MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2002; **25**:595–613 (DOI: 10.1002/mma.305)
MOS subject classification: 35 B 30; 35 J 05

Domain sensitivity analysis of the acoustic far-field pattern

Marius Bochniak[1,*,†] and Fioralba Cakoni[2,‡]

[1]*Abteilung Analysis, Universität Ulm, 89069 Ulm, Germany*
[2]*Department of Mathematical Sciences, University of Delaware, Newark, DE, U.S.A.*

Communicated by P. Werner

SUMMARY

We consider acoustic scattering problems described by the mixed boundary value problem for the scalar Helmholtz equation in the exterior of a 2D bounded domain or in the exterior of a crack. The boundary of the domain is assumed to have a finite set of corner points where the scattered wave may have singular behaviour. The paper is concerned with the sensitivity of the far-field pattern with respect to small perturbations of the shape of the scatterer. Using a modification of the method of adjoint problems, we obtain an integral representation for the Gâteaux derivative which contains only boundary values of functions easily computable by standard BEM and which depends explicitly on the perturbation of the boundary. In some cases, we show the direct influence of the singularities of the solution on the sensitivity of the far-field pattern. In this way, we generalize the domain sensitivity analysis developed earlier for smooth domains by Hettlich, Kirsch, Kress, Potthast and others. Finally, we show that the same approach can be applied to scattering from 3D domains with smooth edges. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: domain sensitivity analysis; inverse scattering problems; acoustic far-field pattern

1. INTRODUCTION

One of the inverse acoustic scattering problems is the problem of recovering the shape of the scattering object from the knowledge of the far-field pattern u_∞ of the scattered wave. In practice, such inverse problems are often solved by some iterative Newton-type algorithms, where the approximations of the unknown boundary are improved successively in such a way that its far-field pattern fits better the given far-field pattern u_∞. This approach requires precise knowledge about the Gâteaux derivative of the far-field pattern with respect to small perturbations of the boundary. Such form sensitivity analysis is well known for scatterers with a smooth boundary and for various kinds of boundary conditions (see References [1, 2] for Dirichlet conditions, Reference [3] for Neumann conditions, Reference [4] for Robin conditions). On the other hand, very little is known about scattering from non-smooth surfaces

*Correspondence to: Marius Bochniak, Abteilung Analysis, Universität Ulm, 89069 Ulm, Germany
†E-mail: bochniak@mathematik.uni-ulm.de
‡E-mail: cakoni@math.udel.edu

Contract/grant sponsor: German Science Foundation DFG; contract/grant number: SFB 404

 Received 27 June 2001

except the work of Kress [5] on acoustic scattering from smooth 2D cracks. If the boundary is not smooth, or if the boundary conditions change along the boundary, then the derivatives of the solution can be unbounded and the investigation of the form sensitivity requires a more careful approach.

Form sensitivity problems appear quite often in engineering applications: form optimization of bounded elastic structures, propagation of cracks in elastic materials, etc. In this case, the sensitivity analysis is developed much better than for scattering problems in exterior domains. Recently, the first author investigated together with A.-M. Sändig the form sensitivity of elastic structures in general bounded piecewise smooth domains allowing mixed boundary conditions and cracks of arbitrary smooth shape [6, 7]. This was performed using the material derivative approach (see e.g. References [8, 9]) and a modification of the method of adjoint problems [10, 11].

In this paper, we use similar methods as in References [6, 7] to analyse the form sensitivity for acoustic scattering from 2D domains with corners under mixed boundary conditions and from 2D piecewise smooth cracks. Furthermore, we show similar results for scattering from 3D domains with smooth edges.

In Section 2, we give a short mathematical formulation of the problem and define a class of admissible perturbations of the boundary, whereas in Section 3, we provide some theorems about the existence and the regularity of solutions to the exterior mixed boundary value problems for the Helmholtz equation. Then we prove, in Section 4, the existence and the regularity of the material and of the shape derivative of the perturbed acoustic field. The analysis is performed directly for the exterior mixed boundary value problem in the differential form, contrary to References [1–5], where a boundary integral or a variational formulation is used. The existence results are formulated in weighted Sobolev spaces, which take into account the behaviour of functions at infinity and at corner points of the boundary. Based on these results, we obtain a simple representation for the Gâteaux derivative of the far-field pattern. In Section 5, we use a modification of the method of adjoint problems to obtain another representation for the Gâteaux derivative which is better suitable for a numerical realization, and which shows in some cases directly the influence of the singularities of the solution on the sensitivity of the far-field pattern. It turns out that, in case of scattering from a bounded domain with pure Dirichlet or pure Neumann conditions imposed on the boundary, the Gâteaux derivative of the far-field pattern depends only on the perturbation of the boundary in the normal direction and the Cauchy data of the unperturbed scattered wave u_0 and the solution w of an appropriately defined adjoint exterior problem. This result holds, when the boundary conditions change at corner points (the so-called collision points) with opening angle smaller than π, too. If the boundary conditions change at a smooth boundary point P (i.e. the opening angle is π), then the formula for the Gâteaux derivative contains additional terms describing the perturbation of the collision point P in the tangential and in the normal direction and on some constants describing the singular behaviour of u_0 and w near P. This knowledge can be used for the solution of the inverse problem for determining the kind of boundary conditions. Similarly, we show that in case of acoustic scattering from a crack, the Gâteaux derivative depends on the perturbation of the boundary in the normal direction, on the perturbation of the crack tips in the tangential direction and on the constants describing the singular behaviour of u_0 and w at the crack tips. In this way, we generalize the results from Reference [5], where severe restrictions were imposed on admissible perturbations of the crack. We mention, that in case of smooth domains our formulas reduce to the ones proved in Reference [12]. Finally, we show in Section 6 that similar results are valid in case of scattering from 3D domains with smooth edges.

2. FORMULATION OF THE PROBLEM

We consider exterior domains $\Omega := \mathbb{R}^2 \backslash \bar{\Omega}_i$ of two types:

- (B) Ω_i is a bounded domain with a piecewise smooth boundary $\Gamma = \Gamma^{\mathrm{D}} \cup \Gamma^{\mathrm{N}} \cup S$. Here $S = \{P_1, \ldots, P_Q\}$ is a finite set of boundary points, such that $\Gamma \backslash S$ is smooth and $\overline{\Gamma^{\mathrm{D}}} \cap \overline{\Gamma^{\mathrm{N}}} \subset S$. It contains all corner points of the boundary and all points where the boundary conditions change. Furthermore, we assume that Ω is locally diffeomorph in the neighbourhood of every corner point P_q to an infinite cone C_q with the opening angle $\omega_q^0 \notin \{0, 2\pi\}$. The unit normal vector $\mathbf{n} = (n_1, n_2)$ on Γ is directed towards Ω_i (see Figure 1).
- (C) $\Omega_i = \Gamma$ is a crack, i.e. a piecewise smooth curve with a finite set $S = \{P_1, \ldots, P_Q\}$ consisting of two crack tips and of interior corner points which satisfy the angle condition $\omega_q^0 \neq \{0, 2\pi\}$. In this case, we assume that either $\Gamma = \Gamma^{\mathrm{D}}$ or $\Gamma = \Gamma^{\mathrm{N}}$, i.e. the boundary conditions do not change along the crack. The direction of the unit normal vector $\mathbf{n}$ on Γ is chosen arbitrarily but is fixed along the crack (see Figure 2).

The mathematical modelling of the scattering of time-harmonic acoustic plane waves from an obstacle Ω_i, surrounded by a homogeneous isotropic medium Ω, leads to the exterior boundary value problem for the Helmholtz equation with a positive wave number k (see Reference [13] for a detailed description of the model)

$$\Delta u(\mathbf{x}) + k^2 u(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \Omega \tag{1}$$

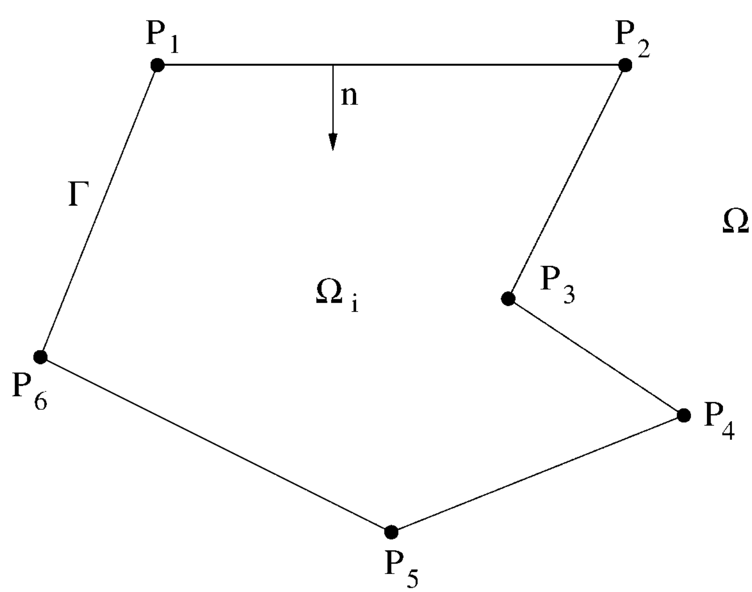

Figure 1. The exterior of a bounded domain.

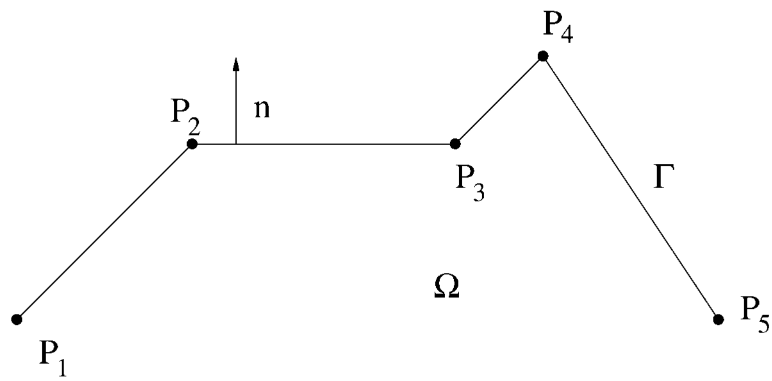

Figure 2. The exterior of a crack.

The total wave $u = u^s + u^i$ is decomposed into a given incident plane wave $u^i(\mathbf{x}) = e^{ik\hat{\mathbf{d}}\cdot\mathbf{x}}$ with $\hat{\mathbf{d}} \in \mathbb{R}^2$ being a unit vector giving the direction of propagation, and the unknown scattered wave u^s. The scattered wave u^s is requested to satisfy the following boundary conditions:

$$\begin{aligned} u^s(\mathbf{x}) &= -u^i(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma^D \\ \frac{\partial u^s(\mathbf{x})}{\partial \mathbf{n}} &= -\frac{\partial u^i(\mathbf{x})}{\partial \mathbf{n}} \quad \text{for } \mathbf{x} \in \Gamma^N \end{aligned} \tag{2}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |\mathbf{x}| \tag{3}$$

uniformly in all directions $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$. For crack problems (C) the boundary conditions have to be posed on both sides of the crack, i.e.

$$\begin{aligned} u^s_\pm(\mathbf{x}) &= -u^i(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma^D \quad \text{or} \\ \frac{\partial u^s_\pm(\mathbf{x})}{\partial \mathbf{n}} &= -\frac{\partial u^i(\mathbf{x})}{\partial \mathbf{n}} \quad \text{for } \mathbf{x} \in \Gamma^N \end{aligned} \tag{4}$$

Here is

$$u^s_\pm(\mathbf{x}) = \lim_{h\to 0+} u^s(\mathbf{x} \pm h\mathbf{n}) \tag{5}$$

$$\frac{\partial u^s_\pm(\mathbf{x})}{\partial \mathbf{n}} = \lim_{h\to 0+} \mathbf{n}\cdot\nabla u^s(\mathbf{x} \pm h\mathbf{n}) \tag{6}$$

The boundary conditions describe the scattering properties of the obstacle. In case of Dirichlet conditions the scatterer is sound-soft, while for Neumann conditions the scatterer is sound-hard.

From now on, we denote by B_R the open ball $\{\mathbf{x}\colon |\mathbf{x}| < R\}$ and by S_R the corresponding sphere $\{\mathbf{x}\colon |\mathbf{x}| = R\}$. It is known that there exists a unique solution of the direct problem

(1)–(3), which belongs to $H^1(\Omega \cap B_R)$ and satisfies the following identity for big enough R [14, 15]:

$$u^s(\mathbf{x}) = \int_{S_R} \left[u^s(\mathbf{y}) \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_\mathbf{y}} - \Phi(\mathbf{x},\mathbf{y}) \frac{\partial u^s(\mathbf{y})}{\partial \mathbf{n}_\mathbf{y}} \right] \mathrm{d}s_\mathbf{y} \tag{7}$$

Here $\partial/\partial \mathbf{n}_\mathbf{x}$ denotes the outward normal derivative on the sphere S_R at the point $\mathbf{x}$ and $\Phi(\mathbf{x},\mathbf{y})$ is the fundamental solution of the Helmholtz equation given by

$$\Phi(\mathbf{x},\mathbf{y}) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)$$

with $H_0^{(1)}$ being the Hankel function of the first kind.

Applying radiation condition (3) and the asymptotic behaviour of the fundamental solution Φ to integral identity (7), we obtain the following behaviour of the scattered wave u^s as $r \to \infty$ and uniformly in $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ (see Reference [13])

$$u^s(\mathbf{x}) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}} \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} \mathscr{F}(\Gamma)(\hat{\mathbf{x}}) + \mathrm{o}(1/\sqrt{r}) \tag{8}$$

The function $\mathscr{F}(\Gamma)$, which depends on the angle variable $\hat{\mathbf{x}} \in S_1$, is called the far-field pattern or the scattering amplitude of the scattered wave u^s. From the existence and the uniqueness of the solution of the direct scattering problem follows that the far-field pattern is uniquely determined by the boundary Γ. The far-field pattern is given by the following integral formula:

$$\mathscr{F}(\Gamma)(\hat{\mathbf{x}}) = \int_{S_R} \left[u^s(\mathbf{y}) \frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}}}{\partial \mathbf{n}_\mathbf{y}} - \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}} \frac{\partial u^s(\mathbf{y})}{\partial \mathbf{n}_\mathbf{y}} \right] \mathrm{d}s_\mathbf{y} \tag{9}$$

Although the boundary of the domain representing the scattering object is non-smooth, integral representation (9) shows that the far-field pattern is an analytic function on the unit sphere S_1.

Our main concern in this work is to study how the perturbation of the domain influences the far field $\mathscr{F}(\Gamma)$. The goal is to derive integral representations for the Gâteaux derivative of the far-field pattern with respect to an admissible class of domain perturbations where the integrand

- contains only boundary values of functions, which are easily computable by standard BEM or FEM methods, and which
- depends explicitly on the perturbation of the boundary, but not on the perturbation of the domain Ω.

3. EXTERIOR BOUNDARY VALUE PROBLEMS FOR THE HELMHOLTZ EQUATION

Let us formulate existence and regularity results for solutions of the mixed boundary value problem

$$\Delta u + k^2 u = f \quad \text{in } \Omega \tag{10}$$

$$\left.\begin{aligned} u &= g \quad \text{on } \Gamma^{\mathrm{D}} \\ \frac{\partial u}{\partial \mathbf{n}} &= h \quad \text{on } \Gamma^{\mathrm{N}} \end{aligned}\right\} \text{(B)} \tag{11}$$

$$\left.\begin{aligned} u_{\pm} &= g_{\pm} \quad \text{on } \Gamma \text{ or} \\ \frac{\partial u_{\pm}}{\partial \mathbf{n}} &= h_{\pm} \quad \text{on } \Gamma \end{aligned}\right\} \text{(C)} \tag{12}$$

which satisfy radiation condition (3) at infinity.

To this end we introduce weighted Sobolev spaces which take into account the singular behaviour of the functions near the singular points $P_q \in S$ and at infinity.

Definition 3.1. Let $\Omega = \mathbb{R}^2 \backslash \Omega_i$ with Ω_i being of type (B) as defined in Section 2 and let $0 \in \Omega_i$. We choose for every singular point $P_q \in S$ a cut-off function $\eta_q \in C_0^\infty(\mathbb{R}^2)$ with support in a neighbourhood of P_q and set $\eta_0 = 1 - \sum_{q=1}^{Q} \eta_q$. For $d \in \mathbb{N}_0$, $\vec{\beta} = (\beta_1, \ldots, \beta_Q) \in \mathbb{R}^Q$ we define the space $V^d_{\vec{\beta},\gamma}(\Omega)$ of all generalized functions which have the finite norm

$$\|u\|_{V^d_{\vec{\beta},\gamma}(\Omega)} := \|(1 + |\mathbf{x}|^2)^{-\gamma/2}(\eta_0 u)\|_{H^d(\Omega)} + \sum_{q=1}^{Q} \sum_{|p| \leqslant d} \|r_q^{\beta_q - d + |p|} D^p(\eta_q u)\|_{L_2(\Omega)}$$

where $r_q = \operatorname{dist}(\mathbf{x}, P_q)$. For $d = 1, 2, \ldots,$ we denote by $V^{d-1/2}_{\vec{\beta}}(\tilde{\Gamma})$ the space of traces on $\tilde{\Gamma} \backslash S \subset \Gamma$ of functions in $V^d_{\vec{\beta},\gamma}(\Omega)$. For domains of the type (C) the weighted spaces $V^d_{\vec{\beta},\gamma}(\Omega), V^{d-1/2}_{\vec{\beta}}(\tilde{\Gamma})$ are defined analogously.

Let us denote by ω_q^{o} the opening angle of the cone C_{P_q} corresponding to the singular boundary point $P_q \in S$. We decompose the set $S = S_{\mathrm{D}} \cup S_{\mathrm{N}} \cup S_{\mathrm{M}}$ in such a way that u satisfies in the vicinity of $P \in S_{\mathrm{D}}, S_{\mathrm{N}}, S_{\mathrm{M}}$ the Dirichlet, the Neumann or the mixed boundary conditions, respectively.

Theorem 3.2 (*Existence and regularity result*). Let $d \in \mathbb{N}_0$, $\gamma > 1$ and $\vec{\beta} = (\beta_1, \ldots, \beta_Q) \in \mathbb{R}^Q$ with

$$\begin{aligned} d + 1 - \beta_q &\in (0, \pi/\omega_q^{\mathrm{o}}) \qquad \text{for } P_q \in S_{\mathrm{D}} \cup S_{\mathrm{N}} \\ d + 1 - \beta_q &\in (0, \pi/(2\omega_q^{\mathrm{o}})) \quad \text{for } P_q \in S_{\mathrm{M}} \end{aligned}$$

Suppose that $f \in V^d_{\vec{\beta},-\gamma}(\Omega), g \in V^{d+3/2}_{\vec{\beta}}(\Gamma^{\mathrm{D}})$ and $h \in V^{d+1/2}_{\vec{\beta}}(\Gamma^{\mathrm{N}})$. Then there exist a unique solution $u \in V^{d+2}_{\vec{\beta},\gamma}(\Omega)$ and the following *a priori* estimate is valid

$$\|u\|_{V^{d+2}_{\vec{\beta},\gamma}(\Omega)} \leqslant c\{\|f\|_{V^d_{\vec{\beta},-\gamma}(\Omega)} + \|g\|_{V^{d+3/2}_{\vec{\beta}}(\Gamma^{\mathrm{D}})} + \|h\|_{V^{d+1/2}_{\vec{\beta}}(\Gamma^{\mathrm{N}})}\} \tag{13}$$

Proof. With the *a priori* estimates for the solutions of boundary value problems in the exterior of a smooth domain [16] and in bounded domains with corners [17–19], we can prove by means of a partition of unity that

$$\|u\|_{V^{d+2}_{\vec{\beta},\gamma}(\Omega)} \leqslant c\{\|f\|_{V^d_{\vec{\beta},-\gamma}(\Omega)} + \|g\|_{V^{d+3/2}_{\vec{\beta}}(\Gamma^{\mathrm{D}})} + \|h\|_{V^{d+1/2}_{\vec{\beta}}(\Gamma^{\mathrm{N}})} + \|u\|_{V^{d+1}_{\vec{\beta},\gamma}(\Omega \cap B_R)}\}$$

with some real constant c and some positive R. The last norm on the right-hand side of the above estimate can be omitted similar as in Reference [20, Lemma III 3.10] because the kernel of the problem is trivial. □

In case of a smooth domain Theorem 3.2 reduces to the result proved in Reference [16].

Remark 3.1. The assumptions of Theorem 3.2 put some restrictions on the behaviour of the right-hand sides f, g, h. In fact, $f = O(r^{\alpha_q-2})$, $g = O(r^{\alpha_q})$, $h = O(r^{\alpha_q-1})$ with some $\alpha_q \in (0, \pi/\omega_q^0)$ near the singular point P_q. In particular, $g \to 0$ as $r \to 0$, an assumption which is not satisfied for plane waves.

Theorem 3.3 (*Asymptotic behaviour at corners*). Let $P_q \in S$ and let η_1, η_2 be cut-off functions with support in some neighbourhood of P_q and $\operatorname{supp} \eta_2 \subset \operatorname{supp} \eta_1$. Furthermore, let $d \in \mathbb{N}_0$ and $\beta_q \in \mathbb{R}$ satisfy

$$\begin{aligned} d + 1 - \beta_q &\in (\pi/\omega_q^{\circ}, 2\pi/\omega_q^{\circ}) \quad &&\text{if } P_q \in S_{\mathrm{D}} \cup S_{\mathrm{N}} \\ d + 1 - \beta_q &\in (\pi/(2\omega_q^{\circ}), \pi/\omega_q^{\circ}) \quad &&\text{if } P_q \in S_{\mathrm{M}} \end{aligned}$$

Suppose that $\eta_1 f \in V^d_{\vec{\beta},-\gamma}(\Omega)$, $\eta_1 g \in V^{d+3/2}_{\vec{\beta}}(\Gamma^{\mathrm{D}})$ and $\eta_1 h \in V^{d+1/2}_{\vec{\beta}}(\Gamma^{\mathrm{N}})$. Then the solution u of Equations (10–12) has in the vicinity of P_q the following asymptotic behaviour as $r = |\mathbf{x} - P_q| \to 0$:

$$\eta_2 u(\mathbf{x}) = \begin{cases} k_1 r^{\pi/\omega_q^{\circ}} \sin(\frac{\pi}{\omega_q^0}\omega) + w(\mathbf{x}) & \text{if } P_q \in S_{\mathrm{D}} \\ k_2 r^{\pi/\omega_q^{\circ}} \cos(\frac{\pi}{\omega_q^0}\omega) + w(\mathbf{x}) & \text{if } P_q \in S_{\mathrm{N}} \\ k_3 r^{\pi/(2\omega_q^{\circ})} \sin(\frac{\pi}{2\omega_q^0}\omega) + w(\mathbf{x}) & \text{if } P_q \in S_{\mathrm{M}} \end{cases} \tag{14}$$

with real constants $k_1, \ldots, k_3$ and $\eta_2 w \in V^{d+2}_{\vec{\beta},\gamma}(\Omega)$. Here, (r, ω) are polar co-ordinates with origin in P_q. The angle variable ω is oriented in such a way that the tangential vectors on the boundary in P_q correspond to the angles 0 and ω_q°, respectively. Moreover, if $P_q \in S_{\mathrm{M}}$ then we assume that $\omega = 0$ corresponds to Dirichlet boundary conditions.

Proof. The assertion is a simple application of results from the theory of general elliptic problems in domains with corners [17–19]. The singular terms of the asymptotic expansions for the Helmholtz operator coincide with the singular terms for the Laplace operator which are well known (see e.g. Reference [18, Section 6.1.8]). □

4. DOMAIN SENSITIVITY OF ACOUSTIC FIELDS

4.1. Description of the domain perturbation

In order to describe the shape sensitivity of exterior boundary value problems, i.e. the influence of the shape of the domain on the solution, we introduce a family of perturbed domains Ω_ε, $\varepsilon \in [0, \varepsilon_0]$, as the image of a fixed domain Ω under a family of diffeomorphism

$$\{\Psi_\varepsilon = I + \varepsilon\Psi \in [C^{d+2}(\bar{\Omega})]^2;\ \varepsilon \in [0, \varepsilon_0]\}, \quad d \in \mathbb{N}_0 \tag{15}$$

Thus, we have

$$\Omega_\varepsilon := \Psi_\varepsilon(\Omega), \quad \Gamma_\varepsilon^{\mathrm{D}} := \Psi_\varepsilon(\Gamma^{\mathrm{D}}), \quad \Gamma_\varepsilon^{\mathrm{N}} := \Psi_\varepsilon(\Gamma^{\mathrm{N}})$$

$$\Gamma_\varepsilon = \Gamma_\varepsilon^{\mathrm{D}} \cup \Gamma_\varepsilon^{\mathrm{N}} := \Psi_\varepsilon(\Gamma), \quad S_\varepsilon := \Psi_\varepsilon(S)$$

Since we are interested in the perturbation of the boundary Γ we can assume that

$$\exists R: \ \Psi_\varepsilon(\mathbf{x}) = \mathbf{x} \quad \forall |\mathbf{x}| > R \tag{16}$$

Remark 4.1. The regularity assumption $\Psi_\varepsilon \in [C^2(\bar{\Omega})]^2$ excludes perturbations which change the number of singular points.

4.2. Form sensitivity of the solution

We consider the following exterior mixed boundary value problem:

$$\begin{aligned} \Delta u_\varepsilon(\mathbf{x}_\varepsilon) + k^2 u_\varepsilon(\mathbf{x}_\varepsilon) = f_\varepsilon(\mathbf{x}_\varepsilon) \quad & \text{in } \Omega_\varepsilon \\ u_\varepsilon(\mathbf{x}_\varepsilon) = g_\varepsilon(\mathbf{x}_\varepsilon) \quad & \text{on } \Gamma_\varepsilon^{\mathrm{D}} \\ \nabla u_\varepsilon(\mathbf{x}_\varepsilon) \cdot \mathbf{n}_\varepsilon(\mathbf{x}_\varepsilon) = h_\varepsilon(\mathbf{x}_\varepsilon) \quad & \text{on } \Gamma_\varepsilon^{\mathrm{N}} \end{aligned} \tag{17}$$

for a scalar field u_ε satisfying radiation condition (3) and f_ε having a compact support.

Let us investigate the existence and the regularity of the material derivative

$$\dot{u} := \left. \frac{\mathrm{d}(u_\varepsilon \circ \Psi_\varepsilon)}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} \tag{18}$$

and the shape derivative

$$u' := \dot{u} - \nabla u_0 \cdot \Psi \tag{19}$$

of the perturbed function u_ε. To this end we transform problem (17) onto the reference configuration by means of a change of variables $\mathbf{x}_\varepsilon = \Psi_\varepsilon(\mathbf{x})$ and obtain in this way a boundary value problem for the transformed field $u_\varepsilon \circ \Psi_\varepsilon$

$$\begin{aligned} \Delta^\varepsilon(u_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x}) + k^2(u_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x}) = (f_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x}) \quad & \text{in } \Omega \\ (u_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x}) = (g_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x}) \quad & \text{on } \Gamma^{\mathrm{D}} \\ \nabla^\varepsilon(u_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x}) \cdot (\mathbf{n}_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x}) = (h_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x}) \quad & \text{on } \Gamma^{\mathrm{N}} \end{aligned} \tag{20}$$

where the operators Δ^ε and ∇^ε are given by

$$\Delta^\varepsilon u = \frac{\operatorname{div}(\det(D\Psi_\varepsilon) D\Psi_\varepsilon^{-1} D\Psi_\varepsilon^{-\top} Du)}{\det D\Psi_\varepsilon}$$

$$\nabla^\varepsilon u \cdot (\mathbf{n}_\varepsilon \circ \Psi_\varepsilon) = (D\Psi_\varepsilon^{-\top} \cdot Du) \cdot \frac{D\Psi_\varepsilon^{-\top}\mathbf{n}}{\|D\Psi_\varepsilon^{-\top}\mathbf{n}\|}$$

Since the coefficients of Δ^ε and ∇^ε depend smoothly on ε, they admit the Taylor expansions

$$\Delta^\varepsilon = \Delta + \varepsilon\tilde{\Delta} + \varepsilon^2 \Delta^R(\varepsilon)$$
$$\nabla^\varepsilon = \nabla + \varepsilon\tilde{\nabla} + \varepsilon^2 \nabla^R(\varepsilon)$$

with $\tilde{\Delta}$ and $\tilde{\nabla}$ given by

$$\tilde{\Delta} u = \operatorname{div}([\operatorname{div}\Psi \cdot I - (D\Psi^\top + D\Psi)] \cdot Du) - \operatorname{div}\Psi \cdot \Delta u$$
$$\tilde{\nabla} u = -D\Psi^\top u$$

Let us assume that the transformed right-hand sides $f_\varepsilon \circ \Psi_\varepsilon, g_\varepsilon \circ \Psi_\varepsilon, h_\varepsilon \circ \Psi_\varepsilon$ depend smoothly on ε

$$f_\varepsilon \circ \Psi_\varepsilon = f_0 + \varepsilon\dot{f} + \varepsilon^2 f_R \tag{21}$$
$$g_\varepsilon \circ \Psi_\varepsilon = g_0 + \varepsilon\dot{g} + \varepsilon^2 g_R \tag{22}$$
$$h_\varepsilon \circ \Psi_\varepsilon = h_0 + \varepsilon\dot{h} + \varepsilon^2 h_R \tag{23}$$

Inserting these expansions together with the formal ansatz

$$(u_\varepsilon \circ \Psi_\varepsilon)(\mathbf{x}) = u_0(\mathbf{x}) + \varepsilon\dot{u}(\mathbf{x}) + O(\varepsilon^2) \tag{24}$$

into Equation (20) and comparing the terms of the order $O(\varepsilon)$ we obtain an exterior boundary value problem for the material derivative $\dot{u}$

$$\begin{aligned} \Delta\dot{u} + k^2\dot{u} &= \dot{f} - \tilde{\Delta}u_0 && \text{in } \Omega \\ \dot{u} &= \dot{g} && \text{on } \Gamma^{\mathrm{D}} \\ \nabla\dot{u}\cdot\mathbf{n} &= \dot{h} - \tilde{\nabla}u_0\cdot\mathbf{n} - \nabla u_0\cdot\dot{\mathbf{n}} && \text{on } \Gamma^{\mathrm{N}} \end{aligned} \tag{25}$$

where u_0 is the solution of Equation (17) with $\varepsilon = 0$ and

$$\dot{\mathbf{n}} = (\mathbf{n}\cdot D\Psi^\top\mathbf{n})\mathbf{n} - D\Psi^\top\mathbf{n}$$

Ansatz (24) has to be justified, i.e. we have to show that the function $\dot{u}$ in Equation (24) coincides with the material derivative $\dot{u}$ defined by Equation (18). The correctness of Equation (24) can be easily proved with the help of the *a priori* estimate (13). Indeed, the following theorem holds.

Theorem 4.1. Let $d \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$ and $\vec{\beta} \in \mathbb{R}^Q$ be defined as in Theorem 3.2. Suppose that the Taylor expansions (21, 22) and (23) are valid with $f_\varepsilon \circ \Psi_\varepsilon, f_0, \dot{f}, f_R \in V^d_{\vec{\beta},-\gamma}(\Gamma)$, $g_\varepsilon \circ \Psi_\varepsilon, g_0, \dot{g}, g_R \in V^{d+3/2}_{\vec{\beta},}(\Gamma)$ and $h_\varepsilon \circ \Psi_\varepsilon, h_0, \dot{h}, h_R \in V^{d+1/2}_{\vec{\beta}}(\Gamma)$. Furthermore, we assume that f_ε has a compact support. Then the following estimate is valid:

$$\|u_\varepsilon \circ \Psi_\varepsilon - u_0 - \varepsilon\dot{u}\|_{V^{d+2}_{\vec{\beta},\gamma}(\Omega)} \leqslant c\varepsilon^2 \tag{26}$$

with a positive real constant c.

Proof. It can be easily checked that the function $v := u_\varepsilon \circ \Psi_\varepsilon - u_0 - \varepsilon \dot{u}$ solves the following mixed boundary value problem:

$$\begin{aligned} \Delta^\varepsilon v + k^2 v &= \varepsilon^2 \left(f_R - \Delta^R u_0 - \tilde{\Delta} \dot{u} \right) + O(\varepsilon^3) && \text{in } \Omega \\ v &= \varepsilon^2 g_R + O(\varepsilon^3) && \text{on } \Gamma^{\mathrm{D}} \\ \nabla^\varepsilon v \cdot (\mathbf{n}_\varepsilon \circ \Psi_\varepsilon) &= \varepsilon^2 (h_R - \nabla \dot{u} \cdot \dot{\mathbf{n}} - \nabla u_0 \cdot \mathbf{n}_R \\ &\quad - \tilde{\nabla} u_0 \cdot \dot{\mathbf{n}} - \tilde{\nabla} \dot{u} \cdot \mathbf{n} - \nabla^R u_0 \cdot \mathbf{n}) + O(\varepsilon^3) && \text{on } \Gamma^{\mathrm{N}} \end{aligned} \tag{27}$$

We note, that the operator of the above boundary value problem is a small perturbation of the operator corresponding to problem (10–12) and that Δ^ε coincides with Δ for big $|\mathbf{x}|$. Therefore, Theorem 3.2 is applicable to Equation (27) for small enough ε. Applying Theorem 3.2 to the unperturbed problem (10–12) and to Equation (25) we conclude that $u_0, \dot{u} \in V^{d+2}_{\vec{\beta},\gamma}(\Omega)$. Since Δ^R and $\tilde{\Delta}$ are operators of second order, whose coefficients have compact support, so $\Delta^R u_0$ and $\tilde{\Delta}\dot{u}$ have support compact, too, and belong to $V^{d}_{\vec{\beta},-\gamma}(\Omega)$. Furthermore, $\nabla \dot{u}, \nabla u_0, \tilde{\nabla} u_0, \tilde{\nabla} \dot{u}, \nabla^R u_0 \in V^{d+1/2}_{\vec{\beta}}(\Gamma)$. Applying *a priori* estimate (13) to the function v we obtain the assertion. □

The existence and the regularity of the shape derivative u' follows directly from definition (19) of u' and the preceding theorem.

Corollary 4.2. Let the assumptions of Theorem 4.1 be satisfied. Then the shape derivative u' exists in $V^{d+1}_{\vec{\beta},\gamma}(\Omega)$.

Remark 4.2. The domain sensitivity analysis performed above cannot be applied directly to the perturbed scattered wave $u^{\mathrm{s}}_\varepsilon$, which satisfies problem (17) with $g_\varepsilon = -u^{\mathrm{i}}|_{\Gamma^{\mathrm{D}}_\varepsilon}$ and $h_\varepsilon = -\partial u^{\mathrm{i}}/\partial \mathbf{n}|_{\Gamma^{\mathrm{N}}_\varepsilon}$ because in this case the right-hand sides $g_\varepsilon, h_\varepsilon$ do not satisfy the assumptions of Theorem 4.1 (see Remark 3.1). Therefore, we perform the form sensitivity analysis for the acoustic field

$$U_\varepsilon := u^{\mathrm{s}}_\varepsilon + \eta u^{\mathrm{i}}$$

where η is a cut-off function with support in the vicinity of the boundary Γ. We note that U_ε satisfies the Helmholtz equation with a right-hand side having a compact support and vanishing in the neighbourhood of Γ_ε and satisfies homogeneous boundary conditions (2) or (4). Furthermore, $U_\varepsilon = u^{\mathrm{s}}_\varepsilon$ outside some neighbourhood of Γ_ε.

A boundary value problem for the shape derivative U' of U_ε can be obtained by inserting Equation (19) into Equation (25) [8,21]. By long but straightforward calculations, we find out that U' solves the following exterior mixed boundary value problem:

$$\begin{aligned} \Delta U' + k^2 U' &= 0 && \text{in } \Omega \\ U' &= -\Psi \cdot \mathbf{n} \frac{\partial U_0}{\partial \mathbf{n}_{\mathbf{x}}} && \text{on } \Gamma^{\mathrm{D}} \\ \frac{\partial U'}{\partial \mathbf{n}_{\mathbf{x}}} &= -\Psi \cdot \mathbf{n} \frac{\partial^2 U_0}{\partial \mathbf{n}_{\mathbf{x}}^2} + \nabla_\Gamma (\Psi \cdot \mathbf{n}) \cdot \nabla_{\mathbf{x}} U_0 && \text{on } \Gamma^{\mathrm{N}} \end{aligned} \tag{28}$$

and the Sommerfeld's radiation condition at infinity. Here, $\partial^2/\partial \mathbf{n}_{\mathbf{x}}^2 = \sum_{i,j} n_i n_j \partial^2/\partial x_i \partial x_j$, and $\nabla_\Gamma f$ is the tangential gradient given by

$$\nabla_\Gamma f = \nabla f - (\mathbf{n} \cdot \nabla f)\mathbf{n}$$

Remember that in case of cracks, the boundary condition must be imposed on both sides of the crack as in Equation (4).

4.3. *Form sensitivity of the far-field pattern*

Let Ω be an exterior domain of type (B) or (C). The perturbed scattered wave $u_\varepsilon^{\mathrm{s}}$ has at infinity asymptotics (8) with the far-field pattern $\mathscr{F}(\Gamma_\varepsilon)$ given by an analogue of formula (9)

$$\mathscr{F}(\Gamma_\varepsilon)(\hat{\mathbf{x}}) = \int_{S_R} \left[u_\varepsilon^{\mathrm{s}}(\mathbf{y}) \frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}} - \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}} \frac{\partial u_\varepsilon^{\mathrm{s}}(\mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} \right] \mathrm{d}s_{\mathbf{y}} \tag{29}$$

Let us calculate the Gâteaux derivative

$$\mathrm{d}\mathscr{F}(\Gamma, \Psi) = \lim_{\varepsilon \to 0} \frac{\mathscr{F}(\Psi_\varepsilon(\Gamma)) - \mathscr{F}(\Gamma)}{\varepsilon} = \left. \frac{\mathrm{d}\mathscr{F}(\Gamma_\varepsilon)}{\mathrm{d}\varepsilon} \right|_{\varepsilon = 0} \tag{30}$$

For big enough $R = |\mathbf{x}|$ we have $\Psi_\varepsilon|_{S_R} = I$ and $u_\varepsilon^{\mathrm{s}}|_{S_R} = U_\varepsilon \circ \Psi_\varepsilon|_{S_R}$. Thus,

$$\mathscr{F}(\Gamma_\varepsilon)(\hat{\mathbf{x}}) = \int_{S_R} \left[(U_\varepsilon \circ \Psi_\varepsilon)(\mathbf{y}) \frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}} - \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}} \frac{\partial (U_\varepsilon \circ \Psi_\varepsilon)(\mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} \right] \mathrm{d}s_{\mathbf{y}} \tag{31}$$

Differentiating both sides of the above equation by ε and taking $\varepsilon = 0$ we obtain immediately

$$\mathrm{d}\mathscr{F}(\Gamma, \Psi)(\hat{\mathbf{x}}) = \int_{S_R} \left[\dot{U}(\mathbf{y}) \frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}} - \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}} \frac{\partial \dot{U}(\mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} \right] \mathrm{d}s_{\mathbf{y}} \tag{32}$$

Since $\dot{U}(\mathbf{x}) = U'(\mathbf{x})$ for big enough $|\mathbf{x}|$, we get

$$\mathrm{d}\mathscr{F}(\Gamma, \Psi)(\hat{\mathbf{x}}) = \int_{S_R} \left[U'(\mathbf{y}) \frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}}}{\partial \mathbf{n}_{\mathbf{y}}} - \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}} \frac{\partial U'(\mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} \right] \mathrm{d}s_{\mathbf{y}} \tag{33}$$

Representation (33) is not suitable for numerical realization because U', in general, cannot be defined as a variational solution of boundary value problem (28). The asymptotic analysis shows that the solution U_0 behaves near the singular points $P_q \in S$ as $|\mathbf{x} - P_q|^{\alpha_q}$ with $\alpha_q \geqslant 1/4$. Therefore, $U_0 \in H^{1+\alpha}(\Omega), \nabla U_0 \in H^\alpha(\Omega)$ with

$$\alpha = \min_{P_q \in S} \alpha_q$$

and consequently $U' \in H^\alpha(\Omega)$ due to Equation (19). If the domain is not convex then we have $\alpha < 1$ and thus $U' \notin H^1(\Omega)$. Furthermore, U' is in general not uniquely defined by Equation (28). For example, in case of a straight crack perturbed along a straight line, U' satisfies the Helmholtz equation with homogeneous right-hand sides. Therefore, U' cannot be computed in general by solving Equation (28) numerically with the help of the standard boundary element or finite element methods.

In order to overcome this difficulty, we derive in the next section from Equation (33) a representation for $\mathrm{d}\mathscr{F}(\Gamma,\Psi)(\hat{\mathbf{x}})$, which is better suitable for a numerical realization. We use the method of adjoint problems [10, 11], which consists in applying the second Green formula to the shape derivative U' and to the solution w of an appropriately defined adjoint problem. This leads to an expression in which only U_0 and the adjoint field w appear.

5. THE METHOD OF ADJOINT PROBLEMS

5.1. Exterior of a bounded domain

Let us first assume that Ω is the exterior of bounded domain. We define w as the solution of the following mixed boundary value problem:

$$\begin{aligned} \Delta w(\mathbf{y}) + k^2 w(\mathbf{y}) &= 0 && \text{in } \Omega \\ w(\mathbf{y}) &= \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}} && \text{on } \Gamma^{\mathrm{D}} \\ \frac{\partial w(\mathbf{y})}{\partial \mathbf{n_y}} &= \frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}}}{\partial \mathbf{n_y}} && \text{on } \Gamma^{\mathrm{N}} \end{aligned} \tag{34}$$

which satisfies Sommerfeld radiation condition (3) at infinity.

Using second Green's formula for the functions U' and w in the domain $B_R \cap \Omega$, passing to the limit as $R \to +\infty$ and taking into account that U', w satisfy Equation (3) we obtain

$$0 = \int_{S_R} \left[w(\mathbf{y}) \frac{\partial U'(\mathbf{y})}{\partial \mathbf{n_y}} - U'(\mathbf{y}) \frac{\partial w(\mathbf{y})}{\partial \mathbf{n_y}} \right] \mathrm{d}s_{\mathbf{y}} \tag{35}$$

Summing up expressions (35) and (33) we get

$$\mathrm{d}\mathscr{F}(\Gamma,\Psi)(\hat{\mathbf{x}}) = \int_{S_R} \left[W(\mathbf{y}) \frac{\partial U'(\mathbf{y})}{\partial \mathbf{n_y}} - U'(\mathbf{y}) \frac{\partial W(\mathbf{y})}{\partial \mathbf{n_y}} \right] \mathrm{d}s_{\mathbf{y}} \tag{36}$$

with $W := w - \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}}$. We note that the normal vector $\mathbf{n}$ on S_R is directed outwards.

Let $B_\delta(P_q)$ be a ball with centre in P_q and radius δ. Inserting U' and W into second Green's formula in $(\Omega \cap B_R) \setminus \bigcup_{q=1}^{Q} B_\delta(P_q)$, we obtain from (36)

$$\begin{aligned} \mathrm{d}\mathscr{F}(\Gamma,\Psi)(\hat{\mathbf{x}}) = &\int_{\Gamma \setminus \bigcup_{q=1}^{Q} B_\delta(P_q)} \left\{ U'(\mathbf{y}) \frac{\partial W(\mathbf{y})}{\partial \mathbf{n_y}} - \frac{\partial U'(\mathbf{y})}{\partial \mathbf{n_y}} W(\mathbf{y}) \right\} \mathrm{d}s_{\mathbf{y}} \\ &+ \int_{\bigcup_{q=1}^{Q} \partial B_\delta(P_q) \cap \Omega} \left\{ U'(\mathbf{y}) \frac{\partial W(\mathbf{y})}{\partial \mathbf{n_y}} - \frac{\partial U'(\mathbf{y})}{\partial \mathbf{n_y}} W(\mathbf{y}) \right\} \mathrm{d}s_{\mathbf{y}} \end{aligned} \tag{37}$$

Let us formally pass to the limit as $\delta \to 0$ on both sides of Equation (37) and rewrite it as

$$\mathrm{d}\mathscr{F}(\Gamma,\Psi)(\hat{\mathbf{x}}) = \int_{\Gamma} \left\{ U'(\mathbf{y}) \frac{\partial W(\mathbf{y})}{\partial \mathbf{n_y}} - \frac{\partial U'(\mathbf{y})}{\partial \mathbf{n_y}} W(\mathbf{y}) \right\} \mathrm{d}s_{\mathbf{y}} + \lim_{\delta \to 0} \int_{\bigcup_{q=1}^{Q} \partial B_\delta(P_q) \cap \Omega} \left\{ U'(\mathbf{y}) \frac{\partial W(\mathbf{y})}{\partial \mathbf{n_y}} - \frac{\partial U'(\mathbf{y})}{\partial \mathbf{n_y}} W(\mathbf{y}) \right\} \mathrm{d}s_{\mathbf{y}} \quad (38)$$

Finally, substituting the boundary values of U' in the first integral of Equation (38), we obtain the following formal expression for the far-field derivative:

$$\mathrm{d}\mathscr{F}(\Gamma,\Psi)(\hat{\mathbf{x}}) = -\int_{\Gamma^{\mathrm{D}}} \Psi \cdot \mathbf{n} \frac{\partial U_0}{\partial \mathbf{n_y}} \frac{\partial W}{\partial \mathbf{n_y}} \mathrm{d}s_{\mathbf{y}} + \int_{\Gamma^{\mathrm{N}}} \left(\Psi \cdot \mathbf{n} \frac{\partial^2 U_0}{\partial \mathbf{n}_{\mathbf{y}}^2} - \nabla_\Gamma (\Psi \cdot \mathbf{n}) \cdot \nabla_{\mathbf{y}} U_0 \right) W \, \mathrm{d}s_{\mathbf{y}} + \lim_{\delta \to 0} \sum_{q=1}^{Q} \int_{\partial B_\delta(P_q) \cap \Omega} \left\{ U'(\mathbf{y}) \frac{\partial W(\mathbf{y})}{\partial \mathbf{n_y}} - \frac{\partial U'(\mathbf{y})}{\partial \mathbf{n_y}} W(\mathbf{y}) \right\} \mathrm{d}s_{\mathbf{y}} \quad (39)$$

In order to justify the limit passage in Equation (39) we have to investigate the behaviour of the integrand as $\delta \to 0$ for every singular point $P_q \in S$. The existence of the limit for every $P_q \in S$ depends on the corresponding opening angle and the kind of boundary conditions near P_q. In the following, we denote by L_q:

$$L_q := \lim_{\delta \to 0} \int_{\partial B_\delta(P_q) \cap \Omega} \left\{ U'(\mathbf{y}) \frac{\partial W(\mathbf{y})}{\partial \mathbf{n_y}} - \frac{\partial U'(\mathbf{y})}{\partial \mathbf{n_y}} W(\mathbf{y}) \right\} \mathrm{d}s_{\mathbf{y}} \quad (40)$$

Theorem 5.1. (a) Let $P_q \in S_{\mathrm{D}} \cup S_{\mathrm{N}}$. Then $L_q = 0$.
(b) Let $P_q \in S_{\mathrm{M}}$. Then

$$L_q = \begin{cases} 0 & \text{if } \omega_q^o < \pi \\ k_q(U_0) k_q(W) (\frac{1}{4} \Psi_t(P_q) - \frac{1}{2} \Psi_n(P_q)) & \text{if } \omega_q^o = \pi \\ \infty & \text{if } \omega_q^o > \pi \end{cases}$$

Here we denote by $\Psi_t(P_q), \Psi_n(P_q)$ the tangential and the normal components of Ψ at P_q, respectively.

Proof. (a) Let us assume first that $P_q \in S_{\mathrm{D}} \cup S_{\mathrm{N}}$. According to Equation (14) the functions U_0 and W behave in the neighbourhood of P_q as

$$U_0(\mathbf{x}) = O(r^{\pi/\omega_q^o}), \quad W(\mathbf{x}) = O(r^{\pi/\omega_q^o}) \quad (41)$$

Consequently, $U'(\mathbf{x}) = O(r^{(\pi/\omega_q^o)-1})$ due to Equation (19) and so the integrand of Equation (40) behaves as $O(r^{(2\pi/\omega_q^o)-2})$. Since $0 < \omega_q^o < 2\pi$ then $2\pi/\omega_q^o - 2 > -1$, which implies that $L_q = 0$.

(b) Let us suppose now that $P_q \in S_{\mathrm{M}}$. Then according to Equation (14) the functions U_0 and W behave in the neighbourhood of P_q as $O(r_q^{\pi/2\omega_q^o})$, and consequently, $U'(\mathbf{x}) = O(r_q^{(\pi/2\omega_q^o)-1})$. Therefore, the integrand of Equation (40) behaves as $O(r_q^{(\pi/\omega_q^o)-2})$. Thus, $L_q = 0$ if $\omega_q^o < \pi$ and $L_q = \infty$ if $\omega_q^o > \pi$.

Let us investigate the case $\omega_q^o = \pi$, i.e. the boundary Γ is smooth in the vicinity of P_q. We can assume without loss of generality that the tangential vector at the singular point P_q has the direction of the x_1-axis.

From Equation (14) we know that

$$U_0(\mathbf{x}) = k_q(U_0) r^{1/2} \sin(\tfrac{1}{2}\omega) + O(r)$$
$$W(\mathbf{x}) = k_q(W) r^{1/2} \sin(\tfrac{1}{2}\omega) + O(r) \tag{42}$$

near P_q.

Direct calculation shows that

$$\partial_{x_1} U_0(\mathbf{x}) = -\tfrac{1}{2} k_q(U_0) r^{-1/2} \sin(\tfrac{1}{2}\omega) + O(1)$$
$$\partial_{x_2} U_0(\mathbf{x}) = \tfrac{1}{2} k_q(U_0) r^{-1/2} \cos(\tfrac{1}{2}\omega) + O(1)$$

and thus

$$U'(\mathbf{x}) = \tfrac{1}{2} k_q(U_0) r^{-1/2} (\Psi_t(P_q) \sin(\tfrac{1}{2}\omega) - \Psi_n(P_q) \cos(\tfrac{1}{2}\omega)) + O(1) \tag{43}$$

because of Equation (19). Inserting Equations (43) and (42) into Equation (40) we can calculate easily the limit and obtain

$$L_q = k_q(U_0) k_q(W) (\tfrac{1}{4}\Psi_t(P_q) - \tfrac{1}{2}\Psi_n(P_q)) \tag{44}$$

□

Remark 5.1. The method of adjoint problems used above fails if the boundary contains a collision point $P_q \in S_{\mathrm{M}}$ with an exterior opening angle $\omega_q^o > \pi$, unless P_q is not perturbed, i.e. $\Psi(P_q) = 0$.

5.2. Exterior domains with cracks

The above considerations along with formula (37) can be repeated with some obvious changes for problems in the exterior of a crack. Note, that for crack problems we assumed that the boundary conditions do not change along the crack, i.e. $S = S_{\mathrm{D}}$ or S_{N}. The adjoint backscattered wave is defined as the solution of the following boundary value problem in the exterior of the curve:

$$\Delta w(\mathbf{y}) + k^2 w(\mathbf{y}) = 0 \qquad \text{in } \Omega$$
$$w_\pm(\mathbf{y}) = \mathrm{e}^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}} \qquad \text{on } \Gamma^{\mathrm{D}} \backslash S \text{ or}$$
$$\frac{\partial w_\pm(\mathbf{y})}{\partial \mathbf{n_y}} = \frac{\partial \mathrm{e}^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}}}{\partial \mathbf{n_y}} \qquad \text{on } \Gamma^{\mathrm{N}} \backslash S$$

satisfying the Sommerfeld radiation condition (3) at infinity.

Formula (39) reads now

$$\mathrm{d}\mathscr{F}(\Gamma, \Psi)(\hat{\mathbf{x}}) = -\int_\Gamma \Psi \cdot \mathbf{n} \left[\left[\frac{\partial U_0}{\partial \mathbf{n_y}} \frac{\partial W}{\partial \mathbf{n_y}} \right]\right] \mathrm{d}s_{\mathbf{y}} + \sum_{q=1}^{Q} L_q \tag{45}$$

in case of Dirichlet conditions and

$$d\mathscr{F}(\Gamma,\Psi)(\hat{\mathbf{x}}) = \int_\Gamma \left[\!\left[\left(\Psi\cdot\mathbf{n}\frac{\partial^2 U_0}{\partial \mathbf{n}_{\mathbf{y}}^2} - \nabla_\Gamma(\Psi\cdot\mathbf{n})\cdot\nabla_{\mathbf{y}} U_0 \right) W \right]\!\right] ds_{\mathbf{y}} + \sum_{q=1}^{Q} L_q \tag{46}$$

if Neumann conditions are prescribed. Here, $[\![\cdot]\!]$ denotes the jump across the crack and U_0 is defined as in Remark 5.1.

The limits L_q can be calculated similar as for problems in the exterior of a bounded domain.

Let us denote by $\Psi_t(P_q), \Psi_n(P_q)$ the tangential and the normal components of Ψ at the crack tip P_q, respectively.

Theorem 5.2. Let P_q be the tips of the crack. Then

$$L_q = (\pi/2)k_q(U_0)k_q(W)\Psi_t(P_q) \qquad \text{if } P_q \in S_D$$
$$L_q = -(\pi/2)k_q(U_0)k_q(W)\Psi_t(P_q) \quad \text{if } P_q \in S_N$$

If P_q is an interior corner of the crack, then $L_q = 0$.

Proof. For interior corner points the proof is the same as in Theorem 5.1 for $P_q \in S_D \cup S_N$. Let us therefore suppose that P_q is a crack tip. From Equation (14) we know that

$$\begin{aligned} U_0(\mathbf{x}) &= k_q(U_0)r^{1/2}\sin(\tfrac{1}{2}\omega) + O(r) \\ W(\mathbf{x}) &= k_q(W)r^{1/2}\sin(\tfrac{1}{2}\omega) + O(r) \end{aligned} \tag{47}$$

near P_q for $P_q \in S_D$ and

$$\begin{aligned} U_0(\mathbf{x}) &= k_q(U_0)r^{1/2}\cos(\tfrac{1}{2}\omega) + O(r) \\ W(\mathbf{x}) &= k_q(W)r^{1/2}\cos(\tfrac{1}{2}\omega) + O(r) \end{aligned} \tag{48}$$

for $P_q \in S_N$.

Direct calculation shows that

$$\partial_{x_1} U_0 = -\tfrac{1}{2}k_q(U_0)r^{-1/2}\sin(\tfrac{1}{2}\omega) + O(1) \quad \text{for } P_q \in S_D$$
$$\partial_{x_2} U_0 = \tfrac{1}{2}k_q(U_0)r^{-1/2}\cos(\tfrac{1}{2}\omega) + O(1) \quad \text{for } P_q \in S_D$$
$$\partial_{x_1} U_0 = \tfrac{1}{2}k_q(U_0)r^{-1/2}\cos(\tfrac{1}{2}\omega) + O(1) \quad \text{for } P_q \in S_N$$
$$\partial_{x_2} U_0 = \tfrac{1}{2}k_q(U_0)r^{-1/2}\sin(\tfrac{1}{2}\omega) + O(1) \quad \text{for } P_q \in S_N$$

and thus

$$U' = \tfrac{1}{2}k_q(U_0)r^{-1/2}(\Psi_t(P_q)\sin(\tfrac{1}{2}\omega) - \Psi_n(P_q)\cos(\tfrac{1}{2})) + O(1) \qquad \text{for } P_q \in S_D$$
$$U' = -\tfrac{1}{2}k_q(U_0)r^{-1/2}(\Psi_t(P_q)\cos(\tfrac{1}{2}\omega) + \Psi_n(P_q)\sin(\tfrac{1}{2})) + O(1) \quad \text{for } P_q \in S_N$$

because of Equation (19). Inserting these asymptotics and Equations (47, 48) into Equation (40) we can calculate easily the limit and obtain the assertion. □

6. 3D DOMAIN WITH SMOOTH EDGES

In the last section, we want to show that scattering from three-dimensional domains with smooth edges can be treated in a similar manner as two-dimensional problems in domains with corners. Since proofs are very similar as in the 2D case, we will state only the main results. The interested reader can find the complete proofs in Reference [22]. Furthermore, we will consider only two geometrical configurations where the harmonic fields behave like the square root of the distance from the edges, i.e. only those cases where we get an explicit dependence of the shape derivative $d\mathscr{F}(\Gamma,\Psi)$ on the singularities of the harmonic fields.

Let us consider the mixed boundary value problem (1,2,3) in a three-dimensional domain $\Omega := \mathbb{R}^3\backslash\bar{\Omega}_i$ which is of one of the following types:

- (B) Ω_i is a bounded domain with a smooth boundary $\Gamma=\Gamma^{\mathrm{D}}\cup\Gamma^{\mathrm{N}}\cup S$, where S is a smooth non-intersecting curve. The unit normal vector $\mathbf{n}=(n_1,n_2,n_3)$ on Γ is directed towards Ω_i.
- (C) $\Omega_i=\Gamma$ is a crack, i.e. a smooth bounded open surface bounded by a smooth edge M. In this case, we assume that either $\Gamma=\Gamma^{\mathrm{D}}$ or Γ^{N}, i.e. the boundary conditions do not change on the crack surface. The direction of the unit normal vector $\mathbf{n}$ on Γ is chosen arbitrarily but is fixed on Γ.

It is well known (see e.g. Reference [13]), that the 3D scattered wave u^{s} has the following behaviour as $r\to\infty$:

$$u^{\mathrm{s}}(\mathbf{x})=\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\mathscr{F}(\Gamma)(\hat{\mathbf{x}})+o(1/r) \tag{49}$$

As in the two-dimensional case, the far-field pattern $\mathscr{F}(\Gamma)$ is given by

$$\mathscr{F}(\Gamma)(\hat{\mathbf{x}})=\int_{S_R}\left[u^{\mathrm{s}}(\mathbf{y})\frac{\partial\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}}}{\partial\mathbf{n}_{\mathbf{y}}}-\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}}\frac{\partial u^{\mathrm{s}}(\mathbf{y})}{\partial\mathbf{n}_{\mathbf{y}}}\right]\,\mathrm{d}s_{\mathbf{y}}$$

The asymptotic behaviour of harmonic waves in the vicinity of the edge M can be best described in curvilinear coordinates (r,ω,s) which are defined in a toroidal neighbourhood of M (see Figure 3). Here, $s\in[0,L_M)$ denotes the arc length and (r,ω) are the polar co-ordinates in the plane perpendicular to M, i.e. $r=\mathrm{dist}(\mathbf{x},M)$.

Theorem 6.1 (*Asymptotic behaviour at edges*). Suppose that u satisfies the homogeneous Helmholtz equation and homogeneous boundary conditions in the neighbourhood of the edge M. Then u has in the vicinity of M the following asymptotic behaviour as $r\to 0$:

$$u(\mathbf{x})=k(s)r^{1/2}\sin(\omega/2)+O(r)$$

in case of a crack with Dirichlet boundary conditions and a mixed boundary value problem in the exterior of a bounded domain, and

$$u(\mathbf{x})=k(s)r^{1/2}\cos(\omega/2)+O(r)$$

for a crack with Neumann boundary conditions. Furthermore, $k\in C^{\infty}(M)$.

The assertion is a simple application of results from the theory of general elliptic problems in domains with edges [19]. We note that the only difference between the asymptotics of

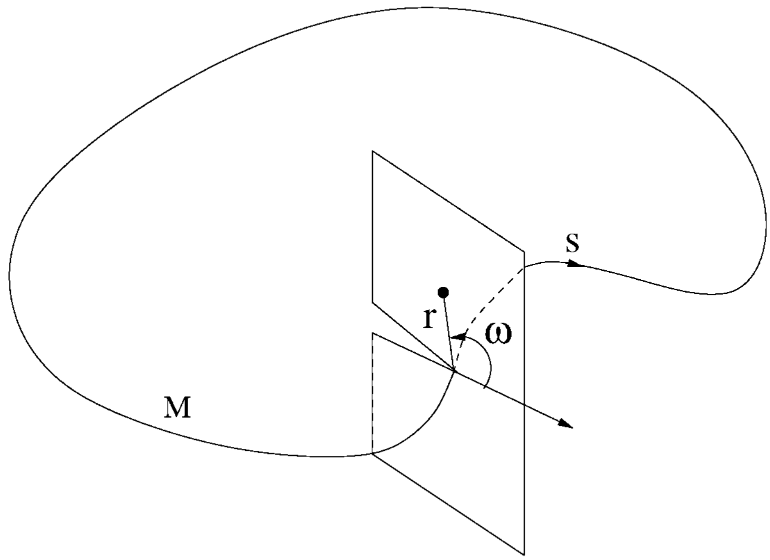

Figure 3. Curvilinear co-ordinates (r, ω, s).

harmonic fields for 2D domains with corners and for 3D domains considered here lies in the dependence of the coefficients k and the arc length s.

Let us assume first, that Ω is the exterior of a bounded smooth domain. The previous calculations for two-dimensional domains with corners can be repeated here, taking into account that U' has now a uniform $r^{-1/2}$ singularity along the whole edge M. Therefore, we can show that

$$\mathrm{d}\mathscr{F}(\Gamma,\Psi)(\hat{\mathbf{x}}) = -\int_{\Gamma^{\mathrm{D}}} \Psi\cdot\mathbf{n}\frac{\partial U_0}{\partial \mathbf{n_y}}\frac{\partial W}{\partial \mathbf{n_y}}\,\mathrm{d}s_{\mathbf{y}} + \int_{\Gamma^{\mathrm{N}}}\left(\Psi\cdot\mathbf{n}\frac{\partial^2 U_0}{\partial \mathbf{n}_{\mathbf{y}}^2} - \nabla_\Gamma(\Psi\cdot\mathbf{n})\cdot\nabla_{\mathbf{y}}U_0\right)W\,\mathrm{d}s_{\mathbf{y}}$$
$$+\lim_{\delta\to 0}\sum_{q=1}^{Q}\int_{\partial T_\delta(M)\cap\Omega}\left\{U'(\mathbf{y})\frac{\partial W(\mathbf{y})}{\partial \mathbf{n_y}} - \frac{\partial U'(\mathbf{y})}{\partial \mathbf{n_y}}W(\mathbf{y})\right\}\mathrm{d}s_{\mathbf{y}} \tag{50}$$

Here $W = w - \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{x}}\cdot\mathbf{y}}$, w is the solution of the three-dimensional adjoint problem (34) and

$$T_\delta(M) := \{(r,\omega,s)\in\mathbb{R}^3:\ r\in[0,\delta],\ \omega\in[0,2\pi],\ s\in[0,L_M)\}$$

is a toroidal neighbourhood of the edge M with the length L_M.

Let us denote by L the limit in Equation (50) and introduce curvilinear coordinates (t,n,s) near M, where s is defined as before and (t,n) are Cartesian coordinates in the plane perpendicular to M related to the polar coordinates (r,ω) by

$$t = r\cos(\omega), \quad n = r\sin(\omega)$$

Thus, the vector $(t,0,0)$ is tangential to Γ, whereas $(0,n,0)$ is normal to Γ.

Let us denote by $\Psi_t(M), \Psi_n(M), \Psi_s(M)$ the traces of the t,n and s component of the mapping Ψ on M. We remark that they are functions depending on the variable s.

Similar as in the 2D case, we calculate the asymptotics of U' from the asymptotics of U_0 using Equation (19) and obtain

$$U'(\mathbf{x}) = \tfrac{1}{2}k(U_0)(s)r^{-1/2}(\Psi_t(M)(s)\sin(\tfrac{1}{2}\omega) - \Psi_n(M)(s)\cos(\tfrac{1}{2}\omega)) + O(1) \tag{51}$$

Inserting Equation (51) into Equation (50) and calculating the limit we obtain

$$L = \int_M k(U_0)k(W)\left(\frac{1}{4}\Psi_t(M) - \frac{1}{2}\Psi_n(M)\right) \mathrm{d}s$$

Performing the same calculations for the exterior of a smooth crack we get

$$\mathrm{d}\mathscr{F}(\Gamma,\Psi)(\hat{\mathbf{x}}) = -\int_\Gamma \Psi\cdot\mathbf{n}\left[\left[\frac{\partial U_0}{\partial \mathbf{n_y}}\,\frac{\partial W}{\partial \mathbf{n_y}}\right]\right] \mathrm{d}s_{\mathbf{y}} + \frac{\pi}{2}\int_M k(U_0)k(W)\Psi_t(M)\,\mathrm{d}s$$

in case of Dirichlet conditions and

$$\mathrm{d}\mathscr{F}(\Gamma,\Psi)(\hat{\mathbf{x}}) = \int_\Gamma \left[\left[\left(\Psi\cdot\mathbf{n}\frac{\partial^2 U_0}{\partial \mathbf{n}_{\mathbf{y}}^2} - \nabla_\Gamma(\Psi\cdot\mathbf{n})\cdot\nabla_{\mathbf{y}}U_0\right)W\right]\right] \mathrm{d}s_{\mathbf{y}}$$
$$- \frac{\pi}{2}\int_M k(U_0)k(W)\Psi_t(M)\,\mathrm{d}s$$

if Neumann conditions are prescribed.

ACKNOWLEDGEMENTS

The work of the first author was supported by the German Science Foundation DFG in the framework of the SFB 404. The paper was written during the stay of the second author at the University of Stuttgart as a Alexander von Humboldt fellow. Furthermore, the authors would like to thank Prof. W. L. Wendland for making this joint work possible.

REFERENCES

1. Kirsch A. The domain derivative and two applications in inverse scattering theory. *Inverse Problems* 1993; **9**:81–96.
2. Potthast R. Fréchet differentiability of boundary integral operators in inverse acoustic scattering. *Inverse Problems* 1994; **10**:431–447.
3. Potthast R. Fréchet differentiability of the solution to the acoustic Neumann scattering problem with respect to the domain. *Journal of Inverse Ill-Posed Problems* 1996; **4**:67–84.
4. Hettlich F. Fréchet derivatives in inverse obstacle scattering. *Inverse Problems* 1995; **11**:371–382.
5. Kress R. Inverse scattering from an open arc. *Mathematical Methods in the Applied Sciences* 1995; **18**:267–293.
6. Bochniak M, Sändig A-M. Sensitivity analysis of elastic structures in presence of stress singularities. *Archives of Mechanics* 1999; **51**:155–171.
7. Bochniak M, Sändig A-M. Sensitivity analysis of stress intensity factors in 2d coupled elastic structures. *Asymptotic Analysis* 2001; **25**:299–328.
8. Simon J. Differentiation with respect to the domain in boundary value problems. *Numerical Functional Analysis and Optimization* 1980; **2**:649–687.
9. Sokolowski J, Zolesio JP. *Introduction to Shape Optimization*. Springer: Berlin, 1992.
10. Dems K, Mróz Z. Variational approach by means of adjoint systems to structural optimization and sensitivity analysis II. Structure shape variation. *International Journal of Solids and Structures* 1984; **20**:527–552.
11. Dems K, Mróz Z. Shape sensitivity in mixed Dirichlet–Neumann boundary-value problems and associated class of path-independent integrals. *European Journal of Mechanics A/Solids* 1995; **14**:169–203.

12. Hettlich F. *The Domain Derivative in Inverse Obstacle Problems*. Habilitationsschrift: Erlangen, 1999.
13. Colton D, Kress R. *Inverse Acoustic and Electromagnetic Scattering Theory*. Springer: Berlin, 1992.
14. Ramm AG. Existence and uniqueness of scattering solutions in non-smooth domains. *Journal of Mathematical Analysis and Application* 1996; **201**:329–338.
15. Wendland WL, Stephan EP. A hypersingular boundary integral method for two-dimensional screen and crack problems. *Archives in Rational Mechanics and Analysis* 1990; **112**:363–390.
16. Vainberg BR. *Asymptotic Methods in Equations of Mathematical Physics*. Gordon and Breach Science Publishers: London, 1982.
17. Kondrat'ev VA. Boundary problems for elliptic equations in domains with conical or angular points. *Transactions of the Moscow Mathematical Society* 1967; **16**:209–292.
18. Kozlov VA, Maz'ya V, Rossmann J. *Elliptic Boundary Value Problems in Domains with Point Singularities.* American Mathematical Society: Providence, RI, 1997.
19. Nazarov SA, Plamenevsky BA. *Elliptic Problems in Domains with Piecewise Smooth Boundaries*. Walter de Gruyter: Berlin, 1994.
20. Berezanskii JuM. *Expansions in Eigenfunctions of Selfadjoint Operators*, *Translations of Mathematical Monographs*, vol. 17. American Mathematical Society: Providence, RI, 1968.
21. Simon J. Optimum design for Neumann condition and for related boundary value conditions. *Preprint no.* 86001, Paris 6, 1988.
22. Bochniak M, Cakoni F. The method of adjoint problems for the domain sensitivity analysis of the acoustic far field pattern. Part II: 3D domains with edges. *Preprint* 2000/07, *SFB* 404, University of Stuttgart, 2000.

