1. Answer the following items.

   a) How many different words can be formed using all the letters from the word COMMITTEE?

   b) How many of those words start and end with the same letter? For example, TCEMOMEIT is one of them, but COMMITTEE itself is not.

\[ \text{a) } 9 \text{ letters with 3 repeated pairs MM, TT, EE} \]

\[ \text{Answer is } \binom{9}{2,2,2,1,1,1} = \frac{9!}{2!2!2!}. \]

\[ \text{b) } 3 \text{ ways to choose the letter pair at start and end. Once that's chosen, letters in the middle form a word with 7 letters and 2 repeated pairs.} \]

\[ \text{Answer is } 3 \cdot \binom{7}{2,2,1,1,1} = 3 \cdot \frac{7!}{2!2!}. \]
2. A store owner has 40 cell-phones that she wishes to give as presents to her 10 employees. All phones must be given out.

   a) In how many ways can the distribution happen if all phones are of the same model, each employee must receive at least 1, and Jack (the employee of the month) must receive exactly 5?

   b) In how many ways can the distribution happen if all phones are of different models, but the only restrictions are that all phones must be given out and that Jack receives exactly 5?

a) Give 5 to Jack. 35 left.
   Give 1 to each of the other employees. 26 left.
   Distribute 26 freely among 9 (Jack already has his 5).

   Answer: \( \binom{26 + 9 - 1}{9 - 1} = \binom{34}{8} \).

b) Choose the 5 models to give to Jack \( \binom{40}{5} \) ways to do that.

   Then each of the other 35 can go to any of the other 9 employees \( 9 \cdot 9 \cdot 9 \ldots \cdot 9 \) \( 35 \text{ times} \) ways to do that.

   Answer: \( \binom{40}{5} \cdot 9^{35} \).
3. To play in a certain lottery, a player chooses his bets: 4 different numbers from the set \( \{1, 2, \ldots, 15\} \). Then the winning numbers are randomly drawn: 4 random different numbers from the set \( \{1, 2, \ldots, 15\} \). If the player’s 4 bets are the same as the 4 winning numbers, he wins a big prize. If he only has 3 bets that are winning numbers, he still wins a small prize. In all other cases, no prize for him.

a) Frederic chose the numbers 1, 2, 3, 4 as bets. How many drawing results allow him to win the small prize?

b) If Peter chooses his bets randomly, what is the probability that he wins some prize?

a) The drawing must have 3 numbers from \( \{1, 2, 3, 4\} \) and 1 from \( \{5, 6, \ldots, 15\} \) in order for Frederic to get small prize.

Number of drawings is \( \binom{4}{3} \cdot \binom{11}{1} = \binom{44}{1} \).

b) What Peter chooses as his bets does not matter, since the 4 winning numbers are random anyway. We can imagine he chooses \( \{1, 2, 3, 4\} \).

As above, 44 ways to get small prize. Of course there is 1 way to get big prize (must get all numbers right). Sample space size is \( \binom{15}{4} \) (all drawing results). 

Answer: \( \frac{44}{\binom{15}{4}} \).

If we don’t make the simplification of imagining his bets are \( \{1, 2, 3, 4\} \), we can still reach the same answer. This time, sample space size is \( \binom{15}{4} \cdot \binom{4}{4} \) (his numbers and the winning numbers are chosen) and # of ways to win small prize is \( \binom{15}{4} \cdot 45 \) (45 for each of his possible \( \binom{4}{4} \) choices), giving answer of \( \frac{\binom{15}{4} \cdot 45}{\binom{15}{4} \cdot \binom{4}{4}} = \frac{45}{\binom{15}{4}} \).
4. Answer the following items. One of them may be easier by considering the complementary event, and the other by using the inclusion-exclusion principle.

a) A box has 14 balls. They are 2 red, 2 orange, 2 yellow, 2 green, 2 blue, 2 indigo and 2 violet balls. Suppose Jane randomly selects 5 balls. What is the probability that she gets at least 2 balls of the same color?

b) A box has 20 balls. They are 4 black, 4 brown, 4 gray, 4 beige and 4 white balls. Suppose Nicole randomly selects 10 balls. What is the probability that among those selected there are 4 balls of the same color?

a) Complementary event is "All 5 balls of different colors". Number of ways for this to happen is \( \binom{7}{5} \cdot 2^5 \). Sample space size is \( \binom{14}{5} \).

Choose which 5 colors

Choose which ball of each color

Answer: \( 1 - \frac{32 \binom{7}{5}}{\binom{14}{5}} \)

b) Let \( E_1 = \) "Nicole selected all 4 black balls"

\( E_2 = \) "brown"

\( E_3 = \) "gray"

\( E_4 = \) "beige"

\( E_5 = \) "white"

Inclusion-exclusion: \( P(E_1 \cup E_2 \cup \ldots \cup E_5) = \) All terms of form \( P(E_i) \) (5 terms)

- All terms of form \( P(E_i \cap E_j) \) (10 terms)

+ All terms of form \( P(E_i \cap E_j \cap E_k) \) (10 terms)

- All terms of form \( P(E_i \cap E_j \cap E_k \cap E_l) \) (5 terms)

+ \( P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5) \)

\( P(E_i \cap E_j \cap E_k) = 0 \) (no way to select all balls of colors \( i,j,k \))

\( P(E_i \cap E_j \cap E_k \cap E_l) = 0 \), \( P(E_i \cap E_j \cap E_k \cap E_l \cap E_m) = 0 \) (likewise)
5. Three regular dice are rolled.

a) What is the probability that at least one of the results is 6?

b) If we know at least one of the results was 6, what is the probability that all three results were even numbers?

\[ \text{a) Prob. of no sixes is } \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left( \frac{5}{6} \right)^3 \text{ by independence} \]

\[ \text{Prob. of at least one six is } 1 - \left( \frac{5}{6} \right)^3 \]

b) Let \( E = \) "At least one result was 6" \( F = \) "All 3 were even numbers"

\[ \text{Want } P(F|E) = \frac{P(FOE)}{P(E)} \]

It's easy to count \( |FOE| \), because this is the event "all 3 were even and at least one was 6", which means

\[ \text{FOE} = \{666, 266, 466, 626, 646, 662, 664, 226, 246, 426, 446, 262, 264, 462, 464, 622, 624, 642, 644\} \Rightarrow |FOE| = 19 \]

Then \( P(FOE) = \frac{19}{216} = \frac{19}{6^3} \) (sample space size is \( 6 \cdot 6 \cdot 6 = 6^3 \)).

\[ P(E) \text{ we know from item (a).} \]

Answer: \[ P(F|E) = \frac{19/216}{1 - \left( \frac{5}{6} \right)^3} = \left( \frac{19}{91} \right) \]
6. Box A contains a lot of blue balls only. Box B contains a lot of red balls only. Box C contains 20 blue balls and 40 red balls. Charles is going to first throw a die and then draw a random ball from one of the boxes, based on the following alternatives:

- If the result of the die is 1, he picks his ball from box A.
- If the result of the die is 2 or 3, he picks his ball from box B.
- If the result of the die is 4, 5 or 6, he picks his ball from box C.

Answer the following items:

a) What is the probability that Charles picks a blue ball?

b) If we know that Charles picked a blue ball, what is the probability that it came from box C?

\[ \text{Law of total prob.: } P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) \]

\[ = 1 \cdot \frac{1}{6} + 0 \cdot \frac{2}{6} + \frac{20}{60} \cdot \frac{1}{2} = \frac{1}{3}, \]

\[ \text{Bayes: } P(A_3|B) = \frac{P(A_3)P(B|A_3)}{P(B)} = \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{20}{60}}{\frac{1}{3}} = \frac{1}{2} \]
7. The random variable $X$ can only take the values 1, 2 and 3, with probabilities of $\frac{2}{9}, \frac{3}{9}$ and $\frac{4}{9}$ respectively.

a) Draw a graph of its cumulative distribution function.

b) Find $E[X]$ and $\text{Var}(X)$.

\[
\begin{align*}
\text{a) } F_X & \text{ has jumps at } 1, 2, 3 \text{ of sizes } \frac{2}{9}, \frac{3}{9}, \frac{4}{9}. \text{ So its values at these points are } \frac{2}{9}, \frac{2}{9} + \frac{3}{9} = \frac{5}{9}, \text{ and } \frac{5}{9} + \frac{4}{9} = 1. \\
\text{b) } E[X] & = 1 \cdot \frac{2}{9} + 2 \cdot \frac{3}{9} + 3 \cdot \frac{4}{9} = \frac{20}{9} \text{ (values of } X \text{ are } 1, 2, 3) \\
E[X^2] & = 1 \cdot \frac{2}{9} + 4 \cdot \frac{3}{9} + 9 \cdot \frac{4}{9} = \frac{50}{9} \text{ (values of } X^2 \text{ are } 1^2, 2^2, 3^2) \\
\text{Var } (X) & = E[X^2] - E[X]^2 = \frac{50}{9} - \frac{400}{81} = \frac{50}{81}
\end{align*}
\]
8. Suppose that $X$ is a Poisson random variable of parameter $\lambda = 3$.

a) What is the probability mass function of $X$?

b) What is the probability that $X \geq 2$? 

\[ f_X(i) = e^{-3} \frac{3^i}{i!}, \quad i = 0, 1, 2, \ldots \] 

(This is just the formula, nothing to be said here.) 

\[ P(X > 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-3} - 3e^{-3} = 1 - 4e^{-3} \] (And $f_X(x) = 0$ for all other values.)
9. Barbara and Dianne go target shooting. Suppose that when Barbara shoots she has a probability \( p \) of hitting the target, while Dianne’s shot hits her target with probability \( q \). Their shots are independent of each other. Suppose that they shoot simultaneously at the same target duck. If the duck is knocked over (indicating that it was hit), what is the probability that

a) both shots have hit the duck?

b) at least Barbara’s shot has hit the duck?

No animals were harmed during the making of this question. If you have studied the practice questions, you know that this is just a wooden duck.

a) Let \( B = \text{"Barbara hits"} \) and \( D = \text{"Dianne hits"} \). The event “the duck was hit” is \( B \cup D \).

\[
P(B \cap D | B \cup D) = \frac{P((B \cap D) \cap (B \cup D))}{P(B \cup D)} = \frac{P(B \cap D)}{P(B \cup D)}
\]

By independence:

\[
P(B \cap D) = P(B)P(D) = pq
\]

By incl.-excl.: \( P(B \cup D) = P(B) + P(D) - P(B)P(D) = p + q - pq \)

Answer: \( \frac{pq}{p+q-pq} \)

b) The event “at least Barbara hit” is \( B \).

\[
P(B | B \cup D) = \frac{P(B \cap (B \cup D))}{P(B \cup D)} = \frac{P(B)}{P(B \cup D)} = \frac{p}{p+q-pq}
\]
10. Answer the following two items.

a) Xerxes and Yvonne have a die and they want to play a game. Paul is their friend who's there to help. The rules are that Xerxes wins when Paul rolls an odd number, and Yvonne wins when Paul rolls the number 6. Paul is going to keep rolling the die until somebody has won. What is the probability that Xerxes wins this game?

b) Alice and Bob have a die and they want to play a game. There is no friend this time. The rules are that Alice wins when she rolls an odd number, and Bob wins when he rolls the number 6. But Alice doesn't win when Bob rolls, and Bob doesn't win when Alice rolls. Alice starts rolling the die, and if she didn't win then Bob rolls it; if he didn't win then Alice rolls it, and they continue alternating until somebody wins. What is the probability that Alice wins this game?

\[ a) \text{Add the probabilities that } X \text{ wins in each round } 1, 2, 3, 4, \ldots: \]
\[ \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \left(\frac{1}{3}\right)^2 \cdot \frac{1}{2} + \left(\frac{1}{3}\right)^3 \cdot \frac{1}{2} + \ldots = \frac{\frac{1}{2}}{1 - \frac{1}{3}} = \frac{3}{4} \]
(The \( \frac{1}{3} \) comes from prob. that nobody wins at a round)

\[ b) \text{Add the probabilities that } A \text{ wins in each round } 1, 3, 5, 7, \ldots \text{ (she cannot win when Bob plays)}: \]
\[ \frac{1}{2} + \frac{1}{2} \cdot \frac{5}{6} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{5}{6} \cdot \frac{1}{2} \cdot \frac{5}{6} \cdot \frac{1}{2} + \ldots = \frac{\frac{1}{2}}{1 - \frac{5}{12}} = \frac{6}{7} \]
11. (Extra credit - 5pts) Laura has 1000 coins, Maureen has 1001 coins. If they are all flipped, what is the probability that Laura obtains at least as many heads as Maureen?

The events

\[ H = "\text{Laura gets at least as many heads as Maureen}" \]
\[ T = "\text{Laura gets at least as many tails as Maureen}" \]

are complementary! That is, it's not possible that both happen (because Laura doesn't have at least as many coins as Maureen), and it's not possible that both do not happen (because Maureen doesn't have at least 2 coins more than Laura). This means \( H = T^c \) and vice-versa.

So \( P(H) = 1 - P(T) \).

But also \( P(H) = P(T) \) by symmetry.

So \( P(H) = 1 - P(H) \quad \Rightarrow \quad P(H) = \frac{1}{2} \)
12. (Extra credit - 2pts) After taking this exam, what would you say is the probability that you will pass this class with the grade you want? If this is a low probability, what would you say is a topic that you need to study more?

Literally any answer here (even the drawing of a Graveler, for example) gives you the 2 points.