13. Finite element methods for parabolic problems

We consider the parabolic problem: Find $u = u(x, t)$ satisfying

$$u_t - \frac{\partial}{\partial x} \left( p \frac{\partial u}{\partial x} \right) + qu = f, \quad a < x < b, \quad 0 < t \leq T,$$

$$u(a, t) = 0, \quad u(b, t) = 0, \quad 0 < t \leq T; \quad u(x, 0) = \psi(x), \quad a < x < b.$$

Let $V^0 = \{ v \in H^1(a, b) : v(a) = v(b) = 0 \}$. A variational formulation of this problem is to seek $u$ such that $u(x, 0) = \psi(x)$, and for each fixed $t > 0$, $u \in V^0$ satisfies

$$(\partial u/\partial t, v) + a(u, v) = (f, v), \quad v \in V^0,$$

where $(\cdot, \cdot)$ denotes the $L^2$ inner product on $(a, b)$ and now

$$a(u, v) = \int_a^b \left[ \frac{p}{\partial x} \frac{\partial u}{\partial x} + quv \right] dx.$$

13.1. Continuous time Galerkin scheme. We first consider an approximation in which we discretize by finite elements in the spatial variable, but keep time continuous. Thus, we choose a finite dimensional subspace $V^0_h \subset V^0$ and look for an approximation $u_h$ such that $u_h(x, 0) = \psi_h(x, 0)$ (where $\psi_h$ an approximation to $\psi$) and for each fixed $t > 0$, $u_h \in V^0_h$ satisfies

$$(\partial u_h/\partial t, v) + a(u_h, v) = (f, v), \quad v \in V^0_h.$$

To see what is involved in solving this problem, we write $u_h(x, t) = \sum_{j=1}^{m} \alpha_j(t) \phi_j(x)$, where $\{\phi_j\}_{j=1}^m$ is a basis for $V^0_h$. Inserting this into the variational equations, and choosing $v$ to be each of the basis functions $\phi_i$, we get

$$\sum_{j=1}^{m} \alpha'_j(t)(\phi_j, \phi_i) + \sum_{j=1}^{m} \alpha_j(t)a(\phi_j, \phi_i) = (f, \phi_i), \quad i = 1, \ldots, m.$$

Let

$$M_{ij} = (\phi_j, \phi_i), \quad A_{ij} = a(\phi_j, \phi_i), \quad F_i = (f, \phi_i), \quad \alpha = (\alpha_1, \ldots, \alpha_m)^T.$$

Our equations then have the form

$$M \alpha'(t) + A \alpha = F,$$

a first order system of ordinary differential equations. If we write $\psi_h$ in the form $\sum_{j=1}^{m} \beta_j \phi_j$, then we immediately get the initial condition that $\alpha_j(0) = \beta_j$. A simple example is when $V_h$ is chosen to be the space of continuous piecewise linear functions on a uniform mesh of width $h$ on $[a, b]$, and $\psi_h$ is chosen to be the interpolant of $\psi$ in this space. In that case, $\beta_j = \psi(a + jh)$.

The following error estimate is known for this semidiscrete approximation.

**Theorem 36.** If $V_h$ consists of piecewise polynomials of degree $\leq r$, the initial approximation $\psi_h$ satisfies $\|\psi - \psi_h\|_{L^2} \leq C h^{r+1} \|\psi\|_{r+1}$, and $u$ is sufficiently smooth, then for $t \geq 0$,

$$\|u(t) - u_h(t)\|_{L^2} \leq C h^{r+1} \left[ \|\psi\|_{r+1} + \int_0^t \|u_t\|_{r+1} ds \right].$$
13.2. Fully discrete schemes: Finite Differences in Time. One way to get a fully discrete scheme is to combine the use of finite elements to discretize the spatial variable with a finite difference approximation in time. For example, if we approximate \( u_t \) by the backward Euler approximation, we get the scheme: Find \( U^n \in V^0_h \), satisfying \( U^0(x) = \psi_h(x) \) and for \( n = 0, 1, \ldots, N - 1 \) (with \( T = Nk \)),

\[
([U^{n+1} - U^n]/k, v) + a(U^{n+1}, v) = (f^{n+1}, v) \quad v \in V^0_h.
\]

Using the matrices defined previously, and defining \( U^n(x) = \sum_{j=1}^{m} \alpha_j^n \phi_j(x) \), the discrete variational formulation above corresponds to the linear system

\[
(M + kA)\alpha^{n+1} = Ma^n + kF^{n+1}, \quad n = 0, 1, \ldots
\]

Another choice is the Crank-Nicholson-Galerkin method, which has the form: Find \( U^n \in V^0_h \), satisfying \( U^0(x) = \psi_h(x) \) and for \( n = 0, 1, \ldots, N - 1 \),

\[
([U^{n+1} - U^n]/k, v) + a([U^{n+1} + U^n]/2, v) = ([f^{n+1} + f^n]/2), \quad v \in V^0_h.
\]

In this case, we get the linear system

\[
(M + \frac{1}{2}kA)\alpha^{n+1} = (M - \frac{1}{2}kA)\alpha^n + k(F^{n+1} + F^n)/2, \quad n = 0, 1, \ldots
\]

For the backward Euler method, we have the following error estimate \( (t_n = nk) \).

**Theorem 37.** Under the assumptions of the previous theorem, we have

\[
\|u(t_n) - U^n\| \leq Ch^{r+1} \left[ \|\psi\|_{r+1} + \int_{0}^{t_n} \|u_t(s)\|_{r+1} \right] + k \int_{0}^{t_n} \|u_{tt}(s)\| \, ds, \quad n \geq 0.
\]

13.3. Fully discrete schemes: Finite Elements in Time. Instead of obtaining a fully discrete method by discretizing in time using finite differences, we now consider two methods for discretizing in time using finite elements. The first is the continuous Galerkin method: We let \( 0 = t_0 < t_1 < \cdots t_N = T \) be a partition of \([0, T]\) and let \( S_k \) be a finite element space consisting of continuous piecewise polynomials of degree \( \leq q \) in the time variable \( t \). Then define \( W_{h,k} \) to be the tensor product space \( W_{h,k} = V_h \otimes S_k \). For example, if \( q = 1 \) and we consider the time slab \( \Omega \times [t_{n-1}, t_n] \), we can write a function in \( W_{h,k} \) in the form

\[
w^{hk} = [(t - t_{n-1})/k]v^n_h(x) + [(t_n - t)/k]v^{n-1}_h(x).
\]

We then define \( U^{h,k} \in W_{h,k} \) such that

\[
\int_{0}^{T} \left[ (U_t^{h,k}, v_t) + a(U^{h,k}, v_t) \right] dt = \int_{0}^{T} (f, v_t) \, dt, \quad \text{for all } v \in W_{h,k}.
\]

While this appears to be a global problem in time, in fact it is a marching scheme, i.e., we can compute \( U^{h,k} \) on \([t_{n-1}, t_n] \), \( n = 1, 2, \ldots, N \), successively by solving

\[
\int_{t_{n-1}}^{t_n} \left[ (U_t^{h,k}, w) + a(U^{h,k}, w) \right] dt = \int_{t_{n-1}}^{t_n} (f, w) \, dt, \quad \text{for all } w \in V_h \otimes P^{q-1}([t_{n-1}, t_n]),
\]
where \( P^{q-1}(t_{n-1}, t_n) \) denotes the set of polynomials of degree \( \leq q - 1 \) on the interval \([t_{n-1}, t_n]\). To see this, consider the case of \( q = 1 \), piecewise linear in time. If we choose

\[
v = \begin{cases} 
  v^{n-1}(x), & 0 \leq t \leq t_{n-1} \\
  [(t_n - t)/k]v^{n-1}(x) + [(t - t_{n-1})/k]\_v(x), & t_{n-1} \leq t \leq t_n, \\
  v^n(x), & t \geq t_n,
\end{cases}
\]

then \( v_t = [v^n(x) - v^{n-1}(x)]/k \) for \( t_{n-1} \leq t \leq t_n \) and zero elsewhere. Hence, the integral from 0 to \( T \) reduces to an integral over \([t_{n-1}, t_n]\) and by choosing \( v^n(x) \) and \( v^{n-1}(x) \) appropriately, we can get any function \( w \in V_h \otimes P^0 \).

Notice also that in the case of \( q = 1 \), if we write

\[
U^{h,k} = [(t - t_{n-1})/k]U^n_h(x) + [(t_n - t)/k]U^{n-1}_h(x),
\]

then

\[
\int_{t_{n-1}}^{t_n} [(U_{h,k}^n, w) + a(U_{h,k}^n, w)] \, dt = (U^n_h(x) - U^{n-1}_h(x), w) + \frac{k}{2} [a(U^n_h(x), w) + a(U^{n-1}_h(x), w)],
\]

so we get a type of Crank-Nicholson-Galerkin scheme, where the right hand side is averaged.

A second possibility is to use the discontinuous Galerkin approach. Let

\[
w^n_+ = \lim_{t \to t_{n+1}} w(t), \quad w^n_- = \lim_{t \to t_n} w(t), \quad \text{and} \quad [w^n] = w^n_+ - w^n_-.
\]

We now define \( S_k \) as the set of all discontinuous piecewise polynomials of degree \( \leq q \) on the mesh on \([0, T]\) and \( W_{h,k} = V_h \otimes S_k \). Then we seek \( U \in W_{h,k} \) as the solution of

\[
\sum_{n=1}^N \int_{t_{n-1}}^{t_n} [(U_t, w) + a(U, w)] \, dt + \sum_{n=1}^N ([U^{n-1}], w^n_+) + (U^n_-, w^n_-) \\
= (\psi, w^n_+) + \int_0^{t_N} (f, w) \, dt, \quad \text{for all} \ w \in W_{h,k}.
\]

Since the finite element space is discontinuous in time, we can choose \( w \) so that it is non-zero only on the subinterval \([t_{n-1}, t_n]\). We again get a time marching scheme that determines \( U \) successively on \([t_{n-1}, t_n]\) by solving

\[
\int_{t_{n-1}}^{t_n} [(U_t, w) + a(U, w)] \, dt + (U^n_- - U^{n-1}_-, w^n_-) + \int_{t_{n-1}}^{t_n} (f, w) \, dt, \quad \text{for all} \ w \in W_{h,k}.
\]

On the first subinterval, we will have

\[
\int_{t_0}^{t_1} [(U_t, w) + a(U, w)] \, dt + (U^{0}_+, w^0_+) = (\psi, w^0_+) + \int_0^{t_1} (f, w) \, dt.
\]

Note that the true solution will satisfy these equations, since \( w^{n-1}_+ = w^{n-1}_- \).

In the continuous scheme, we have a single value for \( U \) at \( t = t_n \). In the discontinuous scheme, we have two values, one from the minus side and one from the plus side. So, if we choose \( q = 1 \), then on the subinterval \([t_{n-1}, t_n]\), we are writing

\[
U = [(t - t_{n-1})/k]U^n(x) + [(t_n - t)/k]U^{n-1}_+(x).
\]