LOCAL SPACE-PRESERVING DECOMPOSITIONS FOR THE BUBBLE TRANSFORM

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Abstract. The bubble transform is a procedure to decompose differential forms, which are piecewise smooth with respect to a given triangulation of the domain, into a sum of local bubbles. In this paper, an improved version of a construction in the setting of the de Rham complex previously proposed by the authors is presented. The major improvement in the decomposition is that unlike the previous results, in which the individual bubbles were rational functions with the property that groups of local bubbles summed up to preserve piecewise smoothness, the new decomposition is strictly space-preserving in the sense that each local bubble preserves piecewise smoothness. An important property of the transform is that the construction only depends on the given triangulation of the domain and is independent of any finite element space. On the other hand, all the standard piecewise polynomial spaces are invariant under the transform. Other key properties of the transform are that it commutes with the exterior derivative, is bounded in L^2, and satisfies the *stable decomposition property*.

1. Introduction

The present paper is a continuation of the earlier papers [5] and [6] of the authors. While the first paper was devoted to scalar valued functions, the second paper, [6], develops a theory for decomposing differential forms into a sum of functions, or bubbles, which have local support on domains defined by a given simplicial mesh of the domain. The decomposition, which we refer to as *the bubble transform*, commutes with the exterior derivative, and has the additional property that all the standard piecewise polynomial spaces utilized in the finite element exterior calculus (FEEC), cf. [1, 2, 3], are in some sense invariant. However, the piecewise polynomial spaces are not *strictly* invariant for the decomposition constructed in [6]. In general, each individual bubble is a rational function, but with the property that groups of the local bubbles sum up to preserve the desired polynomial structure. The purpose of the theory presented here is to refine the earlier theory, so that we obtain a transform which is strictly *space-preserving* in the sense that each

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local bubble preserves piecewise smoothness and the standard piecewise polynomial spaces of FEEC.

The development of the bubble transform is partly motivated by the hp-finite element method, i.e., where both piecewise polynomials of arbitrary high degree and arbitrary small mesh cells are allowed. While the analysis of finite element methods based on mesh refinements and a fixed polynomial degree, i.e., the h-method, is by now well understood, the corresponding analysis for the p-method, where the polynomial degree is unbounded, is so far less canonical. However, the bubble transform represents a theory where the decomposition itself, and the associated operator bounds, are obtained independently of any finite element space. The entire construction only depends on a given triangulation of the domain. In fact, the decomposition is also stable with respect to mesh refinements, and therefore the results will apply to general hp-methods. As a consequence, the decomposition represents a new tool for understanding hp-methods. As an example, consider the analysis of overlapping Schwarz preconditioners. In [7], it is established how to construct such preconditioners for second order elliptic problems in the setting of hp-refinements. But so far, the corresponding verification for more general Hodge-Laplace problems appears to be open. In fact, the key obstacle for establishing such a bound is to verify the so-called stable decomposition property, i.e., to establish the existence of a bounded decomposition, cf. [9, Chapter 2] and references given therein. Such a bound is simply a special case of the bounds we derive here. Although the discussion in the present paper will be restricted to the basic theory of the bubble transform, a more thorough motivation can be found in [6].

In order to describe the main results of this paper, it is first necessary to introduce some basic notation. Throughout this paper, Ω will be a bounded polyhedral domain in $\mathbb{R}^n$, and for $0 \le k \le n$, the space of smooth differential k-forms on Ω will be denoted $\Lambda^k(\Omega)$. The construction of the bubble transform is based on a simplicial mesh $\mathcal{T}$ of Ω. The corresponding space, $\Lambda^k(\mathcal{T})$, is the space of k–forms on Ω which are piecewise smooth with respect to $\mathcal{T}$. More precisely, the elements of $\Lambda^k(\mathcal{T})$ are smooth on the closed simplices T in the triangulation and have single-valued traces on each subsimplex of $\mathcal{T}$. For the piecewise polynomial subspaces of $\Lambda^k(\mathcal{T})$, we will adopt the standard notation of FEEC, cf. [1, 2, 3], i.e., $\mathcal{P}_r\Lambda^k(\mathcal{T})$ denotes the space of piecewise polynomial forms of degree less than or equal to r, while $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ is the corresponding space of trimmed polynomials. The set of all subsimplices of $\mathcal{T}$ is denoted $\Delta(\mathcal{T})$, while $\Delta_m(\mathcal{T})$ is the set of simplices of dimension m. For each $f \in \Delta(\mathcal{T})$, the macroelement Ω_f consists of the union of all n–simplices in $\Delta(\mathcal{T})$ containing f as a subsimplex. Furthermore, $\mathcal{T}_f$ is the restriction of the mesh $\mathcal{T}$ to the macroelement Ω_f, and $\mathring{\Lambda}^k(\mathcal{T}_f)$ is the subspace of $\Lambda^k(\mathcal{T})$ consisting of forms which have support on Ω_f, i.e., which vanish on $\Omega \setminus \Omega_f$.

For given $u \in \Lambda^k(\mathcal{T})$, the bubble transform leads to a decomposition of the form

$$u = W^k u + \sum_{f \in \Delta(\mathcal{T})} B_f^k u = W^k u + \sum_{m=0}^{n} \sum_{f \in \Delta_m(\mathcal{T})} B_f^k u, \tag{1.1}$$

where the bubbles $B_f^k u$ belong to $\mathring{\Lambda}^k(\mathcal{T}_f)$, and where $W^k u$ is a trimmed piecewise linear k-form. More precisely, we will show how to construct linear operators W^k :

$\Lambda^k(\mathcal{T}) \to \mathcal{P}_1^-\Lambda^k(\mathcal{T})$ and local operators $B_f^k : \Lambda^k(\mathcal{T}) \to \mathring{\Lambda}^k(\mathcal{T}_f)$ which commute with the exterior derivative d, i,e.,

$$dW^k = W^{k+1}d, \quad \text{and } dB_f^k = B_f^{k+1}d, \quad 0 \le k \le n-1.$$

In fact, if we let $\mathcal{B}^k$ denote the collection of all the operators $\{B_f^k\}_{f\in\Delta(\mathcal{T})}$, such that we can view

$$\mathcal{B}^k : \Lambda^k(\mathcal{T}) \to \prod_{f\in\Delta(\mathcal{T})} \mathring{\Lambda}^k(\mathcal{T}_f) := \mathring{\Lambda}^k(\mathcal{T},\Delta),$$

we can summarize and state that the diagram

$$\begin{array}{ccc}
\Lambda^k(\mathcal{T}) & \xrightarrow{d} & \Lambda^{k+1}(\mathcal{T}) \\
\Big\downarrow{\scriptstyle (W^k,\mathcal{B}^k)} & & \Big\downarrow{\scriptstyle (W^{k+1},\mathcal{B}^{k+1})} \\
\mathcal{P}_1^-\Lambda^k(\mathcal{T}) \times \mathring{\Lambda}^k(\mathcal{T},\Delta) & \xrightarrow{d} & \mathcal{P}_1^-\Lambda^{k+1}(\mathcal{T}) \times \mathring{\Lambda}^{k+1}(\mathcal{T},\Delta)
\end{array}$$

commutes. We will also show that the operators W^k and B_f^k can be extended to bounded operators in L^2. Furthermore, the polynomial preserving property of the bubble transform can simply be expressed by the fact that

$$B_f^k(\mathcal{P}_r\Lambda^k(\mathcal{T})) \subset \mathring{\mathcal{P}}_r\Lambda^k(\mathcal{T}_f) \quad \text{and } B_f^k(\mathcal{P}_r^-\Lambda^k(\mathcal{T})) \subset \mathring{\mathcal{P}}_r^-\Lambda^k(\mathcal{T}_f) \tag{1.2}$$

for all $f \in \Delta(\mathcal{T})$ and $r \ge 1$, where $\mathring{\mathcal{P}}_r\Lambda^k(\mathcal{T}_f) = \mathcal{P}_r\Lambda^k(\mathcal{T}) \cap \mathring{\Lambda}^k(\mathcal{T}_f)$ and with corresponding definition for the $\mathcal{P}_r^-$-spaces.

The individual bubbles, $B_f^k u$, introduced above, will not correspond to the bubbles constructed in [6]. However, a key part of the analysis given in [6] is the study of a family of trace preserving operators, $C_m^k : \Lambda^k(\mathcal{T}) \to \Lambda^k(\mathcal{T})$, where $0 \le m \le n-1$. These operators are explicitly given in formula (2.20) below. A key property of these operators is that if $f \in \Delta_m(\mathcal{T})$, where $m \ge k$, then

$$\operatorname{tr}_f C_m^k u = \operatorname{tr}_f u, \tag{1.3}$$

cf. [6, Lemma 2.2]. Here tr_f is the trace operator. Furthermore, these operators commute with the exterior derivative and they preserve piecewise smoothness and the standard spaces of piecewise polynomials, cf. [6, Proposition 7.1]. It was also shown how the global functions $C_m^k u$ can be decomposed into a sum of local bubbles, but these bubbles were rational functions leading to the apparent defect of the theory of [6]. However, in the analysis below, where we will overcome the problems just mentioned, the operators C_m^k will still play an essential part. In fact, the new decomposition of $u \in \Lambda^k(\mathcal{T})$ will be initialized by expressing u as

$$u = (u - C_{n-1}^k u) + C_{n-1}^k u.$$

It follows from (1.3) above that $u - C_{n-1}^k u$ is a global piecewise smooth function which has zero trace on elements of $\Delta_{n-1}(\mathcal{T})$. As a consequence, we can decompose this function as $u - C_{n-1}^k u = \sum_{f\in\Delta_n(\mathcal{T})} B_f^k u$, where each local bubble, $B_f^k u := \operatorname{tr}_f(u - C_{n-1}^k u)$, is piecewise smooth, and with support in $f = \Omega_f$, for $f \in \Delta_n(\mathcal{T})$. Furthermore, the operators $\{B_f^k\}_{f\in\Delta_n(\mathcal{T})}$ will commute with the exterior derivative and preserve the piecewise polynomial spaces. We can conclude that we have decomposed u into

$$u = \sum_{f \in \Delta_n(\mathcal{T})} B_f^k u + C_{n-1}^k u, \quad B_f^k u = \operatorname{tr}_f(u - C_{n-1}^k u), \tag{1.4}$$

where the first part is a sum of local bubbles. The task for the rest of the construction will be to show that the second part, the function $C_{n-1}^k u$, also admits such a decomposition.

The present paper is organized as follows. In the next section, we list some assumptions we will make, recall some standard notation and properties of differential forms and simplicial complexes, and define some of the basic operators that we will use in the construction of our decomposition. We end Section 2 with an outline of the construction we will present in the remainder of the paper, and state the main results, cf. Theorem 2.3. To motivate the general theory, we discuss the decomposition in the case of scalar-valued functions in Section 3. In Section 4, we define various local functions depending on the given mesh $\mathcal{T}$. The delicate recursive construction of these mesh functions, which represents a completely new approach as compared to the construction performed in [6], can be seen as the main new tool utilized to obtain the improved results of this paper. In particular, the new mesh functions are used to define families of order reduction operators in Section 5, and these operators are then used to define the local bubbles B_f^k, cf. Section 6. In Section 7, we focus on the operator $C_m^k - C_{m-1}^k$, and show that this operator admits a decomposition into local bubbles with desired properties. By utilizing a telescoping series argument, we will then obtain a similar decomposition for the operator C_{n-1}^k. In the final section, we show that all the operators of the decomposition (1.1) are bounded in L^2.

2. Preliminaries

In this section, we introduce the basic assumptions, notation, and concepts that will be used in the construction below, and give an outline of the complete theory.

2.1. **Assumptions.** We assume that $\Omega \subset \mathbb{R}^n$ is a bounded polyhedral Lipschitz domain which is partitioned into a finite set of n simplices, $\Delta_n(\mathcal{T})$. The simplicial mesh $\mathcal{T}$ is assumed to be a simplicial decomposition of Ω, i.e., the union of the simplices in $\Delta_n(\mathcal{T})$ is the closure of Ω and the intersection of any two is either empty or a common subsimplex of each. The set of all such simplices of dimension m is denoted $\Delta_m(\mathcal{T})$, while $\Delta(\mathcal{T}) = \bigcup_{0 \le m \le n} \Delta_m(\mathcal{T})$. Furthermore, below we will frequently write Δ instead of $\Delta(\mathcal{T})$ and Δ_m instead of $\Delta_m(\mathcal{T})$.

2.2. **Notation.** The space of smooth differential k-forms on Ω will be denoted $\Lambda^k(\Omega)$. More precisely, for each $x \in \Omega$, $u_x \in \mathrm{Alt}^k$, where Alt^k is the space of alternating k -linear maps $\mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$. We recall that a projection, skw, mapping k-linear forms to Alt^k, is given by

$$(\operatorname{skw} u)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma} \operatorname{sign}(\sigma) u(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

where the sum is over all permutations of $\{1,\dots,k\}$. In our discussion below, we will encounter both the tensor product, $\otimes$, and the wedge product, $\wedge$, of differential forms. If $u_1 \in \Lambda^j(\Omega)$ and $u_2 \in \Lambda^k(\Omega)$, then $u_1 \wedge u_2 \in \Lambda^{j+k}(\Omega)$, and the identity

$$u_1 \wedge u_2 = \binom{k+j}{k} \operatorname{skw}(u_1 \otimes u_2) \tag{2.1}$$

holds. The exterior derivative will be denoted $d = d_k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$. Furthermore, if $F : \mathcal{M} \to \mathcal{M}'$ is a smooth map between manifolds, then the corresponding pullback F^* is a map from $\Lambda^k(\mathcal{M}') \to \Lambda^k(\mathcal{M})$. In particular, when $\mathcal{M}$ is a submanifold of $\mathcal{M}'$, then the pullback by the inclusion map, $\Lambda^k(\mathcal{M}') \to \Lambda^k(\mathcal{M})$, is the trace map $\operatorname{tr}_{\mathcal{M}}$. The notation $H\Lambda^k(\Omega)$ refers to the Sobolev space

$$H\Lambda^k(\Omega) = \{u \in L^2\Lambda^k(\Omega) : du \in L^2\Lambda^{k+1}(\Omega)\},$$

where $u \in L^2\Lambda^k(\Omega)$, if for all vectors $v_1,\dots,v_k \in \mathbb{R}^n$, the function $u_x(v_1,\dots,v_k) \in L^2(\Omega)$ as a function of x. The corresponding space of piecewise smooth k-forms with single valued traces with respect to $\mathcal{T}$, $\Lambda^k(\mathcal{T})$, will then be a subspace of $H\Lambda^k(\Omega)$. Furthermore, the set of piecewise polynomial k-forms of order r, $\mathcal{P}_r\Lambda^k(\mathcal{T})$, and the corresponding space of trimmed piecewise polynomials, $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$, will satisfy

$$\mathcal{P}_{r-1}\Lambda^k(\mathcal{T}) \subset \mathcal{P}_r^-\Lambda^k(\mathcal{T}) \subset \mathcal{P}_r\Lambda^k(\mathcal{T}) \subset \Lambda^k(\mathcal{T}) \subset H\Lambda^k(\Omega).$$

If $f \in \Delta_m(\mathcal{T})$, then f corresponds to an ordered subset of the vertices, $\Delta_0(\mathcal{T})$. We assume that all the vertices are numbered by a set of integers $\mathcal{I} = \{0,1,\dots,N(\mathcal{T})\}$ such that

$$\Delta_0(\mathcal{T}) = \{x_i \,:\, i \in \mathcal{I}\}.$$

Any $f \in \Delta_m(\mathcal{T})$ is of the form $f = [x_{j_0}, x_{j_1}, \dots x_{j_m}]$, where $j_0 < j_1 < \dots < j_m$ and $I(f) := \{j_0, j_1, \dots, j_m\} \subset \mathcal{I}$. Here we have used the notation $[\cdot,\cdots,\cdot]$ to denote convex combinations. Furthermore, the statement $g \in \Delta(f)$ means that g is a subsimplex of f, and with increasingly ordered vertices. We define the map $\sigma_f : \Delta_0(f) \to \{0,\dots,m\}$ by

$$\sigma_f(x_{j_i}) = i.$$

In other words, $\sigma_f(y)$ gives the internal numbering of a vertex y relative to the simplex f. The number of vertices in f is denoted $|f|$, i.e., $|f| = m+1$ if $f \in \Delta_m(\mathcal{T})$. If $e, f \in \Delta(\mathcal{T})$, with a disjoint set of vertices, and such that the union of the vertices defines a simplex, then although e and f are increasingly ordered, the simplex $[e,f]$ will not necessarily be increasingly ordered. We then denote by $\langle e, f\rangle$ the increasingly ordered simplex composed of the vertices of e and f. The set $\bar{\Delta}(f)$ contains the emptyset, $\emptyset$, in addition to the elements of $\Delta(f)$, and $\emptyset$ is the single element of $\Delta_{-1}(\mathcal{T})$. Since the ordering of a simplex $f = [x_{j_0}, x_{j_1}, \dots x_{j_m}] \in \Delta_m(\mathcal{T})$ is inherited from the global numbering of the vertices, the various simplices are not necessarily equally oriented. If $m = n$, we define the orientation of f, $o(f)$, by

$$o(f) = \operatorname{sign}\det\Big(x_{j_1} - x_{j_0}, \dots, x_{j_m} - x_{j_0}\Big),$$

i.e., it is the sign of the determinant of a nonsingular $n \times n$ matrix.

In the analysis that follows, we will make use of some concepts from simplicial complexes, e.g., see [8]. The k-chains defined by the mesh $\mathcal{T}$ is a vector space consisting of linear combinations of the form $\sum_{f\in\Delta_k} c_f f$. We will let $\mathcal{C}_k$ denote the corresponding space of vector representations, i.e., if $c \in \mathcal{C}_k$ then $c = \{c_f\}_{f\in\Delta_k}$, $c_f \in \mathbb{R}$. In fact, in the development of the theory below, we will consider k-chains with values in a finite dimensional vector spaces X, by which we mean spaces of the form $\mathcal{C}_k \otimes X$. In particular, X will be a subspace of $\mathcal{P}_1^- \Lambda^k(\mathcal{T})$. The *boundary operator* is a chain map,

$$f \mapsto \sum_{i \in I(f)} (-1)^{\sigma_f(x_i)} f(\hat{x}_i),$$

where the hat is used to indicate a suppressed argument. Using vector representations, the corresponding operator $\partial_k : \mathcal{C}_k \otimes X \to \mathcal{C}_{k-1} \otimes X$ can be expressed as

$$(\partial_k c)_f = \sum_{i \in \mathcal{I}} c_{[x_i,f]} \equiv \sum_{i \in \mathcal{I}} (-1)^{\sigma_{\langle x_i,f \rangle}(x_i)} c_{\langle x_i,f \rangle}, \quad f \in \Delta_{k-1}(\mathcal{T}),\, 1 \le k \le n,$$

where $c_{\langle x_i,e \rangle} = 0$ if $\langle x_i, e \rangle$ is not an element of $\Delta_k(\mathcal{T})$. We can also identify $\mathcal{C}_{-1} \otimes X$ as X, corresponding to the single element $\emptyset$ of $\Delta_{-1}(\mathcal{T})$, and $\partial_0 : \mathcal{C}_0 \otimes X \to X$ by $\partial_0 c = \sum_{i \in \mathcal{I}} c_{x_i}$.

The corresponding *coboundary operators* are cochain maps $\delta_k : \mathcal{C}_k \otimes X \to \mathcal{C}_{k+1} \otimes X$ given by

$$(\delta_k c)_f = \sum_{i \in I(f)} (-1)^{\sigma_f(x_i)} c_{f(\hat{x}_i)}, \quad f \in \Delta_{k+1}(\mathcal{T}),$$

for $-1 \le k \le n-1$. If X and Y are finite dimensional vector spaces, then these definitions lead to the identity

$$\sum_{f \in \Delta_{k-1}(\mathcal{T})} (\partial_k c)_f \otimes \tilde{c}_f = \sum_{f \in \Delta_k(\mathcal{T})} c_f \otimes (\delta_{k-1} \tilde{c})_f, \quad c \in \mathcal{C}_k \otimes X,\, \tilde{c} \in \mathcal{C}_{k-1} \otimes Y, \tag{2.2}$$

i.e., ∂ is the adjoint of δ with respect to the inner product on $\mathcal{C}_k$. Furthermore, the operators ∂ and δ will satisfy the complex properties $\partial^2 = \delta^2 = 0$.

Associated to any vertex $x_i \in \Delta_0(\mathcal{T})$, we denote by $\lambda_i(x)$ the nonnegative piecewise linear function equal to one at x_i and zero at all other vertices. More generally, if f is a ordered subset of $\Delta_0(\mathcal{T})$, but not necessarily increasingly ordered, then ϕ_f will denote the Whitney form associated to f. More precisely, if $f = [x_{j_0}, x_{j_1}, \ldots, x_{j_m}]$, then ϕ_f is given by

$$\phi_f = m! \sum_{i=0}^{m} (-1)^i \lambda_{j_i} d\lambda_{j_0} \wedge \ldots \wedge \widehat{d\lambda_{j_i}} \wedge \ldots \wedge d\lambda_{j_m}.$$

In particular, for $f = [x_i]$, $\phi_f = \lambda_i$. If $m > 0$, then $\int_g \phi_f$ is plus or minus one if $g = f$, and zero if $g \in \Delta_m(\mathcal{T})$, $g \ne f$. The forms $\{\phi_f\}_{f \in \Delta_m(\mathcal{T})}$ span the space $\mathcal{P}_1^- \Lambda^m(\mathcal{T})$ and they are local with support in Ω_f. It can also be easily checked that

$$d\phi_f = (m+1)! d\lambda_{j_0} \wedge \ldots \wedge d\lambda_{j_m} = \sum_{j \in \mathcal{I}} \phi_{[x_j,f]} = (\partial_{m+1} \phi)_f, \tag{2.3}$$

where $\phi_{[x_j,f]} = 0$ if $[x_j, f]$ does not correspond to a subsimplex of $\mathcal{T}$. Here the operator ∂_{m+1} has the interpretation given above, where $\{\phi_f\}$ for $f \in \Delta_{m+1}(\mathcal{T})$ should be seen as an element of $\mathcal{C}_{m+1} \otimes \mathcal{P}_1 \Lambda^{m+1}(\mathcal{T})$. Furthermore, if $u \in \mathcal{P}_1^- \Lambda^m(\mathcal{T})$ is expanded, such that $u = \sum_{f \in \Delta_m(\mathcal{T})} c_f \phi_f$, then

$$du = \sum_{f \in \Delta_m(\mathcal{T})} c_f (\partial_{m+1} \phi)_f = \sum_{f \in \Delta_{m+1}(\mathcal{T})} (\delta_m c)_f \phi_f. \tag{2.4}$$

It is also a consequence of the definition of ϕ_f and (2.3) that

$$\phi_{[x_i,f]} = \Big(\lambda_i d - m d\lambda_i \wedge\Big)\phi_f, \quad f \in \Delta_{m-1}(\mathcal{T}). \tag{2.5}$$

If $0 \le m \le n$, then associated to the simplex $f = [x_{j_0}, x_{j_1}, \dots, x_{j_m}]$, we define the standard simplex $\mathcal{S}_f$ by

$$\mathcal{S}_f = \big\{\, \lambda = (\lambda_{j_0}, \lambda_{j_1}, \dots, \lambda_{j_m}) \in \mathbb{R}^{m+1} \,:\, \sum_{i=0}^{m} \lambda_{j_i} = 1, \quad \lambda_{j_i} \ge 0 \,\big\},$$

while $\mathcal{S}_f^c$ is the set of all convex combinations between $\mathcal{S}_f$ and the origin, i.e., $\mathcal{S}_f^c = [0, \mathcal{S}_f]$. Alternatively,

$$\mathcal{S}_f^c = \big\{\, \lambda = (\lambda_{j_0}, \lambda_{j_1}, \dots, \lambda_{j_m}) \in \mathbb{R}^{m+1} \,:\, \sum_{i=0}^{m} \lambda_{j_i} \le 1, \quad \lambda_{j_i} \ge 0 \,\big\}.$$

If $f = [x_{j_0}, x_{j_1}, \dots x_{j_m}] \in \Delta_m(\mathcal{T})$, then $L_f : \Omega \to \mathcal{S}_f^c$ will be the map

$$L_f(x) = \big(\lambda_{j_0}(x), \lambda_{j_1}(x), \dots, \lambda_{j_m}(x)\big).$$

In the special case $f = \emptyset$, we let $\mathcal{S}_\emptyset = \mathcal{S}_\emptyset^c = \{0\}$ and $L_f(\Omega) = 0$. In the construction below, we will frequently use the pullback L_f^* mapping $\Lambda^k(\mathcal{S}_f^c)$ to $\Lambda^k(\mathcal{T})$, i.e., L_f^* maps smooth forms on $\mathcal{S}_f^c$ to piecewise smooth forms on Ω, and also polynomial forms to piecewise polynomial forms. More precisely, if $g \in \Delta(f)$, then

$$L_g^*\Big(\mathcal{P}\Lambda^k(\mathcal{S}_f^c)\Big) \subset \mathcal{P}\Lambda^k(\mathcal{T}), \tag{2.6}$$

where $\mathcal{P}$ is either $\mathcal{P}_r$ or $\mathcal{P}_r^-$. Another key tool we will utilize below is the piecewise linear function ρ_f, defined by

$$\rho_f(x) = 1 - \sum_{i \in I(f)} \lambda_i(x),$$

which can be seen as a distance function between $f \in \bar{\Delta}(\mathcal{T})$ and $x \in \Omega$. Note that $0 \le \rho_f(x) \le 1$ and $\rho_f \equiv 1$ if $f = \emptyset$. Alternatively, we have $\rho_f = L_f^* b$, where $b = b_f : \mathcal{S}_f^c \to \mathbb{R}$ is the distance to the origin, i.e., $b(\lambda) = 1 - \sum_{i=0}^m \lambda_{j_i}$.

2.3. **The macroelements.** We recall that for any $f \in \Delta(\mathcal{T})$, we denote by Ω_f the union of the simplices in $\Delta_n(\mathcal{T})$ containing f as a subsimplex, while $\mathcal{T}_f$ is the restriction of the mesh $\mathcal{T}$ to Ω_f. We refer to Ω_f as the macroelement of f, or alternatively, as the star of f. The domain Ω_f is contractible with respect to any $x \in f$. If $f \in \Delta_{m-1}(\mathcal{T})$ and $T \in \Delta_n(\mathcal{T}_f)$, we let $f^*(T) \in \Delta_{n-m}(T)$ be the subsimplex of T opposite f. The link of f, denoted by f^*, is given as $f^* = \bigcup_{T \in \Delta_n(\mathcal{T}_f)} f^*(T)$ (e.g., see [4]). More precisely,

$$f^* = \{x \in \Omega_f : \lambda_i(x) = 0, i \in I(f)\}.$$

The link f^* can be viewed as an $n-m$ dimensional oriented manifold composed of the simplices $f^*(T)$, where $f^*(T)$ has an orientation, $o(f^*(T), T)$, induced by the n-simplex T. More precisely, if $f = [x_{j_0}, \dots, x_{j_{m-1}}]$, then

$$o(f^*(T), T) = o(T) \prod_{i=0}^{m-1} (-1)^{\sigma_{T_i}(x_{j_i})},$$

where $T_i = T(\hat{x}_{j_0}, \ldots, \hat{x}_{j_{i-1}}) \in \Delta_{n-i}(T)$. Any $x \in \Omega_f$ can be written uniquely as a convex combination of the points x_i, $i \in I(f)$, and points in f^*, i.e., $\Omega_f = [f, f^*]$. In the special case when $m = n - 1$, the manifold f^* will be reduced to two vertices, or only one close to the boundary, while in the special case $f = \emptyset$, we have $\Omega_f = f^* = \Omega$. Below we will also encounter the extended macroelement, Ω_f^E, defined by $\Omega_f^E = \cup_{i \in I(f)} \Omega_{x_i}$.

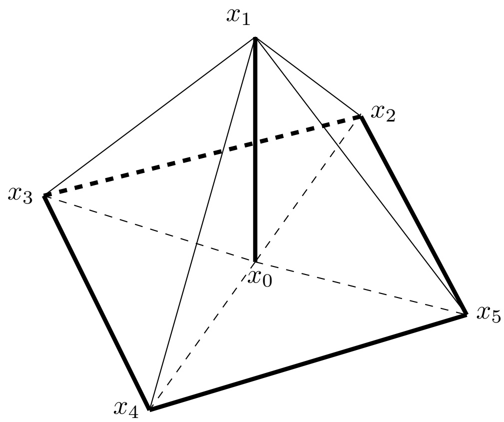

FIGURE 2.1. The macroelement $\Omega_f \subset \mathbb{R}^3$, where $f = [x_0, x_1]$ and f^* is the closed curve connecting the vertices x_2, x_3, x_4, and x_5.

We define $\mathcal{C}_k(f^*)$ as the space of vector representations of k-chains defined on the manifold f^*, i.e., if $c \in \mathcal{C}_k(f^*)$ then $c = \{c_f\}_{f \in \Delta_k(f^*)}$. The corresponding boundary and coboundary operators, $\partial_k(f^*)$ and $\delta_k(f^*)$, are defined as the operators ∂_k and δ_k above, but by restricting to the manifold f^*. In the construction performed later in this paper, we will utilize the fact that for any $f \in \Delta_{m-1}(\mathcal{T})$, $1 \le m \le n-1$, the cochain complex

$$\mathbb{R} \longrightarrow \mathcal{C}_0(f^*) \xrightarrow{\delta} \mathcal{C}_1(f^*) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{C}_{n-m}(f^*) \xrightarrow{\delta} \mathbb{R} \tag{2.7}$$

is exact, where $\delta = \delta(f^*)$, and where the special operator $\delta_{n-m}(f^*) : C_{n-m}(f^*) \to \mathbb{R}$ will be defined below. This exactness is a consequence of the corresponding property for the trimmed linear forms restricted to f^*. Since f^* is a piecewise flat submanifold of the boundary of Ω_f, defined by the mesh $\mathcal{T}$, we can consider the trace spaces $\mathcal{P}_1^-\Lambda^k(f^*)$, defined as $\mathrm{tr}_{f^*} \mathcal{P}_1^-\Lambda^k(\mathcal{T})$. If f is an interior simplex, such that f^* is a manifold without a boundary, the complex

$$\mathbb{R} \longrightarrow \mathcal{P}_1^-\Lambda^0(f^*) \xrightarrow{d} \mathcal{P}_1^-\Lambda^1(f^*) \xrightarrow{d} \cdots \mathcal{P}_1^-\Lambda^{n-m}(f^*) \longrightarrow \mathbb{R} \tag{2.8}$$

is exact, where the final arrow represents the integral over f^*. If f is a boundary simplex, the manifold f^* may have a boundary, and in this case the last arrow in (2.8) is redundant. By expanding the elements of $\mathcal{P}_1^-\Lambda^k(f^*)$ in the basis functions, and using the identity (2.4), we obtain the equivalent complex (2.7). More specifically, the map $l_k : \mathcal{P}_1^-\Lambda^k(f^*) \to \mathcal{C}_k(f^*)$ given by $\sum_{e \in \Delta_k(f^*)} c_e \phi_e \mapsto c_e$ gives the commuting relation $\delta_k(f^*) \circ l_k = l_{k+1} \circ d$. As a consequence, the exactness of the complex (2.7) follows from the exactness of (2.8). The special operator $\delta_{n-m}(f^*)$ is defined by

$$\delta_{n-m}(f^*)c = \sum_{e\in\Delta_{n-m}(f^*)} o(e,T_e)c_e, \quad T_e = \langle e, f\rangle. \tag{2.9}$$

Note that this operator is always well-defined, but that the last arrow in (2.7) may be redundant if f intersects the boundary of Ω. Since ∂ is the adjoint of δ with respect to the inner product of $\mathcal{C}(f^*)$, the following result is a consequence of the exactness of the complex (2.7).

Lemma 2.1. *Let $f \in \Delta_{m-1}(\mathcal{T})$ and $0 \le k < n-m$. If $c \in \mathcal{C}_k(f^*)$ satisfies $\partial_k(f^*)c = 0$, then there is a $\tilde{c} \in \mathcal{C}_{k+1}(f^*)$ such that $c = \partial_{k+1}(f^*)\tilde{c}$. Furthermore, $\tilde{c}$ is uniquely determined if we require $\delta_{k+1}(f^*)\tilde{c} = 0$.*

Below we will use Lemma 2.1 in a generalized sense, where we consider $\mathcal{C}_k$ with values in a finite dimensional vector space X, cf. Sections 3 and 4.1.

2.4. **The average operators and their generalizations.** A key tool for our construction below is a family of average operators, A_f^k, where $f \in \Delta$, which map piecewise smooth k-forms on Ω_f to smooth k-forms on $\mathcal{S}_f^c$. The operators A_f^k will be defined by a function $G_f : \Omega_f \times \mathcal{S}_f^c \to \Omega_f$, given by

$$G_f(y,\lambda) = \sum_{i\in\mathcal{I}(f)} \lambda_i x_i + b(\lambda)y,$$

Note that if $x \in f$ then, since $b(L_f x) = 0$, we have

$$G_f(y, L_f x) = x, \quad x \in f. \tag{2.10}$$

For each fixed $y \in \Omega_f$, $G_f(y,\lambda)$ is linear with respect to λ. The corresponding derivative with respect λ, $DG_f(y,\cdot)$, is therefore an operator mapping tangent vectors of $\mathcal{S}_f^c$, TS_f^c, into $T\Omega_f$ which is independent of λ. It is given by

$$DG_f(y,\cdot) = \sum_{i\in I(f)} (x_i - y)d\lambda_i.$$

Since for each y, the map $G_f(y,\cdot)$ maps $\mathcal{S}_f^c$ to Ω_f, the corresponding pullback, $G_f(y,\cdot)^*$, maps $\Lambda^k(\Omega_f)$ to $\Lambda^k(\mathcal{S}_f^c)$. As a further consequence, an average of these maps over Ω_f with respect to y will also map $\Lambda^k(\Omega_f)$ to $\Lambda^k(\mathcal{S}_f^c)$. In order to define the averages we want, we will introduce a family of piecewise constant n-forms, $z_f \in \mathcal{P}_1^-\Lambda^n(\mathcal{T}_f)$, with the property that z_f has support in Ω_f and

$$\int_{\Omega_f} z_f = \int_\Omega z_f = 1. \tag{2.11}$$

The operator A_f^k is then defined for $f \in \Delta_m$ by

$$A_f^k u = \int_\Omega G_f(y,\cdot)^* u \wedge z_f.$$

Since pullbacks commute with the exterior derivative, so do the operators A_f^k, i.e., $dA_f^k = A_f^{k+1}d$, and from (2.10), we obtain that for $f \in \Delta_m$,

$$\operatorname{tr}_f L_f^* A_f^k = \operatorname{tr}_f, \quad m \ge k.$$

Of course, the properties of the operators A_f^k will also depend on the choice of the functions z_f. For $f \in \Delta_n$, these functions are defined to be

$$z_f = \frac{\kappa_f}{|\Omega_f|}\mathsf{vol}, \tag{2.12}$$

where κ_f is the characteristic function of Ω_f, while for $f \in \Delta_m$, $0 \le m < n$, the functions z_f are defined recursively by the relation

$$z_f = \frac{1}{|f^*|} \sum_{i \in I(f^*)} z_{\langle x_i, f\rangle}. \tag{2.13}$$

We note that it follows by construction that all the functions z_f have support in Ω_f and satisfy relation (2.11). Furthermore, it is a consequence of Lemma 2.1 of [6] that the operators A_f^k map $\Lambda^k(\mathcal{T}_f)$ to $\Lambda^k(S_f^c)$ and also map piecewise polynomial forms to polynomial forms. More precisely, we have for $f \in \Delta$,

$$A_f^k(\mathcal{P}_r\Lambda^k(\mathcal{T}_f)) \subset \mathcal{P}_r\Lambda^k(\mathcal{S}_f^c), \qquad A_f^k(\mathcal{P}_r^-\Lambda^k(\mathcal{T}_f)) \subset \mathcal{P}_r^-\Lambda^k(\mathcal{S}_f^c). \tag{2.14}$$

Remarks. *In the argument given in* [6]*, it was assumed that the functions z_f were given by* (2.12) *for all f. However, the modification we need to cover the more general average functions z_f introduced above is straightforward.*

Consider the case of scalar valued functions, i.e., the case $k = 0$. Since all the functions z_f satisfy the identity (2.11), it follows that for $f \in \Delta$, and $e \in \Delta_1(f)$,

$$(\delta A^0 u)_{e,f} := \sum_{i \in I(e)} (-1)^{\sigma_e(x_i)} A^0_{f(\hat{x}_i)} u$$

will be zero when u is a constant. Therefore, this expression only depends on du. Below we will construct a corresponding operator R^1, mapping one-forms to zero-forms, such that

$$R^1 du = (\delta A^0 u)_{e,f}, \quad e \in \Delta_1(f). \tag{2.15}$$

A natural choice seems to be to label this operator by the simplex pair (e, f). In fact, this was the choice used in [6]. However, for the theory developed in this paper, it appears more appropriate to use the equivalent label, $(e, f \cap e^*)$, where $f \cap e^*$ belongs to e^* and vice versa.

More generally, we introduce the sets $\Delta_{j,m} = \Delta_{j,m}(\mathcal{T})$ of pairs of simplices, given by

$$\Delta_{j,m} := \{(e, f) \,:\, f \in \Delta_m(\mathcal{T}),\, e \in \Delta_j(f^*)\,\}$$

for $-1 \le m \le n$ and $0 \le j < n - m$. For $(e, f) \in \Delta_{j,m}$, $j \ge 0$, we will define operators $R^k_{e,f}$ mapping k-forms to $(k-j)$-forms. We will refer to these operators as order reduction operators. To define them, we recall that the map G_f maps $\Omega_f \times \mathcal{S}_f^c$ to Ω_f, and as a consequence, the corresponding pullback, G_f^*, is a map

$$G_f^* : \Lambda^k(\Omega_f) \to \Lambda^k(\Omega_f \times \mathcal{S}_f^c)$$

and we can express

$$\Lambda^k(\Omega_f \times \mathcal{S}_f^c) = \sum_{j=0}^{k} \Lambda^j(\Omega_f) \otimes \Lambda^{k-j}(\mathcal{S}_f^c).$$

Furthermore, for each $0 \leq j \leq k$, there is a canonical map $\Pi_j : \Lambda^k(\Omega_f \times \mathcal{S}_f^c) \to \Lambda^j(\Omega_f) \otimes \Lambda^{k-j}(\mathcal{S}_f^c)$ such that $\sum_{j=0}^k \Pi_j$ is the identity. We refer to [6, Section 5.1] for more details. As in [6], all the operators $R_{e,f}^k$ will be of the form

$$(R_{e,f}^k u)_\lambda = \int_\Omega (\Pi_j G_f^* u)_\lambda \wedge z_{e,f}, \quad \lambda \in \mathcal{S}_f^c, \tag{2.16}$$

where the functions $z_{e,f}$ are trimmed linear $n-j$ forms for $(e,f) \in \Delta_{j,m}$, and with support in $\Omega_f \cap \Omega_e^E$. The construction of the functions $\{z_{e,f}\}$, given in Section 4 below, will deviate from the corresponding construction given in [6]. In fact, the careful construction of these functions below represents the main tool for obtaining the improved results we derive in this paper, as compared to the results presented in [6].

For pairs $(e,f) \in \Delta_{0,m}$, i.e., when e is a vertex, we define $z_{e,f} = -z_{\langle e,f\rangle}$, where the n-forms $z_{\langle e,f\rangle}$ are defined by (2.12) and (2.13). As a consequence, for any $e \in \Delta_0(f^*)$, we have $R_{e,f}^k = -A_{\langle e,f\rangle}^k$. The functions $\{z_{e,f}\}$ will be constructed to satisfy the relation

$$dz_{e,f} = (-1)^{j+1}(\delta z)_{e,f}, \quad (e,f) \in \Delta_{j,m}, \tag{2.17}$$

for $j \geq 1$, where the generalized coboundary operator defined for pairs of simplices in $\{\Delta_{j,m}\}$ is given by

$$(\delta z)_{e,f} = \sum_{i \in I(e)} (-1)^{\sigma_e(x_i)} z_{e(\hat{x}_i),f}.$$

Remarks. *The identity, $R_{e,f}^k = -A_{\langle e,f\rangle}^k$ and the definition of the δ operator appear to deviate from the corresponding relations in* [6]*. However, as stated above, in the present paper it is more convenient to use a different, but equivalent, labeling of the z functions and R operators as compared to the previous paper. More precisely, the operators $R_{e,f}^k$ introduced above were labeled by the pair $(e, \langle e,f\rangle)$ in* [6]*, and this causes minor differences in the relations above, and also at a few occurrences below.*

As a consequence of the definition of the operators $R_{e,f}^k$, given by (2.16) and (2.17), the operators $R_{e,f}^k$ will satisfy the relation

$$R_{e,f}^{k+1} du = (-1)^j dR_{e,f}^k u - (\delta R^k u)_{e,f}, \quad (e,f) \in \Delta_{j,m},\ 0 \leq j \leq k+1, \tag{2.18}$$

where $\Delta_{j,m}$ is defined to be the emptyset if $j \geq n-m$. Furthermore, $(\delta R^k u)_{e,f}$ is taken to be zero for $e \in \Delta_0$. When $e \in \Delta_{k+1}(f^*)$, $R_{e,f}^k u = 0$ and (2.18) reduces to

$$R_{e,f}^{k+1} du = -(\delta R^k u)_{e,f}, \quad (e,f) \in \Delta_{k+1,m}, \tag{2.19}$$

which is consistent with (2.15). We refer to Section 5.3 of [6] and Section 5 below for more details.

The following lemma generalizes the mapping properties (2.14) of the average operators A_f^k to similar results for the order reduction operators. In fact, this result corresponds to Proposition 5.2 of [6].

Lemma 2.2. *Assume that $(e,f) \in \Delta_{j,m}(\mathcal{T})$.*

i) *If $u \in \Lambda^k(\mathcal{T}_f)$, then $b^{-j} R_{e,f}^k u \in \Lambda^{k-j}(\mathcal{S}_f^c)$.*

ii) *If* $u \in \mathcal{P}_r\Lambda^k(\mathcal{T}_f)$*, then* $b^{-j}R^k_{e,f}u \in \mathcal{P}_r\Lambda^{k-j}(\mathcal{S}^c_f)$*.*
iii) *If* $u \in \mathcal{P}^-_r\Lambda^k(\mathcal{T}_f)$*, then* $b^{-j}R^k_{e,f}u \in \mathcal{P}^-_r\Lambda^{k-j}(\mathcal{S}^c_f)$*.*

Remarks. *The proof of Proposition 5.2 of* [6] *carries over directly to the present setting, even if the definition of the weight functions* $z_{e,f}$ *will be modified below. Therefore, we omit the proof here.*

2.5. **An outline of the construction.** In the present notation, the trace preserving operators C^k_m, introduced in [6, Section 6], admit the representation

$$C^k_m u = \sum_{f\in\Delta_m}\sum_{g\in\bar{\Delta}(f)}(-1)^{|f|-|g|}L^*_g A^k_f u + \sum_{\substack{(e,f)\in\Delta_{j,m-1}\\ 0\le j\le n-m}}(-1)^{j-1}\sum_{g\in\bar{\Delta}(f)}(-1)^{|f|-|g|}\frac{\phi_e}{\rho_g}\wedge L^*_g b^{-j}R^k_{e,f}u. \tag{2.20}$$

where $0 \le m \le n-1$, and where we recall that $R^k_{e,f} = 0$ if $e \in \Delta_j$, $j > k$. The definition of the operator C^k_m contains the rational terms ϕ_e/ρ_g. However, it is established in [6], cf. Lemma 2.2 and Proposition 7.1 of that paper, that the operator C^k_m commutes with the exterior derivative, preserves piecewise smoothness and the piecewise polynomial spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}^-_r\Lambda^k(\mathcal{T})$, and preserves the trace of u on all $f \in \Delta_m$ if $m \ge k$.

Remarks. *The fact that the operator* C^k_m *preserves the trace of* u *on all* $f \in \Delta_m$ *if* $m \ge k$*, can be derived easily from the definition above. In fact, by combining the first term in* (2.20) *and the second term with* $j = 0$*, we obtain the primal operator studied in Section 4 of* [6]*. This operator can be rewritten as*

$$C^k_m(primal) = \sum_{f\in\Delta_m}\sum_{g\in\bar{\Delta}(f)}(-1)^{|f|-|g|}\frac{\rho_f}{\rho_g}L^*_g A^k_f u. \tag{2.21}$$

By arguing as in the proof of [6, Lemma 4.1]*, using the cancellation property with respect to* g*, we can conclude that this operator preserves the trace on all elements of* Δ_m *if* $m \ge k$*. Furthermore, for the second term in* (2.20) *with a fixed* $f \in \Delta_{m-1}$ *and* $e \in \Delta_j(f^*)$*, we can argue in the same way that the resulting function has support in* $\Omega_e \cap \Omega_f = \Omega_{\langle e,f\rangle}$*. As a consequence, if* $j > 0$*, such that the simplex* $\langle e,f\rangle \in \Delta_s$ *for* $s > m$*, these functions have vanishing trace on all* m*-simplices. Hence, the trace property of* C^k_m *follows. The commuting property of the operator* C^k_m *can also be shown directly from the properties of the operators* A^k_f *and* $R^k_{e,f}$*, cf. Section 2.4 above, while the space preserving properties require a deeper analysis, performed in* [6, Section 7]*. In fact, an alternative proof of the space preserving properties follows as a corollary of the analysis given in this paper.*

We will derive the desired decomposition (1.1) of functions $u \in \Lambda^k(\mathcal{T})$ from the telescoping identity

$$u = C^k_0 u + \sum_{m=1}^{n}(C^k_m - C^k_{m-1})u, \tag{2.22}$$

where C_n^k is the identity operator. We have already observed, cf. (1.4), that

$$(C_n^k - C_{n-1}^k)u = u - C_{n-1}^k u = \sum_{f \in \Delta_n} B_f^k u,$$

where $B_f^k u = \mathrm{tr}_f(u - C_{n-1}^k u)$ is a sum of piecewise smooth local bubbles. For the special case $m = 0$, there are no rational functions present in the definition of the operator C_0^k. In fact, by utilizing that the range of the operator $L_\emptyset$ is a single point, i.e., the origin, it follows that $L_\emptyset^*$ maps forms of order greater than zero, to zero. As a consequence, we can represent the operator C_0^k as

$$C_0^k u = \sum_{f \in \Delta_0} \Big(L_f^* A_f^k u - L_\emptyset^* A_f^k u \Big) + (-1)^{k-1} \sum_{e \in \Delta_k} \phi_e \wedge L_\emptyset^* R_{e,\emptyset}^k u. \tag{2.23}$$

In particular, from the definition of the operator $R_{e,\emptyset}^k$, it follows that the second term in (2.23) can be expressed as

$$W^k u := (-1)^{k-1} \sum_{e \in \Delta_k} \phi_e \Big(\int_\Omega u \wedge z_{e,\emptyset} \Big),$$

which is an element of $\mathcal{P}_1^- \Lambda^k(\mathcal{T})$. In other words, W^k is an operator which maps piecewise smooth k-forms into the simplest class of piecewise polynomial k forms, i.e., into trimmed linear forms. In particular, we recall that for $e \in \Delta_k$, the functions $z_{e,\emptyset}$ are elements of $\mathcal{P}_1^- \Lambda^{n-k}(\mathcal{T})$, with support in Ω_e^E. We define the local operators $K_{0,f}^k$ by

$$K_{0,f}^k u = L_f^* A_f^k u - L_\emptyset^* A_f^k u.$$

The functions $K_{0,f}^k u$ will have support on Ω_f, and by using these operators, the identity (2.23) can be written as

$$C_0^k u = \sum_{f \in \Delta_0} K_{0,f}^k u + W^k u.$$

As a consequence of the fact that the operators A_f^k commute with the exterior derivative, it follows that the operators $K_{0,f}^k$ will also commute with d. Furthermore, the operator W^k also commutes with the exterior derivative. In fact, from (2.3), we have

$$dW^k u = (-1)^{k-1} \sum_{e \in \Delta_k} (\partial \phi)_e \Big(\int_\Omega u \wedge z_{e,\emptyset} \Big),$$

and from (2.2), (2.4), (2.17), and the Leibniz rule, we obtain

$$\begin{aligned}\sum_{e \in \Delta_k} (\partial \phi)_e \Big(\int_\Omega u \wedge z_{e,\emptyset} \Big) &= \sum_{e \in \Delta_{k+1}} \phi_e \Big(\int_\Omega u \wedge (\delta z)_{e,\emptyset} \Big) \\ &= (-1)^k \sum_{e \in \Delta_{k+1}} \phi_e \Big(\int_\Omega u \wedge dz_{e,\emptyset} \Big) = - \sum_{e \in \Delta_{k+1}} \phi_e \Big(\int_\Omega du \wedge z_{e,\emptyset} \Big),\end{aligned}$$

which implies that $dW^k = W^{k+1} d$.

We will also show below in Section 8 that W^k can be extended to a bounded linear operator on L^2.

Since we have obtained desired decompositions of the operators $C_n^k - C_{n-1}^k$ and C_0^k, it remains to decompose the functions $(C_m^k - C_{m-1}^k)u$ for $1 \le m \le n-1$, cf. (2.22). In fact, the main contribution of this paper is to show that these operators admit the representation

$$(C_m^k - C_{m-1}^k)u = \sum_{\substack{f \in \Delta_j \\ j=m,m-1}} K_{m,f}^k u, \quad 1 \le m \le n-1, \tag{2.24}$$

cf. Proposition 7.3 below, where the functions $K_{m,f}^k u$ have support in Ω_f. Furthermore, each operator $K_{m,f}^k$ commutes with the exterior derivative and preserves piecewise smoothness and the piecewise polynomial spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$. Our derivation below of the identity (2.24) depends on a careful construction of the family of operators $\{R_{e,f}^k\}$, or more precisely of the functions $\{z_{e,f}\}$ defining these operators, cf. (2.16). As a consequence of the identity (2.24), we obtain the desired decomposition (1.1), since the function $u - W^k u$ can be decomposed into local bubbles of the form

$$u - W^k u = \sum_{f \in \Delta} B_f^k u,$$

where each function $B_f^k u$ has support on Ω_f. More precisely, if we let $K_{n,f}^k = 0$ for each $f \in \Delta_{n-1}$, then

$$B_f^k = K_{m,f}^k + K_{m+1,f}^k, \quad f \in \Delta_m,\ 0 \le m \le n-1, \tag{2.25}$$

while for $f \in \Delta_n$,

$$B_f^k u = \operatorname{tr}_f(u - C_{n-1}^k u) = \operatorname{tr}_f\Big(u - (W^k u + \sum_{\substack{g \in \Delta_m \\ 0 \le m \le n-1}} B_g^k u)\Big). \tag{2.26}$$

We can summarize the main results we will obtain for the construction outlined above in the following theorem.

Theorem 2.3. *For $0 \le k \le n$, there exist operators $W^k : \Lambda^k(\mathcal{T}) \to \mathcal{P}_1^-\Lambda^k(\mathcal{T})$ and for each $f \in \Delta_m(\mathcal{T})$, $0 \le m \le n$, local operators $B_f^k : \Lambda^k(\mathcal{T}) \to \mathring{\Lambda}^k(\mathcal{T}_f)$ such that the decomposition* (1.1) *holds. The operators B_f^k can be extended to bounded operators from $L^2\Lambda^k(\Omega_f)$ to itself if $0 \le m < n$, and from $L^2\Lambda^k(\Omega_f^E)$ to $L^2\Lambda^k(\Omega_f)$ when $m = n$. Furthermore, all these operators commute with the exterior derivative, and satisfy the invariant property* (1.2). *Finally, the decomposition given by* (1.1) *satisfies the stable decomposition property, detailed in Proposition 8.1.*

3. The case of scalar-valued functions

To motivate the general theory developed later in this paper, we will discuss the decomposition (1.1) in the case $k = 0$. In fact, to derive the decomposition (1.1), we only need to establish the identity (2.24) for $1 \le m \le n-1$. A key ingredient in the derivation of (2.24) is to rely on two slightly different representations of the operator C_m^0. For $k = 0$, the expression (2.20) can be written as

$$C_m^0 u = \sum_{f \in \Delta_m} \sum_{g \in \bar{\Delta}(f)} (-1)^{|f|-|g|} \Big[L_g^* A_f^0 u - \sum_{i \in I(f \cap g^*)} \frac{\lambda_i}{\rho_g} L_g^* A_f^0 u \Big], \tag{3.1}$$

where we recall that the operator $R^k_{e,f}u = -A^k_{\langle e,f\rangle}u$ when $(e,f) \in \Delta_{0,m-1}$. Alternatively, we also have

$$C^0_m u = \sum_{f\in\Delta_m} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \frac{\rho_f}{\rho_g} L^*_g A^0_f u, \tag{3.2}$$

cf. (2.21). Motivated by the first term in (3.1), we will define the operators $K^0_{m,f}$ by

$$K^0_{m,f}u = \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} L^*_g A^0_f u, \quad f \in \Delta_m(\mathcal{T}). \tag{3.3}$$

From the properties of the average operators, cf. (2.14), it follows that the operators $K^0_{m,f}u$ have domain of dependence Ω_f, preserve piecewise smoothness and the piecewise polynomial spaces, and have support on Ω_f. The latter property follows from a standard cancellation argument. To see this, consider an index $i \in I(f)$. For each $g \in \Delta(f)$ such that $i \notin I(g)$, we have that the two terms

$$L^*_g A^0_f u, \quad \text{and } L^*_{\langle x_i,g\rangle} A^0_f u$$

cancel, when $\lambda_i(x) = 0$. By repeating this argument, we see that $K^0_{m,f}u \equiv 0$ for all x such that $\lambda_i(x) = 0$ for all $i \in I(f)$. However, this means that the function $K^0_{m,f}u$ has support on Ω_f. From (3.1), it follows that

$$C^0_m u - \sum_{f\in\Delta_m} K^0_{m,f}u = -\sum_{g\in\bar{\Delta}} \sum_{\substack{f\in\Delta_m\\ f\supset g}} (-1)^{|f|-|g|} \sum_{i\in I(f\cap g^*)} \frac{\lambda_i}{\rho_g} L^*_g A^0_f u.$$

However, for each fixed $g \in \bar{\Delta}$, we have

$$\sum_{\substack{f\in\Delta_m\\ f\supset g}} \sum_{i\in I(f\cap g^*)} = \sum_{i\in I(g^*)} \sum_{\substack{f\in\Delta_m\\ f\supset x_i,g}} = \sum_{i\in I(g^*)} \sum_{\substack{f'\in\Delta_{m-1}(x^*_i)\\ f'\supset g}} = \sum_{\substack{f'\in\Delta_{m-1}\\ f'\supset g}} \sum_{i\in I((f')^*)},$$

where we have introduced $f' = f(\hat{x}_i)$. As a consequence, we obtain

$$C^0_m u - \sum_{f\in\Delta_m} K^0_{m,f}u = \sum_{f\in\Delta_{m-1}} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \sum_{i\in I(f^*)} \frac{\lambda_i}{\rho_g} L^*_g A^0_{\langle x_i,f\rangle} u.$$

Furthermore, from (2.13) and (3.2), we also have

$$C^0_{m-1}u = \sum_{f\in\Delta_{m-1}} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \frac{\rho_f}{|f^*|\rho_g} \sum_{i\in I(f^*)} L^*_g A^0_{\langle x_i,f\rangle} u.$$

By combining these representations of C^0_m and C^0_{m-1}, we obtain the identity

$$\begin{aligned}(C^0_m - C^0_{m-1})u - \sum_{f\in\Delta_m} K^0_{m,f}u \\ = \sum_{f\in\Delta_{m-1}} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \rho_g^{-1} \sum_{i\in I(f^*)} \Big[\Big(\lambda_i - \frac{\rho_f}{|f^*|}\Big) L^*_g A^0_{\langle x_i,f\rangle} u\Big].\end{aligned} \tag{3.4}$$

Therefore, if we define the operators $K^0_{m,f}$ by

$$K^0_{m,f}u = \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \rho_g^{-1} \sum_{i\in I(f^*)} \Big[\Big(\lambda_i - \frac{\rho_f}{|f^*|}\Big) L^*_g A^0_{\langle x_i,f\rangle} u\Big], \tag{3.5}$$

for $f \in \Delta_{m-1}$, we obtain that (3.4) simply reads

$$(C^0_m - C^0_{m-1})u = \sum_{\substack{f\in\Delta_j\\ j=m,m-1}} K^0_{m,f}u, \tag{3.6}$$

which is the desired identity (2.24) for $k = 0$.

It remains to see that the operators $K^0_{m,f}$, $f \in \Delta_{m-1}$, have the desired properties. Again, by the properties of the map A^0_f, the operators $L^*_g A^0_{\langle x_i,f\rangle}$ and hence $K^0_{m.f}$ will have domain of dependence Ω_f. In addition, we need to show that the operators $K^0_{m,f}$, given by (3.5), preserve piecewise smoothness and piecewise polynomials, and that the target function has local support. In fact, the property that the functions $K^0_{m,f}u$ are supported on the macroelements Ω_f follows by a cancellation argument similar to the one given above. However, formula (3.5) contains rational functions, and therefore, at first glance, it seems unlikely that the corresponding operators $K^0_{m,f}$ will preserve piecewise smoothness. On the other hand, since $\operatorname{supp} K^0_{m,f}u \subset \Omega_f$, we can restrict the analysis of the functions $K^0_{m,f}u$ to the domain Ω_f, and on this domain we will rely on an alternative representation of the operators.

To derive the alternative representation of $K^0_{m,f}$, we recall that when $f \in \Delta_{m-1}$, the manifold f^* is of dimension $n - m$, and since we assume that $m \le n - 1$, the manifold f^* is of dimension greater or equal to one. For $x_i \in f^*$, we define a corresponding piecewise linear function, $\beta_{x_i} = \beta_{x_i}(f)$ by

$$\beta_{x_i} = \lambda_i - \frac{\rho_f}{|f^*|}.$$

The collection $\{\beta_{x_i}(f)\}_{i\in I(f^*)}$ can be seen as an element in $\mathcal{C}_0(f^*) \otimes \mathcal{P}_1(\mathcal{T})$. Furthermore, on the domain Ω_f, we have that

$$\partial_0\beta = \sum_{i\in I(f^*)} \beta_{x_i} = \Big(\sum_{i\in I(f^*)} \lambda_i\Big) - \rho_f = 0,$$

where $\partial_0 = \partial_0(f^*)$. From Lemma 2.1, we therefore conclude that there is a unique element $\mu = \mu(f) = \{\mu_e(f)\}_{e\in\Delta_1(f^*)} \in \mathcal{C}_1(f^*) \otimes \mathcal{P}_1(\mathcal{T})$, such that the identities

$$(\partial\mu)_{x_i} = \lambda_i - \frac{\rho_f}{|f^*|} = \beta_{x_i}, \quad i \in I(f^*), \quad \text{and } \delta\mu = 0, \tag{3.7}$$

hold on Ω_f, where $\partial = \partial_1(f^*)$ and $\delta = \delta_1(f^*)$. Note that when $n - m = 1$, there is no space $\mathcal{C}_2(f^*)$ (see Figure 2.1) and hence we define $\delta_1(f^*)$ by (2.9).

We can use the piecewise linear functions, $\{\mu_e(f)\}_{e\in\Delta_1(f^*)}$, to obtain an alternative representation of the operators $K^0_{m,f}$. In fact, we have, using (2.2) and (2.19), that

$$\begin{aligned}\sum_{i\in I(f^*)} (\partial\mu)_{x_i} \wedge L^*_g A^0_{\langle x_i,f\rangle}u &= -\sum_{i\in I(f^*)} (\partial\mu)_{x_i} \wedge L^*_g R^0_{x_i,f}u\\ &= -\sum_{e\in\Delta_1(f^*)} \mu_e \wedge L^*_g(\delta R^0u)_{e,f} = \sum_{e\in\Delta_1(f^*)} \mu_e \wedge L^*_g R^1_{e,f}du.\end{aligned}$$

As a consequence, we obtain that the alternative representation,

$$K^0_{m,f}u = \sum_{g\in\bar\Delta(f)} (-1)^{|f|-|g|} \sum_{e\in\Delta_1(f^*)} \Big[\mu_e(f) \wedge L^*_g b^{-1} R^1_{e,f}du\Big],$$

holds on Ω_f. However, from this alternative representation, the mapping properties of the operators $K^0_{m,f}$ follow directly from the corresponding properties of the operators $R^1_{e,f}$, given in Lemma 2.2.

We close the discussion given in this section by summarizing the results we have obtained for the operators $K^0_{m,f}$.

Lemma 3.1. *Let $1 \le m \le n-1$ and $f \in \Delta_j(\mathcal{T})$, where $j = m, m-1$. Assume that the corresponding operators $K^0_{m,f}$ are defined by* (3.3) *and* (3.5). *Then the identity* (3.6) *holds,* $\operatorname{supp} K^0_{m,f} u \subset \Omega_f$, *and the operators $K^0_{m,f}$ have the mapping properties*

$$K^0_{m,f}(\Lambda^0(\mathcal{T}_f)) \subset \Lambda^0(\mathcal{T}_f), \qquad K^0_{m,f}(\mathcal{P}_r(\mathcal{T}_f)) \subset \mathcal{P}_r(\mathcal{T}_f).$$

4. The local structure of the mesh

Key tools for decomposing the operators $C^k_m - C^k_{m-1}$ into a sum of local operators with local target space will be various local functions derived from the given mesh, $\mathcal{T}$. To describe these functions, we introduce the space $\mathcal{P}^-_1\Lambda^k(\mathcal{T}, f^*)$ as the subset of $\mathcal{P}^-_1\Lambda^k(\mathcal{T})$ corresponding to degrees of freedom on f^*. More precisely,

$$\mathcal{P}^-_1\Lambda^k(\mathcal{T}, f^*) = \operatorname{span}_{e\in\Delta_k(f^*)} \phi_e.$$

4.1. The functions $\mu_e(f)$.

Let m be an index, $1 \le m \le n$, and $f \in \Delta_{m-1}$, such that the associated manifold, f^*, will be of dimension $n - m$. In the special case when $m = n$, the manifold f^* will be of dimension zero, and consist of one or two points depending on the location of f relative to the boundary of Ω. Since this case is special, we will first assume that $m \le n - 1$ such that the manifold f^* is a least one dimensional.

In the previous section, we constructed functions $\{\mu_e(f)\}_{e\in\Delta_1(f^*)}$ such that the identity (3.7) holds, where we can consider the collection $\{\mu_e(f)\}$ as an element of $\mathcal{C}_1(f^*) \otimes \mathcal{P}^-_1\Lambda^0(\mathcal{T}, f^*)$. In fact, we will construct the collection $\{\mu_e(f)\}_{e\in\Delta_j(f^*)} \in \mathcal{C}_j(f^*) \otimes \mathcal{P}^-_1\Lambda^{j-1}(\mathcal{T}, f^*)$ for $1 \le j \le n$. This will be done by an inductive process with respect to j. For $e \in \Delta_0(f^*)$, we define $\mu_e(f)$ to be the constant $-1/|f^*|$, such that $\{\mu_e(f)\}_{e\in\Delta_0(f^*)}$ can be viewed as an element of $\mathcal{C}_0(f^*) \otimes \mathcal{P}^-_1\Lambda^{-1}(\mathcal{T}, f^*)$, where $\mathcal{P}^-_1\Lambda^{-1}(\mathcal{T}, f^*)$ is identified as $\mathbb{R}$. Below we will apply the difference operators, $\partial(f^*)$ and $\delta(f^*)$, to elements of $\mathcal{C}_j(f^*) \otimes \mathcal{P}^-_1\Lambda^k(\mathcal{T}, f^*)$. This is done with respect to $\mathcal{C}_j(f^*)$, while the polynomial space $\mathcal{P}^-_1\Lambda^k(\mathcal{T}, f^*)$ is considered fixed. For example, the operator $\partial(f^*)$ maps elements of $\mathcal{C}_j(f^*) \otimes \mathcal{P}^-_1\Lambda^k(\mathcal{T}, f^*)$ to $\mathcal{C}_{j-1}(f^*) \otimes \mathcal{P}^-_1\Lambda^k(\mathcal{T}, f^*)$. On the other hand, the exterior derivative, d, will map elements of $\mathcal{C}_j(f^*) \otimes \mathcal{P}^-_1\Lambda^k(\mathcal{T}, f^*)$ to $\mathcal{C}_j(f^*) \otimes \mathcal{P}^-_1\Lambda^{k+1}(\mathcal{T}, f^*)$.

In addition to the collection of functions $\{\mu_e(f)\}$, we will introduce the associated collection of functions $\{\beta_e\} = \{\beta_e(f)\}_{e\in\Delta_j(f^*)}$ as elements of $\mathcal{C}_j(f^*) \otimes \mathcal{P}^-_1\Lambda^j(\mathcal{T})$ for $0 \le j \le n - m$. The function $\beta_e(f)$ is defined from the corresponding function $\mu_e(f)$ by

$$\beta_e := \rho_f^{j+1} d\Big(\frac{\mu_e}{\rho_f^j}\Big) + (-1)^j \phi_e, \quad e \in \Delta_j(f^*), \tag{4.1}$$

if $1 \le j \le n-m$. In fact, if we let the exterior derivative d_{-1} be the inclusion map from constants to $\mathcal{P}_1\Lambda^0(\mathcal{T}, f^*)$, then the definition (4.1) also holds when $j = 0$, cf. (3.7). We observe that if we restrict to the domain Ω_f, such that $\rho_f = \sum_{i \in I(f^*)} \lambda_i$, then $\beta_e(f)$ also admits the representation

$$\beta_e = \sum_{i \in I(f^*)} (\lambda_i d - j d\lambda_i \wedge)\mu_e + (-1)^j \phi_e, \quad e \in \Delta_j(f^*). \tag{4.2}$$

It then follows that

$$\mu_e \in \mathcal{P}_1^- \Lambda^{j-1}(\mathcal{T}, f^*) \implies \mathrm{tr}_{\Omega_f} \beta_e \in \mathcal{P}_1^- \Lambda^j(\mathcal{T}_f, f^*),$$

where $\mathcal{P}_1^- \Lambda^j(\mathcal{T}_f, f^*) = \mathrm{tr}_{\Omega_f} \mathcal{P}_1^- \Lambda^j(\mathcal{T}, f^*)$. Furthermore, the relation (3.7) can be rephrased as $\beta_e = (\partial(f^*)\mu)_e$ on Ω_f for $e \in \Delta_0(f^*)$.

Lemma 4.1. *Let $f \in \Delta_{m-1}(\mathcal{T})$, where $1 \le m \le n-1$, and j an index such that $1 \le j \le n-m$. Assume that $\{\mu_e\} = \{\mu_e(f)\} \in \mathcal{C}_s(f^*) \otimes \mathcal{P}_1^- \Lambda^{s-1}(\mathcal{T}, f^*)$ has been defined for $s = j-1, j$ such that the identity*

$$\beta_e(f) = (\partial(f^*)\mu)_e, \tag{4.3}$$

holds on Ω_f for $e \in \Delta_{j-1}(f^*)$. Then $(\partial(f^*)\beta)_e = 0$ for all $e \in \Delta_{j-1}(f^*)$, and if $j < n-m$, there exist $\{\mu_e\} = \{\mu_e(f)\} \in \mathcal{C}_{j+1}(f^*) \otimes \mathcal{P}_1^- \Lambda^j(\mathcal{T}, f^*)$ such that (4.3) holds on Ω_f for all $e \in \Delta_j(f^*)$. Furthermore, $\{\mu_e\} \in C_{j+1}(f^*) \otimes \mathcal{P}_1^- \Lambda^j(\mathcal{T}, f^*)$ is uniquely determined by the condition $\delta(f^*)\mu = 0$.

Proof. Throughout the proof, all the identities should be considered to hold on the domain Ω_f, and the operators ∂ and δ are defined with respect to f^*. By assumption, we have

$$\beta_e = \rho^j d\Big(\frac{\mu_e}{\rho^{j-1}}\Big) + (-1)^{j-1}\phi_e = (\partial_j \mu)_e, \quad e \in \Delta_{j-1}(f^*),$$

where $\rho = \rho_f$. Since $d^2 = 0$ and d commutes with ∂_j, this gives

$$(\partial_j d\Big(\frac{\mu}{\rho^j}\Big))_e = (-1)^{j-1} d\Big(\frac{\phi_e}{\rho^j}\Big), \quad e \in \Delta_{j-1}(f^*).$$

Hence, it follows from (4.1) that for $e \in \Delta_{j-1}(f^*)$

$$(\partial_j \beta)_e = \rho^{j+1}(\partial_j d\Big(\frac{\mu}{\rho^j}\Big))_e + (-1)^j(\partial_j \phi)_e = (-1)^{j-1}[\rho^{j+1} d\Big(\frac{\phi_e}{\rho^j}\Big) - (\partial_j \phi)_e].$$

However, a direct computation, using (2.5) and $\rho_f = \sum_{i \in I(f^*)} \lambda_i$ on Ω_f, shows

$$\rho^{(j+1)} d\Big(\frac{\phi_e}{\rho^j}\Big) = \sum_{i \in I(f^* \cap e^*)} (\lambda_i d - j d\lambda_i \wedge)\phi_e = \sum_{i \in I(f^* \cap e^*)} \phi_{[x_i, e]} = (\partial_j \phi)_e.$$

As a consequence, $(\partial_j \beta)_e = 0$ for $e \in \Delta_{j-1}(f^*)$. If $j < n-m$, it follows from Lemma 2.1 that there exist a uniquely determined $\{\mu_e\} \in \mathcal{C}_{j+1}(f^*) \otimes \mathcal{P}_1^- \Lambda^j(\mathcal{T}_f, f^*)$ such that $\beta_e = (\partial_{j+1}\mu)_e$ on Ω_f, for all $e \in \Delta_j(f^*)$ and $\delta_{j+1}\mu = 0$. Furthermore, each function $\mu_e \in \mathcal{P}_1^- \Lambda^j(\mathcal{T}_f, f^*)$ can be uniquely extended to a function in $\mathcal{P}_1^- \Lambda^j(\mathcal{T}, f^*)$. □

It remains to discuss the case $m = n$. In this case, $f \in \Delta_{n-1}$, so f^* will only consist of one or two points, and there is no element in $\Delta_1(f^*)$. In fact, we will have $|f^*| = 1$ if f is a boundary simplex, and otherwise $|f^*| = 2$. On the other hand, by adopting the interpretation above of d_{-1} as the inclusion operator and $\mu_e(f) = -1/|f^*|$ for $e \in \Delta_0(f^*)$, we obtain

$$\beta_e(f) := -\frac{\rho_f}{|f^*|} + \phi_e, \quad e \in \Delta_0(f^*),$$

and $\partial_0(f^*)\beta = 0$.

From Lemma 4.1 and an induction argument with respect to j, we obtain the following result.

Corollary 4.2. *Let $f \in \Delta_{m-1}(\mathcal{T})$, where $1 \le m \le n$. For all $e \in \Delta_j(f^*)$, $0 \le j \le n-m$, there exists functions $\mu_e(f) \in \mathcal{P}_1^- \Lambda^{j-1}(\mathcal{T}, f^*)$ and $\beta_e(f) \in \mathcal{P}_1^- \Lambda^j(\mathcal{T})$, uniquely defined by the inductive procedure above, satisfying $\partial_j(f^*)\beta = 0$ on Ω_f. Furthermore, if $j < n-m$, then the identity (4.3) holds on Ω_f for all $e \in \Delta_j(f^*)$.*

Remarks. *An alternative view of the construction of the functions $\{\mu_e(f)\}$ given above can be given by expanding the function $\mu_e \in \mathcal{P}_1^- \Lambda^{j-1}(\mathcal{T}, f^*)$ in the form*

$$\mu_e = \sum_{e' \in \Delta_{j-1}(f^*)} a_{e,e'} \phi_{e'}, \tag{4.4}$$

where the real coefficients $\{a_{e,e'}\}$ can be identified with an element $a_e \in \mathcal{C}_{j-1}$. It follows from (2.3), (2.5), and (4.2), that if we restrict to Ω_f, then the function $\beta_e = \beta_e(f)$ admits the representation

$$\beta_e - (-1)^j \phi_e = \sum_{\substack{e' \in \Delta_j(f^*) \\ i \in I(e')}} (-1)^{\sigma_{e'}(x_i)} a_{e,e'(\hat{x}_i)} \phi_{e'} = \sum_{e' \in \Delta_j(f^*)} (\delta a_{e,\cdot})_{e'} \phi_{e'}, \tag{4.5}$$

for $e \in \Delta_j(f^*)$. As a consequence, the equation (4.3) for $e \in \Delta_j(f^*)$ can be represented by the algebraic system

$$(\partial_{j+1} a_{\cdot,e'})_e = (\delta_{j-1} a_{e,\cdot})_{e'} + (-1)^j 1_{e,e'}, \quad e' \in \Delta_j(f^*), \tag{4.6}$$

where $\partial = \partial(f^*)$, and $1_{e,e'} = 1$ if $e' = e$ and equals zero otherwise. Furthermore, the condition $\delta_{j+1}(f^*)\mu = 0$ is equivalent to

$$\delta_{j+1} a_{\cdot,e'} = 0, \quad e' \in \Delta_j(f^*). \tag{4.7}$$

If $(e, f) \in \Delta_{j,m-1}$ and $g \in \bar{\Delta}(f)$, then $\rho_g - \rho_f = \sum_{i \in I(f \cap g^*)} \lambda_i$, which leads to

$$\rho_g^{j+1} d\Big(\frac{\mu_e(f)}{\rho_g^j}\Big) = \rho_f^{j+1} d\Big(\frac{\mu_e(f)}{\rho_f^j}\Big) + \sum_{i \in I(f \cap g^*)} \big(\lambda_i d - j d\lambda_i \wedge\big) \mu_e(f).$$

In the analysis below, the functions $\psi_{e,g}(f) \in \mathcal{P}_1^- \Lambda^j(\mathcal{T}, g^*)$, defined by

$$\psi_{e,g}(f) = (-1)^{j-1} \sum_{i \in I(f \cap g^*)} \big(\lambda_i d - j d\lambda_i \wedge\big) \mu_e(f), \tag{4.8}$$

will be useful. We observe that $\psi_{e,f}(f) = 0$, and it follows from (4.1), that

$$\rho_g^{j+1} d\Big(\frac{\mu_e(f)}{\rho_g^j}\Big) + (-1)^j [\phi_e + \psi_{e,g}(f)] = \beta_e(f). \tag{4.9}$$

Also observe that in the special case when $e \in \Delta_0(f^*)$, then

$$\psi_{e,g}(f) = \frac{1}{|f^*|} \sum_{i \in I(f \cap g^*)} \lambda_i = \frac{1}{|f^*|}(\rho_g - \rho_f). \tag{4.10}$$

4.2. **Construction of the weight functions $z_{e,f}$.** We recall from Section 2.4 that the weight functions $z_{e,f}$ are an essential ingredient for the construction of the order reduction operators $R^k_{e,f}$. This section is devoted to an inductive process for constructing the functions $z_{e,f}$. In fact, as a preliminary step, we will first construct a family of local functions $w_{e,f}$, and then the functions $z_{e,f}$ will be constructed as

$$z_{e,f} = (\delta^+ w)_{e,f}. \tag{4.11}$$

Here the operator δ^+ is a variant of the coboundary operator defined for pairs of simplices, given by

$$(\delta^+ w)_{e,f} = \sum_{i \in I(e)} (-1)^{\sigma_e(x_i)} w_{e(\hat{x}_i),\langle x_i, f\rangle}, \quad (e,f) \in \Delta_{j,m}.$$

This operator will satisfy the complex property, $(\delta^+)^2 = 0$, and from [6, Lemma 5.2], we recall that $\delta \circ \delta^+ = -\delta^+ \circ \delta$. For the construction below, we will utilize exactness of the complex of trimmed linear forms with support on Ω_f. Recall that in Section 1, we introduced the space $\mathring{\mathcal{P}}_1^- \Lambda^k(\mathcal{T}_f)$ as the subspace of $\mathcal{P}_1^- \Lambda^k(\mathcal{T})$ consisting of functions which vanish on $\Omega \setminus \Omega_f$. However, if f is a boundary simplex, then functions in this space will in general not have vanishing trace on the boundary of Ω_f. Therefore, we introduce the notation $\mathring{\mathring{\mathcal{P}}}{}_1^- \Lambda^k(\mathcal{T}_f)$ to denote the subspace of $\mathring{\mathcal{P}}_1^- \Lambda^k(\mathcal{T}_f)$ with vanishing trace on $\partial\Omega_f$. The two spaces are equal if f is an interior simplex, but in general $\mathring{\mathring{\mathcal{P}}}{}_1^- \Lambda^k(\mathcal{T}_f) \subset \mathring{\mathcal{P}}_1^- \Lambda^k(\mathcal{T}_f)$. We also recall from (2.4) that if $w = \sum_{g \in \Delta_j} c_g \phi_g \in \mathcal{P}_1^- \Lambda^j(\mathcal{T})$, then $dw = \sum_{g \in \Delta_{j+1}} (\delta_j c)_g \phi_g$. In particular, if w has support on Ω_f for $f \in \Delta$, then the sum can be restricted to all simplices g such that $g \supset f$. Motivated by this, we define $d_f^* : \mathring{\mathring{\mathcal{P}}}{}_1^- \Lambda^j(\mathcal{T}_f) \to \mathring{\mathring{\mathcal{P}}}{}_1^- \Lambda^{j-1}(\mathcal{T}_f)$, by

$$d_f^* w = \sum_{\substack{g \in \Delta_{j-1} \\ g \supset f}} (\partial_j c)_g \phi_g. \tag{4.12}$$

Below we will utilize exactness of the complex $(\mathring{\mathring{\mathcal{P}}}{}_1^- \Lambda(\mathcal{T}_f), d)$ to conclude that a function $w \in \mathring{\mathring{\mathcal{P}}}{}_1^- \Lambda^j(\mathcal{T}_f)$ is uniquely determined by dw and $d_f^* w$.

The functions $w_{e,f}$ will be defined inductively with respect to m for all pairs $(e,f) \in \Delta_{j,m}$, for $0 \le m \le n$ and $-1 \le j < n - m$ as functions in $\mathring{\mathring{\mathcal{P}}}{}_1^- \Lambda^{n-j-1}(\mathcal{T}_f)$. We start the induction process with $m = n$, and hence only $j = -1$ is allowed. The set $\Delta_{-1,n}$ consists of pairs of the form $(\emptyset, f)$, where $f \in \Delta_n$. In this case, we define $w_{\emptyset,f} = -z_f = -(\kappa_f/|\Omega_f|)\mathsf{vol}$. For the general case, when $0 \le m < n$, we will use a variational approach utilizing tensor product spaces of the form $\mathcal{P}_1^- \Lambda^j(\mathcal{T}, f^*) \otimes \mathcal{P}_1^- \Lambda^{n-j-1}(\mathcal{T})$, i.e., we consider products of elements in the two functions spaces and with independent spatial variables. We assume, as an induction hypothesis, that $w_{e,f} \in \mathring{\mathring{\mathcal{P}}}{}_1^- \Lambda^{n-j-1}(\mathcal{T}_f)$ for all $(e,f) \in \Delta_{j,m}$ and $-1 \le j < n - m$, have already been constructed. Furthermore, we assume that these functions satisfy

$$(\delta^+ dw)_{e,f} \in \text{range}(\delta), \quad (e,f) \in \Delta_{n-m,m-1}(f^*). \tag{4.13}$$

In the case $m = n$, the functions of the form $w_{\emptyset,\langle x_i,f\rangle}$, involved in (4.13), have support in $\Omega_{\langle x_i,f\rangle} \subset \Omega_f$. Since we will utilize exactness of the complex consisting of trimmed differential forms with boundary conditions, the exterior derivative in (4.13), in the case $m = n$, should be interpreted as the integral, and for $(x_i, f) \in \Delta_{0,n-1}$, we have

$$(\delta^+ dw)_{x_i,f} = d(\delta^+ w)_{x_i,f} = \int_\Omega w_{\emptyset,\langle x_i,f\rangle} = -1 = \int_\Omega w_{\emptyset,f} = (\delta dw)_{x_i,f}.$$

Therefore, property (4.13) holds initially.

For a fixed $f \in \Delta_{m-1}$ and $-1 \le j < n-m$, $w_{e,f} \in \mathcal{P}_1^-\Lambda^{n-j-1}(\mathcal{T})$ is defined by

$$\sum_{e\in\Delta_j(f^*)} \phi_e \otimes w_{e,f} = (-1)^j \sum_{e\in\Delta_{j+1}(f^*)} \mu_e(f) \otimes (\delta^+ w)_{e,f}, \tag{4.14}$$

where we observe that all functions on the right hand side are already constructed. Since $\mu_e(f) \in \mathcal{P}_1^-\Lambda^j(\mathcal{T}, f^*)$ for $e \in \Delta_{j+1}(f^*)$, we can view the right hand side of (4.14) as an element of $\mathcal{P}_1^-\Lambda^j(\mathcal{T}, f^*) \otimes \mathcal{P}_1^-\Lambda^{n-j-1}(\mathcal{T})$, i.e, it is a trimmed linear j-form with values in $\mathcal{P}_1^-\Lambda^{n-j-1}(\mathcal{T})$. Therefore, the coefficients $w_{e,f}$ of the left hand side are uniquely determined as functions in $\mathcal{P}_1^-\Lambda^{n-j-1}(\mathcal{T})$. Furthermore, since all the domains of the form $\{\Omega_{x_i,f}\}$, $i \in I(f^*)$ are contained in Ω_f, we can conclude from the induction hypothesis that $(\delta^+ w)_{e,f} \in \mathring{\mathcal{P}}_1^-\Lambda^{n-j-1}(\mathcal{T}_f)$ and hence the function $w_{e,f} \in \mathring{\mathcal{P}}_1^-\Lambda^{n-j-1}(\mathcal{T}_f)$ for $(e,f) \in \Delta_{j,m-1}$. In particular, since $\mu_e(f) = -1/|f^*|$ for $e \in \Delta_0(f^*)$, we obtain that the function $w_{\emptyset,f}$, $f \in \Delta_{m-1}$ satisfies the recurrence relation

$$w_{\emptyset,f} = \frac{1}{|f^*|} \sum_{i\in I(f^*)} w_{\emptyset,\langle x_i,f\rangle}.$$

Let $0 \le j \le n-m$. For the discussion below, it will be useful to observe that the function just defined satisfies

$$\begin{aligned}
\sum_{e\in\Delta_j(f^*)} \phi_e \otimes (\delta w)_{e,f} &= \sum_{e\in\Delta_{j-1}(f^*)} (\partial\phi)_e \otimes w_{e,f} \\
&= \sum_{i\in I(f^*)} (\lambda_i d - j d\lambda_i \wedge) \sum_{e\in\Delta_{j-1}(f^*)} \phi_e \otimes w_{e,f} \\
&= (-1)^{j-1} \sum_{i\in I(f^*)} (\lambda_i d - j d\lambda_i \wedge) \sum_{e\in\Delta_j(f^*)} \mu_e \otimes (\delta^+ w)_{e,f},
\end{aligned}$$

where we have used (2.2) and (2.5), in addition to (4.14). Alternatively, from (4.2) we have

$$\sum_{e\in\Delta_j(f^*)} \Big[\phi_e \otimes (\delta w)_{e,f} - \Big(\phi_e + (-1)^{j-1}\beta_e\Big) \otimes (\delta^+ w)_{e,f}\Big] = 0, \tag{4.15}$$

where $f \in \Delta_{m-1}$ and $0 \le j \le n-m$, and where the spatial variable for ϕ_e and β_e is restricted to Ω_f.

The set up above defines the functions $w_{e,f}$ for all $(e,f) \in \Delta_{j,m-1}$, where $-1 \le j < n-m$, from the corresponding functions defined for elements in $\Delta_{j,m}$. However, we will also need the functions $w_{e,f}$ for $(e,f) \in \Delta_{n-m,m-1}$. In this case, the right hand side of (4.14) is not well defined since there are no elements in $\Delta_{n-m,m}$. Instead, for $f \in \Delta_{m-1}(\mathcal{T})$ and $j = n - m$, we will require $w_{e,f}$ to satisfy

$$\sum_{e\in\Delta_j(f^*)} \phi_e \otimes dw_{e,f} = \sum_{e\in\Delta_j(f^*)} \beta_e(f) \otimes (\delta^+ w)_{e,f}, \tag{4.16}$$

and $d_f^* w_{e,f} = 0$. In (4.16), the spatial variables are restricted to $\Omega_f \times \Omega$, i.e., ϕ_e and $\beta_e(f)$ mean the restrictions of these quantities to Ω_f. It follows from the induction hypothesis and the fact that $\mathrm{tr}_{\Omega_f} \beta_e(f) \in \mathcal{P}_1^- \Lambda^{n-m}(\mathcal{T}_f, f^*)$, that the right hand side of (4.16) can be viewed as an element of $\mathcal{P}_1^- \Lambda^{n-m}(\mathcal{T}_f, f^*) \otimes \mathcal{P}_1^- \Lambda^m(\mathcal{T})$. Furthermore, from the hypothesis (4.13), combined with (2.2), we obtain that

$$\sum_{e\in\Delta_j(f^*)} \beta_e(f) \otimes (\delta^+ dw)_{e,f} = 0,$$

since $\partial\beta(f) = 0$. As a consequence, from the exactness of the complex $(\mathring{\mathcal{P}}_1^- \Lambda(\mathcal{T}_f), d)$, we obtain that there exist elements $w_{e,f} \in \mathring{\mathcal{P}}_1^- \Lambda^{m-1}(\mathcal{T}_f)$ such that (4.16) holds. Finally, since we require $d_f^* w_{e,f} = 0$, the functions $w_{e,f}$ are uniquely determined.

To complete the induction argument, we need to verify that the assumption (4.13) is preserved by the induction step, i.e., that the identity (4.13) holds with m replaced by $m - 1$. However, by combining (4.15) and (4.16), we obtain that the identity

$$dw_{e,f} = (-1)^{j+1}[(\delta w)_{e,f} - (\delta^+ w)_{e,f}] \tag{4.17}$$

holds for $(e, f) \in \Delta_{j,m-1}$, $j = n - m$. As a further consequence of the complex property of δ^+, and the identity $\delta^+ \circ \delta = -\delta \circ \delta^+$, we then obtain

$$(\delta^+ dw)_{e,f} = (-1)^{n-m+1}(\delta \circ \delta^+ w)_{e,f},$$

for $(e, f) \in \Delta_{n-m+1,m-2}$. Hence, property (4.13) at level $m - 1$ is verified.

We summarize the properties of the construction above.

Lemma 4.3. *The inductive procedure above uniquely specifies the functions* $w_{e,f} \in \mathring{\mathcal{P}}_1^- \Lambda^{n-j-1}(\mathcal{T}_f)$ *for all* $(e, f) \in \Delta_{j,m}$*, where* $0 \le m \le n$ *and* $-1 \le j < n - m$*.*

Next we will establish that the functions $w_{e,f}$, introduced above, satisfy the identity (4.17) more generally, i.e., not only for $(e, f) \in \Delta_{n-m,m-1}$.

Lemma 4.4. *The functions* $w_{e,f}$ *satisfy the identity* (4.17) *for all* $(e, f) \in \Delta_{j,m}$*, where* $0 \le m \le n - 1$ *and* $0 \le j < n - m$*.*

Proof. The proof will be done by induction with respect to m. For $m = n - 1$, the only possible value of j is $j = 0$. In this case, the desired identity has already been verified above. Next, we assume that the identity holds for all $(e, f) \in \Delta_{j,m}$, where $0 \le j < n - m$. Note that the case $(e, f) \in \Delta_{n-m,m-1}$ is already established above. Therefore, we can assume that $j < n - m$, and from the support property of the functions $w_{e,f}$, it is enough to show this identity on Ω_f. It follows from (4.15) that the desired identity will follow if we can show that

$$\sum_{e\in\Delta_j(f^*)} \beta_e \otimes (\delta^+ w)_{e,f} = \sum_{e\in\Delta_j(f^*)} \phi_e \otimes dw_{e,f}, \quad f \in \Delta_{m-1}, \tag{4.18}$$

when ϕ_e and β_e are restricted to Ω_f. If $0 \le j < m-n$, then from Corollary 4.2 we have the identity $\beta_e = (\partial\mu)_e$ which gives

$$\sum_{e\in\Delta_j(f^*)} \beta_e \otimes (\delta^+ w)_{e,f} = \sum_{e\in\Delta_j(f^*)} (\partial\mu)_e \otimes (\delta^+ w)_{e,f}$$
$$= -\sum_{e\in\Delta_{j+1}(f^*)} \mu_e \otimes (\delta^+ \circ \delta w)_{e,f} = (-1)^j \sum_{e\in\Delta_{j+1}(f^*)} \mu_e \otimes (\delta^+ dw)_{e,f}$$
$$= \sum_{e\in\Delta_j(f^*)} \phi_e \otimes dw_{e,f},$$

where we have used the fact that $\delta \circ \delta^+ = -\delta^+ \circ \delta$, (4.14), and the induction hypothesis (4.17) with $f' = \{x_i, f\} \in \Delta_m$ and $e' = e(\hat{x}_i) \in \Delta_j((f')^*)$. This establishes the identity (4.18), and therefore the proof is completed. □

The desired weight functions $z_{e,f}$ are defined from the corresponding w functions by the relation (4.11). More precisely, the functions $z_{e,f}$ are defined by (4.11) for $(e,f) \in \Delta_{j,m}$ for $-1 \le m < n$ and $0 \le j < n-m$. In particular, for $(e,f) \in \Delta_{0,n-1}$, we have $z_{e,f} = -z_{\langle e,f\rangle}$. Recall also that a key property of these functions is that they satisfy the identity (2.17), i.e., $dz = (-1)^{j+1}\delta z$ for $(e,f) \in \Delta_{j,m}$.

Lemma 4.5. *The functions $z_{e,f}$, where $(e,f) \in \Delta_{j,m}$ for $-1 \le m \le n-1$ and $0 \le j < n-m$ belong to $\mathcal{P}_1^-\Lambda^{n-j}(\mathcal{T})$, and with support in $\Omega_f \cap \Omega_e^E$. In addition, $z_{e,f}$ has vanishing trace on the boundary of Ω. Furthermore, for $j > 0$, the identities* (2.17) *and $(\delta^+ z)_{e,f} = 0$ hold.*

Proof. That the functions $z_{e,f}$ belong to $\mathcal{P}_1^-\Lambda^{n-j}(\mathcal{T})$ follows from the fact that $w_{e,f}$ are elements of $\mathcal{P}_1^-\Lambda^{n-j-1}(\mathcal{T})$. Furthermore, since δ^+ satisfies the complex property, we obtain that $\delta^+ z = 0$, and from (4.17) we have

$$(dz)_{e,f} = (\delta^+ dw)_{e,f} = (-1)^j(\delta^+ \circ \delta w)_{e,f} = (-1)^{j+1}(\delta \circ \delta^+ w)_{e,f} = (-1)^{j+1}(\delta z)_{e,f},$$

where, as above, we have used the fact that $\delta \circ \delta^+ = -\delta^+ \circ \delta$.

From the support property of the w functions given in Lemma 4.3, we have for $(e,f) \in \Delta_{j,m}$ and $i \in I(e)$ that

$$\operatorname{supp} w_{e(\hat{x}_i),\langle x_i,f\rangle} \subset \Omega_{\langle x_i,f\rangle} = \Omega_f \cap \Omega_{x_i},$$

which implies that

$$\operatorname{supp} z_{e,f} \subset \bigcup_{i\in I(e)} \left(\Omega_f \cap \Omega_{x_i}\right) = \Omega_f \cap \Omega_e^E.$$

Finally, since all functions $\{w_{e,f}\}$ have vanishing trace on the boundary of Ω, this property will also hold for all the functions $\{z_{e,f}\}$. □

Recall that the functions $\psi_{e,g}(f)$, defined by (4.8), where $(e,f) \in \Delta_{j,m-1}$ and $g \in \bar{\Delta}(f)$, satisfy the relation (4.9). The following relation between the functions $\psi_{e,g}(f)$ and the weight functions $z_{e,f}$ will be crucial in our analysis below.

Lemma 4.6. *Let $0 \le m \le n-1$ and $0 \le j < n-m$. For any $g \in \Delta_s(\mathcal{T})$, where $-1 \le s \le m-1$, the identity*

$$\sum_{\substack{(e,f)\in\Delta_{j,m}\\ f\supset g}} \psi_{e,g}(f) \otimes z_{e,f} = \sum_{\substack{(e,f)\in\Delta_{j,m-1}\\ f\supset g}} \phi_e \otimes z_{e,f}, \tag{4.19}$$

holds.

Proof. Since $z = \delta^+ w$, we can reformulate the right hand side of the identity as

$$\sum_{\substack{(e,f)\in\Delta_{j,m-1}\\ f\supset g}} \sum_{i\in I(e)} (-1)^{\sigma_e(x_i)} \phi_e \otimes w_{e(\hat{x}_i),\langle x_i, f\rangle} = \sum_{\substack{(e,f)\in\Delta_{j-1,m}\\ f\supset g}} \sum_{i\in I(f\cap g^*)} \phi_{[x_i,e]} \otimes w_{e,f}.$$

On the other hand, using the definition of $\psi_{e,g}(f)$, (2.5), and (4.14), for $e \in \Delta_{j-1}$, the left hand side can be rewritten as

$$\begin{aligned}(-1)^{j-1} \sum_{\substack{f\in\Delta_m\\ f\supset g}} \sum_{i\in I(f\cap g^*)} (\lambda_i d - j d\lambda_i \wedge) \sum_{e\in\Delta_j(f^*)} \mu_e \otimes (\delta^+ w)_{e,f} \\ = \sum_{\substack{f\in\Delta_m\\ f\supset g}} \sum_{i\in I(f\cap g^*)} (\lambda_i d - j d\lambda_i \wedge) \sum_{e\in\Delta_{j-1}(f^*)} \phi_e \otimes w_{e,f} \\ = \sum_{\substack{(e,f)\in\Delta_{j-1,m}\\ f\supset g}} \sum_{i\in I(f\cap g^*)} \phi_{[x_i,e]} \otimes w_{e,f},\end{aligned}$$

and hence the desired identity is verified. ∎

5. The order reduction operators

We recall from Section 2.4 above that the order reduction operators, $R^k_{e,f}$, are defined for $(e,f) \in \Delta_{j,m}$ from corresponding functions $z_{e,f}$ by

$$(R^k_{e,f} u)_\lambda = \int_\Omega (\Pi_j G_f^* u)_\lambda \wedge z_{e,f}, \quad \lambda \in \mathcal{S}_f^c,$$

cf. (2.16). The functions $z_{e,f}$ will be taken to be the weight functions constructed in the previous section. In particular, $z_{e,f} \in \mathcal{P}_1^- \Lambda^{n-j}(\mathcal{T}_f)$ for $(e,f) \in \Delta_{j,m}$, and with support in $\Omega_f \cap \Omega_e^E$, while the function G_f, defined above, maps the product $\Omega_f \times \mathcal{S}_f^c$ to Ω_f. Since G_f is defined on a product space, the target space for the pullback, G_f^*, can be represented as the sum of tensor products, and Π_j is the canonical map of $\Lambda^k(\Omega_f \times \mathcal{S}_f^c)$ to $\Lambda^j(\Omega_f) \otimes \Lambda^{k-j}(\mathcal{S}_f^c)$. By construction, the operators $R^k_{e,f}$ map k-forms to $(k-j)$-forms for $(e,f) \in \Delta_{j,m}$, $0 \le j \le k$. Furthermore, $R^k_{e,f} \equiv 0$ for $j > k$.

The commuting property $dG_f^* u = G_f^* du$, where u is k-form on Ω_f, can in the present setting be expressed by

$$d_\Omega \Pi_{j-1} G_f^* u + (-1)^j d_S \Pi_j G_f^* u = \Pi_j d G_f^* u = \Pi_j G_f^* du, \quad j = 1, \ldots, k, \tag{5.1}$$

where d_Ω and d_S denote the exterior derivative with respect to the spaces Ω and $\mathcal{S}_f^c$, respectively. By combining this with property (2.17), cf. Lemma 4.5, we derive, exactly as in the proof of [6, Proposition 5.4], that the operators $R^k_{e,f}$ satisfy the fundamental relation (2.18). Furthermore, from the fact that $\delta^+ z = 0$, cf. Lemma 4.5, we obtain that $(\delta^+ R^k)_{e,f} = 0$.

Next we will use Lemma 4.6 to obtain the following property of the operators $R^k_{e,f}$.

Lemma 5.1. *Let $0 \le m \le n-1$ and $0 \le j < n-m$. For any $g \in \Delta_s(\mathcal{T})$, where $-1 \le s \le m-1$, the identity*

$$\sum_{\substack{(e,f)\in\Delta_{j,m}\\ f\supset g}} \psi_{e,g}(f) \wedge L^*_g R^k_{e,f} u = \sum_{\substack{(e,f)\in\Delta_{j,m-1}\\ f\supset g}} \phi_e \wedge L^*_g R^k_{e,f} u, \tag{5.2}$$

holds.

Proof. We consider the simplex g to be fixed. Since $\psi_{e,g}(f) \in \mathcal{P}^-_1\Lambda^j(\mathcal{T}, g^*)$, it follows that there exist constants $\{a_{e,e'}(f)\}$ such that $\psi_{e,g}(f) = \sum_{e'\in\Delta_j(g^*)} a_{e,e'}(f)\phi_{e'}$. As a consequence, the identity (4.19) can be expressed as

$$\sum_{\substack{(e,f)\in\Delta_{j,m-1}\\ f\supset g}} \phi_e \otimes z_{e,f} = \sum_{e\in\Delta_j(g^*)} \phi_e \otimes \sum_{\substack{(e',f)\in\Delta_{j,m}\\ f\supset g}} a_{e',e}(f) z_{e',f},$$

which implies that

$$\sum_{\substack{f\in\Delta_{m-1}(e^*)\\ f\supset g}} z_{e,f} = \sum_{\substack{(e',f)\in\Delta_{j,m}\\ f\supset g}} a_{e',e}(f) z_{e',f}, \quad e \in \Delta_j(g^*).$$

From the definition of the order reduction operators $R^k_{e,f}$, we then obtain the corresponding identity

$$\sum_{\substack{f\in\Delta_{m-1}(e^*)\\ f\supset g}} R^k_{e,f} = \sum_{\substack{(e',f)\in\Delta_{j,m}\\ f\supset g}} a_{e',e}(f) R^k_{e',f}, \quad e \in \Delta_j(g^*).$$

By applying L^*_g and then reversing the steps above, we obtain the desired identity. □

In addition to the operators $R^k_{e,f}$, we will also use the operators $Q^k_{e,f}$, defined from the functions $w_{e,f}$. More precisely,

$$(Q^k_{e,f} u)_\lambda = \int_\Omega (\Pi_{j+1} G^*_f u)_\lambda \wedge w_{e,f}, \quad (e,f) \in \Delta_{j,m},$$

where $0 \le m \le n$ and $-1 \le j < n-m$. We recall that for $e \in \Delta_j(f^*)$, the functions $w_{e,f}$ are trimmed linear $(n-j-1)$-forms with support in Ω_f, and as a consequence, the operator $Q^k_{e,f}$ maps k-forms to $(k-j-1)$-forms. In particular, if $k < j+1$, then $Q^k_{e,f} \equiv 0$. Since $z_{e,f} = (\delta^+ w)_{e,f}$, we also have ,

$$(\delta^+ Q^k u)_{e,f} = R^k_{e,f} u. \tag{5.3}$$

Since the operators $Q^k_{e,f}$ are constructed from a similar procedure as the operators $R^k_{e,f}$, the new operators will also preserve piecewise smoothness and the piecewise polynomial spaces. In particular, we have for $(e, f) \in \Delta_{j,m}$ that

$$b^{-(j+1)} Q^k_{e,f}(\mathcal{P}\Lambda^k(\mathcal{T}_f)) \subset \mathcal{P}\Lambda^k(\mathcal{S}^c_f), \tag{5.4}$$

where $\mathcal{P}$ can either be $\mathcal{P}_r$ or $\mathcal{P}^-_r$. Here, the extra factor b^{-1}, as compared to the mapping properties of the corresponding operators $R^k_{e,f}$ given in Lemma 2.2, is due to the fact that the order of the forms $w_{e,f}$ are reduced by one as compared to the forms $z_{e,f}$. We refer to the proof of Proposition 5.2 of [6] for further details.

Lemma 5.2. *Let* $1 \le m \le n$ *and* $j = n - m$. *Assume that* $f \in \Delta_{m-1}(\mathcal{T})$ *and* $g \in \bar{\Delta}(f)$. *The identity*

$$\sum_{e \in \Delta_j(f^*)} \phi_e \wedge L^*_g\Big((-1)^{j+1} Q^{k+1}_{e,f} du - dQ^k_{e,f} u\Big) = \sum_{e \in \Delta_j(f^*)} \beta_e(f) \wedge L^*_g R^k_{e,f} u,$$

holds on Ω_f.

Proof. For a fixed $e \in \Delta_j(f^*)$, we have

$$\begin{aligned}(-1)^{j+1} Q^{k+1}_{e,f} du - dQ^k_{e,f} u &= \int_\Omega \Big((-1)^{j+1}\Pi_{j+1} G^*_f du - d_S \Pi_{j+1} G^*_f u\Big) \wedge w_{e,f} \\ &= (-1)^{j+1} \int_\Omega d_\Omega \Pi_j G^*_f u \wedge w_{e,f} = \int_\Omega \Pi_j G^*_f u \wedge dw_{e,f},\end{aligned}$$

where we have used the identity (5.1), and the local support of $w_{e,f}$. However, from (4.16) we have that

$$\sum_{e \in \Delta_j(f^*)} \phi_e \otimes L^*_g \int_\Omega \Pi_j G^*_f u \wedge dw_{e,f} = \sum_{e \in \Delta_j(f^*)} \beta_e(f) \otimes L^*_g R^k_{e,f} u$$

on Ω_f, and hence the desired identity follows from (2.1). □

6. The local operators $K^k_{m,f}$

We recall that the operators $\{B^k_f\}_{f\in\Delta}$, appearing in the decomposition (1.1), will be defined by the operator W^k, mapping into the space of trimmed linear k-forms, and from the local operators $\{K^k_{m,f}\}_{f\in\Delta_j}$, $j = m, m-1$, by formulas (2.25)–(2.26). The purpose of this section is to define the operators $\{K^k_{m,f}\}$. For each value of m, $1 \le m \le n-1$, we define the operators $K^k_{m,f}$, for $f \in \Delta_m$, by

$$K^k_{m,f} u = \sum_{g \in \bar{\Delta}(f)} (-1)^{|f|-|g|} L^*_g A^k_f u. \tag{6.1}$$

This is the obvious generalization of the corresponding operators defined for $k = 0$ in Section 3 above. From the properties of the operators A^k_f, cf. Section 2.4, it follows that the operator $L^*_g A^k_f$, and hence $K^k_{m,f}$, has domain of dependence Ω_f. Furthermore, the operator (6.1) commutes with the exterior derivative, preserves piecewise smoothness and the piecewise polynomial spaces, and by the cancellation argument, cf. Section 3, we obtain that the functions $K^k_{m,f} u$ have support on Ω_f.

For $f \in \Delta_{m-1}$, the generalization of the operators $K^0_{m,f}$, introduced in Section 3, to the case of k-forms is less obvious. In this case, we define $K^k_{m,f}$ by an alternating sum of the form

$$K^k_{m,f} = \sum_{g \in \bar{\Delta}(f)} (-1)^{|f|-|g|} K^k_{m,f,g}, \tag{6.2}$$

where

$$K^k_{m,f,g} u = -L^*_g A^k_f u + \sum_{j=0}^{n-m} \frac{(-1)^{j-1}}{\rho_g^{j+1}} \sum_{e \in \Delta_j(f^*)} \Big(\phi_e + \psi_{e,g}(f)\Big) \wedge L^*_g R^k_{e,f} u \\ + \sum_{\substack{e \in \Delta_{n-m}(f^*) \\ j=n-m}} (-1)^j \Big(d \frac{\phi_e}{\rho_g^{j+1}}\Big) \wedge L^*_g Q^k_{e,f} u. \tag{6.3}$$

These operators reduce to the corresponding operators $K^0_{m,f}$ if $k = 0$. To see this, observe that all the operators $Q^0_{e,f}$ are identically zero. Similarly, $R^0_{e,f} = 0$ if $(e, f) \in \Delta_{j,m}$, $j > 0$, and if $j = 0$ then $R^0_{e,f} = -A^0_{\langle e,f \rangle}$. As a consequence, we obtain from (6.2) and (6.3) that

$$\begin{aligned} K^0_{m,f} u &= \sum_{g \in \bar{\Delta}(f)} (-1)^{|f|-|g|} \Big[- L^*_g A^0_f u + \rho_g^{-1} \sum_{i \in I(f^*)} \Big(\lambda_i + \psi_{x_i,g}(f)\Big) \wedge L^*_g A^0_{\langle x_i, f \rangle} u \Big] \\ &= \sum_{g \in \bar{\Delta}(f)} (-1)^{|f|-|g|} \rho_g^{-1} \Big[\sum_{i \in I(f^*)} \Big(\lambda_i - \frac{\rho_f}{|f^*|}\Big) \wedge L^*_g A^0_{\langle x_i, f \rangle} u \Big], \end{aligned}$$

where we have used (2.13) and (4.10). Hence, we can conclude that the definition above agrees with the definition given in Section 3 when $k = 0$, cf. formula (3.5). We note that the domain of dependence of all the operators in (6.3) is Ω_f and hence the domain of dependence of $K^k_{m,f}$ is Ω_f. Furthermore, it follows from (4.8) that

$$\operatorname{tr}_{\lambda_i=0} \psi_{e,g}(f) = \operatorname{tr}_{\lambda_i=0} \psi_{e,\langle x_i, g \rangle}(f), \quad x_i \in f \cap g^*.$$

By the cancellation argument introduced in Section 3, it is then easy to see that the functions $K^k_{m,f} u$ have support on Ω_f. In fact, if $g \in \bar{\Delta}(f)$ and $i \in I(f \cap g^*)$, then

$$\operatorname{tr}_{\lambda_i=0} \Big(K^k_{m,f,\langle x_i, g \rangle} u - K^k_{m,f,g} u\Big) = 0,$$

which shows that $K^k_{m,f} u$ has support on Ω_{x_i}. Furthermore, since $i \in I(f)$ is arbitrary, the support of $K^k_{m,f} u$ must be limited to

$$\Omega_f = \bigcap_{i \in I(f)} \Omega_{x_i}.$$

A key step to show that the operator $K^k_{m,f}$ commutes with the exterior derivative and preserves piecewise smoothness is the following alternative expression of the first part of the operator $K^k_{m,f,g}$.

Lemma 6.1. *Assume that $1 \le m \le n-1$. For each $f \in \Delta_{m-1}(\mathcal{T})$ and $g \in \bar{\Delta}(f)$, the identity*

$$- Lg^* A^k_f u + \sum_{j=0}^{n-m} \frac{(-1)^{j-1}}{\rho_g^{j+1}} \sum_{e \in \Delta_j(f^*)} \Big(\phi_e + \psi_{e,g}\Big) \wedge L^*_g R^k_{e,f} u \tag{6.4}$$

$$= \sum_{j=1}^{n-m} \sum_{e \in \Delta_j(f^*)} \Big[d(\mu_e \wedge L_g^* b^{-j} R_{e,f}^k u) + \mu_e \wedge L_g^* b^{-j} R_{e,f}^{k+1} du \Big] - \sum_{\substack{e \in \Delta_j(f^*) \\ j=n-m}} \frac{\beta_e}{\rho_g^{j+1}} \wedge L_g^* R_{e,f}^k u$$

holds on Ω_f. Here, the functions μ_e, β_e, $\psi_{e,g}$ are all associated to the simplex f.

The proof of the lemma above is partly technical. Therefore, we delay the proof, and we will first use the identity to prove the following key result.

Lemma 6.2. *Let $1 \le m \le n-1$ and assume $f \in \Delta_j(\mathcal{T})$, $j = m, m-1$. The function $K_{m,f}^k u$ has domain of dependence Ω_f and support on Ω_f. Furthermore, the operator $K_{m,f}^k$ commutes with the exterior derivative and maps the spaces $\Lambda^k(\mathcal{T}_f)$, $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_f)$, and $\mathcal{P}_r\Lambda^k(\mathcal{T}_f)$ to themselves.*

Proof. Following the discussion above, it remains to show that the operator defined by (6.2) and (6.3) preserves piecewise smoothness and, in particular, the piecewise polynomial spaces, and that this operator commutes with the exterior derivative. In fact, we will show that each of the operators $K_{m,f,g}^k$ has these properties, where $g \in \bar{\Delta}(f)$. Furthermore, due to the support property of the functions $K_{m,f}^k u$ already verified, it is enough to establish these properties on the domain Ω_f.

It is a consequence of the identity of Lemma 6.1 that the operator $K_{m,f,g}^k$ admits the alternative representation

$$K_{m,f,g}^k u = \sum_{j=1}^{n-m} \sum_{e \in \Delta_j(f^*)} \Big(d(\mu_e \wedge L_g^* b^{-j} R_{e,f}^k u) + \mu_e \wedge L_g^* b^{-j} R_{e,f}^{k+1} du \Big) + \sum_{\substack{e \in \Delta_j(f^*) \\ j=n-m}} \Big((-1)^j \Big(d\frac{\phi_e}{\rho_g^{j+1}} \Big) \wedge L_g^* Q_{e,f}^k u - \frac{\beta_e}{\rho_g^{j+1}} \wedge L_g^* R_{e,f}^k u \Big).$$

However, by Lemma 5.2, the two last terms can be rewritten in the form

$$\sum_{\substack{e \in \Delta_j(f^*) \\ j=n-m}} \Big[(-1)^j \Big(d\frac{\phi_e}{\rho_g^{j+1}} \Big) \wedge L_g^* Q_{e,f}^k u - \frac{\phi_e}{\rho_g^{j+1}} \wedge L_g^* \Big((-1)^{j+1} Q_{e,f}^{k+1} du - dQ_{e,f}^k u \Big) \Big]$$
$$= \sum_{\substack{e \in \Delta_j(f^*) \\ j=n-m}} (-1)^j \Big(d\Big(\frac{\phi_e}{\rho_g^{j+1}} \wedge L_g^* Q_{e,f}^k u \Big) + \frac{\phi_e}{\rho_g^{j+1}} \wedge L_g^* Q_{e,f}^{k+1} du \Big).$$

As a consequence, it follows that the operator $K_{m,f,g}^k$ can be expressed as

$$K_{m,f,g}^k u = \sum_{j=1}^{n-m} \sum_{e \in \Delta_j(f^*)} \Big(d(\mu_e(f) \wedge L_g^* b^{-j} R_{e,f}^k u) + \mu_e(f) \wedge L_g^* b^{-j} R_{e,f}^{k+1} du \Big) + \sum_{\substack{e \in \Delta_j(f^*) \\ j=n-m}} (-1)^j \Big(d\Big(\phi_e \wedge L_g^* b^{-(j+1)} Q_{e,f}^k u \Big) + \phi_e \wedge L_g^* b^{-(j+1)} Q_{e,f}^{k+1} du \Big).$$

From this representation, commuting with the exterior derivative is obvious, and the space preserving properties follow from (2.6) and the space preserving properties of the operators $R^k_{e,f}$ and $Q^k_{e,f}$ given in Lemma 2.2 and (5.4). In particular, for the trimmed piecewise polynomial spaces, we use the fact that $\mathcal{P}_1^-\Lambda^j \wedge \mathcal{P}_{r-1}^-\Lambda^{k-j} \subset \mathcal{P}_r^-\Lambda^k$, cf. [2, Section 3.3]. $\square$

Next, we will establish Lemma 6.1.

Proof. (of Lemma 6.1) From the identity (4.9), we obtain

$$\begin{aligned}
&\sum_{j=0}^{n-m} \sum_{e\in\Delta_j(f^*)} \frac{(-1)^{j-1}}{\rho_g^{j+1}} \Big(\phi_e + \psi_{e,g}\Big) \wedge L_g^* R^k_{e,f} u \\
&\qquad = \sum_{j=0}^{n-m} \sum_{e\in\Delta_j(f^*)} \Big[d\Big(\frac{\mu_e}{\rho_g^j}\Big) - \frac{\beta_e}{\rho_g^{j+1}}\Big] \wedge L_g^* R^k_{e,f} u \\
&\qquad = \sum_{\substack{e\in\Delta_j(f^*)\\ j=n-m}} \Big[d\Big(\frac{\mu_e}{\rho_g^j}\Big) - \frac{\beta_e}{\rho_g^{j+1}}\Big] \wedge L_g^* R^k_{e,f} u \\
&\qquad\qquad + \sum_{j=0}^{n-m-1} \sum_{e\in\Delta_j(f^*)} \Big[d\Big(\frac{\mu_e}{\rho_g^j}\Big) - \frac{(\partial\mu)_e}{\rho_g^{j+1}}\Big] \wedge L_g^* R^k_{e,f} u.
\end{aligned}$$

where we have used the identity $\beta_e = (\partial\mu)_e$ from Lemma 4.1, and where $\partial = \partial(f^*)$. Recall also that for $e \in \Delta_0(f^*)$, we obtain from (2.13) that

$$\sum_{e\in\Delta_0(f^*)} d\mu_e \wedge L_g^* R^k_{e,f} u = L_g^* A^k_f u,$$

since $d\mu_e = -1/|f^*|$ and $R^k_{e,f} u = -A^k_{\langle e,f\rangle}$. Hence, the left hand side of (6.4) is given by

$$\begin{aligned}
(6.5)\qquad &\sum_{\substack{e\in\Delta_j(f^*)\\ j=n-m}} \Big[d\Big(\frac{\mu_e}{\rho_g^j}\Big) - \frac{\beta_e}{\rho_g^{j+1}}\Big] \wedge L_g^* R^k_{e,f} u - \sum_{e\in\Delta_0(f^*)} \frac{(\partial\mu)_e}{\rho_g} \wedge L_g^* R^k_{e,f} u \\
&\qquad + \sum_{j=1}^{n-m-1} \sum_{e\in\Delta_j(f^*)} \Big[d\Big(\frac{\mu_e}{\rho_g^j}\Big) - \frac{(\partial\mu)_e}{\rho_g^{j+1}}\Big] \wedge L_g^* R^k_{e,f} u.
\end{aligned}$$

On the other hand, it follows from (2.18) and the Leibniz rule for the wedge product, that

$$\begin{aligned}
&d(\mu_e \wedge L_g^* b^{-j} R^k_{e,f} u) + \mu_e \wedge L_g^* b^{-j} R^{k+1}_{e,f} du \\
&\qquad = \Big(d\frac{\mu_e}{\rho_g^j}\Big) \wedge L_g^* R^k_{e,f} u + \mu_e \wedge L_g^* b^{-j}((-1)^{j-1} dR^k_{e,f} u + R^{k+1}_{e,f} du) \\
&\qquad\qquad = \Big(d\frac{\mu_e}{\rho_g^j}\Big) \wedge L_g^* R^k_{e,f} u - \mu_e \wedge L_g^* b^{-j} (\delta R^k u)_{e,f},
\end{aligned}$$

for $e \in \Delta_j(f^*)$, $0 \le j \le n-m$. As a consequence, the right hand side of (6.4) is given by

$$= \sum_{j=1}^{n-m-1} \sum_{e\in\Delta_j(f^*)} \Big(d\frac{\mu_e}{\rho_g^j}\Big) \wedge L_g^* R_{e,f}^k u + \sum_{\substack{e\in\Delta_j(f^*)\\ j=n-m}} \Big[\Big(d\frac{\mu_e}{\rho_g^j}\Big) - \frac{\beta_e}{\rho_g^{j+1}}\Big] \wedge L_g^* R_{e,f}^k u$$
$$- \sum_{j=0}^{n-m-1} \sum_{e\in\Delta_j(f^*)} \partial\mu_e \wedge L_g^* b^{-(j+1)}(R^k u)_{e,f},$$

where we have used (2.2) to rewrite the last term. If we combine terms and compare this with (6.5), we obtain (6.4). □

The definitions of the operators $\{K_{m,f}^k\}$, given above, combined with the operators $\{W^k\}$ introduced in Section 2.5, complete the construction of all the operators required for the decomposition (1.1), cf. (2.25) and (2.26).

Proposition 6.3. *The operators $\{B_f^k\}$, defined by* (2.25) *and* (2.26)*, have domain of dependence Ω_f if $f \in \Delta_m$, $m < n$, and Ω_f^E if $m = n$. Furthermore, the functions $\mathcal{B}_f^k u$ have support in Ω_f.*

Proof. All these properties follow directly from the corresponding properties of the operators $\{K_{m,f}\}$ given in Lemma 6.2, except for the domain of dependence property in the case $f \in \Delta_n$. However, it is a consequence (2.26) that in this case the operator B_f^k has a domain of dependence included in

$$\Big(\bigcup_{e\in\Delta_k(f)} \Omega_e^E\Big) \cup \Big(\bigcup_{g\in\Delta(f)} \Omega_g\Big) \subset \Omega_f^E.$$

□

To complete the proof of the main theorem of the paper, Theorem 2.3, it remains to verify the identity (2.24) and to establish the desired operator bounds. This will be done in the two next sections.

7. Verifying the fundamental identity

The purpose of this section is to establish the fundamental identity (2.24), i.e.,

$$C_m^k u - \sum_{\substack{f\in\Delta_j\\ j=m,m-1}} K_{m,f}^k u = C_{m-1}^k u, \quad 1 \le m \le n-1.$$

Therefore, we have to study sums of the operators $K_{m,f}^k$ introduced in the previous section. As a preliminary step, we study sums of expressions corresponding to a part of the definition (6.3).

Lemma 7.1. *Assume that $1 \le m \le n-1$. Then*

$$(7.1) \quad \sum_{\substack{(e,f)\in\Delta_{j.m-1}\\ 0\le j\le n-m}} (-1)^{j-1} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \frac{\psi_{e,g}(f)}{\rho_g} \wedge L_g^* b^{-j} R_{e,f}^k u$$

$$= - \sum_{\substack{(e,f)\in\Delta_{j.m-2}\\ 0\le j\le n-m}} (-1)^{j-1} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} R_{e,f}^k u.$$

Proof. Recall from (4.8) that $\psi_{e,g}(f) = 0$ if $g = f$. Therefore, by changing the order of summation and then using (5.2), the left hand side of (7.1) can be rewritten as

$$\sum_{\substack{g\in\Delta_s\\ -1\le s\le m-2}} (-1)^{m-|g|} \sum_{j=0}^{n-m} \frac{(-1)^{j-1}}{\rho_g^{j+1}} \sum_{\substack{(e,f)\in\Delta_{j,m-1}\\ f\supset g}} \psi_{e,g}(f) \wedge L_g^* R_{e,f}^k u$$

$$= \sum_{\substack{g\in\Delta_s\\ -1\le s\le m-2}} (-1)^{m-|g|} \sum_{j=0}^{n-m} (-1)^{j-1} \sum_{\substack{(e,f)\in\Delta_{j,m-2}\\ f\supset g}} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} R_{e,f}^k u,$$

where the right hand side corresponds exactly to the right hand side of (7.1). □

Next, we consider a corresponding sum of the last term of the definition (6.3).

Lemma 7.2. *Assume that* $1 \le m \le n-1$. *Then*

$$\sum_{\substack{(e,f)\in\Delta_{j,m-1}\\ j=n-m}} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \Big(d\frac{\phi_e}{\rho_g^{j+1}}\Big) \wedge L_g^* Q_{e,f}^k u \tag{7.2}$$

$$= - \sum_{\substack{(e,f)\in\Delta_{j,m-2}\\ j=n-m+1}} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} R_{e,f}^k u.$$

Proof. Let $f \in \Delta_{m-1}$ and $g \in \bar{\Delta}(f)$ be fixed. Since $\Omega_f \subset \Omega_g$, it follows that $\rho_g = \sum_{i\in I(g^*)} \lambda_i$ on Ω_f. Since $\dim f^* = n-m$ and $e \in \Delta_{n-m}(f^*)$, $\phi_{[x_i,e]} = 0$ unless $i \in I(f)$ and so we have

$$d\Big(\frac{\phi_e}{\rho_g^{n-m+1}}\Big) = \frac{1}{\rho_g^{n-m+2}} \sum_{i\in I(f\cap g^*)} \phi_{[x_i,e]}, \quad \text{on } \Omega_f.$$

As a consequence, when we restrict to Ω_f, we can conclude that

$$\sum_{\substack{e\in\Delta_j(f^*)\\ j=n-m}} \Big(d\frac{\phi_e}{\rho_g^{j+1}}\Big) \wedge L_g^* Q_{e,f}^k u$$

$$= \sum_{i\in I(f\cap g^*)} \sum_{\substack{e\in\Delta_j(f(\hat{x}_i)^*)\\ e\supset x_i\\ j=n-m+1}} (-1)^{\sigma_{e(x_i)}} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} Q_{e(\hat{x}_i),f}^k u.$$

Next, if we sum over all $g \in \bar{\Delta}(f)$ we obtain

$$\sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \sum_{\substack{e\in\Delta_j(f^*)\\ j=n-m}} \Big(d\frac{\phi_e}{\rho_g^{j+1}}\Big) \wedge L_g^* Q_{e,f}^k u \tag{7.3}$$

$$= \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \sum_{i\in I(f\cap g^*)} \sum_{\substack{e\in\Delta_j(f(\hat{x}_i)^*)\\ e\supset x_i\\ j=n-m+1}} (-1)^{\sigma_{e(x_i)}} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} Q^k_{e(\hat{x}_i),f} u.$$

This identity obviously holds on Ω_f, and by the cancellation argument, it also holds on $\Omega \setminus \Omega_f$, since both sides of the identity vanish there. Furthermore, if we sum (7.3) over all $f \in \Delta_{m-1}$, and use the fact that $f = \langle x_i, f(\hat{x}_i)\rangle$ for $i \in I(f)$, we obtain that the left hand side of (7.2) can be expressed as

$$\sum_{\substack{g\in\Delta_s\\ -1\le s\le m-2}} (-1)^{m-|g|} \sum_{\substack{f\in\Delta_{m-1}\\ f\supset g}} \sum_{i\in I(f\cap g^*)} \sum_{\substack{e\in\Delta_j(f(\hat{x}_i)^*)\\ e\supset x_i\\ j=n-m+1}} (-1)^{\sigma_{e(x_i)}} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} Q^k_{e(\hat{x}_i),f} u$$

$$= \sum_{\substack{g\in\Delta_s\\ -1\le s\le m-2}} (-1)^{m-|g|} \sum_{\substack{f\in\Delta_{m-2}\\ f\supset g}} \sum_{\substack{e\in\Delta_j(f^*)\\ j=n-m+1}} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} \sum_{i\in I(e)} (-1)^{\sigma_e(x_i)} Q^k_{e(\hat{x}_i),\langle x_i,f\rangle} u$$

$$= - \sum_{\substack{(e,f)\in\Delta_{j,m-2}\\ j=n-m+1}} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} (\delta^+ Q^k u)_{e,f},$$

where we have used the fact that for g fixed, we have

$$\sum_{\substack{f\in\Delta_{m-1}\\ f\supset g}} \sum_{i\in I(f\cap g^*)} \sum_{\substack{e\in\Delta_j(f(\hat{x}_i)^*)\\ e\supset x_i}} = \sum_{\substack{f\in\Delta_{m-2}\\ f\supset g}} \sum_{e\in\Delta_j(f^*)} \sum_{i\in I(e)}.$$

However, by (5.3), the final term above is exactly the right hand side of (7.2). □

The main result of this section now follows from the two lemmas above.

Proposition 7.3. *Let* $1 \le m \le n-1$. *Then the identity* (2.24) *holds.*

Proof. It follows from (2.20) and (6.1) that

$$C^k_m u - \sum_{f\in\Delta_m} K^k_{m,f} u = \sum_{\substack{(e,f)\in\Delta_{j,m-1}\\ 0\le j\le n-m}} (-1)^{j-1} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} R^k_{e,f} u.$$

On the other hand, it follows from (6.2), (6.3), and the two identities (7.1) and (7.2) derived above, that

$$\sum_{f\in\Delta_{m-1}} K^k_{m,f} u = - \sum_{f\in\Delta_{m-1}} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} L_g^* A^k_f u$$
$$+ \sum_{\substack{(e,f)\in\Delta_{j,m-1}\\ 0\le j\le n-m}} (-1)^{j-1} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} R^k_{e,f} u.$$
$$- \sum_{\substack{(e,f)\in\Delta_{j,m-2}\\ 0\le j\le n-m+1}} (-1)^{j-1} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} R^k_{e,f} u.$$

Therefore, by comparing the two formulas above, we obtain that

$$C^k_m u - \sum_{\substack{f\in\Delta_j\\ j=m,m-1}} K^k_{m,f} u = \sum_{f\in\Delta_{m-1}} \sum_{g\in\bar\Delta(f)} (-1)^{|f|-|g|} L^*_g A^k_f u$$

$$+ \sum_{\substack{(e,f)\in\Delta_{j,m-2}\\ 0\le j\le n-m+1}} (-1)^{j-1} \sum_{g\in\bar\Delta(f)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g} \wedge L^*_g b^{-j} R^k_{e,f} u,$$

and by (2.20), the right hand side is exactly $C^k_{m-1}u$. This completes the proof. □

8. Bounding the operator norms

To complete the proof of Theorem 2.3, it remains to show that all the operators, W^k and B^k_f, of the decomposition (1.1) are bounded in $L^2\Lambda^k(\Omega)$, and satisfy a stable decomposition property. This will be achieved by Proposition 8.1 below. In fact, since these operators commute with the exterior derivative, they will also be bounded on the Sobolev space $H\Lambda^k(\Omega)$.

The various constants that appear in the bounds below only depend on the mesh $\mathcal{T}$ through the shape-regularity constant $c_{\mathcal{T}}$, defined by

$$c_{\mathcal{T}} = \max_{T\in\Delta_n(\mathcal{T})} \frac{\operatorname{diam}(T)}{\operatorname{diam}(\mathfrak{B}_T)},$$

where $\mathfrak{B}_T$ is the largest ball contained in T. The consequence of this is that if we consider a family of meshes, $\{\mathcal{T}^h\}$, parametrized by a real parameter $h \in (0,1]$, typically obtained by mesh refinements, the bounds will be uniform with respect to h as long as we restrict to a family with a uniform bound on the constants $\{c_{\mathcal{T}^h}\}$. In the bounds we derive below, the various constants that appear will depend on the space dimension n and the domain Ω, in addition to the dependence explicitly stated. Throughout this section, we will assume that the operators under investigation are applied to piecewise smooth differential forms. However, since the space $\Lambda^k(\mathcal{T})$ is dense in $L^2\Lambda^k(\Omega)$, it a consequence of the domain of dependence result in Proposition 6.3 and the bound obtained in Proposition 8.1 below, that all the operators B^k_f, where $f \in \Delta_m$, can be extended to bounded operators from $L^2\Lambda^k(\Omega_f)$ to itself if $0 \le m < n$, and from $L^2\Lambda^k(\Omega^E_f)$ to $L^2\Lambda^k(\Omega_f)$ when $m = n$. To bound the norms of the operators comprising the new decomposition of the bubble transform developed in this paper, we will basically follow the approach developed in [6, Section 8]. We recall that the decomposition (1.1) takes the form

$$u = W^k u + \sum_{m=0}^{n} \sum_{f\in\Delta_m} B^k_f u,$$

The main result of this section is the following bound.

Proposition 8.1. *There exists a constant c, depending on the shape-regularity constant $c_{\mathcal{T}}$, such that for $0 \le k \le n$, we have*

$$\|W^k u\|_{L^2(\Omega)}, \Big(\sum_{f\in\Delta(\mathcal{T})} \|B^k_f u\|^2_{L^2(\Omega_f)}\Big)^{1/2} \le c\|u\|_{L^2(\Omega)}.$$

To establish the bounds in Proposition 8.1, we will need some preliminary results. We define the overlap of a set of subdomains as the smallest upper bound for the number of domains which will contain any fixed element $T \in \Delta_n$. The overlap of the set of macroelements, $\{\Omega_f\}_{f\in\Delta_m}$, will only depend on m and the space dimension n, while the overlap for the set of extended macroelements, $\{\Omega_f^E\}$, will depend on the mesh $\mathcal{T}$. Another important property of the extended macroelements is the variation of the size of the elements. We define $h_f = \max_{T\in\Delta_n(\mathcal{T}_f^E)} \operatorname{diam}(T)$, where $\mathcal{T}_f^E$ is the restriction of the mesh $\mathcal{T}$ to Ω_f^E. The following result, established in Lemmas 8.2 and 8.3 of [6], shows that these domains allow bounded overlap and local quasi-uniformity in the following sense.

Lemma 8.2. *There is a constant c, depending on $\mathcal{T}$ only through the shape-regularity constant $c_{\mathcal{T}}$, which bounds the overlap of the domains $\{\Omega_f^E\}_{f\in\Delta(\mathcal{T})}$. Furthermore,*

$$h_f \le c \min_{T\in\Delta_n(\mathcal{T}_f^E)} \operatorname{diam}(T), \quad f \in \Delta(\mathcal{T}). \tag{8.1}$$

From the definitions of the operators $K_{m,f}^k$, given by (6.1)–(6.3), we will need appropriate bounds for the functions ϕ_e, $\psi_{e,g}(f)$, and also for the functions $w_{e,f}$ and $z_{e,f}$ which are used to define the order reduction operators $Q_{e,f}^k$ and $R_{e,f}^k$. All these functions are trimmed linear forms with local support. In particular, if $(e,f) \in \Delta_{j,m}$ and $g \in \bar{\Delta}(f)$, then $\psi_{e,g}(f)$ belongs to $\mathcal{P}_1^-\Lambda^j(\mathcal{T}, g^*)$, $w_{e,f} \in \mathring{\mathcal{P}}_1^-\Lambda^{n-j-1}(\mathcal{T}_f)$, and $z_{e,f} \in \mathring{\mathcal{P}}_1^-\Lambda^{n-j}(\mathcal{T}_f) \cap \mathring{\mathcal{P}}_1^-\Lambda^{n-j}(\mathcal{T}_e^E)$. In general, if w is any trimmed linear form, say $w \in \mathcal{P}_1^-\Lambda^j(\mathcal{T})$, then w admits a unique expansion of the form

$$w = \sum_{e\in\Delta_j} c_e\phi_e,$$

where $\{c_e\}$ are real coefficients. If $\max_{e\in\Delta_j} |c_e|$ can be bounded by a quantity which only depends on the mesh $\mathcal{T}$ through the mesh regularity constant, we will say that *w admits a uniformly bounded expansion.* It is a consequence of the bound (8.1) that for $g \in \bar{\Delta}(f)$ and $e \in \Delta_j(f^*)$ we have

$$\|\phi_e/\rho_g\|_{L^\infty(\Omega)} \le c h_e^{-j}, \tag{8.2}$$

where c depends on the shape-regularity constant. Note, in particular, that $g = \emptyset$ gives a bound on the L^∞-norm of ϕ_e. Next, recall that the coefficients $\{a_{e,e'}\} = \{a_{e,e'}(f)\}$ of functions $\{\mu_e(f)\}$, given by (4.4), can be computed recursively with respect to increasing values of j by the algebraic systems (4.6), (4.7). There are no explicit mesh dependent quantities present in the systems (4.6), (4.7). Only the number of equations depends on the mesh through the number of elements in $\Delta_j(f^*)$, and this number can be bounded by the shape-regularity constant. Therefore, since $a_{e,\emptyset}(f) = -1/|f^*|$ for $e \in \Delta_0(f^*)$, we can conclude that all functions $\{\mu_e(f)\}$ admit uniformly bounded expansions. Furthermore, since the functions $\{\operatorname{tr}_{\Omega_f} \beta_e(f)\}$ and $\{\psi_{e,g}(f)\}$ are explicitly defined from $\{\mu_e(f)\}$, through (4.2) and (4.8), the same conclusion holds for these function classes. Hence, it follows from (4.4) (4.8), and (8.2) that the forms $\mu_e \in \mathcal{P}_1^-\Lambda^{j-1}(\mathcal{T}, f^*)$ and $\psi_{e,g}(f) \in \mathcal{P}_1^-\Lambda^j(\mathcal{T}, g^*)$ satisfy

$$\|\mu_e(f)\|_{L^\infty(\Omega)} \le c h_e^{-j+1}, \qquad \|\psi_{e,g}(f)/\rho_g\|_{L^\infty(\Omega)} \le c h_e^{-j}, \tag{8.3}$$

where $(e,f) \in \Delta_{j,m}$ and $g \in \bar{\Delta}(f)$ for $0 \le m \le n-1$, $0 \le j < n-m$.

The following lemma below is a key ingredient to establish Proposition 8.1.

Lemma 8.3. *The trimmed linear differential forms* $w_{e,f}$ *and* $z_{e,f}$ *admit uniformly bounded expansions.*

We will delay the proof of this result to the end of the section. However, from this bound, combined with (8.2) and Lemma 8.2, we immediately obtain the estimates

$$h_f^{-1}\|w_{e,f}\|_{L^\infty(\Omega)}, \|z_{e,f}\|_{L^\infty(\Omega)} \le ch_f^{j-n}, \quad (e,f) \in \Delta_{j,m}, \tag{8.4}$$

where the constant c depends on the shape regularity constant.

Lemma 8.4. *The operator* W^k *maps* $L^2(\Omega)$ *to itself, and with an operator norm bounded by the shape-regularity constant.*

Proof. We recall that the operator W^k is given by

$$W^k u = (-1)^{k-1} \sum_{e\in\Delta_k} \phi_e\Big(\int_\Omega u \wedge z_{e,\emptyset}\Big),$$

Since the function $z_{e,\emptyset}$ is supported on Ω_e^E we obtain from (8.4) that

$$\int_\Omega u \wedge z_{e,\emptyset} \le ch_e^{n/2}\|z_{e,\emptyset}\|_{L^\infty(\Omega_e^E)}\|u\|_{L^2(\Omega_e^E)} \le ch_e^{k-n/2}\|u\|_{L^2(\Omega_e^E)},$$

where here, and below, the constant c depends on the shape-regularity constant, but is not necessarily the same at each occurrence. Furthermore, since the function ϕ_e has support on Ω_e, we obtain from (8.2) that $|\int_\Omega \phi_e^2| \le ch^{n-2k}$. Finally, the fact that $\phi_e = \phi_e\kappa_e$, where κ_e is the characteristic function of Ω_e, implies that

$$\Big(\sum_{e\in\Delta_k} \phi_e\big(\int_\Omega u\wedge z_{e,\emptyset}\big)\Big)^2 \le \sum_{e\in\Delta_k} \phi_e^2\big(\int_\Omega u\wedge z_{e,\emptyset}\big)^2 \sum_{e\in\Delta_k}\kappa_e.$$

Putting this together, we obtain

$$\begin{aligned}\|W^k u\|_{L^2(\Omega)}^2 &\le c\int_\Omega \Big(\sum_{e\in\Delta_k} \phi_e\big(\int_\Omega u\wedge z_{e,\emptyset}\big)\Big)^2 \\ &\le c\big(\sum_{e\in\Delta_k}\|u\|^2_{L^2(\Omega_e^E)}\big)\big(\|\sum_{e\in\Delta_k}\kappa_e\|_{L^\infty(\Omega)}\big) \le c\|u\|^2_{L^2(\Omega)},\end{aligned}$$

where the final inequality follows from the overlap properties of the domains $\{\Omega_e\}$ and $\{\Omega_e^E\}$, cf. Lemma 8.2. This completes the proof. □

In addition to the L^2-bound for the operator W^k, we will need corresponding bounds for all the operators $K^k_{m,f}$. We observe from the definitions (6.1), (6.2), and (6.3) of these operators that we will also need appropriate bounds for operators of the form A_f^k, $R^k_{e,f}$, and $Q^k_{e,f}$, composed with the pullback L_g^* for $g \in \bar\Delta(f)$. In fact, the three operators A, Q, R are all of the same form. In general, let $f \in \Delta_m$, and assume that w is a fixed function in $\mathring\Lambda^{n-j}(\mathcal{T}_f)$. Consider the corresponding operator, $\mathcal{Q}_j^k(w) : \Lambda^k(\mathcal{T}_f) \to \Lambda^{k-j}(\mathcal{S}_f^c)$, given by

$$\mathcal{Q}_j^k(w)u = \int_\Omega \Pi_j G_f^* u \wedge w.$$

By following the steps of the derivation of the bound (8.11) of [6], but where we retain the L^∞-norm of w instead of replacing it by an upper bound, we obtain the following result.

Lemma 8.5. *Assume that $f \in \Delta_m(\mathcal{T})$, $0 \le m \le n-1$, and that $0 \le j \le k$. If $w \in \mathring{\Lambda}^{n-j}(\mathcal{T}_f)$ then the bound*

$$\|L_g^* b^{-j} \mathcal{Q}_j^k(w)u\|_{L^2(\Omega_f)} \le ch_f^n \|w\|_{L^\infty(\Omega_f)} \|u\|_{L^2(\Omega_f)},$$

holds for any $g \in \bar{\Delta}(f)$, where the constant c only depends on $\mathcal{T}$ through the shape-regularity constant $c_\mathcal{T}$.

With the help of the results obtained above, the proof of Proposition 8.1 is straightforward.

Proof. (of Proposition 8.1) Consider a typical term in the definition (6.3) of the operator $K_{m,f}^k$ for $f \in \Delta_{m-1}$ given by

$$\Big(\frac{\phi_e + \psi_{e,g}(f)}{\rho_g}\Big) \wedge L_g^* b^{-j} R_{e,f}^k u,$$

where $1 \le m \le n-1$, $0 \le j \le \min(n-m, k)$, $e \in \Delta_j(f^*)$, and $g \in \bar{\Delta}(f)$. Since the operator $R_{e,f}^k$ can be identified as $\mathcal{Q}_j^k(z_{e,f})$, it follows from (8.2), (8.3), (8.4) and Lemma 8.5 that the $L^2(\Omega_f)$-norm of this term can be bounded by

$$\|\Big(\frac{\phi_e + \psi_{e,g}(f)}{\rho_g}\Big)\|_{L^\infty(\Omega_f)} \cdot \|L_g^* b^{-j} R_{e,f}^k u\|_{L^2(\Omega_f)} \le c\|u\|_{L^2(\Omega_f)},$$

when $g \in \bar{\Delta}(f)$. For each $f \in \Delta$, there are a finite number of such terms in the definition of the operator $K_{m,f}^k$ and it therefore follows that

$$\|K_{m,f}^k u\|_{L^2(\Omega_f)} \le c\|u\|_{L^2(\Omega_f)}.$$

From this bound and the finite overlap property of the domains $\{\Omega_f\}$, we then obtain

$$\sum_{m=0}^{n-1} \sum_{\substack{f \in \Delta_s \\ s=m,m-1}} \|K_{m,f}^k u\|_{L^2(\Omega_f)}^2 \le c \sum_{m=0}^{n-1} \sum_{\substack{f \in \Delta_s \\ s=m,m-1}} \|u\|_{L^2(\Omega_f)}^2 \le c\|u\|_{L^2(\Omega)}^2.$$

However, as a consequence of Lemma 8.4 and (2.25)–(2.26), this bound implies the desired bound on $\sum_f B_f^k$. □

Finally, the proof below will complete the discussion of this section.

Proof. (of Lemma 8.3) Recall that the functions $\{w_{e,f}\}$ and $\{z_{e,f}\}$ are defined inductively with respect to decreasing values of m through the relations (4.11), (4.14), and (4.16). We recall that $w_{\emptyset,f} = -(\kappa_f/\Omega_f)\mathsf{vol}$ for $f \in \Delta_n$, corresponding to the case $m = n$. As an induction hypothesis, we assume that all the functions $\{w_{e,f}\}$, for $(e,f) \in \Delta_{j,m}$, $-1 \le j < n-m$, admit a uniformly bounded expansion. As a consequence of (4.11), we then have that the same property holds for all $\{z_{e,f}\}$

for $(e,f) \in \Delta_{j,m-1}$. Furthermore, by expanding the functions $\mu_e(f)$, we derive from (4.11) and (4.14) that for $(e,f) \in \Delta_{j,m-1}$, $-1 \le j < n-m$,

$$w_{e,f} = (-1)^j \sum_{e' \in \Delta_{j+1}(f^*)} a_{e',e} z_{e',f}, \tag{8.5}$$

where the coefficients $a_{e',e}$ are obtained from $\mu_{e'}(f)$, cf. (4.4), i.e., $\mu_{e'} = \sum_e a_{e',e}\phi_e$. Hence, we can conclude by the induction hypothesis and the fact that μ_e' has a uniformly bounded expansion, that the left hand side of (8.5) also has a uniformly bounded expansion.

It remains to bound $w_{e,f}$ for $(e,f) \in \Delta_{n-m,m-1}$. It follows from (4.5) that the expansion of $\beta_e = \beta_e(f)$ is given from the corresponding expansion for $\mu_e(f)$ by

$$\beta_e = \sum_{e' \in \Delta_j(f^*)} b_{e,e'} \phi_{e'}, \quad b_{e,\mathbf{e}'} = (\delta a_{e,\cdot})_{e'} + (-1)^j 1_{e,e'},$$

where $j = n-m$, and we obtain from (4.16) that

$$dw_{e,f} = \sum_{e' \in \Delta_j(f^*)} b_{e',e} z_{e',f}, \quad j = n-m. \tag{8.6}$$

As above, we already know that the right hand side of (8.6) admits a uniformly bounded expansion. Furthermore, since $w_{e,f} \in \mathring{\mathcal{P}}_1^- \Lambda^{m-1}(\mathcal{T})$, it can be expanded in the form

$$w_{e,f} = \sum_{\substack{g \in \Delta_{m-1} \\ g \supset f}} c_g \phi_g \Longrightarrow dw_{e,f} = \sum_{\substack{g \in \Delta_m \\ g \supset f}} (\delta_{m-1} c)_g \phi_g,$$

where we have used (2.4). As a consequence , the coefficients $\delta_{m-1}c$ are uniformly bounded. Also by definition, $d_f^* w_{e,f} = 0$, which means that $\partial_{m-1} c = 0$, cf. (4.12). Recall that by local exactness, the coefficients $\{c_g\}$ are uniquely determined by ∂c and δc. Furthermore, there are no mesh dependent quantities present in the matrix representation of the operators δ and ∂, just $1, -1, 0$. Therefore, since the number of simplices in $\mathcal{T}_f$ is bounded by the shape-regularity constant, we can conclude that

$$\max_{\substack{g \in \Delta_{m-1} \\ g \supset f}} |c_g| \le c \max_{\substack{g \in \Delta_m \\ g \supset f}} (\delta c)_g,$$

where also the constant c is bounded by the shape-regularity constant. This completes the induction step and hence the proof of the Lemma. □

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