbf{u}_\varepsilon^s(\mathbf{x}_\varepsilon) = -\mathbf{u}^i \quad \text{on } \Gamma_\varepsilon, \quad \text{or}\\
&\mathbf{T}_{\mathbf{n}(\mathbf{x}_\varepsilon)} \mathbf{u}_\varepsilon^s(\mathbf{x}_\varepsilon) = -\mathbf{T}_{\mathbf{n}(\mathbf{x}_\varepsilon)} \mathbf{u}^i \quad \text{on } \Gamma_\varepsilon,
\end{aligned} \tag{24}$$

for the scattered wave $\mathbf{u}_\varepsilon^s$ satisfying the radiation condition (8).

3.2. Form sensitivity of the scattered field

Let $\mathbf{U}_\varepsilon := \mathbf{u}_\varepsilon^s + \eta \mathbf{u}^i$, where η is a cut-off function with support in the vicinity of the boundary Γ. We note that $\mathbf{U}_\varepsilon$ satisfies the Navier equations with a right-hand side having a compact support and vanishing in the neighbourhood of Γ_ε and satisfies homogeneous boundary conditions (2) or (3). Furthermore, $\mathbf{U}_\varepsilon$ coincides with $\mathbf{u}_\varepsilon^s$ outside some neighbourhood of Γ_ε.

The form sensitivity of the vector field $\mathbf{U}_\varepsilon$ can be described with the help of the material derivative

$$\dot{\mathbf{U}} := \frac{d(\mathbf{U}_\varepsilon \circ \Psi_\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} \tag{25}$$

and the shape derivative

$$\mathbf{U}' := \dot{\mathbf{U}} - \nabla \mathbf{U}_0 \cdot \Psi. \tag{26}$$

In order to prove the existence of these derivatives we transform the problem (24) into the reference configuration by means of a change of variables $\mathbf{x}_\varepsilon = \Psi_\varepsilon(\mathbf{x})$ and obtain a boundary value problem for the transformed field $\mathbf{U}_\varepsilon \circ \Psi_\varepsilon$:

$$\begin{aligned}
&\Delta^{*\varepsilon}(\mathbf{U}_\varepsilon \circ \Psi_\varepsilon) + \omega^2(\mathbf{U}_\varepsilon \circ \Psi_\varepsilon) = \Delta^{*\varepsilon}(\eta \mathbf{u}^i \circ \Psi_\varepsilon) + \omega^2(\eta \mathbf{u}^i \circ \Psi_\varepsilon) \quad \text{in } \Omega,\\
&(\mathbf{U}_\varepsilon \circ \Psi_\varepsilon) = 0 \quad \text{on } \Gamma, \quad \text{or}\\
&\mathbf{T}^\varepsilon_{\mathbf{n}_\varepsilon \circ \Psi_\varepsilon}(\mathbf{U}_\varepsilon \circ \Psi_\varepsilon) = 0 \quad \text{on } \Gamma.
\end{aligned}$$

Here, the operators $\Delta^{*\varepsilon}$ and $\mathbf{T}^\varepsilon$ have variable coefficients which depend smoothly on the perturbation parameter ε. Therefore we can apply the theory of regularly perturbed partial differential equations (see, e.g., [15, Section 5.5]) and show that $(\mathbf{U}_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x})$ depends smoothly on ε:

$$(\mathbf{U}_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x}) = \mathbf{U}_0(\mathbf{x}) + \varepsilon \dot{\mathbf{U}}(\mathbf{x}) + O(\varepsilon^2). \tag{27}$$

In fact, using the a priori estimate (19) one can prove the following theorem (see [3] for a detailed proof in case of acoustic scattering).

Theorem 3.1. *Let $d \in \mathbb{N}_0$ and $\vec{\beta} \in \mathbb{R}^Q$ be defined as in Theorem* 2.1. *Then the following estimate is valid*:

$$\|\mathbf{U}_\varepsilon \circ \Psi_\varepsilon - \mathbf{U}_0 - \varepsilon \dot{\mathbf{U}}\|_{\mathbf{V}^{d+2}_{\vec{\beta},\gamma}(\Omega)} \leqslant c\varepsilon^2 \tag{28}$$

with a positive real constant c.

The existence and the regularity of the shape derivative $\mathbf{U}'$ follows directly from the definition (26) of $\mathbf{U}'$ and the preceding theorem.

Corollary 3.1. *Let the assumptions of Theorem* 3.1 *be satisfied. Then the shape derivative $\mathbf{U}'$ exists in $\mathbf{V}^{d+1}_{\vec{\beta},\gamma}(\Omega)$.*

According to [14,18,19] the shape derivative $\mathbf{U}'$ satisfies the radiation condition (8) at infinity and solves the following exterior boundary value problem:

$$\begin{aligned}
&\Delta^* \mathbf{U}' + \omega^2 \mathbf{U}' = 0 \quad \text{in } \Omega,\\
&\mathbf{U}' = -\Psi \cdot \mathbf{n} \frac{\partial \mathbf{U}_0}{\partial \mathbf{n}} \quad \text{on } \Gamma, \quad \text{or}\\
&\mathbf{T}_\mathbf{n} \mathbf{U}' = -\Psi \cdot \mathbf{n} \frac{\partial}{\partial \mathbf{n}} \sigma(\mathbf{U}_0) \cdot \mathbf{n} + \sigma(\mathbf{U}_0) \cdot \nabla_\Gamma (\Psi \cdot \mathbf{n}) \quad \text{on } \Gamma,
\end{aligned} \tag{29}$$

where $\nabla_\Gamma f$ is the tangential gradient given by $\nabla_\Gamma f = \nabla f - (\mathbf{n} \cdot \nabla f)\mathbf{n}$ and $(\partial/\partial\mathbf{n})\sigma(\mathbf{u})$ is the matrix consisting of the normal derivatives of all components of the stress tensor.

Remember that in case of cracks, the boundary condition must be imposed on both sides of the crack as in (5).

3.3. *Form sensitivity of the far field pattern*

Let Ω be an exterior domain of type (B) or (C). The perturbed scattered wave $\mathbf{u}_\varepsilon^s$ has at infinity the asymptotics (10), (11) with the far field pattern $\mathcal{F}(\Gamma_\varepsilon) = (\mathcal{F}_p(\Gamma_\varepsilon), \mathcal{F}_s(\Gamma_\varepsilon))$ given by an analogue of formulae (11), (12) with only the change of $\mathbf{u}^s$ to $\mathbf{u}_\varepsilon^s$. Let us calculate the Gâteaux derivative

$$d\mathcal{F}(\Gamma, \Psi) = \lim_{\varepsilon \to \infty} \frac{\mathcal{F}(\Psi_\varepsilon(\Gamma)) - \mathcal{F}(\Gamma)}{\varepsilon} = \left.\frac{d\mathcal{F}(\Gamma_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}. \tag{30}$$

For big enough $R = |\mathbf{x}|$ we have $\Psi_\varepsilon|_{S_R} = I$ and $\mathbf{u}_\varepsilon^s|_{S_R} = \mathbf{U}_\varepsilon \circ \Psi_\varepsilon|_{S_R}$. Thus

$$\mathcal{F}_p(\Gamma_\varepsilon)(\hat{\mathbf{x}}) = \int_{S_R} \Big\{ \big[\mathbf{T}_{\mathbf{n}(\mathbf{y})} \hat{\mathbf{x}} \otimes \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}}\big]^\top (\mathbf{U}_\varepsilon \circ \Psi_\varepsilon)(\mathbf{y}) - \hat{\mathbf{x}} \otimes \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}} \mathbf{T}_{\mathbf{n}(\mathbf{y})} (\mathbf{U}_\varepsilon \circ \Psi_\varepsilon)(\mathbf{y}) \Big\}\, ds_{\mathbf{y}}, \tag{31}$$

with a similar formula for $\mathcal{F}_s(\Gamma_\varepsilon)$. Differentiating both sides of the above equation by ε and taking $\varepsilon = 0$ we obtain immediately

$$d\mathcal{F}_p(\Gamma)(\hat{\mathbf{x}}) = \int_{S_R} \Big\{ \big[\mathbf{T}_{\mathbf{n}(\mathbf{y})} \hat{\mathbf{x}} \otimes \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}}\big]^\top \dot{\mathbf{U}}(\mathbf{y}) - \hat{\mathbf{x}} \otimes \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}} \mathbf{T}_{\mathbf{n}(\mathbf{y})} \dot{\mathbf{U}}(\mathbf{y}) \Big\}\, ds_{\mathbf{y}}. \tag{32}$$

Since $\dot{\mathbf{U}}(\mathbf{x}) = \mathbf{U}'(\mathbf{x})$ for big enough $|\mathbf{x}|$, we get

$$d\mathcal{F}_p(\Gamma)(\hat{\mathbf{x}}) = \int_{S_R} \Big\{ \big[\mathbf{T}_{\mathbf{n}(\mathbf{y})} \hat{\mathbf{x}} \otimes \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}}\big]^\top \mathbf{U}'(\mathbf{y}) - \hat{\mathbf{x}} \otimes \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}} \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{U}'(\mathbf{y}) \Big\}\, ds_{\mathbf{y}} \tag{33}$$

and

$$d\mathcal{F}_s(\Gamma)(\hat{\mathbf{x}}) = \int_{S_R} \Big\{ \big[\mathbf{T}_{\mathbf{n}(\mathbf{y})} [\mathbf{I} - \hat{\mathbf{x}} \otimes \hat{\mathbf{x}}] e^{-ik_s \hat{\mathbf{x}}\cdot\mathbf{y}}\big]^\top \mathbf{U}'(\mathbf{y}) - [\mathbf{I} - \hat{\mathbf{x}} \otimes \hat{\mathbf{x}}] e^{-ik_s \hat{\mathbf{x}}\cdot\mathbf{y}} \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{U}'(\mathbf{y}) \Big\}\, ds_{\mathbf{y}}. \tag{34}$$

The representations (32), (33) are not well suited for a numerical realization because $\mathbf{U}'$, in general, cannot be defined as a variational solution of the boundary value problem (29). The asymptotic analysis (Theorem 2.2) shows that the solution $\mathbf{U}_0$ behaves near the singular points $P_q \in S$ as $|\mathbf{x} - P_q|^{a_q}$ with $a_q \geqslant 1/2$. Therefore, $\mathbf{U}_0 \in H^{1+a_0}(\Omega)$, $\nabla \mathbf{U}_0 \in H^{a_0}(\Omega)$ and consequently $\mathbf{U}' \in H^{a_0}(\Omega)$ due to (26). If the domain is not convex then we have $a_0 < 1$ and thus $\mathbf{U}' \notin H^1(\Omega)$. Therefore $\mathbf{U}'$ cannot be computed in general by solving (29) numerically with the help of standard boundary element or finite element methods. Furthermore, we are interested in an expression for the derivative of the far field patterns depending only on the perturbation of the boundary.

In order to overcome this difficulty, we derive in the next section from (32) and (33) another representations for both $d\mathcal{F}_p(\Gamma, \Psi)(\hat{\mathbf{x}})$, $d\mathcal{F}_s(\Gamma, \Psi)(\hat{\mathbf{x}})$, which are better suited for a numerical realization. We use the method of adjoint problems [2,3,7], which consists in applying the Betti's formulae to the shape derivative $\mathbf{U}'$ and to the solution $\mathbf{w}$ of an appropriately defined adjoint problem. This leads to an expression in which only $\mathbf{U}_0$ and the adjoint field $\mathbf{w}$ appear.

4. The method of adjoint problems

4.1. Exterior of a bounded domain

Let us assume first that Ω is the exterior of a bounded domain and consider the longitudinal far-field pattern $\mathcal{F}_p$. We define $\mathbf{w}_p$ as the solution of the following mixed boundary value problem:

$$\Delta^* \mathbf{w}_p(\mathbf{y}) + \omega^2 \mathbf{w}_p(\mathbf{y}) = 0 \quad \text{in } \Omega,$$
$$\mathbf{w}_p(\mathbf{y}) = \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}} \quad \text{on } \Gamma, \quad \text{or}$$
$$\mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{w}_p(\mathbf{y}) = \mathbf{T}_{\mathbf{n}(\mathbf{y})} \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}} \quad \text{on } \Gamma,$$

which satisfies the Kupradze radiation condition (8) at infinity. We remark that the adjoint field is the scattered field produced by a longitudinal incident plane wave propagating in the observation direction $\hat{\mathbf{x}}$.

Using Betti's formula for $\mathbf{U}'$ and $\mathbf{w}$ in the domain $B_{R'} \cap B_R \cap \Omega$ with $R' > R$, passing to the limit as $R' \to +\infty$ and taking into account that $\mathbf{U}'$, $\mathbf{w}$ satisfy (8) we obtain

$$0 = \hat{\mathbf{x}} \int_{S_R} \left[\mathbf{w}_p(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{U}'(\mathbf{y}) - \mathbf{U}'(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{w}_p(\mathbf{y})\right] ds_{\mathbf{y}}. \tag{35}$$

Summing up the expressions (35) and (33) we get

$$d\mathcal{F}_p(\Gamma, \Psi)(\hat{\mathbf{x}}) = \hat{\mathbf{x}} \int_{S_R} \left[\mathbf{W}_p(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{U}'(\mathbf{y}) - \mathbf{U}'(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{W}_p(\mathbf{y})\right] ds_{\mathbf{y}}, \tag{36}$$

with $\mathbf{W}_p(\mathbf{y}) := \mathbf{w}_p(\mathbf{y}) - \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}}$. Note that the normal vector $\mathbf{n}$ on S_R is directed outwards.

Let $B_\delta(P_q)$ be a ball with centre in P_q and radius δ. Inserting again $\mathbf{U}'$ and $\mathbf{W}$ into Betti's formula in $\Omega \cap B_R \cap \bigcup_{q=1}^{Q} B_\delta(P_q)$, we obtain from (36)

$$\begin{aligned} d\mathcal{F}_p(\Gamma,\Psi)(\hat{\mathbf{x}}) &= \hat{\mathbf{x}} \int\limits_{\Gamma \setminus \bigcup_{q=1}^{Q} B_\delta(P_q)} \big[\mathbf{U}'(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{W}_p(\mathbf{y}) - \mathbf{W}_p(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{U}'(\mathbf{y})\big]\, ds_{\mathbf{y}} \\ &\quad + \hat{\mathbf{x}} \int\limits_{\bigcup_{q=1}^{Q} \partial B_\delta(P_q) \cap \Omega} \big[\mathbf{U}'(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{W}_p(\mathbf{y}) - \mathbf{W}_p(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{U}'(\mathbf{y})\big]\, ds_{\mathbf{y}}. \end{aligned} \tag{37}$$

Let us pass to the limit as $\delta \to 0$ on both sides of (37) and rewrite it as

$$\begin{aligned} d\mathcal{F}_p(\Gamma,\Psi)(\hat{\mathbf{x}}) &= \hat{\mathbf{x}} \int\limits_{\Gamma} \big[\mathbf{U}'(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{W}_p(\mathbf{y}) - \mathbf{W}_p(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{U}'(\mathbf{y})\big]\, ds_{\mathbf{y}} \\ &\quad + \hat{\mathbf{x}} \lim_{\delta\to 0} \int\limits_{\bigcup_{q=1}^{Q} \partial B_\delta(P_q) \cap \Omega} \big[\mathbf{U}'(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{W}_p(\mathbf{y}) - \mathbf{W}_p(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{U}'(\mathbf{y})\big]\, ds_{\mathbf{y}}. \end{aligned} \tag{38}$$

In the following, we denote by L_q

$$L_q := \lim_{\delta\to 0} \int\limits_{\partial B_\delta(P_q) \cap \Omega} \big[\mathbf{U}'(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{W}_p(\mathbf{y}) - \mathbf{W}_p(\mathbf{y})^\top \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{U}'(\mathbf{y})\big]\, ds_{\mathbf{y}}. \tag{39}$$

Substituting the boundary values of $\mathbf{U}'$ and $\mathbf{W}$ into the first integral in (38), we obtain the following expression for the far-field derivative:

$$d\mathcal{F}_p(\Gamma,\Psi)(\hat{\mathbf{x}}) = -\hat{\mathbf{x}} \int\limits_{\Gamma} \Psi(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \frac{\partial \mathbf{U}_0(\mathbf{y})^\top}{\partial \mathbf{n}_{\mathbf{y}}} \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{W}_p(\mathbf{y})\, ds_{\mathbf{y}} + \hat{\mathbf{x}} \sum_{q=1}^{Q} L_q \tag{40}$$

if the Dirichlet problem is considered, and

$$\begin{aligned} d\mathcal{F}_p(\Gamma,\Psi)(\hat{\mathbf{x}}) = \hat{\mathbf{x}} \sum_{q=1}^{Q} L_q \\ + \hat{\mathbf{x}} \int\limits_{\Gamma} \mathbf{W}_p(\mathbf{y})^\top \bigg[\Psi(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \frac{\partial}{\partial \mathbf{n}_{\mathbf{y}}} \sigma\big(\mathbf{U}_0(\mathbf{y})\big) \cdot \mathbf{n}(\mathbf{y}) \end{aligned}$$

$$- \sigma\big(\mathbf{U}_0(\mathbf{y})\big) \cdot \nabla_\Gamma\big(\Psi(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})\big)\Big] \, ds_{\mathbf{y}} \tag{41}$$

if the Neumann problem is considered.

In order to justify the passage to the limit in (37) we have to investigate the behaviour of the integrand as $\delta \to 0$ for every singular point $P_q \in S$.

Theorem 4.1. *Let Ω be the exterior of a bounded domain. Then $L_q = 0$.*

Proof. According to Theorem 2.2, the functions $\mathbf{U}_0$ and $\mathbf{W}_p$ behave in the neighbourhood of P_q as

$$\mathbf{U}_0(\mathbf{x}) = O(r^{a_q}), \qquad \mathbf{W}_p(\mathbf{x}) = O(r^{a_q}). \tag{42}$$

Consequently, $\mathbf{U}'(\mathbf{x}) = O(r^{a_q - 1})$ due to (26) and so the integrand of (39) behaves as $O(r^{2a_q-2})$. Since $a_q > 1/2$ for $\varphi_q^0 < 2\pi$, then $2a_q - 2 > -1$, which implies that $L_q = 0$. □

Next we consider the far field of the transversal wave. We start with the representation (33) for $d\mathcal{F}_s(\Gamma)$. Since $\mathbf{I} - \hat{\mathbf{x}} \otimes \hat{\mathbf{x}} = \hat{\mathbf{z}} \otimes \hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is a unit vector perpendicular to the direction of observation $\hat{\mathbf{x}}$, we naturally define the corresponding adjoint field $\mathbf{w}_s$ as the solution of the following exterior boundary value problem:

$$\begin{aligned}
&\Delta^* \mathbf{w}_s(\mathbf{y}) + \omega^2 \mathbf{w}_s(\mathbf{y}) = 0 \quad \text{in } \Omega, \\
&\mathbf{w}_s(\mathbf{y}) = \hat{\mathbf{z}} e^{-ik_s \hat{\mathbf{x}} \cdot \mathbf{y}} \quad \text{on } \Gamma, \quad \text{or} \\
&\mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{w}_s(\mathbf{y}) = \mathbf{T}_{\mathbf{n}(\mathbf{y})} \hat{\mathbf{z}} e^{-ik_s \hat{\mathbf{x}} \cdot \mathbf{y}} \quad \text{on } \Gamma,
\end{aligned}$$

with $\hat{\mathbf{z}} \perp \hat{\mathbf{x}}$ and satisfying the Kupradze radiation condition (8) at infinity. Hence in this case the adjoint field is the scattered field produced by a shear plane wave propagating in the observation direction $\hat{\mathbf{x}}$ and polarized in the direction $\hat{\mathbf{z}}$.

Now, by repeating the above calculations for $d\mathcal{F}_s(\Gamma)$ given by (34), $\mathbf{U}'$ and $\mathbf{W}_s := \mathbf{w}_s - \hat{\mathbf{z}} e^{-ik_s \hat{\mathbf{x}} \cdot \mathbf{y}}$, and noting that the Theorem 4.1 remains valid in this case, we obtain the following formula for the shape derivative of the transversal far field:

$$d\mathcal{F}_s(\Gamma, \Psi)(\hat{\mathbf{x}}) = -\hat{\mathbf{z}} \int_\Gamma \Psi(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \frac{\partial \mathbf{U}_0(\mathbf{y})^\top}{\partial \mathbf{n}_{\mathbf{y}}} \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{W}_s(\mathbf{y}) \, ds_{\mathbf{y}} \tag{43}$$

if the Dirichlet problem is considered, and

$$\begin{aligned}
d\mathcal{F}_s(\Gamma, \Psi)(\hat{\mathbf{x}}) = \hat{\mathbf{z}} \int_\Gamma \mathbf{W}_s(\mathbf{y})^\top \Big[&\Psi(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \frac{\partial}{\partial \mathbf{n}_{\mathbf{y}}} \sigma\big(\mathbf{U}_0(\mathbf{y})\big) \cdot \mathbf{n}(\mathbf{y}) \\
&- \sigma\big(\mathbf{U}_0(\mathbf{y})\big) \cdot \nabla_\Gamma\big(\Psi(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})\big)\Big] \, ds_{\mathbf{y}}
\end{aligned} \tag{44}$$

if the Neumann problem is considered.

4.2. Exterior domains with cracks

The above considerations along with formula (37) can be repeated with some obvious changes for problems in the exterior of a crack. The longitudinal adjoint scattered wave is defined as the solution of the following boundary value problem in the exterior of the curve:

$$\begin{aligned} &\Delta^* \mathbf{w}_p(\mathbf{y}) + \omega^2 \mathbf{w}_p(\mathbf{y}) = 0 \quad \text{in } \Omega, \\ &\mathbf{w}_{p\pm}(\mathbf{y}) = \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}} \quad \text{on } \Gamma, \quad \text{or} \\ &\mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{w}_{p\pm}(\mathbf{y}) = \mathbf{T}_{\mathbf{n}(\mathbf{y})} \hat{\mathbf{x}} e^{-ik_p \hat{\mathbf{x}}\cdot\mathbf{y}} \quad \text{on } \Gamma, \end{aligned} \tag{45}$$

satisfying the Kupradze radiation condition at infinity.

Formulae (40), (41) read now

$$\begin{aligned} d\mathcal{F}_p(\Gamma, \Psi)(\hat{\mathbf{x}}) = &-\hat{\mathbf{x}} \int_\Gamma \Psi(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \left[\!\left[\frac{\partial \mathbf{U}_0(\mathbf{y})^\top}{\partial \mathbf{n}_\mathbf{y}} \mathbf{T}_{\mathbf{n}(\mathbf{y})} \mathbf{W}_p(\mathbf{y}) \right]\!\right] ds_\mathbf{y} \\ &+ \hat{\mathbf{x}} \sum_{q=1}^{Q} L_q \end{aligned} \tag{46}$$

in case of Dirichlet conditions, and

$$\begin{aligned} d\mathcal{F}_p(\Gamma, \Psi)(\hat{\mathbf{x}}) = \hat{\mathbf{x}} &\sum_{q=1}^{Q} L_q \\ &+ \hat{\mathbf{x}} \int_\Gamma \left[\!\left[\mathbf{W}_p(\mathbf{y})^\top \Big(\Psi(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \frac{\partial}{\partial \mathbf{n}_\mathbf{y}} \sigma\big(\mathbf{U}_0(\mathbf{y})\big) \cdot \mathbf{n}(\mathbf{y}) \right. \right. \\ &\left. \left. - \sigma\big(\mathbf{U}_0(\mathbf{y})\big) \cdot \nabla_\Gamma \big(\Psi(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})\big) \Big) \right]\!\right] ds_\mathbf{y} \end{aligned} \tag{47}$$

if Neumann conditions are prescribed. Here, $[\![\cdot]\!]$ denotes the jump across the crack, $\mathbf{U}_0 := \mathbf{U}_\epsilon|_{\epsilon=0}$, and $\mathbf{W}_p := \mathbf{w}_p - \hat{\mathbf{z}} e^{-ik_s \hat{\mathbf{x}}\cdot\mathbf{y}}$ with $\mathbf{w}_p$ the solution of (45).

The limits L_q can be calculated similar as for problems in the exterior of a bounded domain. If P_q is an interior corner of the crack, we are exactly in the situation of Theorem 4.1 and therefore $L_q = 0$. Once the tips of the crack are considered, a more detailed asymptotic analysis has to be employed since the elastic displacement field assumes higher singularity near the crack tips.

Let P_q, $q = 1, 2$, be the tips of the crack. It is known (see, e.g., [16]) that

$$\Sigma(\mathcal{A}_q) := \big\{ j/2\colon\ j \in \mathbb{Z} \backslash \{0\} \big\}$$

for Dirichlet boundary conditions, and

$$\Sigma(\mathcal{A}_q) := \{ j/2\colon\ j \in \mathbb{Z} \}$$

for Neumann boundary conditions. Therefore the singular decomposition (21) of the solutions $\mathbf{U}_0$ and $\mathbf{W}_p$ near the tip P_q takes the form

$$\mathbf{U}_0(r_q,\varphi_q) = \big[K_q^1(\mathbf{U}_0)\Phi_1(\varphi_q) + K_q^2(\mathbf{U}_0)\Phi_2(\varphi_q)\big]r^{1/2} + O(r_q), \tag{48}$$

$$\mathbf{W}_p(r_q,\varphi_q) = \big[K_q^1(\mathbf{W}_p)\Phi_1(\varphi_q) + K_q^2(\mathbf{W}_p)\Phi_2(\varphi_q)\big]r^{1/2} + O(r_q), \tag{49}$$

where Φ_1, Φ_2 have the following form:

$$\Phi_1(\varphi_q) = \begin{bmatrix} -(2\kappa-1)\cos\big(\frac{3}{2}\varphi_q\big) + (2\kappa-1)\cos\big(\frac{1}{2}\varphi_q\big) \\ (2\kappa-1)\sin\big(\frac{3}{2}\varphi_q\big) - (2\kappa+1)\sin\big(\frac{1}{2}\varphi_q\big) \end{bmatrix}, \tag{50}$$

$$\Phi_2(\varphi_q) = \begin{bmatrix} -(2\kappa+1)\sin\big(\frac{3}{2}\varphi_q\big) + (2\kappa-1)\sin\big(\frac{1}{2}\varphi_q\big) \\ -(2\kappa+1)\cos\big(\frac{3}{2}\varphi_q\big) + (2\kappa+1)\cos\big(\frac{1}{2}\varphi_q\big) \end{bmatrix} \tag{51}$$

for the Dirichlet crack, and

$$\Phi_1(\varphi_q) = \begin{bmatrix} 3\cos\big(\frac{3}{2}\varphi_q\big) + (2\kappa-1)\cos\big(\frac{1}{2}\varphi_q\big) \\ -3\sin\big(\frac{3}{2}\varphi_q\big) - (2\kappa+1)\sin\big(\frac{1}{2}\varphi_q\big) \end{bmatrix}, \tag{52}$$

$$\Phi_2(\varphi_q) = \begin{bmatrix} \sin\big(\frac{3}{2}\varphi_q\big) + (2\kappa-1)\sin\big(\frac{1}{2}\varphi_q\big) \\ \cos\big(\frac{3}{2}\varphi_q\big) + (2\kappa+1)\cos\big(\frac{1}{2}\varphi_q\big) \end{bmatrix} \tag{53}$$

for the Neumann crack. Here κ is a material constant given by $\kappa = (\lambda+3\mu)/(\lambda+\mu)$.

Let us denote by $\Psi_n(P_q)$ and $\Psi_t(P_q)$ the normal and the tangential component of the perturbation at the crack tip P_q, respectively.

Theorem 4.2. *Let $q = 1, 2$. Then we have*

$$\begin{aligned} L_q = {} & \big(K_q^1(\mathbf{U}_0)K_q^1(\mathbf{W}_p) + K_q^2(\mathbf{U}_0)K_q^2(\mathbf{W}_p)\big) \\ & \times \frac{-16\pi\mu(\lambda+3\mu)(\lambda+2\mu)}{(\lambda+\mu)^2}\Psi_t(P_q) \\ & + \big(K_q^1(\mathbf{U}_0)K_q^2(\mathbf{W}_p) + K_q^2(\mathbf{U}_0)K_q^1(\mathbf{W}_p)\big)\frac{-16\pi\mu(\lambda+2\mu)}{(\lambda+\mu)}\Psi_n(P_q) \end{aligned}$$

in case of Dirichlet conditions, and

$$\begin{aligned} L_q = {} & \big(K_q^1(\mathbf{U}_0)K_q^1(\mathbf{W}_p) + K_q^2(\mathbf{U}_0)K_q^2(\mathbf{W}_p)\big)\frac{16\pi\mu(\lambda+2\mu)}{(\lambda+\mu)}\Psi_t(P_q) \\ & + \big(K_q^1(\mathbf{U}_0)K_q^2(\mathbf{W}_p) + K_q^2(\mathbf{U}_0)K_q^1(\mathbf{W}_p)\big)\frac{-16\pi\mu(\lambda+2\mu)}{(\lambda+\mu)}\Psi_n(P_q) \end{aligned}$$

for Neumann conditions.

Proof. We use (26) by means of (48) to express $\mathbf{U}'$ in terms of stress intensity factors of the unperturbed field $K_q^1(\mathbf{U}_0)$, $K_q^2(\mathbf{U}_0)$ and the perturbation of the tip $\Psi_n(P_q)$, $\Psi_t(P_q)$. Hence, we obtain the following asymptotics for $\mathbf{U}'$:

$$\begin{aligned}\mathbf{U}'(r_q,\varphi_q) = r_q^{-1/2}\Big[&\big(K_q^1(\mathbf{U}_0)\widetilde{\Phi}_{1t}(\varphi_q) + K_q^2(\mathbf{U}_0)\widetilde{\Phi}_{2t}(\varphi_q)\big)\Psi_t(P_q) \\ &+ \big(K_q^1(\mathbf{U}_0)\widetilde{\Phi}_{1n}(\varphi_q) + K_q^2(\mathbf{U}_0)\widetilde{\Phi}_{2n}(\varphi_q)\big)\Psi_t(P_q)\Big] \\ &+ O\big(r_q^{1/2}\big),\end{aligned} \tag{54}$$

with

$$\widetilde{\Phi}_{1t}(\varphi_q) = \frac{1}{2}\begin{bmatrix} (2\kappa+1)\cos\big(\frac{3}{2}\varphi_q\big) - (2\kappa+1)\cos\big(\frac{1}{2}\varphi_q\big) \\ -(2\kappa-1)\sin\big(\frac{3}{2}\varphi_q\big) + (2\kappa+1)\sin\big(\frac{1}{2}\varphi_q\big)\end{bmatrix}, \tag{55}$$

$$\widetilde{\Phi}_{2t}(\varphi_q) = \frac{1}{2}\begin{bmatrix} (2\kappa+1)\sin\big(\frac{3}{2}\varphi_q\big) - (2\kappa-1)\sin\big(\frac{1}{2}\varphi_q\big) \\ (2\kappa-1)\cos\big(\frac{3}{2}\varphi_q\big) - (2\kappa-1)\cos\big(\frac{1}{2}\varphi_q\big)\end{bmatrix}, \tag{56}$$

$$\widetilde{\Phi}_{1n}(\varphi_q) = \frac{1}{2}\begin{bmatrix} (2\kappa+1)\sin\big(\frac{3}{2}\varphi_q\big) + (2\kappa-3)\sin\big(\frac{1}{2}\varphi_q\big) \\ (2\kappa-1)\cos\big(\frac{3}{2}\varphi_q\big) + (2\kappa-3)\cos\big(\frac{1}{2}\varphi_q\big)\end{bmatrix}, \tag{57}$$

$$\widetilde{\Phi}_{2n}(\varphi_q) = \frac{1}{2}\begin{bmatrix} -(2\kappa+1)\cos\big(\frac{3}{2}\varphi_q\big) - (2\kappa+3)\cos\big(\frac{1}{2}\varphi_q\big) \\ (2\kappa-1)\sin\big(\frac{3}{2}\varphi_q\big) + (2\kappa+3)\sin\big(\frac{1}{2}\varphi_q\big)\end{bmatrix} \tag{58}$$

for the Dirichlet crack, and

$$\widetilde{\Phi}_{1t}(\varphi_q) = \frac{1}{2}\begin{bmatrix} (2\kappa+1)\cos\big(\frac{3}{2}\varphi_q\big) + \cos\big(\frac{1}{2}\varphi_q\big) \\ -(2\kappa-1)\sin\big(\frac{3}{2}\varphi_q\big) - \sin\big(\frac{1}{2}\varphi_q\big)\end{bmatrix}, \tag{59}$$

$$\widetilde{\Phi}_{2t}(\varphi_q) = \frac{1}{2}\begin{bmatrix} (2\kappa+1)\sin\big(\frac{3}{2}\varphi_q\big) + 3\sin\big(\frac{1}{2}\varphi_q\big) \\ (2\kappa-1)\cos\big(\frac{3}{2}\varphi_q\big) + 3\cos\big(\frac{1}{2}\varphi_q\big)\end{bmatrix}, \tag{60}$$

$$\widetilde{\Phi}_{1n}(\varphi_q) = \frac{1}{2}\begin{bmatrix} (2\kappa+1)\sin\big(\frac{3}{2}\varphi_q\big) - 5\sin\big(\frac{1}{2}\varphi_q\big) \\ (2\kappa-1)\cos\big(\frac{3}{2}\varphi_q\big) - 5\cos\big(\frac{1}{2}\varphi_q\big)\end{bmatrix}, \tag{61}$$

$$\widetilde{\Phi}_{2n}(\varphi_q) = \frac{1}{2}\begin{bmatrix} -(2\kappa+1)\cos\big(\frac{3}{2}\varphi_q\big) - \cos\big(\frac{1}{2}\varphi_q\big) \\ (2\kappa-1)\sin\big(\frac{3}{2}\varphi_q\big) + \sin\big(\frac{1}{2}\varphi_q\big)\end{bmatrix} \tag{62}$$

for the Neumann crack.

Finally, we insert the expressions (54), (49) for both $\mathbf{U}'$ and $\mathbf{W}_p$, respectively, into the integral (39). Then, straightforward calculations give the assertions of the theorem. □

Finally, the same considerations can be repeated for the shape derivative of the shear far field $d\mathcal{F}_s(\Gamma)$ if the adjoint field is defined by

$$\begin{aligned}&\Delta^*\mathbf{w}_s(\mathbf{y}) + \omega^2\mathbf{w}_s(\mathbf{y}) = 0 \quad \text{in } \Omega, \\ &\mathbf{w}_{s\pm}(\mathbf{y}) = \hat{\mathbf{z}}e^{-ik_p\hat{\mathbf{x}}\cdot\mathbf{y}} \quad \text{on } \Gamma, \quad \text{or} \\ &\mathbf{T}_{\mathbf{n}(\mathbf{y})}\mathbf{w}_{s\pm}(\mathbf{y}) = \mathbf{T}_{\mathbf{n}(\mathbf{y})}\hat{\mathbf{z}}e^{-ik_p\hat{\mathbf{x}}\cdot\mathbf{y}} \quad \text{on } \Gamma,\end{aligned} \tag{63}$$

satisfying the Kupradze radiation condition at infinity. In this case we have

$$d\mathcal{F}_s(\Gamma,\Psi)(\hat{\mathbf{x}}) = -\hat{\mathbf{z}}\int_{\Gamma}\Psi(\mathbf{y})\cdot\mathbf{n}(\mathbf{y})\left[\!\left[\frac{\partial\mathbf{U}_0(\mathbf{y})^{\top}}{\partial\mathbf{n}_{\mathbf{y}}}\mathbf{T}_{\mathbf{n}(\mathbf{y})}\mathbf{W}_s(\mathbf{y})\right]\!\right]ds_{\mathbf{y}} + \hat{\mathbf{z}}(L_1+L_2) \tag{64}$$

in case of Dirichlet conditions, and

$$d\mathcal{F}_s(\Gamma,\Psi)(\hat{\mathbf{x}}) = \hat{\mathbf{z}}(L_1+L_2) + \hat{\mathbf{z}}\int_{\Gamma}\left[\!\left[\mathbf{W}_s(\mathbf{y})^{\top}\left(\Psi(\mathbf{y})\cdot\mathbf{n}(\mathbf{y})\frac{\partial}{\partial\mathbf{n}_{\mathbf{y}}}\sigma\big(\mathbf{U}_0(\mathbf{y})\big)\cdot\mathbf{n}(\mathbf{y}) - \sigma\big(\mathbf{U}_0(\mathbf{y})\big)\cdot\nabla_{\Gamma}\big(\Psi(\mathbf{y})\cdot\mathbf{n}(\mathbf{y})\big)\right)\right]\!\right]ds_{\mathbf{y}} \tag{65}$$

if Neumann conditions are prescribed, where $\mathbf{W}_s := \mathbf{w}_s - \hat{\mathbf{z}}e^{-ik_s\hat{\mathbf{x}}\cdot\mathbf{y}}$ with $\mathbf{w}$ the solution of (63). The value of L_1 and L_2 at the tips P_1 and P_2, respectively, are the same as those of the Theorem 4.2 where $K_q^{1(2)}(\mathbf{W}_p)$ are replaced by $K_q^{1(2)}(\mathbf{W}_s)$.

5. Conclusions

The sensitivity analysis performed in this paper shows that in case of problems in the exterior of a bounded domain the sensitivity of the far field pattern depends only on the perturbation of the boundary in the normal direction. In case of problems in the exterior of a curve, the sensitivity depends also on the tangential perturbation of the end points of the curve. The formulae (40), (41), (46), and (47) are well suited for a numerical realization by using boundary element methods because they require only the knowledge of the solution of the original and of the adjoint exterior Dirichlet or Neumann problem with boundary data given by traces of plane waves.

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