Finite Element Methods for Linear Elasticity

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Equations of linear elasticity:

$$A\sigma = \varepsilon(u), \qquad \operatorname{div}\sigma = f \quad \text{in } \Omega.$$

Stress σ takes values in $\mathbb{S} = \mathbb{R}^{n\times n}_{\text{sym}}$.
Displacement u takes values in $\mathbb{V} = \mathbb{R}^n$.

f = body force, $(\varepsilon(u))_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$,
divergence operator applied row-wise.

Compliance tensor $A = A(x) : \mathbb{S} \to \mathbb{S}$ bounded, symmetric, uniformly positive definite operator reflecting material properties.

Isotropic case: Let $\lambda(x), \mu(x)$ positive scalar coefficients (Lamé coefficients), tr = trace. Then

$$A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + n\lambda}\operatorname{tr}(\sigma)I\right).$$

Boundary condition $u = 0$ on $\partial\Omega$ (clamped case). Modifications needed for other B.C., e.g., traction boundary conditions $\sigma n = 0$.

When A invertible, i.e., $\sigma = A^{-1}\varepsilon(u) = C\varepsilon(u)$, then for isotropic elasticity,

$$C\tau = 2\mu(\tau + \lambda \operatorname{tr} \tau I).$$

Then formulate elasticity system weakly as:
Find $\sigma \in L^2(\Omega, \mathbb{S})$, $u \in \mathring{H}^1(\Omega; \mathbb{V})$ such that

$$\int_\Omega \sigma : \tau \, dx - \int_\Omega C\varepsilon(u) : \tau \, dx = 0, \ \tau \in L^2(\Omega, \mathbb{S}),$$
$$\int_\Omega \sigma : \varepsilon(v) \, dx = \int_\Omega f \cdot v \, dx, \ v \in \mathring{H}^1(\Omega; \mathbb{V}),$$

where $\sigma : \tau = \sum_{i,j=1}^n \sigma_{ij}\tau_{ij}$.

In this case, may eliminate σ completely to obtain pure displacement formulation: Find $u \in \mathring{H}^1(\Omega; \mathbb{V})$ such that

$$\int_\Omega C\varepsilon(u) : \varepsilon(v)\, dx = \int_\Omega f \cdot v\, dx, \ \ v \in \mathring{H}^1(\Omega; \mathbb{V}).$$

As material becomes incompressible, i.e., $\lambda \to \infty$, not a good formulation, since operator norm of C also approaching infinity. Instead, consider formulation involving u and new variable $p = (\lambda/[2\mu + n\lambda]) \operatorname{tr} \sigma$.

Taking trace of: $A\sigma = \varepsilon(u)$, get $\operatorname{div} u = \lambda^{-1} p$.
Then write $\sigma = 2\mu\varepsilon(u) + pI$ to obtain: Find $u \in \mathring{H}^1(\Omega; \mathbb{V})$, $p \in L^2_0(\Omega) = \{p \in L^2(\Omega) : \int_\Omega p = 0\}$ such that

$$\int_\Omega 2\mu\varepsilon(u) : \varepsilon(v)\, dx + \int_\Omega p \operatorname{div} v\, dx = \int_\Omega f \cdot v\, dx, \ \ v \in \mathring{H}^1(\Omega; \mathbb{V}),$$
$$\int_\Omega \operatorname{div} u q\, dx = \int_\Omega \lambda^{-1} pq\, dx, \ \ q \in L^2_0(\Omega).$$

This formulation makes sense even for limit $\lambda \to \infty$, giving stationary Stokes equations (to be considered in other lectures).

Consider other weak formulations involving both σ and u.

Strongly imposed symmetry:
Find $\sigma \in H(\operatorname{div}, \Omega; \mathbb{S})$ and $u \in L^2(\Omega; \mathbb{V})$, satisfying

$$\int_\Omega (A\sigma : \tau + \operatorname{div} \tau \cdot u)\, dx = 0, \ \ \tau \in H(\operatorname{div}, \Omega; \mathbb{S}),$$
$$\int_\Omega \operatorname{div} \sigma \cdot v \, dx = \int_\Omega f \cdot v \, dx, \ \ v \in L^2(\Omega; \mathbb{V}).$$

Weakly imposed symmetry: Find $\sigma \in H(\operatorname{div}, \Omega; \mathbb{M})$, $u \in L^2(\Omega; \mathbb{V})$, and $p \in L^2(\Omega; \mathbb{K})$ satisfying

$$\int_\Omega (A\sigma : \tau + \operatorname{div} \tau \cdot u + \tau : p)\, dx = 0, \quad \tau \in H(\operatorname{div}, \Omega; \mathbb{M}),$$
$$\int_\Omega \operatorname{div} \sigma \cdot v\, dx = \int_\Omega f \cdot v\, dx, \quad v \in L^2(\Omega; \mathbb{V}),$$
$$\int_\Omega \sigma : q\, dx = 0, \quad q \in L^2(\Omega; \mathbb{K}),$$

$\mathbb{M} = n \times n$ matrices, $\mathbb{K}$ skew-symmetric matrices, compliance tensor $A(x)$ now symmetric and positive definite operator mapping $\mathbb{M}$ into $\mathbb{M}$.

First consider finite element methods based on variational formulation with strongly imposed symmetry.

Let $\mathbf{\Sigma}_h \subset H(\mathrm{div}, \Omega; \mathbb{S})$ and $\boldsymbol{V}_h \subset L^2(\Omega; \mathbb{V})$ and seek $\sigma_h \in \mathbf{\Sigma}_h$ and $u_h \in \boldsymbol{V}_h$ satisfying

$$\int_\Omega (A\sigma_h : \tau + \mathrm{div}\,\tau \cdot u_h)\,dx = 0,\ \ \tau \in \mathbf{\Sigma}_h,$$
$$\int_\Omega \mathrm{div}\,\sigma_h \cdot v\,dx = \int_\Omega f \cdot v\,dx,\ \ v \in \boldsymbol{V}_h.$$

Can apply standard analysis of mixed finite element theory (e.g., Brezzi, Brezzi-Fortin, Falk-Osborn, Douglas-Roberts). For isotropic elasticity, if we write $\sigma = \sigma_D + (1/n)\,\mathrm{tr}\,\sigma I$, where $\mathrm{tr}\,\sigma_D = 0$, then $\|\sigma\|_0^2 = \|\sigma_D\|_0^2 + (1/n)\|\,\mathrm{tr}\,\sigma\|_0^2$ and so

$$\int_\Omega A\sigma : \sigma\,dx = \int_\Omega \left[\frac{1}{2\mu}\sigma_D : \sigma_D + \frac{1}{2\mu + n\lambda}(\mathrm{tr}\,\sigma)^2\right] dx.$$

This form not uniformly coercive as $\lambda \to \infty$ (only coerce σ_D).

However, for all σ satisfying

$$\int_\Omega \operatorname{tr}\sigma\,dx = 0, \qquad \operatorname{div}\sigma = 0, \tag{1}$$

can show that $\|\operatorname{tr}\sigma\|_0 \le C\|\sigma_D\|_0$, and hence $(A\sigma,\sigma) \ge \alpha\|\sigma\|^2_{H(\mathrm{div})}$ for all σ satisfying (1), with α independent of λ. Implies first Brezzi condition with constant independent of λ. Then analyze methods using following result.

Theorem: Suppose for every $\tau \in \boldsymbol{H}^1(\Omega)$, there exists $\Pi_h\tau \in \boldsymbol{\Sigma}_h$ satisfying

$$\int_\Omega \operatorname{div}(\tau - \Pi_h\tau)\cdot v\,dx = 0, \quad v \in \boldsymbol{V}_h, \qquad \|\Pi_h\tau\|_{H(\mathrm{div})} \le C\|\tau\|_{H(\mathrm{div})}.$$

Further suppose that for all $\tau \in \boldsymbol{\Sigma}_h$ satisfying $\int_\Omega \operatorname{div}\tau\cdot v\,dx = 0$, $v \in \boldsymbol{V}_h$, that $\operatorname{div}\tau = 0$. Then for all $v_h \in \boldsymbol{V}_h$,

$$\|\sigma - \sigma_h\|_0 \le C\|\sigma - \Pi_h\sigma\|_0, \qquad \|u - u_h\|_0 \le C(\|u - v_h\|_0 + \|\sigma - \sigma_h\|_0).$$

Methods using Composite Elements

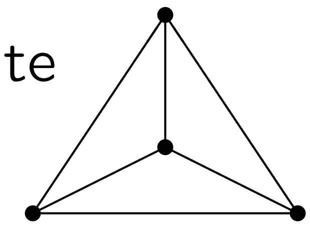
Let $\mathcal{P}_k(X, Y)$ denote space of polynomial functions on X of degree at most k and taking values in Y.

One of first methods based on symmetric formulation: method of Watwood-Hartz analyzed in Johnson-Mercier. Describe triangular element (also similar quadrilateral element).

Basic idea: approximate stress by composite finite element. Starting from mesh $\mathcal{T}_h$ of triangles, connect barycenter of each triangle K to three vertices to form a composite element made up of three triangles, i.e., $K = T_1 \cup T_2 \cup T_3$. Define:

$$\boldsymbol{\Sigma}_h = \{\tau \in H(\mathrm{div}, \Omega; \mathbb{S}) : \tau|_{T_i} \in \mathcal{P}_1(T_i, \mathbb{S})\},$$
$$\boldsymbol{V}_h = \{v \in \boldsymbol{L}^2(\Omega) : v|_K \in \mathcal{P}_1(K, \mathbb{R}^2\}.$$

Composite Element



To construct $\Sigma_h|_K$, start from space of 27 degrees of freedom. Impose at most 12 constraints that require τn be continuous across each of three internal edges of K (all independent). Then, show on each K, τ uniquely determined by 15 degrees of freedom:

(i) the values of $\tau \cdot n$ at two points on each edge of K and
(ii) $\int_K \tau_{ij}\,dx$, $i,j = 1,2$.

Check that if $\int_K \operatorname{div}\tau \cdot v\,dx = 0$ for $v \in \mathcal{P}_1(K,\mathbb{R}^2)$, then $\operatorname{div}\tau = 0$.

Defining Π_h to correspond to degrees of freedom, easy to check $\int_K \operatorname{div}(\tau - \Pi_h\tau) \cdot v\,dx = 0$ for $v \in \mathcal{P}_1(K,\mathbb{R}^2)$.

After establishing $H(\mathrm{div},\Omega)$ norm bound on $\Pi_h\sigma$, get error estimates:

$$\|\sigma-\sigma_h\|_0 \le Ch^2\|\sigma\|_2, \qquad \|u-u_h\|_0 \le Ch^2(\|\sigma\|_2+\|u\|_2).$$

Use of composite finite elements extended to a family in Arnold-Douglas-Gupta. For $k\ge 2$,

$$\boldsymbol{\Sigma}_h = \{\tau \in H(\mathrm{div},\Omega;\mathbb{S}) : \tau|_{T_i} \in \mathcal{P}_k(T_i,\mathbb{S})\},$$
$$\boldsymbol{V}_h = \{v \in \boldsymbol{L}^2(\Omega) : v|_K \in \mathcal{P}_{k-1}(K,\mathbb{R}^2)\}.$$

Non-composite Elements – Arnold-Winther approach

Based on use of discrete differential complexes and close relation between construction of stable mixed finite element methods for Laplace's equation and discrete versions of de Rham complex, with a commuting diagram, i.e.,

$$\begin{array}{ccccccc}
\mathbb{R} & \xrightarrow{\subseteq} & C^\infty(\Omega) & \xrightarrow{\mathbf{curl}} & C^\infty(\Omega,\mathbb{R}^2) & \xrightarrow{\mathrm{div}} & C^\infty(\Omega) \to 0 \\
{\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle I_h}\big\downarrow & & {\scriptstyle \Pi_h}\big\downarrow & & {\scriptstyle P_h}\big\downarrow \\
\mathbb{R} & \xrightarrow{\subseteq} & Q_h & \xrightarrow{\mathbf{curl}} & \Sigma_h & \xrightarrow{\mathrm{div}} & V_h \to 0
\end{array}.$$

I_h, Π_h, P_h natural interpolation operators into finite element spaces Q_h, Σ_h, and V_h.

Simplest case: $Q_h \sim C^0\mathcal{P}_1$, $\Sigma_h \sim \boldsymbol{RT}_0$, $V_h \sim \mathcal{P}_0$. Right half of diagram, involving Π_h and P_h is key result in establishing second Brezzi stability condition.

If Ω simply connected, sequences are exact (range of each map the kernel of following one).

Starting point of Arnold-Winther: elasticity differential complex

$$0 \to P_1(\Omega) \xrightarrow{\subset} C^\infty(\Omega) \xrightarrow{J} C^\infty(\Omega, \mathbb{S}) \xrightarrow{\text{div}} C^\infty(\Omega, \mathbb{R}^2) \to 0, \qquad (2)$$

where Airy stress function

$$Jw = \begin{pmatrix} \partial^2 w/\partial y^2 & -\partial^2 w/\partial x \partial y \\ -\partial^2 w/\partial x \partial y & \partial^2 w/\partial x^2 \end{pmatrix}.$$

If Ω simply-connected, this sequence also exact.

Analogous results hold for functions with less smoothness, e.g.,

$$0 \to P_1(\Omega) \xrightarrow{\subset} H^2(\Omega) \xrightarrow{J} H(\text{div}, \Omega; \mathbb{S}) \xrightarrow{\text{div}} L^2(\Omega, \mathbb{R}^2) \to 0 \qquad (3)$$

is also exact. Implies $\text{div}\, H(\text{div}, \Omega; \mathbb{S}) = L^2(\Omega, \mathbb{R}^2)$.

Stable pairs of finite element spaces $(\boldsymbol{\Sigma}_h, \boldsymbol{V}_h)$ introduced by Arnold-Winther satisfy $\operatorname{div} \boldsymbol{\Sigma}_h = \boldsymbol{V}_h$, i.e., short sequence

$$\boldsymbol{\Sigma}_h \xrightarrow{\operatorname{div}} \boldsymbol{V}_h \to 0 \tag{4}$$

is exact.

Moreover, if there are projections $P_h : C^\infty(\Omega, \mathbb{R}^2) \mapsto \boldsymbol{V}_h$ and $\Pi_h : C^\infty(\Omega, \mathbb{S}) \mapsto \boldsymbol{\Sigma}_h$ defined by degrees of freedom determining finite element spaces, then following diagram commutes:

$$\begin{array}{ccc} C^\infty(\Omega, \mathbb{S}) & \xrightarrow{\operatorname{div}} & C^\infty(\Omega, \mathbb{R}^2) \\ \Pi_h \Big\downarrow & & P_h \Big\downarrow \\ \boldsymbol{\Sigma}_h & \xrightarrow{\operatorname{div}} & \boldsymbol{V}_h \end{array} \tag{5}$$

Stability of mixed method follows from exactness of (4), commutativity of (5), and well-posedness of continuous problem.

Information about construction of such finite element spaces gained by completing sequence (4) to longer sequence,

Set $Q_h = \{q \in H^2(\Omega) : Jq \in \boldsymbol{\Sigma}_h\}$. There is interpolation operator $I_h : C^\infty(\Omega) \mapsto Q_h$ so that following diagram commutes:

$$
\begin{array}{ccccccccc}
0 \to P_1(\Omega) & \xrightarrow{\subseteq} & C^\infty(\Omega) & \xrightarrow{J} & C^\infty(\Omega, \mathbb{S}) & \xrightarrow{\mathrm{div}} & C^\infty(\Omega, \mathbb{R}^2) \to 0 \\
\mathrm{id}\big\downarrow & & I_h\big\downarrow & & \Pi_h\big\downarrow & & P_h\big\downarrow \\
0 \to P_1(\Omega) & \xrightarrow{\subseteq} & Q_h & \xrightarrow{J} & \boldsymbol{\Sigma}_h & \xrightarrow{\mathrm{div}} & \boldsymbol{V}_h \to 0
\end{array}
$$

Existence of stable spaces $(\boldsymbol{\Sigma}_h, \boldsymbol{V}_h)$ approximating $H(\mathrm{div}, \Omega; \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$, implies existence of finite element subspace Q_h of $H^2(\Omega)$ related to $\boldsymbol{\Sigma}_h$ and $\boldsymbol{V}_h$ through above diagram.

Difficulty: Q_h requires $C^1(\Omega)$ finite elements. Simplest choice: Argyris space of C^1 piecewise quintic polynomials.

Since $JQ_h \subset \mathbf{\Sigma}_h$, $\mathbf{\Sigma}_h$ must be piecewise cubic space. Since Argyris space has second derivatie d.o.f. at vertices, d.o.f. of $\mathbf{\Sigma}_h$ with include d.o.f. at vertices, not usually expected for subspaces of $\boldsymbol{H}(\operatorname{div}, \Omega)$.

Simplest element defined locally by:

$$
\begin{aligned}
\Sigma_T &= \mathcal{P}_2(T, \mathbb{S}) + \{\tau \in \mathcal{P}_3(T, \mathbb{S}) : \operatorname{div} \tau = 0\} \\
&= \{\tau \in \mathcal{P}_3(T, \mathbb{S}) : \operatorname{div} \tau \in \mathcal{P}_1(T, \mathbb{R}^2)\}, \qquad V_T = \mathcal{P}_1(T, \mathbb{R}^2).
\end{aligned}
$$

Family of elements developed in Arnold-Winther chooses for $k \geq 1$, locally defined by:

$$\begin{aligned}\Sigma_T &= \mathcal{P}_{k+1}(T,\mathbb{S}) + \{\tau \in \mathcal{P}_{k+2}(T,\mathbb{S}) : \operatorname{div}\tau = 0\} \\ &= \{\tau \in \mathcal{P}_{k+2}(T,\mathbb{S}) : \operatorname{div}\tau \in \mathcal{P}_k(T,\mathbb{R}^2)\}, \qquad V_T = \mathcal{P}_k(T,\mathbb{R}^2).\end{aligned}$$

Unisolvent set of local degrees of freedom given by:

• values of 3 components of $\tau(x)$ at each vertex x of T (9 degrees of freedom)

• values of moments of degree at most k of the two normal components of τ on each edge e of T ($6k+6$ degrees of freedom)

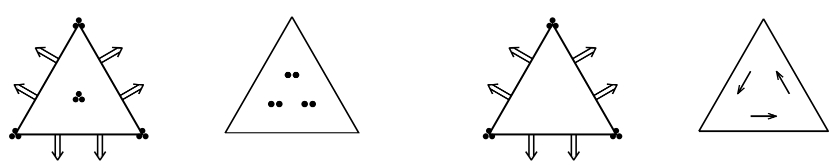
• value of moments $\int_T \tau : \phi \, dx$, $\phi \in \mathcal{P}_k(T,\mathbb{R}^2) + \operatorname{airy}(b_T^2 \mathcal{P}_{k-2}(T,\mathbb{R}))$.

For this family of elements, shown by Arnold-Winther that

$$\|\sigma - \sigma_h\|_0 \le Ch^r \|\sigma\|_r, \quad 1 \le r \le k+2,$$
$$\|\operatorname{div}(\sigma - \sigma_h)\|_0 \le Ch^r \|\operatorname{div}\sigma\|_r, \quad 0 \le r \le k+1,$$
$$\|u - u_h\|_0 \le Ch^r \|u\|_{r+1}, \quad 1 \le r \le k+1.$$

Variant of lowest degree ($k = 1$) element involving fewer degrees of freedom. Choose V_T space of infinitesimal rigid motions on T, i.e., vector functions of form $(a - by, c + bx)$. Then $\Sigma_T = \{\tau \in \mathcal{P}_3(T, \mathbb{S}) : \operatorname{div}\tau \in V_T\}$.

Element diagram for choice $k = 1$ and a simplified element are depicted below.



Nonconforming elements: fewer degrees of freedom; avoid vertex degrees of freedom (Arnold-Winther)

Corresponding to choice $V_T = \mathcal{P}_1(T, \mathbb{R}^2)$, choose for stress shape functions:

$$\Sigma_T = \{\tau \in \mathcal{P}_2(T, \mathbb{S}) : n \cdot \tau n \in \mathcal{P}_1(e, \mathbb{R}), \text{ for each edge } e \text{ of } T\}.$$

Space Σ_T has dimension 15, with degrees of freedom given by:

- values of moments of degree 0 and 1 of two normal components of τ on each edge e of T (12 degrees of freedom),

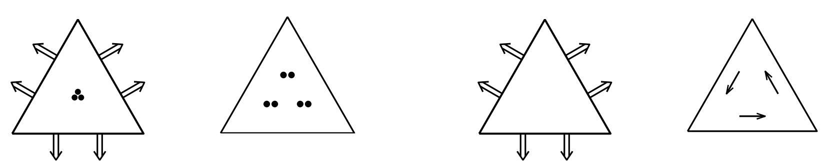
- value of three components of moment of degree 0 of τ on T (3 degrees of freedom).

Nonconforming approximation of $H(\mathrm{div},\Omega;\mathbb{S})$, since although $t\cdot\tau n$ may be quadratic on an edge, only its two lowest order moments are determined on each edge. Hence, τn may not be continuous across element boundaries.

Simplified nonconforming element:
Displacement space chosen to be piecewise rigid motions.

Stress space reduced by requiring that divergence be a rigid motion on each triangle.

Local dimension is 12 and first two moments of normal traction on each edge form unisolvent set of degrees of freedom.



For $k = 1$, saw corresponding space Q_h is Argyris space consisting of C^1 piecewise quintic polynomials.

Also an analogous relationship for composite elements discussed earlier. For Johnson-Mercier element, Q_h is Clough-Tocher composite H^2 element and for family of Arnold-Douglas-Gupta, Q_h spaces are higher order composite elements of Douglas-Dupont-Percell-Scott.

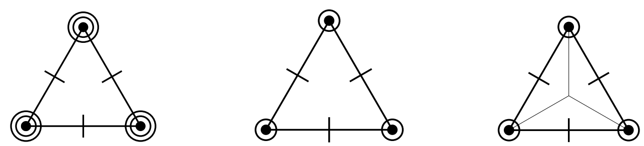


Figure 3: Q_h spaces for $k = 1$ conforming element, nonconforming element, and composite element of Johnson-Mercier.

Weakly Symmetric Finite Element Methods

Advantage: can approximate stress tensor by two copies of standard finite element approximations of $\boldsymbol{H}(\mathrm{div}, \Omega)$ used to discretize scalar second order elliptic problems.

Exploit many close connections between elasticity and scalar elliptic problems.

Structure of these connections most clearly seen in language of exterior calculus. Give only basic notation and connection to language of vectors and differential operators in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Differential Forms

Suppose Ω an open subset of $\mathbb{R}^n$. For $0 \le k \le n$, let $\Lambda^k(\Omega)$ denote space of smooth differential k-forms of Ω.

When $n = 2$, $\omega \in \Lambda^k(\Omega)$ will have forms

$$w, \qquad w_1 dx_1 + w_2 dx_2, \qquad w dx_1 \wedge dx_2, \qquad k = 0, 1, 2.$$

Can identify $w \in \Lambda^0(\Omega)$ or $w dx_1 \wedge dx_2 \in \Lambda^2(\Omega)$ with function $w \in C^\infty(\Omega)$ and $w_1 dx_1 + w_2 dx_2 \in \Lambda^1(\Omega)$ with vector (w_1, w_2) or vector $(-w_2, w_1) \in C^\infty(\Omega, \mathbb{R}^2)$.

When $n = 3$, $\omega \in \Lambda^k(\Omega)$ will have forms (for $k = 0, 1, 2, 3$)

$$w, \qquad w_1 dx_1 + w_2 dx_2 + w_3 dx_3,$$
$$w_1 dx_2 \wedge dx_3 - w_2 dx_1 \wedge dx_3 + w_3 dx_1 \wedge dx_2, \qquad w dx_1 \wedge dx_2 \wedge dx_3.$$

Can identify $w \in \Lambda^0(\Omega)$ or $w dx_1 \wedge dx_2 \wedge dx_3 \in \Lambda^3(\Omega)$
with function $w \in C^\infty(\Omega)$
and
$w_1 dx_1 + w_2 dx_2 + w_3 dx_3 \in \Lambda^1(\Omega)$
or
$w_1 dx_2 \wedge dx_3 - w_2 dx_1 \wedge dx_3 + w_3 dx_1 \wedge dx_2 \in \Lambda^2(\Omega)$
with vector $(w_1, w_2, w_3) \in C^\infty(\Omega, \mathbb{R}^3)$.

Key object: exterior derivative $d = d_k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ defined by

$$d \sum a_\sigma dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)} = \sum_\sigma \sum_{i=1}^n \frac{\partial a_\sigma}{\partial x_i} dx_i \wedge dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)}.$$

Wedge product $dx_i \wedge dx_j$ satisfies: $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

So $dx_i \wedge dx_i = 0$.

d corresponds to differential operators $\mathbf{grad}$, $\mathbf{curl}$, div, and rot.

$n = 2$: $\omega \in \Lambda^0(\Omega)$, $d_0\omega = \partial w/\partial x_1 dx_1 + \partial w/\partial x_2 dx_2 \in \Lambda^1(\Omega)$.
Identifying $\partial w/\partial x_1 dx_1 + \partial w/\partial x_2 dx_2$ with $(\partial w/\partial x_1, \partial w/\partial x_2)$,
$d_0 \sim \mathbf{grad}$.

Identifying $\partial w/\partial x_1 dx_1 + \partial w/\partial x_2 dx_2$ with $(-\partial w/\partial x_2, \partial w/\partial x_1)$,
$d_0 \sim \mathbf{curl}$.

$\mu = w_1 dx_1 + w_2 dx_2 \in \Lambda^1(\Omega)$.
$d_1\mu = (\partial w_2/\partial x_1 - \partial w_1/\partial x_2) dx_1 \wedge dx_2 \in \Lambda^2(\Omega)$.
Identifying $w_1 dx_1 + w_2 dx_2$ with (w_1, w_2), $d_1 \sim$ rot.
Identifying $w_1 dx_1 + w_2 dx_2$ with $(-w_2, w_1)$, $d_1 \sim -$div.

$n = 3$. $\omega \in \Lambda^0(\Omega)$,
$d_0\omega = \partial w/\partial x_1 dx_1 + \partial w/\partial x_2 dx_2 + \partial w/\partial x_3 dx_3 \in \Lambda^1(\Omega)$.
Identifying $\partial w/\partial x_1 dx_1 + \partial w/\partial x_2 dx_2 + \partial w/\partial x_3 dx_3$
with $(\partial w/\partial x_1, \partial w/\partial x_2, \partial w/\partial x_3 dx_3)$, $d_0 \sim \mathbf{grad}$.

$\mu = w_1 dx_1 + w_2 dx_2 + w_3 dx_3 \in \Lambda^1(\Omega)$.
$d_1\mu = (\partial w_3/\partial x_2 - \partial w_2/\partial x_3) dx_2 \wedge dx_3 - (\partial w_1/\partial x_3 - \partial w_3/\partial x_1) dx_1 \wedge dx_3 + (\partial w_2/\partial x_1 - \partial w_1/\partial x_2) dx_1 \wedge dx_2 \in \Lambda^2(\Omega)$.
Identifying $w_1 dx_1 + w_2 dx_2 + w_3 dx_3$ with (w_1, w_2, w_3), $d_1 \sim \mathbf{curl}$.

$\mu = w_1 dx_2 \wedge dx_3 - w_2 dx_1 \wedge dx_3 + w_3 dx_1 \wedge dx_2 \in \Lambda^2(\Omega)$.
$d_2\mu = (\partial w_1/\partial x_1 + \partial w_2/\partial x_2 + \partial w_3/\partial x_3) dx_1 \wedge dx_2 \wedge dx_3 \in \Lambda^3(\Omega)$.
Identifying μ with (w_1, w_2, w_3), $d_2 \sim$ div.

Important role in our analysis played by de Rham sequence, sequence of spaces and mappings given by:

$$0 \to \Lambda^0(\Omega) \xrightarrow{d_0} \Lambda^1(\Omega) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^n(\Omega) \to 0.$$

or L^2 version

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d_0} H\Lambda^1(\Omega) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} H\Lambda^n(\Omega) \to 0,$$

where $H\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega) : d_k\omega \in L^2\Lambda^{k+1}(\Omega)\}$.

In 3-D, we have the correspondences:

k	$\Lambda^k(\Omega)$	$H\Lambda^k(\Omega)$	$d\omega$
0	$C^\infty(\Omega)$	$H^1(\Omega)$	$\mathbf{grad}\, w$
1	$C^\infty(\Omega;\mathbb{R}^3)$	$H(\mathbf{curl},\Omega;\mathbb{R}^3)$	$\mathbf{curl}\, w$
2	$C^\infty(\Omega;\mathbb{R}^3)$	$H(\mathrm{div},\Omega;\mathbb{R}^3)$	$\mathrm{div}\, w$
3	$C^\infty(\Omega)$	$L^2(\Omega)$	0

For $\Omega \subset \mathbb{R}^3$, de Rham complex becomes

$$0 \to C^\infty(\Omega) \xrightarrow{\mathbf{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\mathbf{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\mathrm{div}} C^\infty(\Omega) \to 0,$$

L^2 de Rham complex:

$$0 \to H^1(\Omega) \xrightarrow{\mathbf{grad}} H(\mathbf{curl},\Omega;\mathbb{R}^3) \\ \xrightarrow{\mathbf{curl}} H(\mathrm{div},\Omega;\mathbb{R}^3) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0.$$

For $\Omega \subset \mathbb{R}^2$, de Rham complex becomes

$$0 \to C^\infty(\Omega) \xrightarrow{\mathbf{grad}} C^\infty(\Omega;\mathbb{R}^2) \xrightarrow{\mathrm{rot}} C^\infty(\Omega) \to 0,$$

or

$$0 \to C^\infty(\Omega) \xrightarrow{\mathbf{curl}} C^\infty(\Omega;\mathbb{R}^2) \xrightarrow{\mathrm{div}} C^\infty(\Omega) \to 0,$$

depending on whether we identify $\omega_1 dx_1 + \omega_2 dx_2 \in \Lambda^1(\Omega)$ with the vector (ω_1,ω_2) or the vector $(-\omega_2,\omega_1)$.

Basic Finite Element Spaces and Their Properties

Define $\mathcal{P}_r$ as space of polynomials in n variables of degree at most r and $\mathcal{P}_r\Lambda^k$ as space of differential k-forms with coefficients belonging to $\mathcal{P}_r$.

Define $\mathcal{P}_r^-\Lambda^k \subset \mathcal{P}_r\Lambda^k$ by

$$\mathcal{P}_r^-\Lambda^k = \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{P}_{r-1}\Lambda^{k+1},$$

where for $\omega = \sum_\sigma a_\sigma dx_{\sigma(1)}\wedge\cdots\wedge dx_{\sigma(k+1)} \in \Lambda^{k+1}$, Koszul operator $\kappa = \kappa_{k+1} : \mathcal{P}_{r-1}\Lambda^{k+1} \to \mathcal{P}_r\Lambda^k$ defined by:

$$\kappa\omega = \sum_\sigma \sum_{i=1}^{k+1} (-1)^{i+1} a_\sigma x_{\sigma(i)} dx_{\sigma(1)} \wedge \cdots \wedge \widehat{dx}_{\sigma(i)} \wedge \cdots dx_{\sigma(k+1)}.$$

Term $\widehat{dx}_{\sigma(i)}$ is omitted.

Note κ decreases degree of form and increases polynomial degree. Can show Koszul complex

$$0 \to \mathcal{P}_{r-n}\Lambda^n(\Omega) \xrightarrow{\kappa_n} \mathcal{P}_{r-n+1}\Lambda^{n-1}(\Omega) \xrightarrow{\kappa_{n-1}} \cdots \xrightarrow{\kappa_1} \mathcal{P}_r\Lambda^0(\Omega) \to 0$$

is exact. For $\Omega \subset \mathbb{R}^3$, complex becomes

$$0 \to \mathcal{P}_{r-3}(\Omega) \xrightarrow{x} \mathcal{P}_{r-2}(\Omega;\mathbb{R}^3) \xrightarrow{\times x} \mathcal{P}_{r-1}(\Omega;\mathbb{R}^3) \xrightarrow{\cdot x} \mathcal{P}_r(\Omega) \to 0.$$

Let $\mathcal{T}_h$ be a triangulation of Ω by $n+1$ simplices T and set

$$\mathcal{P}_r\Lambda^k(\mathcal{T}_h) = \{\omega \in H\Lambda^k(\Omega) : \omega|_T \in \mathcal{P}_r\Lambda^k(T)\ \forall T \in \mathcal{T}_h\}, \quad r \geq 0$$
$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) = \{\omega \in H\Lambda^k(\Omega) : \omega|_T \in \mathcal{P}_r^-\Lambda^k(T)\ \forall T \in \mathcal{T}_h\}, \quad r \geq 1,$$

We note that

$$\mathcal{P}_r\Lambda^0(\mathcal{T}_h) = \mathcal{P}_r^-\Lambda^0(\mathcal{T}_h),\ r \geq 1, \quad \mathcal{P}_r\Lambda^n(\mathcal{T}_h) = \mathcal{P}_{r+1}^-\Lambda^n(\mathcal{T}_h),\ r \geq 0.$$

k	$\Lambda_h^k(\Omega)$	Classical finite element space
0	$\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$	Lagrange elements of degree $\le r$
1	$\mathcal{P}_r\Lambda^1(\mathcal{T}_h)$	B-D-M $H(\mathrm{div})$ elements of degree $\le r$
2	$\mathcal{P}_r\Lambda^2(\mathcal{T}_h)$	discontinuous elements of degree $\le r$
0	$\mathcal{P}_r^-\Lambda^0(\mathcal{T}_h)$	Lagrange elements of degree $\le r$
1	$\mathcal{P}_r^-\Lambda^1(\mathcal{T}_h)$	R-T $H(\mathrm{div})$ elements of order $r-1$
2	$\mathcal{P}_r^-\Lambda^2(\mathcal{T}_h)$	discontinuous elements of degree $\le r-1$

Correspondences between finite element differential forms and the classical finite element spaces for $n = 2$.

k	$\Lambda_h^k(\Omega)$	Classical finite element space
0	$\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$	Lagrange elements of degree $\leq r$
1	$\mathcal{P}_r\Lambda^1(\mathcal{T}_h)$	Nédélec 2nd-kind $H(\mathbf{curl})$ elements degree $\leq r$
2	$\mathcal{P}_r\Lambda^2(\mathcal{T}_h)$	Nédélec 2nd-kind $H(\mathrm{div})$ elements degree $\leq r$
3	$\mathcal{P}_r\Lambda^3(\mathcal{T}_h)$	discontinuous elements of degree $\leq r$
0	$\mathcal{P}_r^-\Lambda^0(\mathcal{T}_h)$	Lagrange elements of degree $\leq r$
1	$\mathcal{P}_r^-\Lambda^1(\mathcal{T}_h)$	Nédélec 1st-kind $H(\mathbf{curl})$ elements order $r-1$
2	$\mathcal{P}_r^-\Lambda^2(\mathcal{T}_h)$	Nédélec 1st-kind $H(\mathrm{div})$ elements order $r-1$
3	$\mathcal{P}_r^-\Lambda^3(\mathcal{T}_h)$	discontinuous elements degree $\leq r-1$

Correspondences between finite element differential forms and the classical finite element spaces for $n = 3$.

Key property: spaces form discrete de Rham sequences. In n dimensions, exactly 2^{n-1} distinct sequences. When $n = 2$ and $r \geq 0$, these are

$$0 \to \mathcal{P}_{r+2}\Lambda^0(\mathcal{T}_h) \xrightarrow{d_0} \mathcal{P}_{r+1}\Lambda^1(\mathcal{T}_h) \xrightarrow{d_1} \mathcal{P}_r\Lambda^2(\mathcal{T}_h) \to 0,$$
$$0 \to \mathcal{P}_{r+1}\Lambda^0(\mathcal{T}_h) \xrightarrow{d_0} \mathcal{P}^-_{r+1}\Lambda^1(\mathcal{T}_h) \xrightarrow{d_1} \mathcal{P}_r\Lambda^2(\mathcal{T}_h) \to 0.$$

When $n = 3$ and $r \geq 0$, we have four sequences:

$$0 \to \mathcal{P}_{r+3}\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_{r+2}\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_{r+1}\Lambda^2(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_r\Lambda^3(\mathcal{T}_h) \to 0,$$
$$0 \to \mathcal{P}_{r+2}\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_{r+1}\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}^-_{r+1}\Lambda^2(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_r\Lambda^3(\mathcal{T}_h) \to 0,$$
$$0 \to \mathcal{P}_{r+2}\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}^-_{r+2}\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_{r+1}\Lambda^2(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_r\Lambda^3(\mathcal{T}_h) \to 0,$$
$$0 \to \mathcal{P}_{r+1}\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}^-_{r+1}\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}^-_{r+1}\Lambda^2(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_r\Lambda^3(\mathcal{T}_h) \to 0.$$

First and last involve only $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ or $\mathcal{P}^-_r\Lambda^k(\mathcal{T}_h)$ spaces alone; middle two mix two spaces. Middle two used for elasticity.

To each $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$, associate canonical projection operator $\Pi(=\Pi_{\mathcal{T}_h}) : C^0\Lambda^k(\Omega) \to \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ defined by d.o.f:

$$\int_f \operatorname{Tr}_f \Pi\omega \wedge \nu = \int_f \operatorname{Tr}_f \omega \wedge \nu, \quad \nu \in \mathcal{P}^-_{r-j+k}\Lambda^{j-k}(f), \quad f \in \Delta_j(\mathcal{T}),$$

for $k \le j \le \min(n, r+k-1)$. ($n = 3$: $j = 0, 1, 2, 3$. $\Delta_j(\mathcal{T})$ denote vertices, edges, faces, tetrahedron).

To each $\mathcal{P}^-_r\Lambda^k(\mathcal{T}_h)$, associate canonical projection operator $\Pi(=\Pi_{\mathcal{T}_h}) : C^0\Lambda^k(\Omega) \to \mathcal{P}^-_r\Lambda^k(\mathcal{T}_h)$ defined by d.o.f:

$$\int_f \operatorname{Tr}_f \Pi\omega \wedge \nu = \int_f \operatorname{Tr}_f \omega \wedge \nu, \quad \nu \in \mathcal{P}_{r-j+k-1}\Lambda^{j-k}(f), \quad f \in \Delta_j(\mathcal{T}),$$

for $k \le j \le \min(n, r+k-1)$.

Note: d.o.f of each space use forms from other space.

Key property of these projection operators: they commute with exterior derivative, i.e., following four diagrams commute.

$$\begin{array}{ccc} \Lambda^k(\Omega) & \xrightarrow{d_k} & \Lambda^{k+1}(\Omega) \\ \Pi\big\downarrow & & \Pi\big\downarrow \\ \mathcal{P}_r\Lambda^k(\mathcal{T}) & \xrightarrow{d_k} & \mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}) \end{array} \qquad \begin{array}{ccc} \Lambda^k(\Omega) & \xrightarrow{d_k} & \Lambda^{k+1}(\Omega) \\ \Pi\big\downarrow & & \Pi\big\downarrow \\ \mathcal{P}_r\Lambda^k(\mathcal{T}) & \xrightarrow{d_k} & \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T}) \end{array}$$

$$\begin{array}{ccc} \Lambda^k(\Omega) & \xrightarrow{d_k} & \Lambda^{k+1}(\Omega) \\ \Pi\big\downarrow & & \Pi\big\downarrow \\ \mathcal{P}_r^-\Lambda^k(\mathcal{T}) & \xrightarrow{d_k} & \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T}) \end{array} \qquad \begin{array}{ccc} \Lambda^k(\Omega) & \xrightarrow{d_k} & \Lambda^{k+1}(\Omega) \\ \Pi\big\downarrow & & \Pi\big\downarrow \\ \mathcal{P}_r^-\Lambda^k(\mathcal{T}) & \xrightarrow{d_k} & \mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}). \end{array}$$

These properties play essential role in constructing stable mixed finite element methods for equations of elasticity.

Differential forms with values in a vector space

Let V and W be finite dimensional vector spaces. Define space $\Lambda^k(V;W)$ of differential forms on V with values in W. Examples: $V = \mathbb{V} = \mathbb{R}^n$ and $W = \mathbb{V}$ or $W = \mathbb{K}$, set of anti-symmetric matrices.

When $n = 2$, $\omega \in \Lambda^k(\mathbb{V};\mathbb{V})$, $k = 0, 1, 2$, given by

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad \begin{pmatrix} w_{11} \\ w_{21} \end{pmatrix} dx_1 + \begin{pmatrix} w_{12} \\ w_{22} \end{pmatrix} dx_2, \qquad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} dx_1 \wedge dx_2,$$

while $\omega \in \Lambda^k(\mathbb{V};\mathbb{K})$ given by

$$w\boldsymbol{\chi},\ w_1\boldsymbol{\chi}dx_1 + w_2\boldsymbol{\chi}dx_2,\ w\boldsymbol{\chi}dx_1 \wedge dx_2, \quad \text{where } \boldsymbol{\chi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Recall: 1-form $w_1 dx_1 + w_2 dx_2$ can be identified either with vector (w_1, w_2) or vector $(-w_2, w_1)$. Similar choices for vector or matrix-valued forms.

Choosing second identification, identify $\begin{pmatrix} w_{11} \\ w_{21} \end{pmatrix} dx_1 + \begin{pmatrix} w_{12} \\ w_{22} \end{pmatrix} dx_2 \in \Lambda^1(\mathbb{V};\mathbb{V})$ with matrix

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} -w_{12} & w_{11} \\ -w_{22} & w_{21} \end{pmatrix},$$

and $w_1 \boldsymbol{\chi} dx_1 + w_2 \boldsymbol{\chi} dx_2 \in \Lambda^1(\mathbb{V};\mathbb{K})$ with vector $(-w_2, w_1)$.

When $n = 3$, $\mu_k \in \Lambda^k(\mathbb{V}; \mathbb{V})$ given by:

$$\mu_0 = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \ \mu_1 = \begin{pmatrix} w_{11} \\ w_{21} \\ w_{31} \end{pmatrix} dx_1 + \begin{pmatrix} w_{12} \\ w_{22} \\ w_{32} \end{pmatrix} dx_2 + \begin{pmatrix} w_{13} \\ w_{23} \\ w_{33} \end{pmatrix} dx_3$$

$$\mu_2 = \begin{pmatrix} w_{11} \\ w_{21} \\ w_{31} \end{pmatrix} dx_2 \wedge dx_3 - \begin{pmatrix} w_{12} \\ w_{22} \\ w_{32} \end{pmatrix} dx_1 \wedge dx_3 + \begin{pmatrix} w_{13} \\ w_{23} \\ w_{33} \end{pmatrix} dx_1 \wedge dx_2$$

$$\mu_3 = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} dx_1 \wedge dx_2 \wedge dx_3,$$

Identify μ_0 and μ_3 with vector (w_1, w_2, w_3).
Identify μ_1 and μ_2 with 3×3 matrix $W_{ij} = w_{ij}$.

To describe $\Lambda^k(\mathbb{V};\mathbb{K})$, define operator Skw taking a 3-vector to a skew-symmetric matrix. i.e.,

$$\mathrm{Skw}(w_1, w_2, w_3) = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}.$$

Then $\mu_k \in \Lambda^k(\mathbb{V};\mathbb{K})$ will have the respective forms:

$$\mu_0 = \mathsf{Skw}(w_1, w_2, w_3),$$

$$\mu_1 = \mathsf{Skw}(w_{11}, w_{21}, w_{31})dx_1 + \mathsf{Skw}(w_{12}, w_{22}, w_{32})dx_2 \\ + \mathsf{Skw}(w_{13}, w_{23}, w_{33})dx_3,$$

$$\mu_2 = \mathsf{Skw}(w_{11}, w_{21}, w_{31})dx_2 \wedge dx_3 - \mathsf{Skw}(w_{12}, w_{22}, w_{32})dx_1 \wedge dx_3 \\ + \mathsf{Skw}(w_{13}, w_{23}, w_{33})dx_1 \wedge dx_2,$$

$$\mu_3 = \mathsf{Skw}(w_1, w_2, w_3)dx_1 \wedge dx_2 \wedge dx_3.$$

Identify μ_0 and μ_3 with 3-dimensional vector (w_1, w_2, w_3) and μ_1 and μ_2 with 3×3 matrix $W_{ij} = w_{ij}$.

In mixed formulation of elasticity, need for $k = n-2$ and $k = n-1$, special operators $S_k : \Lambda^k(\mathbb{V},\mathbb{V}) \to \Lambda^{k+1}(\mathbb{V},\mathbb{K})$ defined as follows: First define $K_k : \Lambda^k(\Omega;\mathbb{V}) \to \Lambda^k(\Omega;\mathbb{K})$ by

$$K_k\omega = X\omega^T - \omega X^T,$$

where $X = (x_1, \cdots, x_n)^T$. Then define
$S_k = d_k K_k - K_{k+1} d_k : \Lambda^k(\Omega;\mathbb{V}) \to \Lambda^{k+1}(\Omega;\mathbb{K})$.

When $n = 2$, we get for $\omega = (w_1, w_2)^T$,

$$K_0\omega = (w_1 x_2 - w_2 x_1)\boldsymbol{\chi}$$

and after a simple computation,

$$S_0\omega = (d_0K_0 - K_1d_0)\omega = -w_2\boldsymbol{\chi}dx_1 + w_1\boldsymbol{\chi}dx_2.$$

Note that S_0 is invertible with

$$S_0^{-1}[\mu_1\boldsymbol{\chi}dx_1 + \mu_2\boldsymbol{\chi}dx_2] = (\mu_2, -\mu_1)^T.$$

If $\omega \in \Lambda^1(\mathbb{V};\mathbb{V})$ is given by:

$$\omega = w_1 dx_1 + w_2 dx_2, \qquad w_1 = (w_{11}, w_{21})^T, \quad w_2 = (w_{12}, w_{22})^T,$$

then

$$S_1\omega = -(w_{11} + w_{22})\boldsymbol{\chi} dx_1 \wedge dx_2.$$

If we identity ω with a matrix W by

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} -w_{12} & w_{11} \\ -w_{22} & w_{21} \end{pmatrix},$$

then we can identify $S_1\omega$ with the matrix

$$\begin{pmatrix} 0 & W_{12} - W_{21} \\ W_{21} - W_{12} & 0 \end{pmatrix} = 2\,\mathrm{skw}\,W.$$

In general, S_{n-1} can be identified with $(-1)^n 2\,\mathrm{skw}$.

When $n = 3$, we get for $\omega = w_1 dx_1 + w_2 dx_2 + w_3 dx_3$,

$$\begin{aligned} S_1\omega = \mathsf{Skw}(-w_{33} - w_{22}, w_{12}, w_{13}) dx_2 \wedge dx_3 \\ - \mathsf{Skw}(w_{21}, -w_{11} - w_{33}, w_{23}) dx_1 \wedge dx_3 \\ + \mathsf{Skw}(w_{31}, w_{32}, -w_{11} - w_{22}) dx_1 \wedge dx_2. \end{aligned}$$

Identify $\omega \in \Lambda^1(\mathbb{V}; \mathbb{V})$ with matrix W by $W_{ij} = w_{ij}$, and identify $S_1\omega \in \Lambda^2(\mathbb{V}; \mathbb{K})$ with matrix U given by

$$U = \begin{pmatrix} -w_{33} - w_{22} & w_{21} & w_{31} \\ w_{12} & -w_{11} - w_{33} & w_{32} \\ w_{13} & w_{23} & -w_{11} - w_{22} \end{pmatrix}.$$

Then, W and U related by equations

$$U = \Xi W = W^T - \mathrm{tr}(W) I, \qquad W = \Xi^{-1} U = U^T - \frac{1}{2}\mathrm{tr}(U) I.$$

Hence, S_1 is invertible.

Easily get from fact $d_{k+1}d_k = 0$ and definition

$$S_k = d_k K_k - K_{k+1} d_k : \Lambda^k(\Omega; \mathbb{V}) \to \Lambda^{k+1}(\Omega; \mathbb{K}),$$

$$\boxed{d_{k+1} S_k + S_{k+1} d_k = 0} \tag{6}$$

This identify, for $k = n - 2$, i.e., $d_{n-1}S_{n-2} + S_{n-1}d_{n-2} = 0$ is key identity in establishing stability of continuous and discrete variational formulations of elasticity with weak symmetry.

Formula more complicated stated in terms of proxy fields. When $n = 2$ and $k = 0$, if we identify $\omega = (w_1, w_2)^T \in \Lambda^0(\Omega; \mathbb{V})$ with vector W, then formula $(d_1 S_0 + S_1 d_0)\omega = 0$ becomes

$$(\operatorname{div} W)\boldsymbol{\chi} + 2 \operatorname{skw} \mathbf{curl}\, W = 0, \qquad \operatorname{skw} M = (M - M^T)/2.$$

When $n = 3$ and $k = 1$, if we identify $\omega \in \Lambda^1(\Omega; \mathbb{V})$ with matrix W, then formula $(d_2 S_1 + S_2 d_1)\omega = 0$ becomes

$$\operatorname{Skw} \operatorname{div}(\Xi W) - 2 \operatorname{skw} \mathbf{curl}\, W = 0.$$

Mixed Formulation of Elasticity with Weak Symmetry
(in notation of exterior calculus)

Assume Ω contractible domain in $\mathbb{R}^n$, $\mathbb{V} = \mathbb{R}^n$, and $\mathbb{K}$ skew-symmetric matrices. Since $S = S_{n-1} : \Lambda^{n-1}(\Omega; \mathbb{V}) \to \Lambda^n(\Omega; \mathbb{K})$ corresponds (up to factor ± 2) to taking skew-symmetric part of its argument, elasticity problem with weak symmetry becomes:

Find $(\sigma, u, p) \in H\Lambda^{n-1}(\Omega; \mathbb{V}) \times L^2\Lambda^n(\Omega; \mathbb{V}) \times L^2\Lambda^n(\Omega; \mathbb{K})$ such that

$$\langle A\sigma, \tau\rangle + \langle d\tau, u\rangle - \langle S\tau, p\rangle = 0, \quad \tau \in H\Lambda^{n-1}(\Omega; \mathbb{V}),$$
$$\langle d\sigma, v\rangle = \langle f, v\rangle, \quad v \in L^2\Lambda^n(\Omega; \mathbb{V}),$$
$$\langle S\sigma, q\rangle = 0, \quad q \in L^2\Lambda^n(\Omega; \mathbb{K}).$$

$\boxed{d \sim \mathrm{div}}$ $\boxed{S \sim \mathrm{skw}}$

Problem well-posed in sense that, for each $f \in L^2\Lambda^n(\Omega;\mathbb{V})$, there exists a unique solution $(\sigma, u, p) \in H\Lambda^{n-1}(\Omega;\mathbb{V}) \times L^2\Lambda^n(\Omega;\mathbb{V}) \times L^2\Lambda^n(\Omega;\mathbb{K})$, and solution operator is bounded from

$$L^2\Lambda^n(\Omega;\mathbb{V}) \to H\Lambda^{n-1}(\Omega;\mathbb{V}) \times L^2\Lambda^n(\Omega;\mathbb{V}) \times L^2\Lambda^n(\Omega;\mathbb{K}).$$

Follows from general theory of saddle point problems once we establish two conditions: For some positive constants c_1 and c_2,

(W1) $\|\tau\|_{H\Lambda}^2 \le c_1 \langle A\tau, \tau\rangle$ whenever $\tau \in H\Lambda^{n-1}(\Omega;\mathbb{V})$ satisfies

$$\langle d\tau, v\rangle = 0 \ \forall v \in L^2\Lambda^n(\Omega;\mathbb{V}) \text{ and } \langle S\tau, q\rangle = 0 \ \forall q \in L^2\Lambda^n(\Omega;\mathbb{K}),$$

(W2) for all nonzero $(v, q) \in L^2\Lambda^n(\Omega;\mathbb{V}) \times L^2\Lambda^n(\Omega;\mathbb{K})$, there exists nonzero $\tau \in H\Lambda^{n-1}(\Omega;\mathbb{V})$ with

$$\langle d\tau, v\rangle - \langle S\tau, q\rangle \ge c_2 \|\tau\|_{H\Lambda}(\|v\| + \|q\|).$$

First condition is obvious (and does not even utilize orthogonality of $S\tau$). Second condition more subtle.

Next consider finite element discretization. Choose families of finite-dimensional subspaces $\Lambda_h^{n-1}(\mathbb{V}) \subset H\Lambda^{n-1}(\Omega;\mathbb{V})$, $\Lambda_h^n(\mathbb{V}) \subset L^2\Lambda^n(\Omega;\mathbb{V})$, and $\Lambda_h^n(\mathbb{K}) \subset L^2\Lambda^n(\Omega;\mathbb{K})$, indexed by h, and seek discrete solution $(\sigma_h, u_h, p_h) \in \Lambda_h^{n-1}(\mathbb{V}) \times \Lambda_h^n(\mathbb{V}) \times \Lambda_h^n(\mathbb{K})$ such that

$$\langle A\sigma_h, \tau\rangle + \langle d\tau, u_h\rangle - \langle S\tau, p_h\rangle = 0, \quad \tau \in \Lambda_h^{n-1}(\mathbb{V}),$$
$$\langle d\sigma_h, v\rangle = \langle f, v\rangle, \quad v \in \Lambda_h^n(\mathbb{V}), \qquad \langle S\sigma_h, q\rangle = 0, \quad q \in \Lambda_h^n(\mathbb{K}).$$

In analogy with well-posedness of continuous problem, stability of approximation scheme ensured by Brezzi stability conditions:

(S1) $\|\tau\|^2_{H\Lambda} \le c_1(A\tau, \tau)$ whenever $\tau \in \Lambda_h^{n-1}(\mathbb{V})$ satisfies

$$\langle d\tau, v\rangle = 0 \ \forall v \in \Lambda_h^n(\mathbb{V}) \text{ and } \langle S\tau, q\rangle = 0 \ \forall q \in \Lambda_h^n(\mathbb{K}),$$

(S2) for all nonzero $(v, q) \in \Lambda_h^n(\mathbb{V}) \times \Lambda_h^n(\mathbb{K})$, there exists nonzero $\tau \in \Lambda_h^{n-1}(\mathbb{V})$ with

$$\langle d\tau, v\rangle - \langle S\tau, q\rangle \ge c_2\|\tau\|_{H\Lambda}(\|v\| + \|q\|),$$

where now constants c_1 and c_2 must be independent of h.

Difficulty: design finite element spaces satisfying these conditions.

To prove stability of discrete system, first consider proof of stability of continuous system.

For this, use close, but non-obvious, connection between elasticity complex and de Rham complex, described by Eastwood and related to general construction given by Bernstein-Gelfand-Gelfand, called BGG resolution.

Elasticity complex discussed previously related to strong symmetry formulation of elasticity equations. Can also derive elasticity complex related to weak symmetry formulation from de Rham complex.

Omit details of derivation, but discuss key connections needed for stability proof.

Start with two vector-valued de Rham sequences, one with values in $\mathbb{V}$ and one with values in $\mathbb{K}$.

$$\Lambda^{n-2}(\Omega;\mathbb{K}) \xrightarrow{d_{n-2}} \Lambda^{n-1}(\Omega;\mathbb{K}) \xrightarrow{d_{n-1}} \Lambda^{n}(\mathbb{K}) \to 0,$$

$$\Lambda^{n-3}(\Omega;\mathbb{V}) \xrightarrow{d_{n-3}} \Lambda^{n-2}(\Omega;\mathbb{V}) \xrightarrow{d_{n-2}} \Lambda^{n-1}(\Omega;\mathbb{V}) \xrightarrow{d_{n-1}} \Lambda^{n}(\mathbb{V}) \to 0,$$

Can show if de Rham sequences exact, then sequence

$$\Lambda^{n-3}(\mathbb{W}) \xrightarrow{(d_{n-3},-S_{n-3})} \Lambda^{n-2}(\Omega;\mathbb{K}) \xrightarrow{d_{n-2}\circ S_{n-2}^{-1}\circ d_{n-2}} \Lambda^{n-1}(\Omega;\mathbb{V})$$
$$\xrightarrow{(-S_{n-1},d_{n-1})^T} \Lambda^{n}(\mathbb{W}) \to 0$$

is exact, where $\mathbb{W} = \mathbb{K} \times \mathbb{V}$.

Call this: *elasticity sequence with weak symmetry*.

Key fact crucial to construction:

$S_{n-2} : H^1\Lambda^{n-2}(\Omega;\mathbb{V}) \to H^1\Lambda^{n-1}(\Omega;\mathbb{K})$ is an isomorphism.

Interpret this sequence in language of differential operators in two and three dimensions. When $n = 2$,

$$\Lambda^0(\Omega;\mathbb{K}) \xrightarrow{d_0 \circ S_0^{-1} \circ d_0} \Lambda^1(\Omega;\mathbb{V}) \xrightarrow{(-S_1, d_1)^T} \Lambda^2(\mathbb{W}) \to 0.$$

If we identify $w\boldsymbol{\chi} \in \Lambda^0(\Omega;\mathbb{K})$ with scalar function w, then

$$d_0 S_0^{-1} d_0(w\boldsymbol{\chi}) = \begin{pmatrix} \partial^2 w/\partial x_1 \partial x_2 \\ -\partial^2 w/\partial x_1^2 \end{pmatrix} dx_1 + \begin{pmatrix} \partial^2 w/\partial x_2^2 \\ -\partial^2 w/\partial x_1 \partial x_2 \end{pmatrix} dx_2.$$

We then identity this vector-valued 1-form with the matrix

$$\begin{pmatrix} -\partial^2 w/\partial x_2^2 & \partial^2 w/\partial x_1 \partial x_2 \\ \partial^2 w/\partial x_1 \partial x_2 & -\partial^2 w/\partial x_1^2 \end{pmatrix} \equiv -Jw = -\operatorname{airy} w.$$

To translate second part of sequence, we identify
$\omega = \begin{pmatrix} w_{11} \\ w_{21} \end{pmatrix} dx_1 + \begin{pmatrix} w_{12} \\ w_{22} \end{pmatrix} dx_2 \in \Lambda^1(\mathbb{V};\mathbb{V})$ with the matrix

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} -w_{12} & w_{11} \\ -w_{22} & w_{21} \end{pmatrix}.$$

Saw that $-S_1\omega$ corresponds to $-2\,\mathsf{skw}\,W$. Now

$$d_1\omega = \begin{pmatrix} \partial w_{12}/\partial x_1 - \partial w_{11}/\partial x_2 \\ \partial w_{22}/\partial x_1 - \partial w_{21}/\partial x_2 \end{pmatrix} dx_1 \wedge dx_2 = -\,\mathsf{div}\, W\, dx_1 \wedge dx_2.$$

Hence, modulo some constants, get elasticity sequence:

$$C^\infty(\Omega) \xrightarrow{J} C^\infty(\Omega;\mathbb{M}) \xrightarrow{(\mathsf{skw},\mathsf{div})^T} C^\infty(\Omega, \mathbb{K} \times \mathbb{V}) \to 0.$$

When $n = 3$, we have

$$\Lambda^0(\mathbb{W}) \xrightarrow{(d_0,-S_0)} \Lambda^1(\Omega;\mathbb{K}) \xrightarrow{d_1\circ S_1^{-1}\circ d_1} \Lambda^2(\Omega;\mathbb{V}) \xrightarrow{(-S_2,d_2)^T} \Lambda^3(\mathbb{W}) \to 0.$$

Making the identifications discussed previously, obtain (modulo some unimportant constants), elasticity sequence:

$$C^\infty(\mathbb{V}\times\mathbb{K}) \xrightarrow{(\mathbf{grad},I)} C^\infty(\mathbb{M}) \xrightarrow{\mathbf{curl}\,\Xi^{-1}\,\mathbf{curl}} C^\infty(\mathbb{M}) \xrightarrow{(\mathsf{skw},\mathsf{div})^T} C^\infty(\mathbb{K}\times\mathbb{V}) \to 0.$$

Well-posedness of Weak-Symmetry Formulation of Elasticity

To establish well-posedness of elasticity problem with weakly imposed symmetry, suffices to verify condition (W2). Deduced from following theorem, which says map

$$H\Lambda^{n-1}(\Omega;\mathbb{V}) \xrightarrow{(-S_{n-1},d_{n-1})^T} H\Lambda^n(\Omega;\mathbb{K}) \times H\Lambda^n(\Omega;\mathbb{V})$$

is surjective. Proof uses following well-known result from PDEs.

Lemma: Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary. Then, for all $\mu \in L^2\Lambda^n(\Omega)$, there exists $\eta \in H^1\Lambda^{n-1}(\Omega)$ satisfying $d_{n-1}\eta = \mu$. If, in addition, $\int_\Omega \mu = 0$, then we can choose $\eta \in \mathring{H}^1\Lambda^{n-1}(\Omega)$.

Theorem: Given $(\omega, \mu) \in L^2\Lambda^n(\Omega; \mathbb{K}) \times L^2\Lambda^n(\Omega; \mathbb{V})$, there exists $\sigma \in H\Lambda^{n-1}(\Omega; \mathbb{V})$ such that $d_{n-1}\sigma = \mu$, $-S_{n-1}\sigma = \omega$. Moreover, we may choose σ so that

$$\|\sigma\|_{H\Lambda} \leq c(\|\omega\| + \|\mu\|),$$

for a fixed constant c.

Proof: Second sentence follows from first by Banach's theorem, (i.e., if a continuous linear operator between two Banach spaces has an inverse, then inverse operator continuous), so only prove first sentence.

(1) By Lemma, can find $\eta \in H^1\Lambda^{n-1}(\Omega; \mathbb{V})$ with $d_{n-1}\eta = \mu$.

(2) Since $\omega + S_{n-1}\eta \in H\Lambda^n(\Omega; K)$, can apply Lemma second time to find $\tau \in H^1\Lambda^{n-1}(\Omega; \mathbb{K})$ with $d_{n-1}\tau = \omega + S_{n-1}\eta$.

(3) Since S_{n-2} an isomorphism from $H^1\Lambda^{n-2}(\Omega;\mathbb{V})$ onto $H^1\Lambda^{n-1}(\Omega;\mathbb{K})$, $\exists\rho \in H^1\Lambda^{n-2}(\Omega;\mathbb{V})$ with $S_{n-2}\rho = \tau$.

(4) Define $\sigma = d_{n-2}\rho + \eta \in H\Lambda^{n-1}(\Omega;\mathbb{V})$.

(5) From steps (1) and (4), $d_{n-1}\sigma = d_{n-1}d_{n-2}\rho + d_{n-1}\eta = \mu$.

(6) From (4), $-S_{n-1}\sigma = -S_{n-1}d_{n-2}\rho - S_{n-1}\eta$. But, since $d_{n-1}S_{n-2} = -S_{n-1}d_{n-2}$,

$$-S_{n-1}d_{n-2}\rho = d_{n-1}S_{n-2}\rho = d_{n-1}\tau = \omega + S_{n-1}\eta.$$

Hence, $-S_{n-1}\sigma = \omega$.

Key Points.

(1) Although elasticity problem involves 3 spaces $H\Lambda^{n-1}(\Omega;\mathbb{V})$, $L^2\Lambda^n(\Omega;\mathbb{V})$, and $L^2\Lambda^n(\Omega;\mathbb{K})$, proof uses 2 additional spaces from the BGG construction: $H\Lambda^{n-2}(\Omega;\mathbb{V})$ and $H\Lambda^{n-1}(\Omega;\mathbb{K})$.

(2) Although only S_{n-1} appears in method, S_{n-2} plays important role in proof.

(3) Do not need S_{n-2} an isomorphism from $H^1\Lambda^{n-2}(\mathbb{V};\mathbb{V})$ to $H^1\Lambda^{n-1}(\mathbb{V};\mathbb{K})$; only surjection. Important in discrete version of proof.

(4) Other slightly weaker conditions can be used in some places in proof (also exploit in discrete version for some choices of finite element spaces).

Abstract Conditions for Stable Approximation Schemes
Basic idea: Mimic structure of continuous problem.

To establish stability of continuous problem, only used last two spaces in top sequence and last three spaces in bottom sequence.

$$
\begin{array}{ccccc}
 & & \Lambda^{n-1}(\mathbb{K}) & \xrightarrow{d_{n-1}} & \Lambda^n(\mathbb{K}) \to 0 \\
 & \nearrow S_{n-2} & & \nearrow S_{n-1} & \\
\Lambda^{n-2}(\mathbb{V}) & \xrightarrow{d_{n-2}} & \Lambda^{n-1}(\mathbb{V}) & \xrightarrow{d_{n-1}} & \Lambda^n(\mathbb{V}) \to 0.
\end{array}
$$

Thus, look for five finite dimensional spaces connected by a similar structure, i.e., in addition to spaces

$$\Lambda_h^n(\mathbb{K}) \subset H\Lambda^n(\mathbb{K}), \quad \Lambda_h^{n-1}(\mathbb{V}) \subset H\Lambda^{n-1}(\mathbb{V}), \quad \Lambda_h^n(\mathbb{V}) \subset H\Lambda^n(\mathbb{V})$$

used in finite element method, also seek spaces

$$\Lambda_h^{n-1}(\mathbb{K}) \subset H\Lambda^{n-1}(\mathbb{K}), \quad \Lambda_h^{n-2}(\mathbb{V}) \subset H\Lambda^{n-2}(\mathbb{V}).$$

Require finite element spaces also connected by exact sequences, but introduce additional flexibility by inserting L^2 projection operator Π_h^n and using approximations of S_{n-2} and S_{n-1}.

$$
\begin{array}{ccccc}
 & & \Lambda_h^{n-1}(\mathbb{K}) & \xrightarrow{\Pi_h^n d_{n-1}} & \Lambda_h^n(\mathbb{K}) \to 0 \\
 & \nearrow S_{n-2,h} & & \nearrow S_{n-1,h} & \\
\Lambda_h^{n-2}(\mathbb{V}) & \xrightarrow{d_{n-2}} & \Lambda_h^{n-1}(\mathbb{V}) & \xrightarrow{d_{n-1}} & \Lambda_h^n(\mathbb{V}) \to 0.
\end{array}
\tag{7}
$$

Next step: Identify properties of interpolants into each finite element space needed for stability proof.

Define Π_h^n and $\tilde{\Pi}_h^n$ to be L^2 projection operators into $\Lambda_h^n(\mathbb{K})$ and $\Lambda_h^n(\mathbb{V})$, respectively.

Define Π_h^{n-1} and $\tilde{\Pi}_h^{n-1}$ to be interpolation operators mapping $H^1\Lambda^{n-1}(\mathbb{K})$ to $\Lambda_h^{n-1}(\mathbb{K})$ and $H^1\Lambda^{n-1}(\mathbb{V})$ to $\Lambda_h^{n-1}(\mathbb{V})$, respectively, and satisfying

$$\Pi_h^n d_{n-1}\Pi_h^{n-1}\tau = \Pi_h^n d_{n-1}\tau, \ \tau \in (\mathring{H}^1 + P^1)\Lambda^{n-1}(\mathbb{K}), \tag{8}$$

$$d_{n-1}\tilde{\Pi}_h^{n-1}\tau = \tilde{\Pi}_h^n d_{n-1}\tau, \ \tau \in H^1\Lambda^{n-1}(\mathbb{V}). \tag{9}$$

$$\|\Pi_h^{n-1}\tau\| \le C\|\tau\|_1, \ \tau \in (\mathring{H}^1 + P^1)\Lambda^{n-1}(\mathbb{K}), \tag{10}$$

$$\|\tilde{\Pi}_h^{n-1}\tau\| \le C\|\tau\|_1, \ \tau \in H^1\Lambda^{n-1}(\mathbb{V}). \tag{11}$$

Define $\tilde{\Pi}_h^{n-2}$ mapping $H^1\Lambda^{n-2}(\mathbb{V})$ to $\Lambda_h^{n-2}(\mathbb{V})$ satisfying

$$\|d_{n-2}\tilde{\Pi}_h^{n-2}\rho\| \le c\|\rho\|_1, \quad \rho \in H^1\Lambda^{n-2}. \tag{12}$$

In (12), d_{n-2} corresponds to $\mathbf{curl}$. In some cases, must modify canonical interpolation operator so defined on spaces of functions will less smoothness than usually assumed.

Next step: Define $S_{n-1,h} : \Lambda_h^{n-1}(\mathbb{V}) \to \Lambda_h^3(\mathbb{K})$ by $S_{n-1,h} = \Pi_h^n S_{n-1}$ as a discrete version of S_{n-1} and $S_{n-2,h} : \Lambda_h^{n-2}(\mathbb{V}) \to \Lambda_h^2(\mathbb{K})$ by $S_{n-2,h} = \Pi_h^{n-1} S_{n-2}$ as a discrete version of S_{n-2}.

With these definitions, can establish discrete version of identity $d_{n-1} S_{n-2} = -S_{n-1} d_{n-2}$, i.e.,

$$\Pi_h^n d_{n-1} S_{n-2,h} = -S_{n-1,h} d_{n-2}. \tag{13}$$

Cannot expect invertibility of $S_{n-2,h}$, but require $S_{n-2,h}$ maps $\Lambda_h^{n-2}(\mathbb{V})$ onto $\Lambda_h^{n-1}(\mathbb{K})$. To ensure this, assume $\Lambda_h^{n-2}(\mathbb{V})$ and $\Lambda_h^{n-1}(\mathbb{K})$ related by:

$$S_{n-2,h} \tilde{\Pi}_h^{n-2} \tau = \Pi_h^{n-1} S_{n-2} \tau, \quad \tau \in H^1 \Lambda^{n-2}(\mathbb{V}). \tag{14}$$

Stability of Finite Element Approximation Schemes

Theorem: Assume finite element subspaces $\Lambda_h^k(\mathbb{K})$ and $\Lambda_h^k(\mathbb{V})$ connected by exact sequences given in (7), that there are projection operators associated with these subspaces satisfying conditions (8), (9) (10), (11), (12), and that condition (14) is satisfied. Then, given $(\omega, \mu) \in \Lambda_h^n(\mathbb{K}) \times \Lambda_h^n(\mathbb{V})$, there exists $\sigma \in \Lambda_h^{n-1}(\mathbb{V})$ such that $d_{n-1}\sigma = \mu$, $-S_{n-1,h}\sigma \equiv -\Pi_h^n S_{n-1}\sigma = \omega$, and

$$\|\sigma\|_{H\Lambda} \leq c(\|\omega\| + \|\mu\|), \tag{15}$$

where constant c independent of ω, μ and h.

Proof: Set $\sigma = d_{n-2}\tilde{\Pi}_h^{n-2}\rho + \tilde{\Pi}^{n-1}\eta$ and follow proof in continuous case.

Theorem: Suppose (σ, u, p) exact solution of elasticity system and (σ_h, u_h, p_h) solution of discrete system, where finite element spaces satisfy hypotheses of stability theorem. Then there is a constant C, independent of h, such that

$$\|\sigma-\sigma_h\|_{H\Lambda}+\|u-u_h\|+\|p-p_h\| \le C \inf(\|\sigma-\tau\|_{H\Lambda}+\|u-v\|+\|p-q\|),$$

where infimum over all $\tau \in \Lambda_h^{n-1}(\mathbb{V})$, $v \in \Lambda_h^n(\mathbb{V})$, and $q \in \Lambda_h^n(\mathbb{K})$. Moreover,

$$\|\sigma - \sigma_h\| + \|p - p_h\| + \|u_h - \tilde{\Pi}_h^n u\| \le C(\|\sigma - \tilde{\Pi}_h^{n-1}\sigma\| + \|p - \Pi_h^n p\|),$$
$$\|u - u_h\| \le C(\|\sigma - \tilde{\Pi}_h^{n-1}\sigma\| + \|p - \Pi_h^n p\| + \|u - \tilde{\Pi}_h^n u\|),$$
$$\|d_{n-1}(\sigma - \sigma_h)\| = \|d_{n-1}\sigma - \tilde{\Pi}_h^n d_{n-1}\sigma\|.$$

Examples of Stable Finite Element Methods for Elasticity
(1) Arnold, Falk, Winther families

For $r \geq 0$, choose:

$$\Lambda_h^{n-2}(\mathbb{V}) = \mathcal{P}_{r+2}^{-}\Lambda^{n-2}(\mathcal{T}_h), \quad \Lambda_h^{n-1}(\mathbb{V}) = \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}_h;\mathbb{V}),$$
$$\Lambda_h^{n}(\mathbb{V}) = \mathcal{P}_{r}\Lambda^{n}(\mathcal{T}_h;\mathbb{V}),$$
$$\Lambda_h^{n-1}(\mathbb{K}) = \mathcal{P}_{r+1}^{-}\Lambda^{n-1}(\mathcal{T}_h;\mathbb{K}), \quad \Lambda_h^{n}(\mathbb{K}) = \mathcal{P}_{r}\Lambda^{n}(\mathcal{T}_h;\mathbb{K}).$$

Sequences

$$\mathcal{P}_{r+1}^{-}\Lambda^{n-1}(\mathcal{T}_h;\mathbb{K}) \xrightarrow{d_{n-1}} \mathcal{P}_{r}\Lambda^{n}(\mathcal{T}_h;\mathbb{K}) \to 0$$
$$\mathcal{P}_{r+2}^{-}\Lambda^{n-2}(\mathcal{T}_h;\mathbb{V}) \xrightarrow{d_{n-2}} \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}_h;\mathbb{V}) \xrightarrow{d_{n-1}} \mathcal{P}_{r}\Lambda^{n}(\mathcal{T}_h;\mathbb{V}) \to 0$$

are final parts of longer exact sequences involving $\mathcal{P}_r$ and $\mathcal{P}_r^{-}$ spaces. Hence, (7) satisfied without additional projection at end of first sequence.

For these spaces, canonical projection operators Π_h^{n-1}, Π_h^n, $\tilde{\Pi}_h^{n-1}$, and $\tilde{\Pi}_h^n$ satisfy conditions (8)-(11).

Although canonical projection operator $\tilde{\Pi}_h^{n-2}$ does not satisfy (12), (not defined on functions in $H^1\Lambda^{n-2}(\mathbb{V})$), can define modified operator, $\tilde{P}_h : \Lambda^{n-2}(\Omega;\mathbb{V}) \to \mathcal{P}_{r+2}^{-}\Lambda^{n-2}(\mathcal{T}_h;\mathbb{V})$ that does satisfy (12). $\tilde{P}_h\omega$ will have same moments as ω on faces of codimension 0 and 1, but with moments of a smoothed approximation of ω on faces of codimension 2. (When $n = 2$, vertex values not defined, but Clement interpolant may be used instead).

To satisfy hypotheses of abstract convergence theorem, remains to show that

$$\Pi_h^{n-1} S_{n-2} \tilde{P}_h = \Pi_h^{n-1} S_{n-2}.$$

Consider simplest case: $n = 2$, $r = 0$. Equivalent to showing

$$\Pi_h^1 S_0 \omega = 0, \quad \forall \omega = (I - \tilde{P}_h)\sigma, \quad \sigma \in \Lambda^0(\mathbb{V}),$$

where Π_h^1 interpolant into $\mathcal{P}_0^- \Lambda^1(\mathcal{T}_h; \mathbb{V})$ ($\boldsymbol{RT}_0$), and $\tilde{P}_h$ interpolant into $\mathcal{P}_2\Lambda^0(\mathcal{T}_h; \mathbb{V})$, i.e., piecewise P_2 vectors.

Let $\omega = (w_1, w_2)^T \in \Lambda^0(\mathbb{V})$. Since $\tilde{P}_h \omega = 0$,

$$\int_f \operatorname{Tr}_f \omega \wedge \zeta = 0, \quad \zeta \in \mathcal{P}_0\Lambda^1(f; \mathbb{V}), \quad f \in \Delta_1(\mathcal{T}_h).$$

which for $\omega = (w_1, w_2)^T$ is condition

$$\int_e w_i \, de = 0, \quad i = 1, 2, \quad e \in \Delta_1(\mathcal{T}_h).$$

Must show this condition implies

$$\int_f \mathrm{Tr}_f\, S_0\omega \wedge \mu = 0, \quad \mu \in \mathcal{P}_0\Lambda^0(f;\mathbb{K}), \quad f \in \Delta_1(\mathcal{T}_h),$$

i.e.,
$$\int_e \mathrm{Tr}_e(-w_2\boldsymbol{\chi}\,dx_1 + w_1\boldsymbol{\chi}\,dx_2) = 0, \quad e \in \Delta_1(\mathcal{T}_h).$$

But if (t^1, t^2) is unit tangent to e, then

$$\int_e \mathrm{Tr}_e(-w_2\boldsymbol{\chi}\,dx_1 + w_1\boldsymbol{\chi}\,dx_2) = \int_e (-w_2 t^1 + w_1 t^2)\boldsymbol{\chi}\,de$$
$$= \boldsymbol{\chi}\int_e (w \cdot n)\,de = 0,$$

i.e., degrees of freedom for $\boldsymbol{RT}_0$ are zero.

Analogous argument works for general r when $n = 2$, and basic outline of proof same when $n = 3$, although operator S_1 more complicated.

Using abstract convergence theorem, straightforward to derive following error estimates, valid for $1 \le k \le r+1$, assuming σ, p, and u sufficiently smooth.

$$\|\sigma - \sigma_h\| + \|p - p_h\| + \|u_h - \tilde{\Pi}_h^n u\| \le Ch^k(\|\sigma\|_k + \|p\|_k),$$
$$\|u - u_h\| \le Ch^k(\|\sigma\|_k + \|p\|_k + \|u\|_k),$$
$$\|d_{n-1}(\sigma - \sigma_h)\| \le Ch^k\|d\sigma\|_k.$$

(2) Arnold, Falk, Winther Reduced Element

Spaces $\Lambda_h^n(\mathbb{V})$, $\Lambda_h^{n-1}(\mathbb{K})$, and $\Lambda_h^n(\mathbb{K})$ remain as before, while $\Lambda_h^{n-2}(\mathbb{V})$ and $\Lambda_h^{n-1}(\mathbb{V})$ are modified. Reduced element has simpler stress space.

Basic idea: in verification of surjectivity condition, did not use all degrees of freedom of space $\mathcal{P}_2^-\Lambda^0(\mathcal{T}_h)$, i.e., did not use vanishing of edge integral of both components of ω, but only combination $-w_2t^1 + w_1t^2$ (normal component).

Instead of $\mathcal{P}_2\Lambda^0(\mathcal{T}_h, \mathbb{V})$, use reduced space obtained by imposing constraint that tangential component on each edge varies only linearly on that edge. Reduced space $\mathcal{P}_{2-}\Lambda^0(\mathcal{T}_h, \mathbb{V})$ used previously to approximate velocity field in stationary Stokes equations (together with piecewise constant pressure).

Elements determined by vertex values and integral of normal component on each edge. To complete construction, provide vector-valued discrete de Rham sequence in which 0-forms are $\mathcal{P}_{2-}\Lambda^0(\mathcal{T}_h;\mathbb{R}^2)$, i.e., sequence:

$$\mathcal{P}_{2-}\Lambda^0(\mathcal{T}_h;\mathbb{V}) \xrightarrow{d_0} \mathcal{P}_{1-}\Lambda^1(\mathcal{T}_h;\mathbb{V}) \xrightarrow{d_1} \mathcal{P}_0\Lambda^2(\mathcal{T}_h;\mathbb{V}) \to 0,$$

where

$$\mathcal{P}_{1-}\Lambda^1(\mathcal{T}_h;\mathbb{V}) = \{\tau \in \mathcal{P}_1\Lambda^1(\mathcal{T}_h;\mathbb{V}) : \mathrm{Tr}_e(\tau)\cdot t \text{ constant}\}.$$

Degrees of freedom:

$$\int_e \mathrm{Tr}_e(\tau)\cdot n\, p_1(s)\, ds, \qquad \int_e \mathrm{Tr}_e(\tau)\cdot t\, ds.$$

Identify $\omega \in \Lambda^1(\mathbb{V};\mathbb{V})$ with matrix W. Then $\omega \in \mathcal{P}_1^- \Lambda^1(\mathcal{T}_h;\mathbb{R}^2)$ if on each edge e with tangent t and normal n, $Wn \cdot t$ constant on e. Defines reduced stress space with 3 degrees of freedom per edge. Together with piecewise constants for displacements and multipliers, gives simple stable choice of elements.

3-D simplified element constructed using similar approach. Starting from $\mathcal{P}_2^- \Lambda^1(\mathcal{T}_h;\mathbb{V})$, do not use all degrees of freedom to satisfy surjectivity condition.

Define reduced space $\mathcal{P}_{2-}^- \Lambda(\mathcal{T}_h;\mathbb{V})$ and space $\mathcal{P}_{1-}\Lambda^2(\mathcal{T}_h;\mathbb{V})$ such that these spaces, together with $\mathcal{P}_0\Lambda^3(\mathcal{T}_h;\mathbb{V})$, form exact sequence

$$\mathcal{P}_{2-}\Lambda^1(\mathcal{T}_h;\mathbb{V}) \xrightarrow{d_1} \mathcal{P}_{1-}\Lambda^2(\mathcal{T}_h;\mathbb{V}) \xrightarrow{d_2} \mathcal{P}_0\Lambda^3(\mathcal{T}_h;\mathbb{V}) \to 0.$$

Then able to replace space $\mathcal{P}_1\Lambda^1(\mathcal{T}_h;\mathbb{V})$, which has 36 degrees of freedom (9 per face), by space $\mathcal{P}_1^-\Lambda^2(\mathcal{T}_h;\mathbb{V})$, which has 24 degrees of freedom (6 per face). If we identify element in reduced space with matrix W as before, get six degrees of freedom on each face:

$$\int_f Wn\,df, \quad \int_f (x\cdot t)n^T Wn\,df, \quad \int_f (x\cdot s)n^T Wn\,df,$$
$$\int_f [(x\cdot t)s^T - (x\cdot s)t^T]Wn\,df,$$

where s and t denote orthogonal unit tangent vectors on face f.

(3) PEERS

In PEERS method, $n = 2$ and we choose

$$\Lambda_h^1(\mathbb{V}) = \mathcal{P}_1^- \Lambda^1(\mathcal{T}_h; \mathbb{V}) + dB_3\Lambda^0(\mathcal{T}_h; \mathbb{V}), \quad \Lambda_h^2(\mathbb{V}) = \mathcal{P}_0\Lambda^2(\mathcal{T}_h; \mathbb{V}),$$
$$\Lambda_h^2(\mathbb{K}) = \mathcal{P}_1\Lambda^2(\mathcal{T}_h; \mathbb{K}) \cap H^1\Lambda^2(\mathbb{K}) \quad \text{[denote by } \mathcal{P}_1^0\Lambda^2(\mathcal{T}_h; \mathbb{K})\text{]}.$$

We then choose the two remaining spaces as

$$\Lambda_h^0(\mathbb{V}) = (\mathcal{P}_1 + B_3)\Lambda^0(\mathcal{T}_h; \mathbb{V}), \quad \Lambda_h^1(\mathbb{K}) = S_0\Lambda_h^0(\mathbb{V}).$$

Easy to see that

$$\Lambda_h^1(\mathbb{K}) = (\mathcal{P}_1 + B_3)\Lambda^1(\mathcal{T}_h; \mathbb{K}) \cap H^1\Lambda^1(\mathbb{K}) \equiv (\mathcal{P}_1^0 + B_3)\Lambda^1(\mathcal{T}_h; \mathbb{K}).$$

Since sequence

$$\mathcal{P}_1\Lambda^0(\mathcal{T}_h; \mathbb{V}) \xrightarrow{d_0} \mathcal{P}_1^-\Lambda^1(\mathcal{T}_h; \mathbb{V}) \xrightarrow{d_1} \mathcal{P}_0\Lambda^2(\mathcal{T}_h; \mathbb{V}) \to 0$$

is exact, so is sequence

$$(\mathcal{P}_1 + B_3)\Lambda^0(\mathcal{T}_h;\mathbb{V}) \xrightarrow{d_0} \mathcal{P}_1^-\Lambda^1(\mathcal{T}_h;\mathbb{V}) + d_0 B_3\Lambda^0(\mathcal{T}_h;\mathbb{V})$$
$$\xrightarrow{d_1} \mathcal{P}_0\Lambda^2(\mathcal{T}_h;\mathbb{V}) \to 0.$$

However, not true that $d_1\Lambda_h^1(\mathbb{K}) = \Lambda_h^2(\mathbb{K})$. Instead, use: $\Pi_h^2 d_1\Lambda_h^1(\mathbb{K}) = \Lambda_h^2(\mathbb{K})$. Allows use of stable Stokes elements.

Proof that combination $(\mathcal{P}_1^0 + B_3)\Lambda^1(\mathcal{T}_h;\mathbb{K})$ and $\mathcal{P}_1^0\Lambda^2(\mathcal{T}_h;\mathbb{K})$ is stable Stokes pair (Mini-element) involves construction of interpolation operator $\Pi_h^1 : H^1\Lambda^1(\mathbb{K}) \mapsto (\mathcal{P}_1^0 + B_3)\Lambda^1(\mathcal{T}_h;\mathbb{K})$ satisfying

$$\langle d_1(\tau - \Pi_h^1\tau), q_h\rangle = 0, \quad q_h \in \Lambda_h^2(\mathbb{K}),$$
$$\|\Pi_h^1\tau\|_1 \le C\|\tau\|_1, \quad \tau \in H^1\Lambda^1(\mathbb{K}),$$

which gives properties (8) and (10).

Properties (9) and (11) satisfied by $\boldsymbol{RT}$ interpolant

$$\tilde{\Pi}_h^1 : H^1\Lambda^1(\mathbb{V}) \mapsto \mathcal{P}_1^-\Lambda^1(\mathcal{T}_h; \mathbb{V}).$$

Easily check that (12) and (14) satisfied if we define

$$\tilde{\Pi}_h^0 : H^1\Lambda^0(\mathbb{V}) \mapsto (\mathcal{P}_1 + B_3)\Lambda^0(\mathcal{T}_h; \mathbb{V})$$
$$\text{by} \qquad \tilde{\Pi}_h^0\tau = S_0^{-1}\Pi_h^1 S_0\tau.$$

Note that surjectivity trivial, since for $\tau \in H^1\Lambda^0(\mathbb{V})$,

$$S_{0,h}\tilde{\Pi}_h^0\tau = \Pi_h^1 S_0 S_0^{-1}\Pi_h^1 S_0\tau = \Pi_h^1 S_0\tau.$$

Applying abstract convergence theorem, and standard approximation and regularity results, obtain:

$$\|\sigma - \sigma_h\|_0 + \|p - p_h\|_0 + \|u - u_h\|_0$$
$$\leq Ch(\|\sigma\|_1 + \|p\|_1 + \|u\|_1) \leq Ch\|f\|_0.$$

(4) A PEERS-like Method with Improved Stress Approximation

In this new method, we change one space used in PEERS element and both auxiliary spaces used in analysis, i.e., choose

$$\Lambda_h^1(\mathbb{V}) = \mathcal{P}_1\Lambda^1(\mathcal{T}_h;\mathbb{V}), \quad \Lambda_h^2(\mathbb{V}) = \mathcal{P}_0\Lambda^2(\mathcal{T}_h;\mathbb{V}),$$
$$\Lambda_h^2(\mathbb{K}) = \mathcal{P}_1^0\Lambda^2(\mathcal{T}_h;\mathbb{K}),$$

and two remaining spaces:

$$\Lambda_h^0(\mathbb{V}) = \mathcal{P}_2\Lambda^0(\mathcal{T}_h;\mathbb{V}),$$
$$\Lambda_h^1(\mathbb{K}) = S_0\Lambda_h^0(\mathbb{V}) \equiv \mathcal{P}_2\Lambda^1(\mathcal{T}_h;\mathbb{K}) \cap H^1\Lambda^1(\mathbb{K}).$$

Change from PEERS' analysis: now use combination of $\mathcal{P}_2\Lambda^1(\mathcal{T}_h;\mathbb{K}) \cap H^1\Lambda^1(\mathbb{K})$ and $\mathcal{P}_1^0\Lambda^2(\mathcal{T}_h;\mathbb{K})$ as stable Stokes pair (Taylor-Hood element).

Method also modification of lowest order A-F-W method: same σ_h and u_h spaces and lower sequence. Changed multiplier space.

Advantage of new method: higher order approximation to stress variable. From convergence theorem: error estimate for $\|\sigma-\sigma_h\|_0$ depends both on $\|\sigma - \tilde{\Pi}_h^{n-1}\sigma\|_0$ and $\|p - \Pi_h^n p\|_0$.

In lowest order A-F-W method, $\|\sigma - \tilde{\Pi}_h^{n-1}\sigma\|_0 \le Ch^2\|\sigma\|_2$ since $\tilde{\Pi}_h^{n-1}\sigma \in \mathcal{P}_1$. But $\|p - \Pi_h^n p\|_0 \le Ch\|p\|_1$ since $\Pi_h^n p \in \mathcal{P}_0$.

Since $\Pi_h^n p \in \mathcal{P}_1$ in new method, recover 2nd order convergence.

Since $\tilde{\Pi}_h^n u \in \mathcal{P}_0$, only obtain $\|u-u_h\|_0 \le Ch$. However, $\|u_h-\tilde{\Pi}_h^n u\|_0$ also $O(h^2)$, so post-processing might produce better result.

Remark: Similar ideas used to develop hybrid methods for elasticity equations. E.g., see Farhloul-Fortin.

(5) Methods of Stenberg

For $r \geq 2$, $n = 2$ or $n = 3$, choose

$$\Lambda_h^{n-1}(\mathbb{V}) = \mathcal{P}_r\Lambda^{n-1}(\mathcal{T}_h;\mathbb{V}) + dB_{r+n}\Lambda^{n-2}(\mathcal{T}_h;\mathbb{V}),$$
$$\Lambda_h^n(\mathbb{V}) = \mathcal{P}_{r-1}\Lambda^n(\mathcal{T}_h;\mathbb{V}), \qquad \Lambda_h^n(\mathbb{K}) = \mathcal{P}_r\Lambda^n(\mathcal{T}_h;\mathbb{K}),$$

where B_{r+n} denotes functions which on each simplex T have form $b_T\mathcal{P}_{r-1}$, where $b_T(x) = \prod_{i=1}^{n+1}\lambda_i(x)$. To fit framework, choose two remaining spaces as

$$\Lambda_h^{n-2}(\mathbb{V}) = (\mathcal{P}_{r+1} + B_{r+n})\Lambda^{n-2}(\mathcal{T}_h;\mathbb{V}),$$
$$\Lambda_h^{n-1}(\mathbb{K}) = (\mathcal{P}_{r+1} + B_{r+n})\Lambda^{n-1}(\mathcal{T}_h;\mathbb{K}) \cap H^1\Lambda^1(\mathbb{K}).$$

Using exactness of sequence

$$\mathcal{P}_{r+1}\Lambda^{n-2}(\mathcal{T}_h;\mathbb{V}) \xrightarrow{d_{n-2}} \mathcal{P}_r\Lambda^{n-1}(\mathcal{T}_h;\mathbb{V}) \xrightarrow{d_{n-1}} \mathcal{P}_{r-1}\Lambda^{n}(\mathcal{T}_h;\mathbb{V}) \to 0,$$

get exactness of sequence

$$\begin{aligned}(\mathcal{P}_{r+1} + B_{r+n})\Lambda^{n-2}(\mathcal{T}_h;\mathbb{V}) &\xrightarrow{d_{n-2}} \mathcal{P}_r\Lambda^{n-1}(\mathcal{T}_h;\mathbb{V}) \\ &+ d_{n-2}B_{r+n}\Lambda^{n-2}(\mathcal{T}_h;\mathbb{V}) \xrightarrow{d_{n-1}} \mathcal{P}_{r-1}\Lambda^{n}(\mathcal{T}_h;\mathbb{V}) \to 0.\end{aligned}$$

Again not true that $d_{n-1}\Lambda_h^{n-1}(\mathbb{K}) = \Lambda_h^n(\mathbb{K})$. Instead, use:

$$\Pi_h^n d_{n-1}\Lambda_h^{n-1}(\mathbb{K}) = \Lambda_h^n(\mathbb{K})$$

and stable Stokes elements.

From definition of S_{n-2}, easy to see when $n=2$,

$$S_0\Lambda_h^0(\mathbb{V}) = (\mathcal{P}_{r+1} + B_{r+n})\Lambda^1(\mathcal{T}_h;\mathbb{K}) \cap H^1\Lambda^1(\mathbb{K}),$$

and when $n=3$,

$$S_1[\Lambda_h^1(\mathbb{V}) \cap H^1\Lambda^1(\mathbb{V})] = (\mathcal{P}_{r+1} + B_{r+n})\Lambda^2(\mathcal{T}_h;\mathbb{K}) \cap H^1\Lambda^2(\mathbb{K}).$$

Proof that combination $(\mathcal{P}_{r+1} + B_{r+n})\Lambda^{n-1}(\mathcal{T}_h;\mathbb{K}) \cap H^1\Lambda^{n-1}(\mathbb{K})$ and $\mathcal{P}_r\Lambda^n(\mathcal{T}_h;\mathbb{K})$ is stable Stokes pair involves construction of interpolation operator $\Pi_h^{n-1} : H^1\Lambda^{n-1}(\mathbb{K}) \mapsto (\mathcal{P}_{r+1} + B_{r+n})\Lambda^{n-1}(\mathcal{T}_h;\mathbb{K}) \cap H^1\Lambda^{n-1}(\mathbb{K})$ satisfying

$$\begin{aligned} &\langle d_{n-1}(\tau - \Pi_h^{n-1}\tau), q_h\rangle = 0, \quad && q_h \in \Lambda_h^n(\mathbb{K}),\\ &\|\Pi_h^{n-1}\tau\|_1 \le C\|\tau\|_1, && \tau \in H^1\Lambda^{n-1}(\mathbb{K}), \end{aligned}$$

which gives properties (8) and (10).

Next observe that canonical interpolant $\tilde{\Pi}_h^{n-1} : H^1\Lambda^{n-1}(\mathbb{V}) \mapsto \mathcal{P}_r\Lambda^{n-1}(\mathcal{T}_h;\mathbb{V})$ satisfies (9) and (11).

Easily check that (12) and (14) satisfied if we define

$$\tilde{\Pi}_h^{n-2} : H^1\Lambda^{n-2}(\mathbb{V}) \mapsto (\mathcal{P}_{r+1} + B_{r+n})\Lambda^{n-2}(\mathcal{T}_h;\mathbb{V}) \cap H^1\Lambda^{n-2}(\mathbb{V})$$
$$\text{by} \qquad \tilde{\Pi}_h^{n-2}\tau = S_{n-2}^{-1}\Pi_h^{n-1}S_{n-2}\tau.$$

When $n = 2$, same analysis carries over to $r = 1$, since combination $(\mathcal{P}_2 + B_3)\Lambda^1(\mathcal{T}_h;\mathbb{K}) \cap H^1\Lambda^1(\mathbb{K})$ and $\mathcal{P}_1\Lambda^2(\mathcal{T}_h;\mathbb{K})$ is stable Stokes pair. Situation more complicated in 3-d, since analogous combination not stable Stokes pair.

Using abstract convergence theorem, and assuming σ, p, and u sufficiently smooth, can show:

$$\|\sigma - \sigma_h\| + \|p - p_h\| + \|u_h - \tilde{\Pi}_h^n u\| \le Ch^k(\|\sigma\|_k + \|p\|_k),$$
$$1 \le k \le r+1,$$
$$\|u - u_h\| \le Ch^k(\|\sigma\|_k + \|p\|_k + \|u\|_k), \qquad 1 \le k \le r,$$
$$\|d_{n-1}(\sigma - \sigma_h)\| \le Ch^k\|d_{n-1}\sigma\|_k, \qquad 1 \le k \le r.$$